Expressive Logics for Coinductive Predicates

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Abstract

The classical Hennessy-Milner theorem says that two states of an image-finite transition system are bisimilar if and only if they satisfy the same formulas in a certain modal logic. In this paper we study this type of result in a general context, moving from transition systems to coalgebras and from bisimilarity to coinductive predicates. We formulate when a logic fully characterises a coinductive predicate on coalgebras, by providing suitable notions of adequacy and expressivity, and give sufficient conditions on the semantics. The approach is illustrated with logics characterising similarity, divergence and a behavioural metric on automata.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics; Theory of computation → Categorical semantics

Keywords and phrases Coalgebra, Fibration, Modal Logic

Digital Object Identifier 10.4230/LIPIcs.CSL.2020.26

Funding This work was partially supported by a Marie Curie Fellowship (grant code 795119).

Acknowledgements We would like to thank Bart Jacobs for useful discussions and the anonymous referees for their very constructive and helpful feedback.

1 Introduction

A prominent example of the deep connection between bisimilarity and modal logic is the *Hennessy-Milner theorem*: two states of an image-finite labelled transition system (LTS) are behaviourally equivalent iff they satisfy the same formulas in a certain modal logic [13]. From left to right, this equivalence is sometimes referred to as *adequacy* of the logic w.r.t. bisimilarity, and from right to left as *expressivity*. By proving both adequacy and expressivity, the Hennessy-Milner theorem thus gives a logical characterisation of behavioural equivalence.

There are numerous variants and generalisations of this kind of result. For instance, a state x of an LTS *simulates* a state y if every formula satisfied by x is also satisfied by y, where the logic only has conjunction and diamond modalities; see [36] for this and many other related results. Another class of examples is logical characterisations of quantitative notions of equivalence, such as probabilistic bisimilarity and behavioural distances (e.g., [27, 8, 35, 19, 24, 37, 7]). In many such cases, including bisimilarity, the comparison between states is *coinductive*, and the problem is thus to characterise a coinductively defined relation (or distance) with a suitable modal logic.

Both coinduction and modal logic can be naturally and generally studied within the theory of *coalgebra*, which provides an abstract, uniform study of state-based systems [32, 18]. Indeed, in the area of *coalgebraic modal logic* [26] there is a rich literature on deriving expressive logics for behavioural equivalence between state-based systems, thus going well beyond labelled transition systems [29, 33, 22]. However, such results focus almost exclusively on behavioural equivalence or bisimilarity – a coalgebraic theory of logics for characterising coinductive predicates other than bisimilarity is still missing. The aim of this paper is to

28th EACSL Annual Conference on Computer Science Logic (CSL 2020).
Editors: Maribel Fernández and Anca Muscholl; Article No. 26; pp. 26:1–26:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

accommodate the study of logical characterisation of coinductive predicates in a general manner, and provide tools to prove adequacy and expressivity.

Our approach is based on universal coalgebra, to achieve results that apply generally to state-based systems. Central to the approach are the following two ingredients.

1. *Coinductive predicates in a fibration.* To characterise coinductive predicates, we make use of fibrations – this approach originates from the seminal work of Hermida and Jacobs [14]. The fibration is used to speak about predicates and relations on states. In this context, liftings of the type functor of coalgebras uniformly determine coinductive predicates and relations on such coalgebras. An important feature of this approach, advocated in [12], is that it covers not only bisimilarity, but also other coinductive predicates including, e.g., similarity of labelled transition systems and other coalgebras [16], behavioural metrics [2, 4, 34], unary predicates such as divergence [5, 12], and many more.
2. *Coalgebraic modal logic via dual adjunctions.* We use an abstract formulation of coalgebraic logic, which originated in [30, 22], building on a tradition of logics via duality (e.g., [25, 6]). This framework is formulated in terms of a contravariant adjunction, which captures the basic connection between states and theories, and a distributive law, which captures the one-step semantics of the logic. It covers classical modal logics of course, but also easily accommodates multi-valued logics, and, e.g., logics without propositional connectives, where formulas can be thought of as basic tests on state-based systems. This makes the framework suitable for an abstract formulation of Hennessy-Milner type theorems, where formulas play the role of tests on state-based systems.

To formulate adequacy and expressivity with respect to general coinductive predicates, we need to know how to compare collections of formulas. For instance, if the coinductive predicate is similarity of LTSs, the associated logical theories of one state should be *included* in the other, not necessarily equal. This amounts to stipulating a *relation* on truth values, that extends to a relation between theories. In the quantitative case, we need a *logical distance* between collections of formulas; this typically arises from a distance between truth values (which, in this case, will typically be an interval in the real numbers). The fibrational setting provides a convenient means for defining such an object for comparing theories.

With this in hand, we arrive at the main contributions of this paper: the formulation of adequacy and expressivity of a coalgebraic modal logic with respect to a coinductive predicate in a fibration, and sufficient conditions on the semantics of the logic that guarantee adequacy and expressivity. We exemplify the approach through a range of examples, including logical characterisations of a simple behavioural distance on deterministic automata, similarity of labelled transition systems, and a logical characterisation of a unary predicate: divergence, the set of states of an LTS which have an infinite path of outgoing τ-steps. The latter is characterised, on image-finite LTSs, by a quantitative logic with only diamond formulas, i.e., the set of formulas is simply the set of words.

Related work

As mentioned above, there are numerous specific results on Hennessy-Milner theorems, which – e.g., in the probabilistic setting as in [7] – can be highly non-trivial. A comprehensive historical treatment is beyond the scope of this paper, which is, instead, broad: it aims at studying these kinds of results in a general, coalgebraic setting.

The case of capturing bisimilarity and behavioural equivalence of coalgebras by modal logics has been very well studied, see [26] for an overview. Expressiveness w.r.t. similarity has been studied in [20], which is close in spirit to our approach, but focuses on the poset case. On a detailed level, the logic for similarity is based on distributive lattices, hence it uses disjunction; this differs from our example, which only uses conjunction and diamond modalities. Expressiveness of multi-valued coalgebraic logics w.r.t. behavioural equivalence is studied in [3]. In [1], notions of equivalence are extracted from a logic through a variant of Λ-bisimulation [11]. To the best of our knowledge, the current work is the first in the area that connects general coinductive predicates in a fibration to coalgebraic logics.

In the recent [9], the authors prove Hennessy-Milner type theorems for coalgebras including, but going significantly beyond bisimilarity. The logics are related to a semantics obtained from graded monads, and the focus is exclusively on semantic equivalence of different types. In that sense, the scope differs substantially from the current paper, which relates logic to coinductive predicates and where it is essential to relate theories in different ways than equivalence (to cover, e.g., similarity, divergence or logical distance). On the one hand, it appears that none of our examples can be covered immediately in *loc. cit.*; on the other hand, trace equivalence of various kinds can be covered in [9] but not in the current paper.

In [37] a characterisation theorem is shown for fuzzy modal logic, and in [24] for a wide class of behavioural metrics. These papers are not aimed at other kinds of coinductive predicates, and they do not cover the examples in Section 4 (including the behavioural metric for deterministic automata, as we use a much simpler logic than in [24]). Conversely, the question whether the logical characterisation results of [24] can be covered in the current framework is left open. These papers also treat game-based characterisations of bisimilarity, which are studied in a general setting in the recent [23]. The latter paper, however, does not yet feature modal logic explicitly; in fact, the connection is posed there as future work.

2 Preliminaries

The category of sets and functions is denoted by Set. The powerset functor is denoted by $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$, and the finite powerset functor by $\mathcal{P}_\omega$. The diagonal relation on a set X is denoted by $\Delta_X = \{(x,x) \mid x \in X\}$.

