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On local integration of Lie brackets

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Abstract. We give a direct, explicit and self-contained construction of a local Lie groupoid integrating a given Lie algebroid which only depends on the choice of a spray vector field lifting the underlying anchor map. This construction leads to a complete account of local Lie theory and, in particular, to a finite-dimensional proof of the fact that the category of germs of local Lie groupoids is equivalent to that of Lie algebroids.

1. Introduction

It is not entirely true that “*Lie algebroids are to Lie groupoids as Lie algebras are to Lie groups*” (as a general reference to Lie algebroid and Lie groupoid theory, see [11]). The main difference is that the Lie functor, which differentiates Lie groupoids to Lie algebroids, is not (essentially) surjective, i.e. there are Lie algebroids which do not admit any integration by a smooth Lie groupoid (see [1, 5]). However, the non-integrability issue is of a global nature, and it can be overcome by considering *local Lie groupoids* instead (i.e. the structure maps are defined only locally, around the units – see Section 2.1). In fact, the following holds:

Theorem 1.1. *Consider the category locLieGrpd whose objects are local Lie groupoids and whose morphisms are germs around the unit section of Lie groupoid morphisms, and the category LieAlgd of Lie algebroids. The Lie functor $\text{Lie} : \text{locLieGrpd} \rightarrow \text{LieAlgd}$ induces an equivalence between these categories.*

This result entails two parts:

- (A) Given two local Lie groupoids, every morphism between their respective Lie algebroids can be integrated to a local Lie groupoid map, and moreover, the germ around the unit section of this integration is unique.
- (B) Every Lie algebroid is (isomorphic to) the Lie algebroid of a local Lie groupoid.

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Validity of Theorem 1.1 has already been assumed since the work of Pradines in the 1960s (see [12–14]), however, part (B) was proven only later. In the setting of Poisson geometry, Coste, Dazord and Weinstein [3] constructed a local symplectic groupoid integrating a given Poisson manifold, by using the existence of symplectic realizations. Most likely, this result, when applied to the linear Poisson structure on the dual of a Lie algebroid, can be used to prove existence of local integrations in general; however, we are not aware of any written account of such an attempt. To our knowledge, the first complete proof of part (B) of Theorem 1.1 appeared in the work of Crainic and Fernandes on integrability of Lie algebroids (see [5, Corollary 5.1]). The proof is a by-product of the main construction of their paper: the *Weinstein groupoid* of a Lie algebroid. Namely, they prove that a submanifold of the Banach manifold of C^1 - A -paths, which is transverse to the A -homotopy foliation and contains the unit section, inherits a local Lie groupoid structure with multiplication coming from concatenation of A -paths.

The main result of this paper (Theorem 3.8) consists of an explicit construction of a local Lie groupoid integrating a given Lie algebroid, yielding a direct proof of part (B) of Theorem 1.1. For completeness, a proof of the more standard part (A) following similar methods is also included (Theorem 2.4).

Given a Lie algebroid $(A, [\cdot, \cdot], \rho)$, the input for our construction is a *Lie algebroid spray* consisting of a vector field $V \in \mathfrak{X}(A)$ which lifts the anchor: at $a \in A$, the tangent vector V_a projects along $q : A \rightarrow M$ to $\rho(a)$. Such a spray can be constructed as the horizontal lift of $\rho(a)$ with respect to an ordinary linear connection on A or, more generally, with respect to an A -connection on A (see [5]). The first step in the construction of the local Lie groupoid is to realize its (right-invariant) Maurer–Cartan form. Concretely, using the spray V , one builds a Lie algebroid morphism

$$\theta : T^q U \rightarrow A, \quad T^q U := \ker(dq) \subset TU,$$

over an open neighborhood $U \subset A$ of the zero section, which is a fiber-wise isomorphism. The needed “realization” θ of A was given via an explicit formula in [17]; here, a different approach to the same construction is presented. The *local spray Lie groupoid* $G_V \rightrightarrows M$ associated to V has as total space an open neighborhood $G_V \subset A$ of the zero section that is sufficiently small so that the following structure maps are defined:

- the unit section $u : M \rightarrow G_V$ is the zero section,
- the source map $\sigma : G_V \rightarrow M$ is the restriction of the bundle projection $q : A \rightarrow M$,
- the target map is $\tau = q \circ \phi_V^1$, where ϕ_V^t denotes the local flow of V ,
- the inversion map is $\iota = -\phi_V^1$,
- the multiplication $\mu : G_V \underset{\sigma}{\times} \underset{\tau}{G_V} \rightarrow U$ is defined as $\mu(a, b) = k_1$, where $k_t \in A_{q(b)}$, $t \in [0, 1]$, is the solution to the ODE

$$\frac{dk_t}{dt} = \theta_{k_t}^{-1}(\phi_V^t(a)), \quad k_0 = b.$$

Theorem 3.8 states that these maps define a local Lie groupoid with Maurer–Cartan form θ , thus, providing an explicit proof for the existence of a local Lie groupoid integrating a given Lie algebroid.

In the case of a Lie algebra $A = \mathfrak{g}$, with spray $V = 0$, Theorem 3.8 produces the local Lie group around $0 \in \mathfrak{g}$ defined by the Baker–Campbell–Hausdorff formula; see Example 3.16.

The spray construction can be used to provide explicit formulas for integrating various *geometric structures* from infinitesimal data on the Lie algebroid to multiplicative local structures on the corresponding local Lie groupoids; this will be detailed in [2].

Spray groupoids vs the space of A -paths. The constructions presented here are connected to the Crainic–Fernandes approach to integrability (see [5]). Namely, any A -connection ∇ on A and any local Lie groupoid $G \rightrightarrows M$ integrating A determine an exponential map $\exp_{\nabla} : U \rightarrow G$ as in [5], which is a diffeomorphism between an open neighborhood $U \subset A$ of M and its image. Transporting back the structure maps of G along \exp_{∇} , one obtains a local Lie groupoid. Remarkably, the germ of this local Lie groupoid is independent of G , and is intrinsic to A and ∇ . Moreover, it coincides with the local spray groupoid $G_V \rightrightarrows M$ (see Corollary 3.21); this fact is non-trivial and is supported by Theorems 3.8 and 2.4. The above argument would yield a proof of Theorem 3.8 if one assumes the existence of integrating local Lie groupoids; however, we do not follow this path, but give explicit formulas for the groupoid operations and prove directly the validity of all the axioms.

On the other hand, a possible candidate for an integrating local Lie groupoid is the one constructed in [5] from the infinite-dimensional manifold $P(A)$ of C^1 - A -paths. Namely, the path-exponential map corresponding to a Lie algebroid spray V ,

$$(1.1) \quad \widehat{\exp}_V : U \rightarrow P(A), \quad a \mapsto (t \mapsto \phi_V^t(a))_{t \in [0,1]},$$

is defined on a neighborhood $U \subset A$ of M . As shown in [5], after shrinking U , $\widehat{\exp}_V$ is transverse to the A -homotopy foliation \mathcal{F} and there is an induced local Lie groupoid structure on the set $G := \widehat{\exp}_V(U)$ integrating A (cf. [5, Corollary 5.1]) with multiplication coming from concatenation of A -paths, as in the construction of the Weinstein groupoid $G(A) = P(A)/\mathcal{F}$ associated to A .

Contents. In Section 2, general terminology related to local Lie groupoids and Lie algebroids is introduced, and Theorem 2.4 on the local integration of Lie algebroid morphisms is proven. Section 3 presents the construction of the spray groupoid (Theorem 3.8) and gives the explicit description of integrations of morphisms whose domain is a spray groupoid (Corollary 3.17).

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2. Local integration of Lie algebroid morphisms

In this section we describe an explicit integration procedure for Lie algebroid morphisms based on *Maurer–Cartan forms* and *tubular structures* for local Lie groupoids. This construction provides a proof of the fact that the Lie functor taking germs of local Lie groupoid maps to Lie algebroid morphisms is full and faithful.

2.1. Local Lie groupoids: notations and conventions. Let us begin by explaining what we mean by a local Lie groupoid. Consider a manifold G , the space of arrows, a closed

embedded submanifold $M \subset G$, the space of units, and a submersion $\sigma : G \rightarrow M$, the source map, such that $\sigma|_M = \text{id}_M$. The structure of a *local Lie groupoid* on (G, M, σ) consists of maps

- target $\tau : G_\tau \rightarrow M$,
- inversion $\iota : G_\iota \rightarrow G$,
- multiplication $\mu : G_\mu \rightarrow G$,

where $G_\tau, G_\iota \subset G$ are open neighborhoods of M , τ satisfies $\tau|_M = \text{id}_M$, $G_\mu \subset G_{\sigma \times_\tau G_\tau}$ is an open neighborhood of $M \cong \{(x, x) : x \in M\}$ such that the axioms of a Lie groupoid are satisfied locally around M . To make this precise, note that each groupoid axiom can be written as the equality of two maps defined on $G^{(k)} = \{(g_1, \dots, g_k) : \sigma(g_i) = \tau(g_{i+1})\}$; for example:

$$\begin{aligned}\tau \circ \iota &= \sigma : G \rightarrow M, \\ \tau \circ \mu &= \tau \circ \text{pr}_1 : G^{(2)} \rightarrow M, \\ \mu \circ (\mu \times \text{id}_G) &= \mu \circ (\text{id}_G \times \mu) : G^{(3)} \rightarrow G.\end{aligned}$$

We are assuming that each axiom holds on a neighborhood of $M \cong \{(x, \dots, x) : x \in M\}$ in $G^{(k)}$.

The more standard definition of a local Lie groupoid [3, 15] assumes that the target map and the inversion map are globally defined, and also that some of the axioms hold globally. However, the germ of any local groupoid (in our sense) can be represented by a local groupoid in the classical sense (of [3, 15]):

Proposition 2.1. *For a local Lie groupoid structure on (G, M, σ) there is an open neighborhood U of M in G satisfying*

$$U \subset G_\tau \cap G_\iota, \quad \iota(U) = U, \quad U_{\sigma \times_\tau U} \subset G_\mu,$$

and such that for $g, h \in U$ the following axioms hold:

$$\begin{aligned}\iota(\iota(g)) &= g, & \tau(g) &= \sigma(\iota(g)), & \mu(\tau(g), g) &= g = \mu(g, \sigma(g)), \\ \mu(g, \iota(g)) &= \tau(g), & \mu(\iota(g), g) &= \sigma(g), & \sigma(\mu(g, h)) &= \sigma(h),\end{aligned}$$

and, letting $U_\mu := \{(g, h) \in U_{\sigma \times_\tau U} : \mu(g, h) \in U\}$, for $(g, h) \in U_\mu$ the following hold:

$$(\iota(h), \iota(g)) \in U_\mu, \quad \iota(\mu(g, h)) = \mu(\iota(h), \iota(g)), \quad \tau(\mu(g, h)) = \tau(g),$$

and if the elements (g, h) , $(\mu(g, h), k)$ and (h, k) belong to U_μ , then also $(g, \mu(h, k))$ belongs to U_μ and we have

$$\mu(g, \mu(h, k)) = \mu(\mu(g, h), k).$$

Moreover, the collection of such open sets U forms a basis of neighborhoods of M in G .