Let $\mathcal{C}$ be a category, and $B\colon \mathcal{C} \to \mathcal{C}$ a functor. A (B)-*coalgebra* is a pair (X, γ) where X is an object in $\mathcal{C}$ and $\gamma\colon X \to BX$ a morphism. A homomorphism from a coalgebra (X, γ) to a coalgebra (Y, θ) is a morphism $h\colon X \to Y$ such that $\theta \circ h = Bh \circ \gamma$. An *algebra* for a functor $L\colon \mathcal{D} \to \mathcal{D}$ on a category $\mathcal{D}$ is a pair (A, α) of an object A in $\mathcal{D}$ and an arrow $\alpha\colon LA \to A$.

▶ **Example 1.** A *labelled transition system (LTS)* over a set of labels A is a coalgebra (X, γ) for the functor $B\colon \mathsf{Set} \to \mathsf{Set}$, $BX = (\mathcal{P}X)^A$. For states $x, x' \in X$ and a label $a \in A$, we sometimes write $x \xrightarrow{a} x'$ for $x' \in \gamma(x)(a)$. Image-finite labelled transition systems are coalgebras for the functor $BX = (\mathcal{P}_\omega X)^A$. A *deterministic automaton* over an alphabet A is a coalgebra for the functor $B\colon \mathsf{Set} \to \mathsf{Set}$, $BX = 2 \times X^A$. For many other examples of state-based systems modelled as coalgebras, see, e.g., [18, 32].

2.1 Coinductive Predicates in a Fibration

We recall the general approach to coinductive predicates in a fibration, starting by briefly presenting how bisimilarity of Set coalgebras arises in this setting (see [12, 14, 18] for details). Let Rel be the category where an object is a pair (X, R) consisting of a set X and a relation $R \subseteq X \times X$ on it, and a morphism from (X, R) to (Y, S) is a map $f\colon X \to Y$ such that $x \mathrel{R} y$ implies $f(x) \mathrel{R} f(y)$, for all $x, y \in X$. Below, we sometimes refer to an object (X, R) only by the relation $R \subseteq X \times X$. Any set functor $B\colon \mathsf{Set} \to \mathsf{Set}$ gives rise to a functor $\mathrm{Rel}(B)\colon \mathrm{Rel} \to \mathrm{Rel}$, defined by *relation lifting*:

$$\mathrm{Rel}(B)(R \subseteq X \times X) = \{((B\pi_1)(z), (B\pi_2)(z)) \in BX \times BX \mid z \in BR\}\,. \tag{1}$$

Given a B-coalgebra (X, γ), a *bisimulation* is a relation $R \subseteq X \times X$ such that $R \subseteq (\gamma \times \gamma)^{-1}(\mathrm{Rel}(B)(R))$, i.e., if $x\ R\ y$ then $\gamma(x)\ \mathrm{Rel}(B)(R)\ \gamma(y)$. *Bisimilarity* is the greatest such relation, and equivalently, the greatest fixed point of the monotone map $R \mapsto (\gamma \times \gamma)^{-1}(\mathrm{Rel}(B)(R))$ on the complete lattice of relations on X, ordered by inclusion.

The functor $\mathrm{Rel}(B)$ is a *lifting* of B: it maps a relation on X to a relation on BX. A first step towards generalisation beyond bisimilarity is obtained by replacing $\mathrm{Rel}(B)$ by an arbitrary lifting $\overline{B}\colon \mathrm{Rel} \to \mathrm{Rel}$ of B. For instance, for $BX = (\mathcal{P}_\omega X)^A$ one may take

$$\overline{B}(R) = \{(t_1, t_2) \mid \forall a \in A. \forall x \in t_1(a). \exists y \in t_2(a).(x, y) \in R\}\,. \tag{2}$$

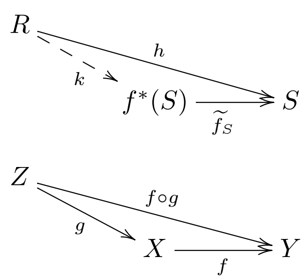
Then, for an LTS $\gamma\colon X \to (\mathcal{P}_\omega X)^A$, the greatest fixed point of the monotone map $R \mapsto (\gamma \times \gamma)^{-1} \circ \overline{B}(R)$ is *similarity*. In the same way, by varying the lifting $\overline{B}$, one can define many different coinductive relations on Set coalgebras.

Yet a further generalisation is obtained by replacing Set by a general category $\mathcal{C}$, and Rel by a category of "predicates" on $\mathcal{C}$. A suitable categorical infrastructure for such predicates on $\mathcal{C}$ is given by the notion of *fibration*. This allows us, for instance, to move beyond (Boolean, binary) relations to quantitative relations (e.g., behavioural metrics) or unary predicates. Such examples follow in Section 4; also see, e.g., [12, 5].

To define fibrations, it will be useful to fix some associated terminology first. Let $p\colon \mathcal{E} \to \mathcal{C}$ be a functor. If $p(R) = X$ then we say R is *above* X, and similarly for morphisms. The collection of all objects R above a given object X and arrows above the identity id_X form a category, called the *fibre above* X and denoted by $\mathcal{E}_X$.

▶ **Definition 2.** *A functor $p\colon \mathcal{E} \to \mathcal{C}$ is a* (poset) fibration *if*

- *each fibre $\mathcal{E}_X$ is a poset category (that is, at most one arrow between every two objects); the corresponding order on objects is denoted by $\leq$;*
- *for every $f\colon X \to Y$ in $\mathcal{C}$ and object S above Y there is a* Cartesian morphism $\widetilde{f}_S\colon f^*(S) \to S$ *above f, with the property that for every arrow $g\colon Z \to X$, every object R above Z and arrow $h\colon R \to S$ above $f \circ g$, there is a unique arrow $k\colon R \to f^*(S)$ above g such that $\widetilde{f}_S \circ k = h$.*



▶ Remark 3. In this paper we only consider poset fibrations, and refer to them simply as fibrations. The usual definition of fibration is more general (e.g., [17]): normally, fibres are not assumed to be posets. Poset fibrations have several good properties, mentioned below. In the application to coinductive predicates, it is customary to work with poset fibrations.

For a morphism $f\colon X \to Y$, the assignment $R \mapsto f^*(R)$ gives rise to a functor $f^*\colon \mathcal{E}_Y \to \mathcal{E}_X$, called *reindexing along f*. (Note that functors between poset categories are just monotone maps.) We use a strengthening of poset fibrations, following [34, 23].

▶ **Definition 4.** *A poset fibration $p\colon \mathcal{E} \to \mathcal{C}$ is called a $\mathsf{CLat}_\wedge$-fibration if $(\mathcal{E}_X, \leq)$ is a complete lattice for every X, and reindexing preserves arbitary meets.*

Any poset fibration p is split: we have $(g \circ f)^* = f^* \circ g^*$ for any morphisms f, g that compose. Further, p is faithful. This captures the intuition that morphisms in $\mathcal{E}$ are morphisms in $\mathcal{C}$ with a certain property; e.g., relation-preserving, or non-expansive (Examples 5, 6). We note that $\mathsf{CLat}_\wedge$-fibrations are instances of topological functors [15]. We use the former, in line with existing related work [12, 23]. This also has the advantage of keeping our results amenable to possible future extensions to a wider class of examples.

▶ **Example 5.** Consider the *relation fibration* $p\colon \mathrm{Rel} \to \mathsf{Set}$, where $p(R \subseteq X \times X) = X$. Reindexing is given by inverse image: for a map $f\colon X \to Y$ and a relation $S \subseteq Y \times Y$, we have $f^*(S) = (f \times f)^{-1}(S)$. The functor p is a $\mathsf{CLat}_\wedge$-fibration.

Closely related is the *predicate fibration* $p\colon \mathrm{Pred} \to \mathsf{Set}$. An object of Pred is a pair (X, Γ) consisting of a set X and a subset $\Gamma \subseteq X$, and an arrow from (X, Γ) to (Y, Θ) is a map $f\colon X \to Y$ such that $x \in \Gamma$ implies $f(x) \in \Theta$. The functor p is given by $p(X, \Gamma) = X$, reindexing is given by inverse image, and p is a $\mathsf{CLat}_\wedge$-fibration as well.

In the relation fibration, we sometimes refer to an object $(X, R \subseteq X^2)$ simply by R, and similarly in the predicate fibration.

▶ **Example 6.** Let $\mathcal{V}$ be a complete lattice. Define the category $\mathrm{Rel}_\mathcal{V}$ as follows: an object is a pair (X, d) where X is a set and a function $d\colon X \times X \to \mathcal{V}$, and a morphism from (X, d) to (Y, e) is a map $f\colon X \to Y$ such that $d(x, y) \leq e(f(x), f(y))$. The forgetful functor $p\colon \mathrm{Rel}_\mathcal{V} \to \mathsf{Set}$ is a $\mathsf{CLat}_\wedge$-fibration, where reindexing along $f\colon X \to Y$ is given by $f^*(Y, e) = (X, e \circ f \times f)$.

For $\mathcal{V} = 2 = \{0, 1\}$ with the usual order $0 \leq 1$, $\mathrm{Rel}_\mathcal{V}$ coincides with Rel. Another example is given by the closed interval $\mathcal{V} = [0, 1]$, with the *reverse* order. Then, a morphism from (X, d) to (Y, e) is a *non-expansive map* $f\colon X \to Y$, that is, s.t. $e(f(x), f(y)) \leq d(x, y)$ (with $\leq$ the usual order, i.e., where 0 is the smallest). This instance will be denoted by $\mathrm{Rel}_{[0,1]}$.

Liftings and Coinductive Predicates

Let $p\colon \mathcal{E} \to \mathcal{C}$ be a fibration, and $B\colon \mathcal{C} \to \mathcal{C}$ a functor. A functor $\overline{B}\colon \mathcal{E} \to \mathcal{E}$ is called a *lifting* of B if $p \circ \overline{B} = B \circ p$. In that case, $\overline{B}$ restricts to a functor $\overline{B}_X\colon \mathcal{E}_X \to \mathcal{E}_{BX}$, for any X in $\mathcal{C}$.

A lifting $\overline{B}$ of B gives rise to an abstract notion of coinductive predicate, as follows. For any B-coalgebra (X, γ) there is the functor, i.e., monotone function defined by $\gamma^* \circ \overline{B}_X\colon \mathcal{E}_X \to \mathcal{E}_X$. We think of post-fixed points of $\gamma^* \circ \overline{B}_X$ as *invariants*, generalising *bisimulations*. If p is a $\mathsf{CLat}_\wedge$-fibration, then $\gamma^* \circ \overline{B}_X$ has a greatest fixed point $\nu(\gamma^* \circ \overline{B}_X)$, which is also the greatest post-fixed point. It is referred to as the *coinductive predicate* defined by $\overline{B}$ on γ.

▶ **Example 7.** First, for a Set functor $B\colon \mathsf{Set} \to \mathsf{Set}$, recall the lifting $\mathrm{Rel}(B)$ of B defined in the beginning of this section. We refer to $\mathrm{Rel}(B)$ as the *canonical relation lifting* of B. For a coalgebra (X, γ), a post-fixed point of the operator $\gamma^* \circ \mathrm{Rel}(B)_X$ is a bisimulation, as explained above. The coinductive predicate $\nu(\gamma^* \circ \mathrm{Rel}(B)_X)$ defined by $\mathrm{Rel}(B)$ is bisimilarity. Another example is given by the lifting $\overline{B}$ for similarity defined in the beginning of this section, which we further study in Section 4. In that section we also define a unary predicate, divergence, making use of the predicate fibration. Coinductive predicates in the fibration $\mathrm{Rel}_{[0,1]}$ can be thought of as *behavioural distances*, providing a quantitative analogue of bisimulations, measuring the distances between states. A simple example on deterministic automata is studied in Section 4.1.

▶ Remark 8. In the quantitative examples, such as $\mathrm{Rel}_{[0,1]}$, one can replace the latter by a category with more structure, such as the category of pseudometrics and non-expansive maps.

Similarly, one can replace Rel by the category of equivalence relations. Defining liftings then requires slightly more work, and since we use fibrations to *define* coinductive predicates, this unnecessarily complicates matters. Therefore, we do not use such categories in our examples.

We sometimes need the notion of *fibration map*: if $\overline{B}$ is a lifting of B, the pair $(\overline{B}, B)$ is called a fibration map if $(Bf)^* \circ \overline{B}_Y = \overline{B}_X \circ f^*$ for any arrow $f\colon X \to Y$ in $\mathcal{C}$. If B preserves weak pullbacks, then $(\mathrm{Rel}(B), B)$ is a fibration map [18] in the relation fibration (Example 5).

2.2 Coalgebraic Modal Logic

We recall a general approach to coalgebraic modal logic, in the context of a contravariant adjunction [30, 22, 19]. We assume the following setting, involving an adjunction $P \dashv Q$ and a natural transformation $\delta\colon BQ \Rightarrow QL$:

$$B \circlearrowright \mathcal{C} \underset{Q}{\overset{P}{\rightleftarrows}} \mathcal{D}^{\mathsf{op}} \circlearrowleft L \quad (P \perp Q) \qquad \text{with} \qquad BQ \overset{\delta}{\Longrightarrow} QL \tag{3}$$

In this context, a *logic* for B-coalgebras is a pair (L, δ) as above. The functor $L\colon \mathcal{D} \to \mathcal{D}$ represents the syntax of the modalities. It is assumed to have an initial algebra $\alpha\colon L\Phi \overset{\cong}{\to} \Phi$, which represents the set (or other structure) of formulas of the logic. The natural transformation δ gives the one-step semantics. It can equivalently be presented in terms of its *mate* $\widehat{\delta}\colon LP \Rightarrow PB$, which is perhaps more common in the literature. However, we will formulate adequacy and expressiveness in terms of the current presentation of δ.

Let (X, γ) be a B-coalgebra. The semantics $[\![\_]\!]$ of a logic (L, δ) arises by initiality of α, making use of the mate $\widehat{\delta}$, as the unique map making the diagram on the left below commute.

$$\begin{array}{ccccc} L\Phi & \xrightarrow{L[\![\_]\!]} & LPX & \xrightarrow{\widehat{\delta}} & PBX \\ {\scriptstyle\alpha}\downarrow & & & & \downarrow{\scriptstyle P\gamma} \\ \Phi & & \xrightarrow{\exists ! [\![\_]\!]} & & PX \end{array} \qquad \begin{array}{ccccc} X & & \xrightarrow{th} & & Q\Phi \\ {\scriptstyle\gamma}\downarrow & & & & \downarrow{\scriptstyle Q\alpha} \\ BX & \xrightarrow{Bth} & BQ\Phi & \xrightarrow{\delta} & QL\Phi \end{array}$$

The *theory map* $th\colon X \to Q\Phi$ is defined as the transpose of $[\![\_]\!]$. It is the unique map making the diagram on the right above commute.

▶ **Example 9.** Let $\mathcal{C} = \mathcal{D} = \mathsf{Set}$, $P = Q = 2^-$ the contravariant powerset functor, and $BX = 2 \times X^A$. We define a simple logic for B-coalgebras, where formulas are just words over A. To this end, let $LX = A \times X + 1$. The initial algebra of L is the set A^* of words. Define $\delta\colon BQ \Rightarrow QL$ on a component X as follows:

$$\delta_X\colon 2 \times (2^X)^A \to 2^{A \times X + 1} \qquad \delta_X(o, t)(u) = \begin{cases} o & \text{if } u = * \in 1 \\ t(a)(x) & \text{if } u = (a, x) \in A \times X \end{cases}$$

For a coalgebra $\langle o, t\rangle\colon X \to 2 \times X^A$, the associated theory map $th\colon X \to 2^{A^*}$ is given by $th(x)(\varepsilon) = o(x)$ and $th(x)(aw) = th(t(x)(a))(w)$ for all $x \in X$, $a \in A$, $w \in A^*$. This is, of course, the usual semantics of deterministic automata.