Proof. The proof is straightforward: start with any open neighborhood of M in G and shrink it step by step such that it satisfies all the properties. For example, to ensure that $\iota(U) = U$, and $\iota^2(g) = g$, take U of the form $U = V \cap \iota(V)$, where V is small enough such that $V, \iota(V) \subset G_\iota$ and $\iota^2|_V = \text{id}_V$. An important point is to notice that open sets of the form $U^{(k)} = G^{(k)} \cap U^k$, where $U \subset G_\tau \cap G_\iota$ is an open neighborhood M , form a basis of neighborhoods of M in $G^{(k)}$; this follows because $G^{(k)}$ has the subset topology of the product topology on $(G_\tau \cap G_\iota)^k$. \square

Any local Lie groupoid on (G, M, σ) has an associated Lie algebroid $A \rightarrow M$, which is constructed exactly as in the case of ordinary Lie groupoids. As a vector bundle $A = \ker(d\sigma)|_M$ the anchor of A is $\rho = d\tau|_A : A \rightarrow TM$, and the Lie bracket is obtained by identifying sections of A with right invariant vector fields on G .

Convention 2.2. In order to keep notation as simple as possible, we will denote maps which are defined only in a neighborhood of M with a dashed arrow \dashrightarrow . In particular, the structure maps of a local groupoid on (G, M, σ) are written as $\tau : G \dashrightarrow M, \iota : G \dashrightarrow G$ and $\mu : G \times_{\sigma} G \dashrightarrow G$.

2.2. The Maurer–Cartan form. We recall the definition of the right-invariant Maurer–Cartan form θ_G of a local Lie groupoid G . Denote the vertical bundle of σ by $T^\sigma G := \ker(d\sigma)$. Given a smooth path $(-\epsilon, \epsilon) \rightarrow G, t \mapsto g_t$, such that $\sigma(g_t) = x$ for all t , its derivative at $t = 0$ gives an element in $T^\sigma G$:

$$\left. \frac{d}{dt} g_t \right|_{t=0} \in T^\sigma_{g_0} G,$$

and all elements in $T^\sigma G$ can be represented in this way. The *right-invariant Maurer–Cartan form* (or Maurer–Cartan form, for brevity) of G , denoted by θ_G , is the following vector bundle map covering the target map:

$$(2.1) \quad \begin{array}{ccc} T^\sigma G & \xrightarrow{\theta_G} & A, \\ \downarrow & & \downarrow \\ G & \xrightarrow{\tau} & M \end{array} \quad \theta_G \left(\left. \frac{d}{dt} g_t \right|_{t=0} \right) := \left. \frac{d}{dt} g_t g_0^{-1} \right|_{t=0} \in A_{\tau(g_0)}.$$

Notice that the domain of θ_G is of the form $T^\sigma U$, where $U \subset G$ is a neighborhood of M , and that it is right invariant:

$$\theta_G \left(\left. \frac{d}{dt} g_t h \right|_{t=0} \right) = \theta_G \left(\left. \frac{d}{dt} g_t \right|_{t=0} \right)$$

for any h such that $\tau(h) = \sigma(g_t) = x$.

The Maurer–Cartan form θ_G is a Lie algebroid map between the vertical distribution $T^\sigma G$ and A (this condition can be written in the form of a Maurer–Cartan equation, see [8]). This Lie algebroid morphism integrates to a local groupoid map, called the *division map*, given by

$$(2.2) \quad \begin{array}{ccc} G \times_{\sigma} G & \xrightarrow{\delta} & G, \\ \Downarrow & & \Downarrow \\ G & \xrightarrow{\tau} & M \end{array} \quad \delta(g, h) = gh^{-1}.$$

Remark 2.3. The division map can be factorized as the following composition of groupoid maps:

$$(2.3) \quad \begin{array}{ccccccc} G \times_{\sigma} G & \hookrightarrow & \sigma!G & \xrightarrow{\text{Ad}_\beta} & \sigma!G & \xrightarrow{\text{pr}_2} & G \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ G & \xrightarrow{\text{id}} & G & \xrightarrow{\iota} & G & \xrightarrow{\sigma} & M, \end{array}$$

where $\sigma^!G = G_{\sigma} \times_{\tau} G_{\sigma} \times_{\sigma} G$ is the pullback groupoid of G via the submersion $\sigma : G \rightarrow M$ with source $\bar{\sigma}(l, g, k) = k$, target $\bar{\tau}(l, g, k) = l$, and multiplication $(l, g, k)(k, h, e) = (l, gh, e)$ (see e.g. [11]); where the inclusion $G_{\sigma} \times_{\sigma} G \hookrightarrow \sigma^!G$ maps (g, h) to $(g, 1_{\sigma(g)}, h)$; and, using the canonical local bisection $\beta : G \dashrightarrow \sigma^!G$, $\beta(g) = (g^{-1}, g, g)$, the local groupoid map Ad_{β} is conjugation by β : $\text{Ad}_{\beta}(l, g, k) = \beta(l)(l, g, k)\beta(k)^{-1} = (l^{-1}, l g k^{-1}, k^{-1})$.

2.3. Tubular structures. In this subsection we discuss special tubular neighborhoods on local Lie groupoids.

Let G be a local groupoid. Along the unit section $M \subset G$ the fibers of the source map give a natural decomposition of the tangent space $TG|_M = TM \oplus A$; in particular, the normal bundle of M in G is canonically identified with A .

A *tubular structure* on the local groupoid G is a tubular neighborhood of M in G along the source fibers, i.e. an open embedding of bundles

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & G \\ q \downarrow & & \downarrow \sigma \\ M & \xrightarrow{\text{id}_M} & M, \end{array}$$

where $q : A \rightarrow M$ denotes the projection, such that $\sigma \circ \varphi = q$, $\varphi(0_x) = x$ for $x \in M$, and

$$\left. \frac{d}{dt} \varphi(ta) \right|_{t=0} = (0, a) \in T_x M \oplus A_x = T_x G$$

for all $a \in A$, where $x = q(a)$.

Note that φ is defined on a neighborhood of M in A . A tubular structure induces a scalar multiplication on G along the source fibers

$$(t, g) \mapsto t g := \varphi(t\varphi^{-1}(g)),$$

which is defined on a neighborhood of $\{0\} \times M$ in $\mathbb{R} \times G$. Moreover, the germ of φ around M can be recovered from this operation: $\varphi(\left. \frac{d}{dt} t g \right|_{t=0}) = g$. Clearly, any local Lie groupoid admits a tubular structure.

2.4. Local integration of Lie algebroid morphisms. A *local Lie groupoid map* between local Lie groupoids G_1 and G_2 is a smooth map

$$F : G_1 \dashrightarrow G_2$$

defined on a neighborhood of $M_1 \subset G_1$ that restricts to a map between the units

$$F_M := F|_{M_1} : M_1 \rightarrow M_2,$$

and that is multiplicative around M_1 , i.e. $F(gh) = F(g)F(h)$ for all (g, h) in an open neighborhood of M_1 in $G_1_{\sigma} \times_{\tau} G_1$. The local groupoid map F induces a Lie algebroid morphism, denoted by

$$\begin{array}{ccc} A_1 & \xrightarrow{F_A} & A_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{F_M} & M_2, \end{array}$$

where $A_1 \rightarrow M_1$ and $A_2 \rightarrow M_2$ are the respective Lie algebroids and $F_A = dF|_{A_1}$. In this case, we say that F *integrates* the Lie algebroid morphism F_A .

Local Lie groupoids and germs of local Lie groupoid maps form a category. The Lie functor, taking G to its Lie algebroid A , and a germ of a local Lie groupoid map F to the induced Lie algebroid map F_A , is an equivalence to the category of Lie algebroids. This fact seems to be well known; however, a complete reference is missing from the literature (see the bibliographical comments in the introduction). Our constructions below provide a detailed proof of this result. First, we show that the functor is full and faithful, i.e. Lie algebroid morphisms can be integrated to local Lie groupoid maps. Moreover, by using a tubular structure on the domain, the integration can be made quite explicit. That the functor is essentially surjective will be shown in the next section, where we present a construction of a local integration of a Lie algebroid.

To address the integration of morphisms, let G_i be local Lie groupoids with Lie algebroids A_i , $i = 1, 2$, and let $f : A_1 \rightarrow A_2$ be a Lie algebroid morphism covering $f_M : M_1 \rightarrow M_2$. Fixing a tubular structure on G_1 and an element $g \in G_1$, the map f together with the Maurer–Cartan forms θ_{G_i} give rise to the following ODE:

$$(2.4) \quad \theta_{G_2} \left(\frac{d}{dt} k_t \right) = f \circ \theta_{G_1} \left(\frac{d}{dt} (tg) \right), \quad k_0 = f_M(\sigma_1(g)),$$

for a curve $t \mapsto k_t \in \sigma_2^{-1}(f_M(\sigma_1(g))) \subset G_2$.

Theorem 2.4. *Let G_1 and G_2 be two local Lie groupoids with Lie algebroids A_1 and A_2 , respectively, and let $f : A_1 \rightarrow A_2$ be a Lie algebroid morphism. There exists a local Lie groupoid map $F : G_1 \dashrightarrow G_2$ integrating f , and any two such integrations coincide around the unit section.*

Explicitly, fixing a tubular structure on G_1 , we have that $F(g) = k_1$, where k_t is the solution of the ODE (2.4) and $g \in G_1$ is taken close enough to the unit section such that the solution is defined for $t \in [0, 1]$.

As an immediate consequence of Theorem 2.4, we recover the well-known fact that the germ of an integration is essentially unique:

Corollary 2.5. *Any two local Lie groupoids integrating a given Lie algebroid are isomorphic around the unit section.*

An essential role in the proof of the above theorem is played by the following extract (Lemma 2.7 below) of [5, Proposition 1.3] which describes variations of Lie algebroid paths. Before stating this result, we recall from [5] that, given a Lie algebroid A , a time-dependent section $(t, x) \mapsto \alpha(t, x) \in A_x$ defines time-dependent flow of Lie algebroid automorphisms $\phi_\alpha^{t,s} \in \text{Aut}(A)$ induced by the inner derivation $\text{ad}_\alpha = [\alpha, -]$ of A :

$$(2.5) \quad \frac{d}{dt} (\phi_\alpha^{t,s})^* \beta = (\phi_\alpha^{t,s})^* [\alpha^t, \beta] \quad \text{for all } \beta \in \Gamma(A), \quad \phi_\alpha^{s,s} = \text{id},$$

where the pullback of sections is defined as

$$(\phi_\alpha^{t,s})^* \beta = \phi_\alpha^{s,t} \circ \beta \circ \phi_{\rho(\alpha)}^{t,s}.$$

The flow has also the multiplicative property

$$\phi_\alpha^{t,s} \circ \phi_\alpha^{s,u} = \phi_\alpha^{t,u}.$$

Remark 2.6. When A is the Lie algebroid of a local groupoid G , the flow of α can be naturally integrated to a flow by local Lie groupoid maps as follows (see [5]). Let $\phi_{\alpha^R}^{t,s}$ denote the flow of the time-dependent right invariant vector field $\alpha^R \in \mathfrak{X}(G)$ induced by α . This flow gives rise to a family of bisections $\phi_{\alpha^R}^{t,s} := \phi_{\alpha^R}^{t,s}|_M$ of G by restricting to $M \subset G$. Then the local Lie groupoid map $\text{Ad}_{\phi_{\alpha^R}^{t,s}} : G \dashrightarrow G$ defined by conjugation by $\phi_{\alpha^R}^{t,s}$ integrates the flow of the time dependent section α of A :

$$(2.6) \quad d(\text{Ad}_{\phi_{\alpha^R}^{t,s}})|_A = \phi_{\alpha^R}^{t,s}.$$

We now state the results on variations of algebroid paths that we need in the proof of Theorem 2.4.