In the above example, the logic does not contain propositional connectives; this is reflected by the choice $\mathcal{D} = \mathsf{Set}$. To add those, one chooses a category of algebras for $\mathcal{D}$. For instance, Boolean algebras are a standard choice for propositional logic, and in Section 4 we use the category of semilattices to represent conjunction. In fact, if one is only interested in defining the semantics of the logic, one can simply work with algebras for a signature; this is supported by the adjunctions presented in the next subsection. We outline in the next subsection how this can be used to represent the propositional part of a real-valued modal logic.

2.3 Contravariant Adjunctions

In this subsection we discuss several adjunctions that we use for presenting coalgebraic logic as above, and will allow us in Section 4 to demonstrate that a large variety of concrete examples is covered by our framework. In all cases, the adjunctions that we use for the logic are generated by an object Ω of "truth values". In fact, we believe all of the dual adjunctions listed in this section are instances of the so-called concrete dualities from [31] where Ω is the dualising object inducing the adjunction.

For a simple but useful class of such adjunctions, let $\mathcal{D}$ be a category with products, and Ω an object in $\mathcal{D}$. Then there is an adjunction

$$P \dashv Q \colon \mathsf{Set} \leftrightarrows \mathcal{D}^{\mathsf{op}} \qquad \text{where } PX = \Omega^X \text{ and } QX = \mathrm{Hom}(X, \Omega)\,, \tag{4}$$

where Ω^X is the X-fold product of Ω.

▶ **Example 10.** To illustrate the usefulness of this simple adjunction, consider the real-valued coalgebraic modal logics from [24]. The set Φ of formulas of these logics is given by the following definition that is indexed by a set $\mathfrak{C}$ of modal operators:

$$\Phi ::= \top \mid [\mathfrak{c}]\varphi, \mathfrak{c} \in \mathfrak{C} \mid \min(\varphi_1, \varphi_2) \mid \neg\varphi \mid \varphi \ominus q, q \in \mathbb{Q} \cap [0, \top]$$

where $\ominus$ is interpreted as truncated subtraction on $[0, \top]$ given by $p \ominus q := \max(p - q, 0)$, min is interpreted as minimum and where negation on $[0, \top]$ is defined as $\neg q := \top - q$. Describing the category of L-algebras that precisely represents a given logic (i.e., where the initial algebra corresponds to the set of formulas modulo equivalence) is in general nontrivial. For studying expressivity, however, it is sufficient to consider formulas and their semantics, i.e., expressivity of a real-valued logic for B-coalgebras for some functor $B : \mathsf{Set} \to \mathsf{Set}$ can be studied by considering the dual adjunction

$$B \circlearrowright \mathsf{Set} \underset{Q = \mathrm{Hom}(-,[0,\top])}{\overset{P = [0,\top]^{-}}{\rightleftarrows}} \mathrm{Alg}(\Sigma)^{\mathsf{op}} \circlearrowleft L \qquad (P \dashv Q)$$

where $\Sigma X = 1 + X^2 + X + X \times (\mathbb{Q} \cap [0, \top])$ and $L(A) = T_\Sigma(\{[\mathfrak{c}]a \mid a \in A, \mathfrak{c} \in \mathfrak{C}\})$ with $T_\Sigma(G)$ denoting the free Σ-algebra over a set G of generators.

Another class of adjunctions we use relates Rel to categories of algebras. To formulate it, we assume:

- $\mathcal{V}$ is a complete lattice of distance values,
- Ω is a bounded poset of truth values,
- $\Sigma \colon \mathsf{Set} \to \mathsf{Set}$ is a functor,
- $a_\Omega \colon \Sigma\Omega \to \Omega$ is a Σ-algebra,
- $(\Omega, R_\Omega \colon \Omega \times \Omega \to \mathcal{V}) \in \mathrm{Rel}_\mathcal{V}$, and
- Σ has a lifting $\overline{\Sigma} \colon \mathrm{Rel}_\mathcal{V} \to \mathrm{Rel}_\mathcal{V}$ such that
  1. there is a morphism $\overline{a}_\Omega \colon \overline{\Sigma} R_\Omega \to R_\Omega$ above a_Ω and
  2. for any $(X, R), (Y, S) \in \mathrm{Rel}_\mathcal{V}$ there is a morphism $\overline{\mathrm{st}}_{R,S} \colon R \times \overline{\Sigma} S \to \overline{\Sigma}(R \times S)$ above the strength map $\mathrm{st}_{X,Y} : X \times \Sigma Y \to \Sigma(X \times Y)$ for the set functor Σ.

▶ **Proposition 11.** *Under the above assumptions there is a dual adjunction*

$$\mathrm{Rel}_\mathcal{V} \underset{\mathrm{Hom}(\_, a_\Omega)}{\overset{\mathrm{Hom}(\_, R_\Omega)}{\rightleftarrows}} \mathrm{Alg}(\Sigma)^{\mathsf{op}} \qquad (\bot) \tag{5}$$

▶ **Corollary 12.** *In the above scenario, assume that* Σ *is a polynomial functor and* $\overline{\Sigma}\colon \mathrm{Rel}_{\mathcal{V}} \to \mathrm{Rel}_{\mathcal{V}}$ *is interpreted to be the canonical lifting of* Σ *that interprets products and coproducts occurring in* Σ *as products and coproducts in* $\mathrm{Rel}_{\mathcal{V}}$*, respectively. Then the condition on* $\mathrm{st}_{R,S}$ *is always satisfied and the dual adjunction from (5) exists if there is a morphism* $\overline{a}_\Omega\colon \overline{\Sigma}R_\Omega \to R_\Omega$ *above* a_Ω*.*

The following remark is obvious, but at the same time useful for concrete examples.

▶ Remark 13. In the above cases, let $\mathcal{C}$ be a full subcategory of $\mathrm{Rel}_{\mathcal{V}}$ and $\mathcal{D}$ a full subcategory of $\mathrm{Alg}(\Sigma)$ such that $\mathrm{Hom}(-, a_\Omega)$ and $\mathrm{Hom}(-, R_\Omega)$ restrict to functors of type $\mathcal{D} \to \mathcal{C}$ and of type $\mathcal{C} \to \mathcal{D}$, respectively. Then the above dual adjunction restricts to a dual adjunction between $\mathcal{C}$ and $\mathcal{D}$.

3 Abstract Framework: Adequacy & Expressivity

In this section, we define when a logic is adequate and expressive with respect to a coinductive predicate, and provide sufficient conditions on the logic. Coinductive predicates are expressed abstractly via fibrations and functor lifting, and logic via a contravariant adjunction. Therefore, we make the following assumptions.

▶ **Assumption 14.** Throughout this section, we assume:
1. *(Type of coalgebra)* An endofunctor $B\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$;
2. *(Coinductive predicate)* A $\mathsf{CLat}_\wedge$-fibration $p\colon \mathcal{E} \to \mathcal{C}$ and a lifting $\overline{B}\colon \mathcal{E} \to \mathcal{E}$ of B;
3. *(Coalgebraic logic)* An adjunction $P \dashv Q\colon \mathcal{C} \leftrightarrows \mathcal{D}^{\mathsf{op}}$, a functor $L\colon \mathcal{D} \to \mathcal{D}$ with an initial algebra $\alpha\colon L(\Phi) \xrightarrow{\cong} \Phi$, and a natural transformation $\delta\colon BQ \Rightarrow QL$.

As explained in the introduction, to formulate adequacy and expressiveness, we need one more crucial ingredient: an object that stipulates how collections of formulas should be compared. In the abstract fibrational setting, we assume an object above $Q\Phi$; more systematically, a functor $\overline{Q}$ above Q.

▶ **Definition 15** (Adequacy and Expressivity)**.** *Let* $\overline{Q}\colon \mathcal{D}^{\mathsf{op}} \to \mathcal{E}$ *be a functor such that* $p \circ \overline{Q} = Q$*. We say the logic* (L, δ) *is*
- adequate *if* $\nu(\gamma^* \circ \overline{B}_X) \leq th^*(\overline{Q}\Phi)$ *for every* B*-coalgebra* (X, γ)*;*
- expressive *if* $\nu(\gamma^* \circ \overline{B}_X) \geq th^*(\overline{Q}\Phi)$ *for every* B*-coalgebra* (X, γ)*.*

When we need to refer to the functors $\overline{Q}$ or $\overline{B}$ explicitly, we speak about adequacy and expressivity *via* $\overline{Q}$ *w.r.t.* $\overline{B}$. Examples follow in Section 3.2, where classical expressivity and adequacy w.r.t. bisimilarity is recovered, and Section 4, where other instances are treated.

▶ Remark 16. Definition 15 can be generalised to arbitrary poset fibrations, not necessarily assuming complete lattice structure on the fibres, as follows. Adequacy means that for any B-coalgebra (X, γ), if $R \leq \gamma^* \circ \overline{B}_X(R)$ then $R \leq th^*(\overline{Q}\Phi)$. Expressivity means that for any B-coalgebra (X, γ), we have $th^*(\overline{Q}\Phi) \leq R$ for some R with $R \leq \gamma^* \circ \overline{B}_X(R)$. In fact, with these definitions, if (L, δ) is both adequate and expressive then $\gamma^* \circ \overline{B}_X$ has a greatest fixed point, given by $th^*(\overline{Q}\Phi)$. We prefer to work with $\mathsf{CLat}_\wedge$-fibrations, since the definition is slightly simpler, and it covers all our examples.

3.1 Sufficient conditions for expressivity and adequacy

The results below give conditions on $\overline{B}$, $\overline{Q}$ and primarily the one-step semantics δ that guarantee expressivity (Theorem 19) and adequacy (Theorem 18). For simplicity we fix the functor $\overline{Q}$.

▶ **Assumption 17.** In the remainder of this section we assume a functor $\overline{Q}\colon \mathcal{D}^{\mathsf{op}} \to \mathcal{E}$ such that $p \circ \overline{Q} = Q$.

For adequacy, the main idea is to require sufficient conditions to lift δ to a logic for $\overline{B}$.

▶ **Theorem 18.** *Suppose that*
1. $\overline{B}\,\overline{Q}X \leq \delta_X^*(\overline{Q}LX)$ *for every object X in $\mathcal{D}$, and*
2. *the functor $\overline{Q}$ has a left adjoint.*

Then (L, δ) is adequate.

Proof. The first assumption yields a natural transformation $\overline{\delta}\colon \overline{B}\,\overline{Q} \Rightarrow \overline{Q}L$, defined on a component X by

$$\overline{\delta}_X = \left(\overline{B}\,\overline{Q}X \longrightarrow \delta_X^*(\overline{Q}LX) \xrightarrow{\widetilde{\delta}} \overline{Q}LX \right)$$

where the left arrow is the inclusion $\overline{B}\,\overline{Q}X \leq \delta_X^*(\overline{Q}LX)$, and the right arrow $\widetilde{\delta}$ is the Cartesian morphism to $\overline{Q}LX$ above δ_X. It follows that $\overline{\delta}_X$ is above δ_X. Further, naturality follows from p being faithful (as it is a poset fibration, see Section 2.1) and naturality of δ. Observe that we have thus established $(L, \overline{\delta})$ as a logic for $\overline{B}$-coalgebras, via the adjunction $\overline{P} \dashv \overline{Q}$.

Now let (X, γ) be a B-coalgebra, and $R = \nu(\gamma^* \circ \overline{B}_X)$. Then, in particular, $R \leq \gamma^* \circ \overline{B}_X(R)$, which is equivalent to a coalgebra $\overline{\gamma}\colon R \to \overline{B}R$ above $\gamma\colon X \to BX$. The logic $(L, \overline{\delta})$ gives us a theory map $\overline{th}$ of $(R, \overline{\gamma})$ as the unique map making the following diagram commute.

$$\begin{array}{ccccc}
R & & \overset{\overline{th}}{\dashrightarrow} & & \overline{Q}\Phi \\
\downarrow{\scriptstyle \overline{\gamma}} & & & & \downarrow{\scriptstyle \overline{Q}\alpha} \\
\overline{B}R & \overset{\overline{B}\,\overline{th}}{\dashrightarrow} & \overline{B}\,\overline{Q}\Phi & \xrightarrow{\overline{\delta}} & \overline{Q}L\Phi
\end{array}$$

Since $p \circ \overline{Q} = Q$ and $p(\overline{\delta}_\Phi) = \delta_\Phi$, it follows that $p(\overline{th})$ equals the theory map th of (X, γ). Hence $R \leq th^*(\overline{Q}\Phi)$ as required. ◀

Expressivity requires the converse inequality of the one in Theorem 18, but only on one component: the carrier Φ of the initial algebra. Further, the conditions include that $(\overline{B}, B)$ is a fibration map. In particular, for the canonical relation lifting $\mathrm{Rel}(B)$ this means that B should preserve weak pullbacks; this case is explained in more detail in Section 3.2.

▶ **Theorem 19.** *Suppose $(\overline{B}, B)$ is a fibration map. If $\delta_\Phi^*(\overline{Q}L\Phi) \leq \overline{B}\,\overline{Q}\Phi$ then (L, δ) is expressive.*

Proof. Let (X, γ) be a B-coalgebra, with th the associated theory map. We show that $th^*(\overline{Q}\Phi)$ is a post-fixed point of $\gamma^* \circ \overline{B}_X$:

$$\begin{aligned}
th^*(\overline{Q}\Phi) &= (Q(\alpha^{-1}) \circ \delta_\Phi \circ B th \circ \gamma)^*(\overline{Q}\Phi) \\
&= \gamma^* \circ (B th)^* \circ \delta_\Phi^* \circ Q(\alpha^{-1})^*(\overline{Q}\Phi) \\
&= \gamma^* \circ (B th)^* \circ \delta_\Phi^*(\overline{Q}L\Phi) && \text{(follows from } \alpha^{-1} \text{ being an iso)} \\
&\leq \gamma^* \circ (B th)^*(\overline{B}\,\overline{Q}\Phi) && \text{(assumption)} \\
&= \gamma^* \circ \overline{B}_X \circ th^*(\overline{Q}\Phi) && \text{((}\overline{B}, B\text{) fibration map)}
\end{aligned}$$

Expressivity follows since $\nu(\gamma^* \circ \overline{B}_X)$ is the greatest post-fixed point. ◀

3.2 Adequacy and Expressivity w.r.t. Bisimilarity

In the setting of coalgebraic modal logic recalled in Section 2.2, Klin [22] proved that
1. the theory map th of a coalgebra (X, γ) factors through coalgebra morphisms from (X, γ);
2. if δ has monic components, then th factors as a coalgebra morphism followed by a mono.

The first item can be seen as adequacy w.r.t. behavioural equivalence (i.e., identification by a coalgebra morphism), and the second as expressivity.