Lemma 2.7 ([5]). *Let $q : A \rightarrow M$ be a Lie algebroid. Consider a smooth map*

$$a : [0, 1] \times J \rightarrow A, \quad (t, \epsilon) \mapsto a_{\epsilon}(t),$$

where J is an interval, and denote the base map by $\gamma_{\epsilon}(t) := q \circ a_{\epsilon}(t)$. Assume that $\gamma_{\epsilon}(0) = x$ for all $\epsilon \in J$, where $x \in M$ is fixed, and that a_{ϵ} is an A -path for all $\epsilon \in J$, i.e.

$$\rho \circ a_{\epsilon}(t) = \frac{d}{dt} \gamma_{\epsilon}(t),$$

where $\rho : A \rightarrow TM$ denotes the anchor. Then there exists a unique smooth map

$$b : [0, 1] \times J \rightarrow A, \quad (t, \epsilon) \mapsto b_{\epsilon}(t),$$

such that $b_{\epsilon}(t) \in A_{\gamma_{\epsilon}(t)}$, $b_{\epsilon}(0) = 0$, and such that the following is a Lie algebroid morphism:

$$f_{a,b} : T([0, 1] \times J) \rightarrow A, \quad f_{a,b} = a_{\epsilon}(t)dt + b_{\epsilon}(t)d\epsilon.$$

Moreover, b can be explicitly constructed as follows:

- (1) Let $\alpha_{\epsilon}(t)$ be a family of compactly supported sections of A depending smoothly on $(t, \epsilon) \in [0, 1] \times J$ such that $\alpha_{\epsilon}(t, \gamma_{\epsilon}(t)) = a_{\epsilon}(t)$. Denote the flow of the time dependent section α_{ϵ} by $\phi_{\alpha_{\epsilon}}^{t,s} : A_x \rightarrow A_y$, $y := \phi_{\rho(\alpha_{\epsilon})}^{t,s}(x)$. Then b is given by the integral formula

$$(2.7) \quad b_{\epsilon}(t) = \int_0^t \phi_{\alpha_{\epsilon}}^{t,s}(\gamma_{\epsilon}(s)) \frac{d\alpha_{\epsilon}}{d\epsilon}(s, \gamma_{\epsilon}(s)) ds.$$

- (2) If A is integrable by a Lie groupoid G , and $g : [0, 1] \times J \rightarrow G$ is the solution to the ODE

$$\theta_G \left(\frac{d}{dt} g_{\epsilon}(t) \right) = a_{\epsilon}(t), \quad g_{\epsilon}(0) = 1_x,$$

where θ_G denotes the Maurer–Cartan form of G , then b is given by

$$b_{\epsilon}(t) = \theta_G \left(\frac{d}{d\epsilon} g_{\epsilon}(t) \right).$$

We now proceed to the proof of Theorem 2.4.

Proof of Theorem 2.4. We divide the proof into three steps.

Step 1. In this step, we show that the germ of F around the unit section is unique. Assume that $F : G_1 \dashrightarrow G_2$ is a local groupoid map integrating f . Consider an open neighborhood U of M_1 in G_1 such that for all $g \in U$ and all $s, t \in [0, 1]$, we have that the following equality holds and, of course, all terms involved in the equality are defined:

$$F((sg)(tg)^{-1}) = F(sg)F(tg)^{-1}.$$

Using that $dF|_{A_1} = F_A = f$, applying $\frac{d}{ds}|_{s=t}$ to this equality, we obtain

$$f \circ \theta_{G_1} \left(\frac{d}{dt}(tg) \right) = \theta_{G_2} \left(\frac{d}{dt}F(tg) \right).$$

This shows that the curve $t \mapsto F(tg)$ is the solution $t \mapsto k_t$ to the ODE (2.4). The fact that $\theta_{G_2} : T_h^\sigma G_2 \rightarrow A_{2,\tau(h)}$ is a linear isomorphism implies uniqueness of the solution to (2.4), and therefore, uniqueness of F ; namely $F(g) = k_1$, where k_t is the (unique) solution of (2.4).

Step 2. We now go back to the definition of F and show that, on an open neighborhood of M_1 , the map F satisfies $F^*(\theta_{G_2}) = f \circ \theta_{G_1}$; in other words we have the following commutative diagram of Lie algebroid maps:

$$(2.8) \quad \begin{array}{ccc} T^\sigma G_1 & \xrightarrow{\theta_{G_1}} & A_1 \\ dF \downarrow & & \downarrow f \\ T^\sigma G_2 & \xrightarrow{\theta_{G_2}} & A_2. \end{array}$$

First, we observe that for $g = x \in M_1$ the solution to the ODE is given by the constant path $k_t = x$, which is defined for all $t \in \mathbb{R}$. Therefore, there is a neighborhood U of M_1 in G , such that for all $g \in U$ the ODE has a solution defined up to time $t = 1$. On U we thus have $F(g) := k_1$ well defined.

Let $g, v \in G_1$ be such that $\sigma_1(g) = \sigma_1(v) = x$. By taking g in a sufficiently small neighborhood of M_1 and a sufficiently small open interval J containing 0, the tubular structure can be used to identify g and elements of the form ϵv , where $\epsilon \in J$, as elements of $A_1|_x$. This, in turn, allows to make sense of the expression $t(g + \epsilon v)$ for any $t \in [0, 1]$, obtaining then the following map:

$$v : [0, 1] \times J \rightarrow G_1, \quad (t, \epsilon) \mapsto v_\epsilon(t) = t(g + \epsilon v).$$

Note that

$$dv_\epsilon(t) = \frac{dv_\epsilon(t)}{dt} dt + \frac{dv_\epsilon(t)}{d\epsilon} d\epsilon$$

is a Lie algebroid morphism from $T([0, 1] \times J)$ to $T^\sigma G_1$. Composing it with the Lie algebroid morphism $\theta_{G_1} : T^\sigma G_1 \dashrightarrow A_1$ and then with $f : A_1 \rightarrow A_2$, we obtain that the following is also a Lie algebroid morphism:

$$(2.9) \quad f \circ \theta_{G_1} \left(\frac{d}{dt}v_\epsilon(t) \right) dt + f \circ \theta_{G_1} \left(\frac{d}{d\epsilon}v_\epsilon(t) \right) d\epsilon : T([0, 1] \times J) \rightarrow A_2.$$

We now want to compute the components of (2.9) in terms of F . Denoting by $k_t(g) \in G_2$ the solution of (2.4), we notice the following rescaling property:

$$k_{ts}(g) = k_s(tg) \quad \text{for } t, s \in [0, 1] \text{ and } g \in U.$$

This follows by applying the chain rule on both sides of (2.4) and by uniqueness of the solution of (2.4) for small g (recall Step 1). Combining this property for $s = 1$ with the definition of F , we get $F(v_\epsilon(t)) = F(t(g + \epsilon v)) = k_t(g + \epsilon v)$ and the first component of (2.9) is then given by

$$f \circ \theta_{G_1} \left(\frac{d}{dt} v_\epsilon(t) \right) = \theta_{G_2} \left(\frac{d}{dt} F(v_\epsilon(t)) \right).$$

Since $v_\epsilon(0) = x$, the second component in (2.9) vanishes for $t = 0$, and therefore, by part (2) of Lemma 2.7 with $G = G_2$ and $g_\epsilon(t) = F(v_\epsilon(t))$, it is given by

$$f \circ \theta_{G_1} \left(\frac{d}{d\epsilon} v_\epsilon(t) \right) = \theta_{G_2} \left(\frac{d}{d\epsilon} F(v_\epsilon(t)) \right).$$

In particular, for $t = 1$ and $\epsilon = 0$, we obtain

$$f \circ \theta_{G_1} \left(\frac{d}{d\epsilon} (g + \epsilon v) \Big|_{\epsilon=0} \right) = \theta_{G_2} \left(\frac{d}{d\epsilon} F(g + \epsilon v) \Big|_{\epsilon=0} \right).$$

Since v was arbitrary in the source fiber of g , this is precisely the commutativity of (2.8) at g .

Step 3. We show that F is multiplicative. Let $g, h \in G_1$ be composable arrows that are close enough to the unit section. We show that the curves $t \mapsto F(tg)F(h)$ and $t \mapsto F((tg)h)$, which start at $F(h)$, both satisfy the ODE

$$\theta_{G_2} \left(\frac{d}{dt} k_t \right) = f \circ \theta_{G_1} \left(\frac{d}{dt} (tg) \right);$$

hence the two curves must coincide for $t = 1$, i.e. $F(gh) = F(g)F(h)$. For this we use right-invariance of both Maurer–Cartan forms and the relation $f \circ \theta_{G_1} = F^*(\theta_{G_2})$ to obtain

$$\theta_{G_2} \left(\frac{d}{dt} F((tg)h) \right) = f \circ \theta_{G_1} \left(\frac{d}{dt} (tg)h \right) = f \circ \theta_{G_1} \left(\frac{d}{dt} (tg) \right)$$

and

$$\theta_{G_2} \left(\frac{d}{dt} F(tg)F(h) \right) = \theta_{G_2} \left(\frac{d}{dt} F(tg) \right) = f \circ \theta_{G_1} \left(\frac{d}{dt} (tg) \right). \quad \square$$

Remark 2.8. The role of the tubular structure on G_1 above is to provide a path $g(t)$ joining $\sigma(g)$ to g within the underlying source fiber, namely, $g(t) = tg$. For any such path, there is an associated ODE generalizing (2.4) which (uniquely) characterizes the value $F(g)$ of an integration of f , for g close enough to the identities.

2.5. Local integration of cochains. In this subsection we show how a tubular structure can be used to integrate Lie algebroid cochains to local Lie groupoid cochains. Let $G \rightrightarrows M$ be a local Lie groupoid endowed with a tubular structure with scalar multiplication $g \mapsto tg$, and let A denote its Lie algebroid. Recall from [4, 10, 16] that Lie algebroid p -cochains are sections of $\wedge^p A^*$, while local Lie groupoid cochains are smooth functions defined on (small) composable arrows $f : G^{(p)} \dashrightarrow \mathbb{R}$. The *van Est map*, applied to such a local Lie groupoid cochain f , is defined as follows [16]:

$$\text{ve}(f)(a_1, \dots, a_p)(x) = \sum_{\pi \text{ perm.}} \text{sign}(\pi) D_{a_{\pi(p)}} \cdots D_{a_{\pi(1)}} f|_x, \quad a_i \in \Gamma(A),$$

where $D_a f : G^{(p-1)} \dashrightarrow \mathbb{R}$ is defined by

$$D_a f(g_2, \dots, g_p) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} f(h(\epsilon), g_2, \dots, g_p)$$

for any curve $h(\epsilon) \in \sigma^{-1}(\tau(g_2))$ starting at $h(0) = \tau(g_2)$ with velocity $h'(0) = a(\tau(g_2))$. If f is *normalized*, i.e. $f(g_1, \dots, g_p) = 0$ whenever one of the g_i is a unit, then $\text{ve}(f)(a_1, \dots, a_p)$ is $C^\infty(M)$ -linear in the sections $a_i \in \Gamma(A)$, so that the map $\text{ve}(f)$ indeed defines an element of $\Gamma(\wedge^p A^*)$.