In the current section we revisit this result for Set functors, as a sanity check of Definition 15. To this end, we focus on the canonical lifting $\mathrm{Rel}(B)\colon \mathrm{Rel} \to \mathrm{Rel}$ of a Set functor B in the relation fibration, so that, for a coalgebra (X, γ), $\nu(\gamma^* \circ \mathrm{Rel}(B)_X)$ is coalgebraic bisimilarity. We have to restrict to weak pullback preserving functors B. The reason is that expressive logics typically capture behavioural equivalence rather than bisimilarity. As is well-known, for weak pullback preserving functors, the two coincide [32].

To obtain the appropriate notion of adequacy and expressivity, we need to compare collections of formulas for equality. Therefore, the functor $\overline{Q}$ in Definition 15 will be instantiated with $\overline{Q}X = (QX, \Delta_{QX})$ where Δ_{QX} denotes the diagonal. Then, for a coalgebra (X, γ), $th^*(\overline{Q}\Phi)$ is the set of all pairs of states (x, y) such that $th(x) = th(y)$. Adequacy then means that for every coalgebra (X, γ), bisimilarity is contained in $th^*(\overline{Q}\Phi)$, i.e., if x is bisimilar to y then $th(x) = th(y)$. Expressivity is the converse implication.

To state and prove the result, let $\mathsf{Eq}\colon \mathsf{Set} \to \mathrm{Rel}$ be the functor given by $\mathsf{Eq}(X) = \Delta_X$. This functor has a left adjoint $\mathsf{Quot}\colon \mathrm{Rel} \to \mathsf{Set}$, which maps a relation $R \subseteq X \times X$ to the quotient of X by the least equivalence relation containing R (cf. [14]).

▶ **Proposition 20** (Adequacy and expressivity w.r.t. bisimilarity)**.** *Consider the relation fibration $p\colon \mathrm{Rel} \to \mathsf{Set}$, let $B\colon \mathsf{Set} \to \mathsf{Set}$ be a weak pullback preserving functor, let $P \dashv Q\colon \mathsf{Set} \leftrightarrows \mathcal{D}^{\mathsf{op}}$ for some category $\mathcal{D}$, $L\colon \mathcal{D} \to \mathcal{D}$ a functor with an initial algebra and $\delta\colon BQ \Rightarrow QL$. Then*
1. *(L, δ) is adequate w.r.t. $\mathrm{Rel}(B)$;*
2. *if δ is componentwise injective, then (L, δ) is expressive w.r.t. $\mathrm{Rel}(B)$,*

via $\overline{Q} = \mathsf{Eq} \circ Q$.

Proof. For adequacy, we use Theorem 18. By composition of adjoints, $P \circ \mathsf{Quot}$ is a left adjoint to $\mathsf{Eq} \circ Q$. It will be useful to simplify $\mathrm{Rel}(B) \circ \mathsf{Eq} \circ QX$ and $\delta_X^*(\mathsf{Eq} \circ Q \circ LX)$:

$$\mathrm{Rel}(B) \circ \mathsf{Eq} \circ QX = \mathrm{Rel}(B)(\Delta_{QX}) = \Delta_{BQX}\,, \tag{6}$$

$$\delta_X^*(\mathsf{Eq} \circ Q \circ LX) = (\delta_X \times \delta_X)^{-1}(\Delta_{QLX})\,, \tag{7}$$

using that $\mathrm{Rel}(B) \circ \mathsf{Eq} = \mathsf{Eq} \circ B$ in the first equality (e.g., [18]). The remaining hypothesis of Theorem 18 is that $\mathrm{Rel}(B) \circ \mathsf{Eq} \circ QX \le \delta_X^*(\mathsf{Eq} \circ Q \circ LX)$ for all X, i.e., $\Delta_{BQX} \subseteq (\delta_X \times \delta_X)^{-1}(\Delta_{QLX})$, which is trivial.

For expressivity, we use Theorem 19. Since B preserves weak pullbacks, $(\mathrm{Rel}(B), B)$ is a fibration map. We need to prove that $\delta_\Phi^*(\mathsf{Eq} \circ Q \circ L\Phi) \le \mathrm{Rel}(B) \circ \mathsf{Eq} \circ Q\Phi$, which amounts to the inclusion

$$(\delta_\Phi \times \delta_\Phi)^{-1}(\Delta_{QL\Phi}) \subseteq \Delta_{BQ\Phi}$$

But this is equivalent to injectivity of δ_Φ. ◀

4 Examples

In this section we instantiate the abstract framework to three concrete examples: a behavioural metric on deterministic automata (Section 4.1), captured by $[0, 1]$-valued tests; a unary predicate on transition systems (Section 4.2); and similarity of transition systems, captured by a logic with conjunction and diamond modalities (Section 4.3).

4.1 Shortest distinguishing word distance

We study a simple behavioural distance on deterministic automata: for two states x, y and a fixed constant c with $0 < c < 1$, the distance is given by c^n, where n is the length of the smallest word accepted from one state but not the other. Following [4], this is referred to as the *shortest distinguishing word distance*, and, for an automaton with state space X, denoted by $d_{sdw} \colon X \times X \to [0,1]$.

Formally, fix a finite alphabet A, and consider the functor $B \colon \mathsf{Set} \to \mathsf{Set}$, $BX = 2 \times X^A$ of deterministic automata. We make use of the fibration $p \colon \mathrm{Rel}_{[0,1]} \to \mathsf{Set}$, and define the lifting $\overline{B} \colon \mathrm{Rel}_{[0,1]} \to \mathrm{Rel}_{[0,1]}$ by

$$\overline{B}(X,d) = \left(BX, ((o_1,t_1),(o_2,t_2)) \mapsto \begin{cases} 1 & \text{if } o_1 \neq o_2 \\ c \cdot \max_{a \in A}\{d(t_1(a), t_2(a))\} & \text{otherwise} \end{cases} \right)$$

The shortest distinguishing word distance d_{sdw} on a deterministic automaton $\gamma \colon X \to 2 \times X^A$ is the greatest fixed point $\nu(\gamma^* \circ \overline{B}_X)$.

For an associated logic, we simply use words over A as formulas, and define a satisfaction relation which is weighted in $[0,1]$. Consider the following setting.

$$B{=}2\times\mathsf{Id}^A \circlearrowright \mathsf{Set} \underset{Q=[0,1]^-}{\overset{P=[0,1]^-}{\rightleftarrows}} \mathsf{Set}^{\mathsf{op}} \circlearrowleft L{=}A\times\mathsf{Id}{+}1 \qquad \text{with} \qquad B([0,1]^-) \overset{\delta}{\Longrightarrow} [0,1]^{L-}$$

The initial algebra of L is the set of words A^*. The natural transformation δ is given by $\delta_X \colon 2 \times ([0,1]^X)^A \to [0,1]^{A \times X + 1}$,

$$\delta_X(o,t)(u) = \begin{cases} o & \text{if } u = * \in 1 \\ c \cdot t(a)(x) & \text{if } u = (a,x) \in A \times X \end{cases}$$

which is a quantitative, discounted version of the Boolean-valued logic in Example 9. The logic (L, δ) defines, for any deterministic automaton $\langle o, t \rangle \colon X \to 2 \times X^A$, a theory map $th \colon X \to [0,1]^{A^*}$, given by

$$th(x)(\varepsilon) = o(x) \qquad \text{and} \qquad th(x)(aw) = c \cdot th(t(x)(a))(w)\,,$$

for all $x \in X$, $a \in A$, $w \in A^*$.

We characterise the shortest distinguishing word distance with the above logic, by instantiating and proving adequacy and expressivity. Define

$$\overline{Q} \colon \mathsf{Set}^{\mathsf{op}} \to \mathrm{Rel}_{[0,1]}\,, \qquad \overline{Q}(X) = \left([0,1]^X, (\phi_1, \phi_2) \mapsto \sup_{x \in X} |\phi_1(x) - \phi_2(x)| \right).$$

Technically, this functor is given by mapping a set X to the X-fold product of the object $\overline{[0,1]} = ([0,1], (r,s) \mapsto |r - s|)$. It follows immediately that $\overline{Q}$ has a left adjoint, mapping (X,d) to $\mathrm{Hom}((X,d), \overline{[0,1]})$, see Equation 4. This will be useful for proving adequacy below.