We now use the tubular structure to produce a right inverse for ve . Given a p -tuple $(g_1, \dots, g_p) \in G^{(p)}$ of small composable arrows in G , we recursively define maps

$$\gamma_{g_1, \dots, g_p} : I^p \rightarrow \sigma^{-1}(\sigma(g_p)) \subset G \quad \text{with } I = [0, 1],$$

by

$$\gamma_{g_1}(t_1) = t_1 g_1, \quad \gamma_{g_1, \dots, g_p}(t_1, \dots, t_p) = t_p \mu(\gamma_{g_1, \dots, g_{p-1}}(t_1, \dots, t_{p-1}), g_p).$$

Let the map $\Psi : \Gamma(\wedge^p A^*) \rightarrow \{f : G^{(p)} \dashrightarrow \mathbb{R}\}$ be defined by

$$\begin{aligned} \Psi(\alpha)(g_1, \dots, g_p) &= \int_{I^p} \alpha(\partial_1 \gamma_{g_1, \dots, g_p}, \dots, \partial_p \gamma_{g_1, \dots, g_p}) dt_1 \dots dt_p \\ &= \int_{I^p} (\theta \circ d\gamma_{g_1, \dots, g_p})^* \alpha, \end{aligned}$$

where $\theta := \theta_G$ is the Maurer–Cartan form of G and $\partial_j \gamma$ is the partial derivative of

$$(t_1, \dots, t_p) \mapsto \gamma(t_1, \dots, t_p)$$

with respect to its j -th variable. Notice that, when one of the g_i is a unit, then

$$\Psi(\alpha)(g_1, \dots, g_p) = 0$$

since γ_{g_1, \dots, g_p} becomes a *degenerate* p -cube as it factors through a map $I^p \rightarrow I^{p-1}$ and $\Omega^p(I^{p-1}) = 0$. Hence, $\Psi(\alpha)$ is normalized for any $\alpha \in \Gamma(\wedge^p A^*)$.

Example 2.9 (1-cochains). Let $\alpha \in \Gamma(A^*)$ be an algebroid 1-cochain. Since $\gamma_g(t) = tg$, we have

$$(2.10) \quad \Psi(\alpha)(g) = \int_0^1 \alpha \circ \theta \left(\frac{d}{dt} tg \right) dt.$$

To differentiate this local groupoid 1-cochain, we consider the curve $h(\epsilon) = \epsilon a$ so that

$$\text{ve}(\Psi(\alpha))(a) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_0^1 \alpha \circ \theta \left(\frac{d}{dt} t\epsilon a \right) dt = \alpha \circ \theta \left(\frac{d}{du} ua \right) \Big|_{u=0} = \alpha(a).$$

When α is *exact*, namely $\alpha = \rho^* df$ ($= d_A f$ below) for $f \in C^\infty(M)$, then

$$\begin{aligned} \Psi(d_A f)(g) &= \int_0^1 df \circ \rho \circ \theta \left(\frac{d}{dt} tg \right) dt \\ &= \int_0^1 \frac{d}{dt} f(\tau(tg)) dt = f(\tau(g)) - f(\sigma(g)), \end{aligned}$$

where, in the last step, we used that the tubular structure φ satisfies $\sigma \circ \varphi = q : A \dashrightarrow M$.

In [16, Theorem 1.3] the authors prove that the van Est map is an isomorphism in degree one cohomology if G is source simply connected, using an explicit formula $\Lambda(\alpha)$ for the inverse

$$\Lambda(\alpha)(g) = \int_0^1 \alpha^R \left(\frac{d}{dt} g(t) \right) dt,$$

where the integration is over any path $g(t)$ in the σ -fiber joining $1_{\sigma(g)}$ and g , and where $\alpha^R \in \Gamma((T^\sigma G)^*)$ is the right invariant foliated 1-form induced by α .¹⁾ Of course the path $g(t) : t \mapsto tg$ does the job and we recover formula (2.10).

The two properties described in the previous example generalize to arbitrary p -cochains:

Proposition 2.10. *The map Ψ defined above satisfies the following:*

- (1) $\Psi(d_A \alpha) = -\delta \Psi(\alpha)$, where d_A denotes the Chevalley–Eilenberg differential associated to A and δ the differentiable cohomology differential for G (see [4] and the proof below).
- (2) $ve(\Psi(\alpha)) = \alpha$ for all $\alpha \in \Gamma(\wedge^p A^*)$.

Proof. To show (1), one first observes that $\theta \circ d\gamma_{g_1, \dots, g_p} : TI^p \rightarrow A$ is an algebroid morphism, being a composition of such; hence, it intertwines d_A and the de Rham differential on $\Omega(I^p)$. Applying Stokes’ theorem in the computation of $\Psi(d_A \alpha)$ yields

$$\Psi(d_A \alpha)(g_1, \dots, g_{p+1}) = \int_{\partial I^{p+1}} (\theta \circ d\gamma_{g_1, \dots, g_{p+1}})^* \alpha.$$

The above integral splits as a sum over the faces of the $(p + 1)$ -cube I^{p+1} which are determined by $t_i = 0$ or $t_i = 1$, for $i = 1, \dots, p + 1$, each one with its induced orientation. Let us then consider the restriction of $\gamma_{g_1, \dots, g_{p+1}}$ to such faces following its definition. The face $t_1 = 0$ yields the (negatively oriented) p -cube $\gamma_{g_2, \dots, g_{p+1}}$ since zero is mapped to the groupoid units through the tubular structure. The face $t_1 = 1$ yields the (positively oriented) p -cube $\gamma_{\mu(g_1, g_2), g_3, \dots, g_{p+1}}$. For $1 < i \leq p + 1$, setting $t_i = 0$ yields a degenerate p -cube since the underlying map does not depend on t_j for $j < i$, the corresponding integrand thus vanishes and hence these terms do not contribute. The faces $t_i = 1$ for $1 < i < p + 1$ yield, using associativity of μ , the p -cubes $\gamma_{g_1, \dots, \mu(g_i, g_{i+1}), \dots, g_{p+1}}$ in which g_i is replaced by $\mu(g_i, g_{i+1})$. Finally, the face $t_{p+1} = 1$ yields $(t_1, \dots, t_p) \mapsto \mu(\gamma_{g_1, \dots, g_p}(t_1, \dots, t_p), g_{p+1})$ which, since θ is right invariant (and taking into account the induced orientation), contributes to the boundary integral as $(-1)^{p+2} \Psi(\alpha)(g_1, \dots, g_p)$.

In conclusion, the sum over boundary terms yields directly $-\delta \Psi(\alpha)$:

$$\begin{aligned} \int_{\partial I^{p+1}} (\theta \circ d\gamma_{g_1, \dots, g_{p+1}})^* \alpha &= -\Psi(\alpha)(g_2, \dots, g_{p+1}) \\ &\quad + \sum_{i=1}^p (-1)^{i+1} \Psi(\alpha)(g_1, \dots, \mu(g_i, g_{i+1}), \dots, g_{p+1}) \\ &\quad + (-1)^{p+2} \Psi(\alpha)(g_1, \dots, g_p) \\ &= -(\delta \Psi(\alpha))(g_1, \dots, g_{p+1}), \end{aligned}$$

and hence (1) is proven.

¹⁾ Due to the conventions for multiplication in [16], left translation in [16] becomes right translation with our conventions.

For (2), we need to recursively differentiate $f = \Psi(\alpha)$ along sections $a_1, \dots, a_p \in \Gamma(A)$. Since both sides of the desired equality are multilinear forms on A , it is enough to show that they agree when evaluated on small enough $a_1, \dots, a_p \in \Gamma(A)$. For such sections, the underlying differentiation is carried out in Lemma A.1 of Appendix A. Taking $k = p$ in Lemma A.1 and noting that by the properties of the tubular structure we have $v^a := \theta(\frac{d}{dt}ta|_{t=0}) = a$ (cf. Definition A.3), we obtain

$$D_{a_p} \cdots D_{a_1} \Psi(\alpha) = \frac{1}{p!} \alpha(a_1, \dots, a_p) \in C^\infty(M).$$

Then (2) follows directly from the above as the $1/p!$ cancels the sum over permutations in $\text{ve}(\Psi(\alpha))$ since α is already alternating. □

The map Ψ above thus provides an explicit geometric description for integration of cochains, inverting differentiation through the van Est map and descending to cohomology. The fact that the van Est map yields an isomorphism in cohomology for any local Lie groupoid was shown in [10] where, moreover, a retraction along the source fibers is shown to induce an inverse for ve at the level of cochains by means of homological perturbation theory.

3. The local Lie groupoid associated to a spray

In this section we describe an explicit, finite-dimensional construction of a local Lie groupoid integrating a given Lie algebroid, which uses a Lie algebroid spray. This construction provides a proof of the fact that the Lie functor, taking local Lie groupoids to Lie algebroids, is essentially surjective, thus, by the previous section, an equivalence of categories.

3.1. Lie algebroid sprays. We first recall the following notion:

Definition 3.1. Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid. Let $q : A \rightarrow M$ denote the bundle projection, and let $m_t : A \rightarrow A$ denote scalar multiplication by $t \in \mathbb{R}$. A *Lie algebroid spray* for A is a vector field V on the manifold A which satisfies:

- (1) V is homogeneous of degree one: $m_t^*(V) = tV$ for all $t \neq 0$.
- (2) V lifts the anchor of A in the sense that $dq(V_a) = \rho(a) \in TM$ for all $a \in A$.

The local flow of the spray will be denoted by $\phi_V^t \equiv \phi_V^{t,0}$. The second condition defining V implies that $a(t) := \phi_V^t(a) \in A$ satisfies the condition

$$\frac{d}{dt}q(a(t)) = \rho(a(t))$$

of an A -path (in the sense of [5]) for any initial condition $a \in A$.

Remark 3.2. (1) The first condition implies that V vanishes along the zero section of A . Hence, its flow is defined up to time $t = 1$ on an open neighborhood of the zero section. We thus obtain the map $\widehat{\text{exp}}_V : A \dashrightarrow P(A)$, $a \mapsto (t \mapsto \phi_V^t(a))$ discussed in the last part of the introduction.

(2) A Lie algebroid spray is equivalent to a torsion-free A -connection on A (for these notions, see [7]). Let x^i denote local coordinates on M and u^a fiberwise linear coordinates on A . Then any spray has the local expression

$$V = \rho_a^i(x)u^a\partial_{x^i} + \Gamma_{ab}^c(x)u^au^b\partial_{u^c},$$

where ρ_a^i denote the local coefficients of the anchor map $\rho : A \rightarrow TM$ and Γ_{ab}^c define the Christoffel symbols for the A -connection. The torsion vanishes since only the symmetric part of these symbols contributes non-trivially to V .

(3) Sprays always exist: for example, define $V_a \in T_aA$ to be the horizontal lift of $\rho(a) \in T_{q(a)}M$ with respect to a fixed linear (TM -)connection ∇ on A .

(4) That V is homogeneous of degree one is equivalent to the following property of its local flow:

$$(3.1) \quad \phi_V^t(sa) = s\phi_V^{st}(a).$$

(5) The flow of a spray fixes points of the zero section $M \subset A$, namely $\phi_V^t(0_x) = 0_x$. Using the natural decomposition $TA|_M = TM \oplus A$ into vectors tangent to the zero section plus vertical vectors, the differential of ϕ_V^t at such points yields

$$d\phi_V^t : TA|_M = TM \oplus A \rightarrow TM \oplus A, \quad (u, a) \mapsto (u + t\rho(a), a),$$

where $d\phi_V^t(a)$ is computed using equation (3.1) and the condition $\frac{d}{dt}q(\phi_V^t(a)) = \rho(\phi_V^t(a))$:

$$\begin{aligned} \left. \frac{d}{ds}\phi_V^t(sa) \right|_{s=0} &= \left. \frac{d}{ds}s\phi_V^{ts}(a) \right|_{s=0} \\ &= \phi_V^0(a) + \left. \frac{d}{ds}q(\phi_V^{ts}(a)) \right|_{s=0} \\ &= a + \left. \frac{d}{ds}q(\phi_V^s(ta)) \right|_{s=0} \\ &= a + \rho(ta). \end{aligned}$$

(6) Consider the pullback Lie algebroid of A via the submersion $q : A \rightarrow M$ (see e.g. [9])

$$q^!A := TA \times_{TM} A = \{(u, a) \in TA \times A : dq(u) = \rho(a)\}.$$

The Lie algebroid $q^!A$ is an extension of A by the vertical foliation $T^qA := \ker(dq)$:

$$\begin{array}{ccccc} T^qA & \hookrightarrow & q^!A & \xrightarrow{\text{pr}_2} & A \\ \downarrow & & \downarrow & & \downarrow q \\ A & \xrightarrow{\text{id}} & A & \xrightarrow{q} & M, \end{array}$$

where the inclusion $T^qA \hookrightarrow q^!A$ sends $v \in T_a^qA$ to $(v, 0_{q(a)}) \in (q^!A)_a$. A spray V on A induces a section of $q^!A$,

$$\widehat{V} \in \Gamma(q^!A), \quad \widehat{V}_a := (V_a, a), \quad a \in A,$$

which satisfies $\text{pr}_2 \circ \widehat{V} = \text{id}_A$. The Euler vector field can be regarded as a section of $q^!A$:

$$E \in \Gamma(T^qA) \subset \Gamma(q^!A), \quad E_a := \left. \frac{d}{dt}(e^t a) \right|_{t=0}, \quad a \in A.$$

The homogeneity of V is expressed as a simple equation in the Lie algebra $\Gamma(q^!A)$:

$$(3.2) \quad [E, \widehat{V}] = \widehat{V}.$$

Example 3.3. Let $A = \mathfrak{g}$ be a Lie algebra (i.e. a Lie algebroid over a point $M = \text{pt}$). Sprays are simply 1-homogeneous vector fields on \mathfrak{g} , since the second condition in the definition of V is empty. Denoting by u^a linear coordinates on \mathfrak{g} , a spray V must be of the form $V = \Gamma_{ab}^c u^a u^b \partial_{u^c}$ with $\Gamma_{ab}^c \in \mathbb{R}$. In particular, $V = 0$ defines a spray for \mathfrak{g} whose flow is the identity $\phi_V^t = \text{id}_{\mathfrak{g}}$.

Example 3.4. An ordinary geodesic spray $V \in \mathfrak{X}(TM)$ associated to a Riemannian metric on M defines a Lie algebroid spray for the tangent Lie algebroid $A = TM$.

3.2. Realization forms. Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid with bundle projection $q : A \rightarrow M$ and let V be a Lie algebroid spray on A . A crucial step in the definition of the multiplication of the local groupoid is the construction of the corresponding right-invariant Maurer–Cartan form associated to V obtained in [17], which is called here the *realization form*, and which plays the role of the Maurer–Cartan form (2.1) of this local groupoid (see Theorem 3.8). We give here an independent account of the construction in [17], as well as a different proof, relying on the Lie algebroid $q^!A \rightarrow A$ and thinking of an infinitesimal analogue of a factorization like (2.3) of the division map.

Let

$$\phi_{\widehat{V}}^t \equiv \phi_{\widehat{V}}^{t,0}$$

denote the flow of the Lie algebroid (time-independent) section $\widehat{V} \in \Gamma(q^!A)$ (recall (2.5)). Since \widehat{V} is sent by the anchor $\text{pr}_1 : q^!A \rightarrow TA$ to V , the flow $\phi_{\widehat{V}}^t$ is a Lie algebroid automorphism covering ϕ_V^t .

Definition 3.5. The *realization form associated to V* , $\theta : T^q A \dashrightarrow A$, is the composition of the following Lie algebroid maps:

$$(3.3) \quad \begin{array}{ccccccc} T^q A & \hookrightarrow & q^!A & \xrightarrow{\phi_{\widehat{V}}^1} & q^!A & \xrightarrow{\text{pr}_2} & A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow q \\ A & \xrightarrow{\text{id}} & A & \xrightarrow{\phi_V^1} & A & \xrightarrow{q} & M. \end{array}$$

Thus, if U an open neighborhood of M in A on which ϕ_V^t is defined for $t \in [0, 1]$, then the realization form is the Lie algebroid morphism covering $\tau = q \circ \phi_V^1 : U \rightarrow M$:

$$\begin{array}{ccc} T^q A|_U & \xrightarrow{\theta} & A, \\ \downarrow & & \downarrow \\ U & \xrightarrow{\tau} & M \end{array} \quad \theta_a(v) := \text{pr}_2 \circ \phi_{\widehat{V}}^1(v)$$

for $a \in U$ and $v \in T_a^q A \cong A_{q(a)}$.

Next, we derive some immediate properties of θ . Using equation (3.2) and that the operation of pulling sections back along $\phi_{\widehat{V}}^t$ behaves like exponentiating the derivation $[\widehat{V}, \cdot]$ (recall equation (2.5)), we obtain

$$\frac{d}{dt} ((\phi_{\widehat{V}}^t)^*(E + t\widehat{V})) = (\phi_{\widehat{V}}^t)^*([\widehat{V}, E] + \widehat{V}) = 0,$$

which implies that

$$(\phi_{\hat{\gamma}}^t)_*(E) = E + t\hat{V}.$$

For $t = 1$, this yields

$$\phi_V^1(E_a, 0_{q(a)}) = (E_{\phi_V^1(a)} + V_{\phi_V^1(a)}, \phi_V^1(a)) \in (q^!A)_{\phi_V^1(a)}$$

for all $a \in U$. Projecting this onto the second component gives

$$(3.4) \quad \theta_a(a) = \phi_V^1(a) \quad \text{for all } a \in U,$$

where we used the identification $T_a^q A = A_{q(a)}$, under which

$$E_a = \left. \frac{d}{dt} \right|_{t=0} (a + ta) \equiv a.$$

Equation (3.4) is a key relation between θ and V to be repeatedly used in this section.

A direct corollary comes from replacing a by ta in equation (3.4), and using (3.1) to obtain

$$(3.5) \quad \theta_{ta} \left(\left. \frac{d}{dt} \right|_{t=0} (ta) \right) = \theta_{ta}(a) = \phi_V^t(a).$$

In the limit $t \rightarrow 0$, we obtain that $\theta_{q(a)}(a) = a$, i.e. θ is the identity along M . Thus, by shrinking U , we may assume that θ is a fibre-wise linear isomorphism.

Compared to [17], our approach of introducing θ using $q^!A$ has the advantage that it makes it obvious that θ is a Lie algebroid map, being a composition of such. To verify that we indeed obtain the same object, one can recall from [17] that equation (3.4) characterizes the Lie algebroid map θ uniquely around M .

The following lemma further characterizes θ as part of a Lie algebroid morphism, and recovers the formula from [17].

Lemma 3.6. *Let U denote an open neighborhood of M in A on which ϕ_V^t is defined for $t \in [0, 1]$, $a \in U$ and $v \in A_{q(a)}$. Consider a small enough interval J around 0 such that $t(a + \epsilon v) \in U$ for all $(t, \epsilon) \in [0, 1] \times J$. Then the map*

$$\phi_V^t(a + \epsilon v) dt + \theta_{t(a+\epsilon v)}(tv) d\epsilon : T([0, 1] \times J) \rightarrow A$$

defines a Lie algebroid morphism over $[0, 1] \times J \rightarrow M$, $(t, \epsilon) \mapsto \gamma_\epsilon(t) := q(\phi_V^t(a + \epsilon v))$. In particular, from (2.7) we get

$$(3.6) \quad \theta_a(v) = \int_0^1 \phi_{\alpha_0}^{1,s}(\gamma_0(s)) \left. \frac{d\alpha_\epsilon}{d\epsilon} \right|_{\epsilon=0} (s, \gamma_0(s)) ds,$$

where α_ϵ is a family of compactly supported, time-dependent sections of A such that

$$\alpha_\epsilon(t, \gamma_\epsilon(t)) = \phi_V^t(a + \epsilon v),$$

with $\phi_{\alpha_\epsilon}^{1,s} : A_{\gamma_\epsilon(s)} \rightarrow A_{\gamma_\epsilon(1)}$ denoting the flow of α_ϵ .

Proof. Differentiating the map $\hat{\gamma} : [0, 1] \times J \rightarrow A_{q(a)}$, $\hat{\gamma}(t, \epsilon) := t(a + \epsilon v)$, we obtain a Lie algebroid morphism covering $\hat{\gamma}$:

$$d\hat{\gamma} = \partial_t \hat{\gamma} dt + \partial_\epsilon \hat{\gamma} d\epsilon : T([0, 1] \times J) \rightarrow T^q A.$$

Composing this with θ , we obtain a Lie algebroid morphism into A :

$$\theta \circ d\hat{\gamma} = \theta_{\hat{\gamma}}(\partial_t \hat{\gamma}) dt + \theta_{\hat{\gamma}}(\partial_\epsilon \hat{\gamma}) d\epsilon : T([0, 1] \times J) \rightarrow A$$

covering $\tau \circ \hat{\gamma}$. The first statement follows because $\theta_{\hat{\gamma}}(\partial_t \hat{\gamma}) = \phi_V^t(a + \epsilon v)$ by equation (3.5) and $\partial_\epsilon \hat{\gamma} = tv$. Formula (3.6) for $\theta_a(v)$ then follows directly by evaluating (2.7) at $t = 1$ and $\epsilon = 0$. \square

Example 3.7. Let \mathfrak{g} be a Lie algebra endowed with the zero spray $V = 0$. We evaluate the associated realization form $\theta : T\mathfrak{g} \dashrightarrow \mathfrak{g}$ using formula (3.6). Given $a \in \mathfrak{g}$ close to zero and $v \in \mathfrak{g}$, since \mathfrak{g} is a Lie algebroid over a single point, then $\alpha_\epsilon(t) = \phi_V^t(a + \epsilon v) = a + \epsilon v$ defines a section that can be used to evaluate (3.6). In particular, $\alpha_0(t) = a$ and $\frac{d}{d\epsilon}|_{\epsilon=0} \alpha_\epsilon(t) = v$ (both time-independent). Moreover, by using the definition (2.5) of the associated flow, it follows that

$$\phi_{\alpha_0}^{t,s} = e^{-(t-s)\text{ad}_a} : \mathfrak{g} \rightarrow \mathfrak{g}.$$

We can then directly compute θ using (3.6), yielding the well-known expression

$$(3.7) \quad \theta_a(v) = \int_0^1 e^{-(1-s)\text{ad}_a}(v) ds = \frac{e^{-\text{ad}_a} - I}{-\text{ad}_a}(v).$$

For this see [6, Section 1.5], where ad_a appears instead of $-\text{ad}_a$ due to a different convention for the Lie bracket; namely, our convention for the Lie bracket comes from right-invariant vector fields, which leads to $[a, b] = -\text{ad}_a b$ for $a, b \in \mathfrak{g}$, where

$$\text{ad}_a b = \frac{\partial}{\partial t \partial \epsilon} \Big|_{t,\epsilon=0} \exp(ta) \exp(\epsilon b) \exp(ta)^{-1}$$

denotes the standard adjoint action (as opposed to [6] which uses left-invariant vector fields).