The functor $\overline{Q}$ yields a "logical distance" between states $x, y \in X$, given by $th^*(\overline{Q}\Phi)$. We abbreviate it by $d_{log} \colon X \times X \to [0,1]$. Explicitly, we have

$$d_{log}(x,y) = \sup_{w \in A^*} |th(x)(w) - th(y)(w)|\,. \tag{8}$$

Instantiating Definition 15, the logic (L, δ) is

- *adequate* if $d_{sdw} \geq d_{log}$, and
- *expressive* if $d_{sdw} \leq d_{log}$.

Here $\leq$ is the usual order on $[0,1]$, with 0 the least element (the order in $\mathrm{Rel}_{[0,1]}$ is reversed).

To prove adequacy and expressivity, we use Theorem 18 and Theorem 19. The functor $\overline{Q}$ has a left adjoint, as explained above. Further, $(\overline{B}, B)$ is a fibration map [4]. We prove the remaining hypotheses of both propositions by showing the equality $\overline{B}\,\overline{Q}X = \delta_X^*(\overline{Q}LX)$ for every object X in $\mathcal{D}$. To this end, we compute (suppressing the carrier set BQX):

$$\begin{aligned}
& \delta_X^*(\overline{Q}LX) \\
= \; & \big(((o_1,t_1),(o_2,t_2)) \mapsto \sup\nolimits_{u \in A \times X + 1} |\delta_X(o_1,t_1)(u) - \delta_X(o_2,t_2)(u)|\big) \\
= \; & \left((o_1,t_1),(o_2,t_2)) \mapsto \begin{cases} 1 & \text{if } o_1 \neq o_2 \\ \sup_{u \in A \times X} |\delta_X(o_1,t_1)(u) - \delta_X(o_2,t_2)(u)|) & \text{otherwise} \end{cases} \right) \\
= \; & \left((o_1,t_1),(o_2,t_2)) \mapsto \begin{cases} 1 & \text{if } o_1 \neq o_2 \\ \sup_{(a,x) \in A \times X} |c \cdot t_1(a)(x) - c \cdot t_2(a)(x)|) & \text{otherwise} \end{cases} \right) \\
= \; & \left((o_1,t_1),(o_2,t_2)) \mapsto \begin{cases} 1 & \text{if } o_1 \neq o_2 \\ c \cdot \max_{a \in A} \sup_{x \in X} |t_1(a)(x) - t_2(a)(x)|) & \text{otherwise} \end{cases} \right) \\
= \; & \overline{B}\,\overline{Q}X
\end{aligned}$$

Hence, the logic (L, δ) is adequate and expressive w.r.t. the shortest distinguishing word distance, i.e., d_{sdw} coincides with the logical distance d_{log} given in Equation 8.

4.2 Divergence of processes

A state of an LTS is said to be *diverging* if there exists an infinite path of τ-transitions starting at that state. To model this predicate, let $B \colon \mathsf{Set} \to \mathsf{Set}$, $BX = (\mathcal{P}_\omega X)^A$, where A is a set of labels containing the symbol $\tau \in A$. Consider the predicate fibration $p \colon \mathrm{Pred} \to \mathsf{Set}$, and define the lifting $\overline{B} \colon \mathrm{Pred} \to \mathrm{Pred}$ by

$$\overline{B}(X, \Gamma) = ((\mathcal{P}_\omega X)^A, \{t \mid \exists x \in \Gamma.\, x \in t(\tau)\})\,.$$

The coinductive predicate defined by $\overline{B}$ on a B-coalgebra (X, γ) is the set of diverging states:

$$\nu(\gamma^* \circ \overline{B}_X) = (X, \{x \mid x \text{ is diverging}\})\,.$$

Now, we want to prove in our framework of adequacy and expressivity that x is diverging iff for every $n \in \mathbb{N}$ there is a finite path of τ-steps starting in x, i.e., $x \models \langle\tau\rangle^n \top$ for every n. The proof relies on two main observations:

- if x satisfies infinitely many formulas of $\langle\tau\rangle^n \top$, then one of its τ-successors does, too;
- if a state x satisfies $\langle\tau\rangle^n \top$ for some n then x satisfies $\langle\tau\rangle^m \top$ for all $0 \leq m \leq n$.

Combined, one can then give a coinductive proof, showing that if the current state satisfies all formulas of the form $\langle\tau\rangle^n \top$ then one of its τ-successors also satisfies all these formulas.

We make this argument precise by casting it into the abstract framework. First, for the logic, we have the following setting:

$$B = (\mathcal{P}_\omega -)^A \circlearrowright \mathsf{Set} \underset{Q = \mathrm{Hom}(-,2)}{\overset{P = 2^-}{\rightleftarrows}} \mathsf{Pos}^{\mathsf{op}} \circlearrowleft L = \mathsf{Id}_\top \qquad \text{with} \qquad B\mathrm{Hom}(-,2) \overset{\delta}{\Longrightarrow} \mathrm{Hom}(L-,2)$$

Here Pos is the category of posets and monotone maps, and $2 = \{0,1\}$ is the poset given by the order $0 \leq 1$. For a poset S, $\mathrm{Hom}(S,2)$ is then the set of *upwards closed* subsets of S.

The functor $LS = S_\top$ is defined on a poset S by adjoining a new top element $\top$, i.e., the carrier is $S + \{\top\}$ and $\top$ is strictly above all elements of S. The initial algebra Φ of L is the set of natural numbers, representing the formulas of the form $\langle\tau\rangle^n \top$, linearly ordered, with 0 the top element. The choice of Pos means that the set $\mathrm{Hom}(\Phi, 2)$ used to represent the theory of a state $x \in X$ consists of upwards closed sets (so closed under lower natural numbers in the usual ordering), corresponding to the second observation above concerning the set of formulas satisfied by x.

The natural transformation δ is given by $\delta_S\colon (\mathcal{P}_\omega \mathrm{Hom}(S, 2))^A \to \mathrm{Hom}(S_\top, 2)$,

$$\delta_S(t)(x) = \begin{cases} 1 & \text{if } x = \top \\ \bigvee_{\phi \in t(\tau)} \phi(x) & \text{otherwise} \end{cases}.$$

To show that this is well-defined, suppose $x, y \in S_\top$ with $x \le y$, and suppose $\delta_S(t)(x) = 1$. If $x = \top$ then $y = \top$, so $\delta_S(t)(y) = 1$. Otherwise, there is $\phi \in \mathrm{Hom}(S, 2)$ such that $\phi \in t(\tau)$ and $\phi(x) = 1$. Since ϕ is upwards closed, $\phi(y) = 1$ and consequently $\delta_S(t)(y) = 1$ as needed.

Now, the theory map $th\colon X \to \mathrm{Hom}(\Phi, 2)$ is given by $th(x)(n) = 1$ iff there exists a path of τ-steps of length n from x. We define

$$\overline{Q}\colon \mathsf{Pos}^{\mathsf{op}} \to \mathrm{Pred}\,, \quad \overline{Q}(S) = (\mathrm{Hom}(S, 2), \{\phi \mid \forall x \in S.\, \phi(x) = 1\})\,.$$

Instantiating Definition 15, *adequacy* means that if x is diverging, then $x \models \langle\tau\rangle^n \top$ for all n; and expressivity is the converse.