3.3. The construction of the spray groupoid. For the rest of this section we let $(A, [\cdot, \cdot], \rho), q : A \rightarrow M$ be a Lie algebroid, and let V be a Lie algebroid spray on A . Using V we shall construct a local Lie groupoid $G_V \rightrightarrows M$ integrating A . The total space is $G_V = A$, the space of units is M identified with the zero section of A , and the source map is the bundle projection $\sigma = q : A \rightarrow M$. The target map and the inversion map are defined on the open neighborhood of M on which the flow of V is defined up to $t = 1$:

$$\begin{aligned} \tau : G_V &\dashrightarrow M, & \tau(a) &:= q(\phi_V^1(a)), \\ \iota : G_V &\dashrightarrow G_V, & \iota(a) &:= -\phi_V^1(a) = \phi_V^{-1}(-a) \quad (\text{see (3.1)}). \end{aligned}$$

The groupoid axioms involving only these maps are easily verified: σ and τ are surjective submersions, and for $x \in M$ and $a \in A$,

$$\begin{aligned} \tau(x) &= q(\phi_V^1(0_x)) = q(0_x) = x, \\ \iota(x) &= -\phi_V^1(0_x) = x, \\ \sigma \circ \iota(a) &= q(-\phi_V^1(a)) = q(\phi_V^1(a)) = \tau(a), \\ \iota \circ \iota(a) &= -\phi_V^1(-\phi_V^1(a)) = \phi_V^{-1} \circ \phi_V^1(a) = a, \end{aligned}$$

where we have used that ϕ_V^t fixes M , and equation (3.1).

The main theorem of this subsection is that G_V carries a natural multiplication, defined by the ODE (3.8) below, that makes it into a local Lie groupoid integrating A , and has Maurer–Cartan form the realization form θ associated to V . The starting point is to observe that there is a neighborhood U_μ of M in $A \times_{\sigma, \tau} A$ such that for all (a, b) in this neighborhood the ODE

$$(3.8) \quad \theta \left(\frac{d}{dt} k_t \right) = \phi_V^t(a), \quad k_0 = b,$$

has its unique solution $t \mapsto k_t = k(t, a, b) \in A_{\sigma(b)}$ defined for all $t \in [0, 1]$. Indeed, we know that there is a neighborhood U of $M \subset A$ on which bundle map $\theta : T^q A|_U \rightarrow A$ is a fiberwise linear isomorphism. Hence, for a, b close enough to M , the ODE has a unique solution defined for t in a neighborhood of 0. Now, when $a = x \in M$, we have that $k_t = b$ is a solution for all $t \in \mathbb{R}$. For $b = x \in M$, $k_t = ta$ is a solution for all t such that θ_{ta} is defined; this follows from equation (3.5). This implies the existence of such an open neighborhood U_μ .

Theorem 3.8. *The maps $\sigma = q$, $\tau = q \circ \phi_V^1$, $\iota = -\phi_V^1$, together with the multiplication map*

$$\mu : G_V \times_{\sigma, \tau} G_V \dashrightarrow G_V, \quad \mu(a, b) := k(1, a, b),$$

where $k(t, a, b)$ is the solution of the ODE (3.8), define the structure maps of a local Lie groupoid on $(G_V, M, \sigma) = (A, M, q)$. The Maurer–Cartan form of this local groupoid coincides with the realization form θ associated to V ; in particular, G_V is a local Lie groupoid integrating A .

Definition 3.9. The local Lie groupoid G_V is called the *spray groupoid* of A associated to V .

First, we detail immediate properties of μ and then we prove Theorem 3.8. As we saw before, $k(t, a, \sigma(a)) = ta$ and $k(t, \tau(b), b) = b$; thus, the following unit axioms hold:

$$\mu(a, \sigma(a)) = a, \quad \mu(\tau(b), b) = b.$$

Since $k_t \in A_{\sigma(b)}$ and $\tau(k_t) = q(\phi_V^t(a))$, at $t = 1$, we obtain the axioms

$$\sigma(\mu(a, b)) = \sigma(b), \quad \tau(\mu(a, b)) = \tau(a).$$

Next, we describe the path $k(t, a, b)$ in terms of μ :

Lemma 3.10. *For $(a, b) \in U_\mu \subset G_V \times_{\sigma, \tau} G_V$ and $t \in [0, 1]$, we have*

$$k(t, a, b) = \mu(ta, b).$$

Proof. Fix $s \in [0, 1]$. Using (3.1), we calculate

$$\theta \left(\frac{d}{dt} k(st, a, b) \right) = s \theta \left(\frac{d}{du} k(u, a, b) \Big|_{u=st} \right) = s \phi_V^{st}(a) = \phi_V^t(sa).$$

Thus, the curves $t \mapsto k(st, a, b)$ and $t \mapsto k(t, sa, b)$ satisfy the same ODE and start at b . So they are equal and, for $t = 1$, we obtain the conclusion $\mu(sa, b) = k(1, sa, b) = k(s, a, b)$. \square

In particular, we obtain

$$(3.9) \quad \theta\left(\frac{d}{dt}\mu(ta, b)\right) = \theta\left(\frac{d}{dt}ta\right) = \phi_V^t(a).$$

The following result shows that θ plays the role of the Maurer–Cartan form of μ :

Lemma 3.11. *The form θ is right invariant for μ :*

$$\theta\left(\frac{d}{d\epsilon}\mu(a + \epsilon v, b)\Big|_{\epsilon=0}\right) = \theta\left(\frac{d}{d\epsilon}(a + \epsilon v)\Big|_{\epsilon=0}\right) = \theta_a(v)$$

for all $(a, b) \in U_\mu$ and all $v \in A_{\sigma(a)}$.

Proof. Let J be a small interval around 0 such that for all $(t, \epsilon) \in [0, 1] \times J$ we have that $\mu(t(a + \epsilon v), b) \in A_{\sigma(b)}$ is defined. Then the map

$$d(\mu(t(a + \epsilon v), b)) = \frac{d}{dt}\mu(t(a + \epsilon v), b)dt + \frac{d}{d\epsilon}\mu(t(a + \epsilon v), b)d\epsilon : T([0, 1] \times J) \rightarrow T^\sigma G_V$$

is a Lie algebroid morphism. Composing this with the Lie algebroid map $\theta : T^\sigma G_V \dashrightarrow A$, and applying (3.9), we obtain that the following is a Lie algebroid morphism:

$$\phi_V^t(a + \epsilon v)dt + \theta\left(\frac{d}{d\epsilon}\mu(t(a + \epsilon v), b)\right)d\epsilon : T([0, 1] \times J) \rightarrow A.$$

The coefficient of $d\epsilon$ vanishes for $t = 0$, so, by the uniqueness property in Lemma 2.7 and by Lemma 3.6, we obtain

$$\theta\left(\frac{d}{d\epsilon}\mu(t(a + \epsilon v), b)\right) = t\theta_{t(a + \epsilon v)}(v),$$

which, for $\epsilon = 0$ and $t = 1$, gives the conclusion. □

Although in Lemma 3.11 we used paths of the particular form $\epsilon \mapsto (a + \epsilon v)$, the result is about the differential of right multiplication $a \mapsto \mu(a, b)$, which is defined on an open set around $\tau(b)$ in $A_{\tau(b)}$. Therefore, we can reformulate the lemma as follows: for any smooth curve $\epsilon \mapsto a_\epsilon$, such that $(a_\epsilon, b) \in U_\mu$, we have

$$\theta\left(\frac{d}{d\epsilon}\mu(a_\epsilon, b)\right) = \theta\left(\frac{d}{d\epsilon}a_\epsilon\right).$$

This implies:

Lemma 3.12. *The map μ is associative on a neighborhood of M in $G_V^{(3)}$.*

Proof. Consider $(a, b, c) \in G_V^{(3)}$, close enough to M such that $\mu(\mu(ta, b), c)$ is defined for all $t \in [0, 1]$. By right invariance of θ , and by (3.9), we have

$$\theta\left(\frac{d}{dt}\mu(\mu(ta, b), c)\right) = \theta\left(\frac{d}{dt}\mu(ta, b)\right) = \phi_V^t(a).$$

Hence the two curves $t \mapsto \mu(\mu(ta, b), c)$ and $t \mapsto \mu(ta, \mu(b, c))$ satisfy the same ODE and start at $\mu(b, c)$; thus they are equal. For $t = 1$, we obtain associativity:

$$\mu(\mu(a, b), c) = \mu(a, \mu(b, c)).$$

□

Finally, we check the law of inverses. To that end, we first prove the following result.

Lemma 3.13. *Given $t, s \in \mathbb{R}$, the identity*

$$\mu(t\phi_V^s(a), sa) = (t + s)a$$

holds for all $a \in A$ close enough to $M \subset A$.

Proof. The idea is to show that both curves $t \mapsto \mu(t\phi_V^s(a), sa)$ and $t \mapsto (t + s)a$ satisfy the same ODE and have the same initial condition at $t = 0$ so that they must agree. (Notice that for the first curve to be defined, we need a close to $M \subset A$.) Taking their velocities and composing with θ we get

$$\theta\left(\frac{d}{dt}(t + s)a\right) = \theta\left(\frac{d}{du}ua\Big|_{u=t+s}\right) = \phi_V^{t+s}(a)$$

and, by (3.9),

$$\theta\left(\frac{d}{dt}\mu(t\phi_V^s(a), sa)\right) = \phi_V^t(\phi_V^s(a)) = \phi_V^{t+s}(a).$$

Taking a close enough to $M \subset A$ so that the curves lie in the domain in which θ is a fiberwise isomorphism, we conclude that both curves satisfy the same ODE (3.8). Since both curves also start at the same point $sa \in A$ for $t = 0$, the lemma follows. \square

Coming back to the law of inverses, since $\iota|_M = \text{id}_M$, we have $(a, \iota(a)), (\iota(a), a) \in U_\mu$ for all a close enough to M . Moreover, we can take such an a close enough to $M \subset A$ so that Lemma 3.13 with $t = -1$ and $s = 1$ also applies yielding

$$\mu(\iota(a), a) = \mu(-\phi_V^1(a), a) = 0 \cdot a = \sigma(a).$$

Analogously, considering $s = -1, t = 1$ in the lemma above and denoting $a = \phi_V^1(b)$,

$$\mu(b, \iota(b)) = \mu(\phi_V^{-1}\phi_V^1(b), -\phi_V^1(b)) = \mu(\phi_V^{-1}(a), -a) = 0 \cdot a = \tau(b).$$

The law of inverses is thus proven.

We conclude that μ endows G_V with the structure of a local groupoid. By Lemma 3.11, and by the fact that $\theta_x = \text{id}_{A_x}$ for all $x \in M$, we have that θ is the Maurer–Cartan form of G_V . Thus, we have proven Theorem 3.8.

Remark 3.14. Notice that the spray groupoid $G_V \rightrightarrows M$ just defined carries the natural tubular structure φ (in the sense of Section 2.3) given by the identity map of $G_V = A$.

Remark 3.15. Let us consider the spray groupoid $G = G_V$ and, in the context of Remark 2.3, the following diagram of local Lie groupoid maps:

$$(3.10) \quad \begin{array}{ccccccc} G & \xrightarrow{\sigma \times \sigma} & G & \hookrightarrow & \sigma^!G & \xrightarrow{\text{Ad}_{\bar{\beta}}} & \sigma^!G & \xrightarrow{\text{pr}_2} & G \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ G & \xrightarrow{\text{id}} & G & \xrightarrow{-\iota} & G & \xrightarrow{\sigma} & M, \end{array}$$

where $\bar{\beta}(a) = (-\iota(a), a, a) = (\phi_V^1(a), a, a)$ defines a bisection of $\sigma^!G$. The above diagram is a slight modification of (2.3) in which β is replaced by $\bar{\beta}$ but the overall composition still yields

the division map (2.2). Moreover, (3.10) integrates the diagram of Lie algebroid morphisms (3.3) thus providing a conceptual explanation for the construction of θ . To see this, the only non-trivial verification needed is that $\text{Ad}_{\bar{\beta}}$ integrates $\phi_{\widehat{V}}^1$, having in mind that q^1A is the Lie algebroid of σ^1G via the identification

$$(3.11) \quad T_b A \times_{TM} A \ni (u, a) \mapsto (u, a, 0_b) \in T_{(b, \sigma(a), b)}(G_\sigma \times_\tau G_\sigma \times_\sigma G).$$

By the general formula (2.6) for differentiating bisections of flows, it is in turn enough to check that $\bar{\beta}(a)$ coincides with the flow up to time $t = 1$ of the right-invariant vector field $\widehat{V}^R \in \mathfrak{X}(\sigma^1G)$, which is induced by the section \widehat{V} of q^1A , when applied to the identity element $(a, \sigma(a), a) \in \sigma^1G$ defined by $a \in G$. Writing $c(t) = (\phi_V^t(a), ta, a)$, Lemma 3.13 implies that

$$c(t + s) = (\phi_V^{t+s}(a), s\phi_V^t(a), \phi_V^t(a)) \cdot c(t),$$

where the \cdot denotes multiplication in σ^1G (cf. Remark 2.3). It then follows that $c(t)$ is the integral curve of \widehat{V}^R starting at $c(0) = (a, \sigma(a), a)$:

$$\left. \frac{d}{ds} \right|_{s=0} c(t + s) = \left. \frac{d}{ds} \right|_{s=0} (\phi_V^s(\phi_V^t(a)), s\phi_V^t(a), \phi_V^t(a)) \cdot c(t) = \widehat{V}^R|_{c(t)},$$

since

$$\widehat{V}^R|_{(b, \sigma(b), b)} = \left. \frac{d}{ds} \right|_{s=0} (\phi_V^s(b), sb, b) \in T_{(b, \sigma(b), b)}\sigma^1G$$

by (3.11). Thus $c(1) = (\phi_V^1(a), a, a) = \bar{\beta}(a)$ as desired.

Example 3.16. Let $A = \mathfrak{g}$ be a Lie algebra, and let us choose the zero spray $V = 0$ as in Examples 3.3 and 3.7. Using the specific formula for the realization form θ computed in (3.7), the ODE (3.8) defining μ on the spray group $G_V = \mathfrak{g}$ becomes

$$\frac{d}{dt}k_t = \frac{-\text{ad}_{k_t}}{e^{-\text{ad}_{k_t}} - I}(a), \quad k_0 = b.$$

It is a standard computation to deduce the Baker–Campbell–Hausdorff series (or Dynkin’s formula) from the above differential equation

$$\mu(a, b) = k_1 = a + b - \frac{1}{2}[a, b] + \dots$$

See [6, Section 1.7] where the bracket appears replaced by its opposite due to the difference in conventions recalled in Example 3.7.

3.4. Local integration of algebroid morphisms to the spray groupoid. Remark 3.14, Theorem 2.4 and equation (3.9) imply the following:

Corollary 3.17. *Let $f : A_1 \rightarrow A_2$ be a Lie algebroid morphism covering the map $f_M : M_1 \rightarrow M_2$. Let $G_V \rightrightarrows M_1$ be the spray groupoid corresponding to a spray V for A_1 , and let $G_2 \rightrightarrows M_2$ be any local Lie groupoid integrating A_2 . The local Lie groupoid map*

$$F : G_V \dashrightarrow G_2$$

integrating f from Theorem 2.4 is given by $F(a) = k(1, a)$, where, for $a \in G_V$ close enough to M_1 , $t \mapsto k_t = k(t, a)$ is the solution of the ODE

$$(3.12) \quad \theta_{G_2} \left(\frac{d}{dt}k_t \right) = f(\phi_V^t(a)), \quad k_t \in \sigma_2^{-1}(f_M(\sigma_1(a))), \quad k_0 = f_M(\sigma_1(a)).$$

Example 3.18. Let $f : A_1 \rightarrow A_2$ be a Lie algebroid morphism, and assume that there are sprays V_1 and V_2 , on A_1 and A_2 , respectively, such that they are f -related. In this case, it follows from equation (3.5) that $k(t, a) = tf(a)$ is a solution to equation (3.12); therefore $F := f : G_{V_1} \dashrightarrow G_{V_2}$ is the local groupoid map integrating f .

However, we remark that for a general Lie algebroid map, related sprays might not exist; for example, one can check that this is the case for $f = dg : T\mathbb{R} \rightarrow T\mathbb{R}$, where $g(x) = x^2$.

Example 3.19 (Integrating 1-cocycles). Let $f : A \rightarrow \mathbb{R}$ be a Lie algebroid morphism, where \mathbb{R} is a trivial Lie algebra, viewed as a Lie algebroid over a point. Note that f represents a 1-cocycle for A . In this case, equation (3.12) reduces to

$$\frac{d}{dt}k_t = f(\phi_V^t(a)), \quad k_0 = 0.$$

Hence, the local groupoid map $F : G_V \dashrightarrow \mathbb{R}$ (\mathbb{R} seen as a 1-dimensional Lie group) is given by

$$F(a) = \int_0^1 f(\phi_V^t(a)) dt,$$

which defines a local 1-cocycle on G_V integrating f . Compare with the inverse (2.10) of the van Est map at the level of 1-cochains: for the spray groupoid $\theta(\frac{d}{dt}ta) = \phi_V^t(a)$, hence the two formulas coincide. Similar formulas hold for multiplicative forms and other multiplicative geometric structures near the identities; these will be detailed elsewhere [2].

For $f = \text{id}$, the induced map F is called the *exponential map* corresponding to the spray:

Definition 3.20. Let A be a Lie algebroid with a spray V . Assume that G is a local Lie groupoid integrating A with Maurer–Cartan form θ_G . For $a \in A_x$, close enough to M , the ODE

$$\theta_G\left(\frac{d}{dt}k_t\right) = \phi_V^t(a), \quad k_0 = x,$$

has a solution $t \mapsto k_t = k(t, a) \in \sigma^{-1}(x)$ for all $t \in [0, 1]$. The *spray exponential* is the map

$$\exp_V : A \dashrightarrow G, \quad \exp_V(a) := k(1, a).$$

Note that \exp_V defines a diffeomorphism near M (its differential along M is the identity). The results of this section imply the following characterization of the spray groupoid structure:

Corollary 3.21. *Let A be a Lie algebroid with a spray V . The germ of the spray groupoid structure on (A, M, q) is uniquely characterized by the property that, for any local groupoid G integrating A , the associated spray exponential $\exp_V : G_V \dashrightarrow G$ defines a local Lie groupoid map (in this case, the germ of \exp_V automatically defines an isomorphism between the germs of G_V and G).*

Remark 3.22. In the case that the Lie algebroid A is integrable, i.e. if the Weinstein groupoid $G(A) = P(A)/\mathcal{F}$ is smooth (see [5]), then the exponential map associated to $G(A)$ and V factors as $\exp_V = \pi \circ \widehat{\exp}_V$, where $\widehat{\exp}_V : A \dashrightarrow P(A)$ was described in equation (1.1) of the introduction, and $\pi : P(A) \rightarrow G(A)$ is the quotient map.

3.5. Tubular structures versus spray exponentials. In this final subsection, we show that a tubular structure $\varphi : A \dashrightarrow G$ naturally induces a Lie algebroid spray V_φ on A . The resulting local Lie groupoid G_{V_φ} can be thought of as providing a “spray groupoid approximation” to G . We provide necessary and sufficient conditions for $\varphi : G_{V_\varphi} \dashrightarrow G$ to be a local Lie groupoid map.

Let $G \rightrightarrows M$ be a local Lie groupoid with Lie algebroid A , and let $\varphi : A \dashrightarrow G$ be a tubular structure as defined in Section 2.3. Following the case of a spray groupoid, in which the identity $A \rightarrow G_V$ defines a tubular structure, we consider

$$\lambda_t : A \dashrightarrow A, \quad \lambda_t(a) := \theta_G \left(\frac{d}{dt} \varphi(ta) \right).$$

The map $(t, a) \mapsto \lambda_t(a)$ is defined on an open subset of $\mathbb{R} \times A$ containing both $\mathbb{R} \times M$ and $\{0\} \times A$. Since φ is a tubular structure on G , we have $\lambda_0 = \text{id}$.

Lemma 3.23. *The derivative of λ_t at $t = 0$*

$$V_\varphi \in \mathfrak{X}(A), \quad V_\varphi(a) := \left. \frac{d}{dt} \lambda_t(a) \right|_{t=0}$$

defines a Lie algebroid spray for A .

Proof. First, note that $\lambda_t(a)$ is an A -path: this is equivalent to $\lambda_t(a)dt : T[0, 1] \rightarrow A$ being a Lie algebroid map, which follows since it is the composition of the Lie algebroid maps

$$\theta_G : T^\sigma G \dashrightarrow A \quad \text{and} \quad \frac{d}{dt}(\varphi(ta)) dt : T[0, 1] \rightarrow T^\sigma G.$$

By using $\sigma \circ \varphi = q$, the A -path condition at $t = 0$ gives the second spray condition:

$$dq(V_\varphi(a)) = \rho(a).$$

Next, note that $\lambda_t(a)$ satisfies the analog of equation (3.1):

$$\lambda_t(sa) = \theta_G \left(\frac{d}{dt} \varphi(sta) \right) = s \theta_G \left(\frac{d}{du} \varphi(ua) \Big|_{u=st} \right) = s \lambda_{st}(a).$$

This implies that

$$m_s^{-1} \circ \lambda_t \circ m_s = \lambda_{st}.$$

Taking the derivative at $t = 0$, we obtain the first spray condition $m_s^*(V_\varphi) = sV_\varphi$. □

Let $\text{exp}_{V_\varphi} : A \dashrightarrow G$ be the spray exponential associated to V_φ (cf. Definition 3.20). The following proposition states criteria for $\varphi : G_{V_\varphi} \dashrightarrow G$ to be a local groupoid map.

Proposition 3.24. *With the above notation, the following are equivalent:*

- (1) $\varphi : G_{V_\varphi} \dashrightarrow G$ is a local groupoid map.
- (2) $\text{exp}_{V_\varphi} = \varphi$ near the zero section.
- (3) The isotopy λ_t coincides with the flow of V_φ near the zero section.

Proof. By Corollary 3.21, (2) implies (1).

Assume that (3) holds: $\lambda_t = \phi_{V_\varphi}^t$. By Lemma 3.23, V_φ is a spray for A . On the other hand, for small enough $a \in A$, we have

$$\theta_G \left(\frac{d}{dt} \varphi(ta) \right) = \lambda_t(a) = \phi_{V_\varphi}^t(a).$$

Therefore $k_t = \varphi(ta)$ satisfies the ODE from Definition 3.20 of the spray exponential map. Taking $t = 1$, we conclude

$$\exp_{V_\varphi}(a) = \varphi(a)$$

for any $a \in A$ close enough to the zero section, thus (2) holds.

Finally, assume that (1) holds. Then φ intertwines the Maurer–Cartan forms of the two groupoids, and so

$$\lambda_t(a) = \theta_G \left(\frac{d}{dt} \varphi(ta) \right) = \theta \left(\frac{d}{dt} (ta) \right) = \phi_{V_\varphi}^t(a),$$

where we have used that φ induces the identity at the level of Lie algebroids. Thus (3) holds. \square

A. Differentiating integrated cochains

In this appendix, we consider the differentiation of the local Lie groupoid cochains obtained by integration through the map Ψ defined in Section 2.5. The main result is Lemma A.1 which is used in the proof of item (2) of Proposition 2.10.

Following the notation of Section 2.5, let $\alpha \in \Gamma(\wedge^p A^*)$ be an algebroid p -cochain and let $\Psi(\alpha) : G^{(p)} \dashrightarrow \mathbb{R}$ be the corresponding local groupoid cochain. In order to apply the van Est map, we consider sections $a_1, \dots, a_k \in \Gamma(A)$ and compute the iterated differentiation $D_{a_p} \cdots D_{a_1} \Psi(\alpha)$. We do this inductively on the number p of derivatives.

Let us fix composable arrows $g_2, \dots, g_p \in G^{(p-1)}$ in a sufficiently small neighborhood of M , and consider $h_1(\epsilon) = \epsilon a_1(\tau(g_2))$ so that

$$\begin{aligned} & D_{a_1} \Psi(\alpha)(g_2, \dots, g_p) \\ &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Psi(\alpha)(h_1(\epsilon), g_2, \dots, g_p) \\ &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \int_{I^p} (\theta^* \alpha)(\partial_1 \gamma_{h_1(\epsilon), g_2, \dots, g_p}, \dots, \partial_p \gamma_{h_1(\epsilon), g_2, \dots, g_p}) dt_1 \dots dt_p. \end{aligned}$$

By using the relations

$$(A.1) \quad \begin{aligned} \gamma_{\epsilon g_1, g_2, \dots, g_p}(t_1, \dots, t_p) &= \gamma_{g_1, \dots, g_p}(\epsilon t_1, t_2, \dots, t_p), \\ \gamma_{\tau(g_2), g_2, \dots, g_p}(t_1, \dots, t_p) &= \gamma_{g_2, \dots, g_p}(t_2, \dots, t_p), \end{aligned}$$

where $\tau(g_2) = \sigma(g_1) = 0 \cdot g_1$, it follows that, for $a \in \Gamma(A)$ and small $(g_2, \dots, g_n) \in G^{(n)}$,

$$(A.2) \quad \begin{aligned} \partial_1 \gamma_{\epsilon a(\tau(g_2)), g_2, \dots, g_n}(t_1, \dots, t_n) &= \epsilon \cdot \partial_1 \gamma_{a(\tau(g_2)), g_2, \dots, g_n}(\epsilon t_1, t_2, \dots, t_n), \\ \partial_j \gamma_{\tau(g_2), g_2, \dots, g_n}(t_1, t_2, \dots, t_n) &= \partial_{j-1} \gamma_{g_2, \dots, g_n}(t_2, \dots, t_n), \quad j > 1. \end{aligned}$$

Applying (A.2), we can compute $\frac{d}{d\epsilon}|_{\epsilon=0}$ (equivalently, apply the substitution $u = \epsilon t_1$ as in Example 2.9) yielding

$$D_{a_1} \Psi(\alpha)(g_2, \dots, g_p) = \int_{I^{p-1}} (\theta^* \alpha)(\partial_1 \gamma_{a_1(\tau(g_2)), g_2, \dots, g_p}(0, t_2, \dots, t_p), \partial_1 \gamma_{g_2, \dots, g_p}(t_2, \dots, t_p), \dots, \partial_{p-1} \gamma_{g_2, \dots, g_p}(t_2, \dots, t_p)) dt_2 \dots dt_p,$$

where $\int_I dt_1 = 1$ has been factored out. Above, the first factor inside α is of a different nature from the rest: it is the only one depending on a_1 while the others only depend on the fixed string g_2, \dots, g_p . We introduce the following general notation for such terms: given $a \in \Gamma(A)$, small $(k_1, \dots, k_n) \in G^{(n)}$ and any $(s_1, \dots, s_n) \in I^n$,

$$(A.3) \quad v_{k_1, \dots, k_n}^a(s_1, \dots, s_n) := \partial_1 \gamma_{a(\tau(k_1)), k_1, \dots, k_n}(0, s_1, \dots, s_n);$$

thus, we have

$$v_{k_1, \dots, k_n}^a(s_1, \dots, s_n) \in T^\sigma G_{\gamma_{k_1, \dots, k_n}(s_1, \dots, s_n)}.$$

Before stating the general inductive result, let us compute one more derivative fixing elements $g_3, \dots, g_p \in G^{(p-2)}$ and letting $h_2(\epsilon) = \epsilon a_2(\tau(g_3))$,

$$\begin{aligned} D_{a_2} D_{a_1} \Psi(\alpha)(g_3, \dots, g_p) &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} D_{a_1} \Psi(\alpha)(h_2(\epsilon), g_3, \dots, g_p) \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{I^{p-1}} (\theta^* \alpha)(v_{h_2(\epsilon), g_3, \dots, g_p}^{a_1}, \partial_1 \gamma_{h_2(\epsilon), g_3, \dots, g_p}, \dots, \partial_{p-1} \gamma_{h_2(\epsilon), g_3, \dots, g_p})(t_1, \dots, t_{p-1}) dt_1 \dots dt_{p-1}, \end{aligned}$$

where we have relabeled the integration variables. Using (A.2) we compute $\frac{d}{d\epsilon}|_{\epsilon=0}$, yielding the integral over $dt_1 \dots dt_{p-1}$ of

$$(\theta^* \alpha)(v_{h_2(0), g_3, \dots, g_p}^{a_1}(t_1, \dots, t_{p-1}), v_{g_3, \dots, g_p}^{a_2}(t_2, \dots, t_{p-1}), \partial_1 \gamma_{g_3, \dots, g_p}(t_2, \dots, t_{p-1}), \dots, \partial_{p-2} \gamma_{g_3, \dots, g_p}(t_2, \dots, t_{p-1})).$$

The key step for recognizing an inductive structure in our computation is to note that (recall that M is identified with the unit section)

$$(A.4) \quad \gamma_{g_1, \tau(g_3), g_3, \dots, g_n}(t_1, t_2, \dots, t_n) = \gamma_{g_1, g_3, \dots, g_n}(t_1 \cdot t_2, t_3, \dots, t_n)$$

for any small composable string $(g_1, g_3, g_4, \dots, g_n) \in G^{(n-1)}$ and any $(t_1, t_2, \dots, t_n) \in I^n$. Using (A.4) and the notation (A.3), we get

$$(A.5) \quad v_{\tau(g_3), g_3, \dots, g_n}^a(t_1, \dots, t_{n-1}) = t_1 \cdot v_{g_3, \dots, g_n}^a(t_2, \dots, t_{n-1})$$

for any $a \in \Gamma(A)$, small $(g_3, \dots, g_n) \in G^{(n-2)}$ and $(t_1, \dots, t_{n-1}) \in I^{n-1}$. Thus, since

$$h_2(0) = \tau(g_3),$$

we have

$$D_{a_2} D_{a_1} \Psi(\alpha)(g_3, \dots, g_p) = \int_{I^{p-1}} t_1 \cdot (\theta^* \alpha)(v_{g_3, \dots, g_p}^{a_1}, v_{g_3, \dots, g_p}^{a_2}, \partial_1 \gamma_{g_3, \dots, g_p}, \dots, \partial_{p-2} \gamma_{g_3, \dots, g_p})(t_2, \dots, t_{p-1}) dt_1 \dots dt_{p-1}$$

from which $\int_I t_1 dt_1 = \frac{1}{2}$ factors out yielding

$$\begin{aligned} & D_{a_2} D_{a_1} \Psi(\alpha)(g_3, \dots, g_p) \\ &= \frac{1}{2} \int_{I^{p-2}} (\theta^* \alpha)(v_{g_3, \dots, g_p}^{a_1}, v_{g_3, \dots, g_p}^{a_2}, \partial_1 \gamma_{g_3, \dots, g_p}, \dots, \partial_{p-2} \gamma_{g_3, \dots, g_p}). \end{aligned}$$

Continuing by induction, we obtain the following:

Lemma A.1. For $a_1, \dots, a_k \in \Gamma(A)$ and $(g_{k+1}, \dots, g_p) \in G^{(p-k)}$ small enough, we have

$$\begin{aligned} (A.6) \quad & D_{a_k} \cdots D_{a_1} \Psi(\alpha)(g_{k+1}, \dots, g_p) \\ &= \frac{1}{k!} \int_{I^{p-k}} (\theta^* \alpha)(v_{g_{k+1}, \dots, g_p}^{a_1}, \dots, v_{g_{k+1}, \dots, g_p}^{a_k}, \\ & \quad \partial_1 \gamma_{g_{k+1}, \dots, g_p}, \dots, \partial_{p-k} \gamma_{g_{k+1}, \dots, g_p}), \end{aligned}$$

where $v_{g_{k+1}, \dots, g_p}^a : I^{p-k} \rightarrow T^\sigma G$ is defined in (A.3).

Proof. We will consider an induction over $k \geq 1$ recalling that the case $k = 1$ was already worked out above. Assume now (A.6) and compute

$$\begin{aligned} & D_{a_{k+1}} \cdots D_{a_1} \Psi(\alpha)(g_{k+2}, \dots, g_p) \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} D_{a_k} \cdots D_{a_1} \Psi(\alpha)(\epsilon a_{k+1}(\tau(g_{k+2})), g_{k+2}, \dots, g_p). \end{aligned}$$

To use (A.6) on the right-hand side above, let us introduce the following notation:

$$w_{g_{k+1}, \dots, g_p}^{a_1, \dots, a_k} := v_{g_{k+1}, \dots, g_p}^{a_1} \wedge \cdots \wedge v_{g_{k+1}, \dots, g_p}^{a_k}$$

and $b := a_{k+1}(\tau(g_{k+2}))$, so that

$$\begin{aligned} (A.7) \quad & k! D_{a_{k+1}} \cdots D_{a_1} \Psi(\alpha)(g_{k+2}, \dots, g_p) \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{I^{p-k}} (\theta^* \alpha)(w_{\epsilon b, g_{k+2}, \dots, g_p}^{a_1, \dots, a_k} \wedge d\gamma_{\epsilon b, g_{k+2}, \dots, g_p}). \end{aligned}$$

By using (A.1), it follows that

$$d\gamma_{\epsilon b, g_{k+2}, \dots, g_p}(t_1, \dots, t_{p-k}) = \epsilon \cdot d\gamma_{b, g_{k+2}, \dots, g_p}(\epsilon t_1, \dots, \epsilon t_{p-k}),$$

and that

$$\gamma_{b, g_{k+2}, \dots, g_p}(0, t_2, \dots, t_{p-k}) = \gamma_{g_{k+2}, \dots, g_p}(t_2, \dots, t_{p-k}).$$

Using these to compute (A.7) gives

$$\begin{aligned} & k! D_{a_{k+1}} \cdots D_{a_1} \Psi(\alpha)(g_{k+2}, \dots, g_p) \\ &= \int_{I^{p-k}} (\theta^* \alpha)(w_{\tau(g_{k+2}), g_{k+2}, \dots, g_p}^{a_1, \dots, a_k}(t_1, \dots, t_{p-k}) \wedge d\gamma_{g_{k+2}, \dots, g_p}(t_2, \dots, t_{p-k})) dt_1 \dots dt_{p-k}. \end{aligned}$$

Applying (A.5), we get

$$w_{\tau(g_{k+2}), g_{k+2}, \dots, g_p}^{a_1, \dots, a_k}(t_1, \dots, t_{p-k}) = t_1^k \cdot w_{g_{k+2}, \dots, g_p}^{a_1, \dots, a_k}(t_2, t_3, \dots, t_{p-k}).$$

The lemma thus follows by noticing that $\int_0^1 t_1^k dt_1 = \frac{1}{k+1}$ factors out. □

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