We start with proving adequacy, using Theorem 18. The left adjoint $\overline{P}$ is given by $\overline{P}(X, \Gamma) = (\mathrm{Hom}((X, \Gamma), (2, \{1\})), \{(\phi_1, \phi_2) \mid \forall x \in X.\, \phi_1(x) \le \phi_2(x)\})$. It remains to prove that $\overline{B}\,\overline{Q}(S) \le \delta_S^*(\overline{Q}LS)$ for all S. To this end, we observe $BQS = (\mathcal{P}_\omega(\mathrm{Hom}(S, 2)))^A$ and compute:

$$\begin{aligned}
\delta_S^*(\overline{Q}LS) &= \{t \mid \delta_S(t) \in \overline{Q}LS\} \\
&= \{t \mid \forall x \in S_\top.\, \delta_S(t)(x) = 1\} \\
&= \{t \mid \forall x \in S.\, \delta_S(t)(x) = 1\} \\
&= \{t \mid \forall x \in S.\, \bigvee_{\phi \in t(\tau)} \phi(x) = 1\}
\end{aligned}$$

and $\overline{B}\,\overline{Q}(S) = \{t \mid (\lambda x.1) \in t(\tau)\}$. The needed inclusion is now trivial.

For expressivity we have to prove the reverse inclusion with $S = \Phi$, i.e.,

$$\{t \in (\mathcal{P}_\omega(\mathrm{Hom}(\Phi, 2)))^A \mid \forall x \in \Phi.\, \bigvee_{\phi \in t(\tau)} \phi(x) = 1\} \subseteq \{t \in (\mathcal{P}_\omega(\mathrm{Hom}(\Phi, 2)))^A \mid (\lambda x.1) \in t(\tau)\}.$$

To this end, let t be an element of the left-hand side, and suppose towards a contradiction that for all ϕ with $\phi \in t(\tau)$, there is an element $x_\phi \in \Phi$ with $\phi(x_\phi) = 0$. Choosing an assignment $\phi \mapsto x_\phi$ of such elements, we get a *finite* set $\{x_\phi \mid \phi \in t(\tau)\}$. Let x_ϕ be the smallest element of that set (w.r.t. the order of Φ, i.e., the largest natural number), and let $\psi \in \mathrm{Hom}(\Phi, 2)$ be such that $\psi(x_\phi) = 1$; such a ψ exists by assumption on t. However, since $x_\phi \le x_\psi$ and ψ is upwards closed we have $\psi(x_\psi) = 1$, which gives a contradiction. Hence, the inclusion holds as required. The lifting $(\overline{B}, B)$ is a fibration map. We thus conclude from Theorem 19 that the logic is expressive.

4.3 Simulation of processes

Let $B\colon \mathsf{Set} \to \mathsf{Set}$, $BX = (\mathcal{P}_\omega X)^A$, and let $\gamma\colon X \to (\mathcal{P}_\omega X)^A$ be B-coalgebra, i.e., a labelled transition system. Denote *similarity* by $\precsim \subseteq X \times X$, defined more precisely below. Consider the logic with the following syntax:

$$\varphi, \psi ::= \langle a \rangle \varphi \mid \varphi \wedge \psi \mid \top \tag{9}$$

where a ranges over A, with the usual interpretation $x \models \varphi$ for states $x \in X$. A classical Hennessy-Milner theorem for similarity is:

$$x \precsim y \text{ iff } \forall \varphi.\, x \models \varphi \to y \models \varphi\,. \tag{10}$$

We show how to formulate and prove this result within our abstract framework.

First, recall from Equation 2 in Section 2.1 the appropriate lifting $\overline{B}\colon \mathrm{Rel} \to \mathrm{Rel}$ in the relation fibration $p\colon \mathrm{Rel} \to \mathsf{Set}$. A simulation on a B-coalgebra (X, γ) is a relation R such that $R \leq \gamma^* \circ \overline{B}_X(R)$, and similarity $\precsim$ is the greatest fixed point of $\gamma^* \circ \overline{B}_X$.

For the logic, to incorporate finite conjunction, we instantiate $\mathcal{D}$ with the category SL of bounded (meet)-semilattices, i.e., sets equipped with an associative, commutative and idempotent binary operator $\wedge$ and a top element $\top$.

To add the modalities $\langle a \rangle$ for each $a \in A$, we proceed as follows. Let $U\colon \mathsf{SL} \to \mathsf{Set}$ be the forgetful functor. It has a left adjoint $\mathcal{F}\colon \mathsf{Set} \to \mathsf{SL}$, mapping a set X to the meet-semilattice $\mathcal{P}_\omega(X)$ with the top element given by $\emptyset$ and the meet by union. The functor $L\colon \mathsf{SL} \to \mathsf{SL}$ is given by $LX = \mathcal{F}(A \times UX)$; its initial algebra Φ consists precisely of the logic presented in Equation 9, quotiented by the semilattice equations. For the adjunction, we use:

$$B{=}(\mathcal{P}_\omega -)^A \circlearrowright \mathsf{Set} \underset{Q=\mathrm{Hom}(-,2)}{\overset{P=2^-}{\rightleftarrows}} \mathsf{SL}^{\mathsf{op}} \circlearrowleft L{=}\mathcal{F}(A \times U-) \qquad \text{with} \quad B\mathrm{Hom}(-,2) \overset{\delta}{\Longrightarrow} \mathrm{Hom}(L-,2)$$

which is an instance of Equation 4. Here $2 = \{0, 1\}$ is the meet-semilattice given by the order $0 \leq 1$. For a semilattice S, the set $\mathrm{Hom}(S, 2)$ of semi-lattice morphisms is isomorphic to the set of *filters* on S: subsets $X \subseteq S$ such that $\top \in X$, and $x, y \in X$ iff $x \wedge y \in X$.

To define the natural transformation $\delta_S\colon (\mathcal{P}_\omega(\mathrm{Hom}(S, 2)))^A \to \mathrm{Hom}(\mathcal{F}(A \times US), 2)$ on a semilattice S, we use that for every map $f\colon A \times US \to 2$ there is a unique semilattice homomorphism $f^\sharp\colon \mathcal{F}(A \times US) \to 2$:

$$\delta_S(t) = ((a, x) \mapsto \bigvee_{\phi \in t(a)} \phi(x))^\sharp = \left(W \mapsto \bigwedge_{(a,x) \in W} \bigvee_{\phi \in t(a)} \phi(x) \right).$$

For an LTS (X, γ), the associated theory map $th\colon X \to \mathrm{Hom}(\Phi, 2)$ maps a state to the formulas in (9) that it accepts, with the usual semantics.

To recover (10), we need to relate logical theories appropriately. Define

$$\overline{Q}\colon \mathsf{SL}^{\mathsf{op}} \to \mathrm{Rel}\,, \qquad \overline{Q}S = (\mathrm{Hom}(S, 2), \{(\phi_1, \phi_2) \mid \forall x \in S.\, \phi_1(x) \leq \phi_2(x)\})\,.$$

Then $th^*(\overline{Q}\Phi) = \{(x, y) \mid \forall \varphi \in \Phi.\, th(x)(\varphi) \leq th(y)(\varphi)\}$, i.e., it relates all (x, y) such that the set of formulas satisfied at x is included in the set of formulas satisfied at y. Thus, instantiating Definition 15, adequacy $\precsim = \nu(\gamma^* \circ \overline{B}_X) \leq th^*(\overline{Q}\Phi)$ is the implication from left to right in Equation 10, and expressivity is the converse.

We prove adequacy and expressivity. The functor $\overline{Q}$ has a left adjoint, given by $\overline{P}(X, R) = \mathrm{Hom}((X, R), \overline{2})$, where $\overline{2} = (2, \{(x, y) \mid x \leq y\})$. This follows by a straightforward computation, or using Proposition 11 with Remark 13, with SL as a full subcategory of the category of all algebras for the corresponding signature.

Given a semilattice S, we compute $\delta_S^*(\overline{Q}LS) \subseteq (BQS)^2 = ((\mathcal{P}_\omega(\mathrm{Hom}(S,2)))^A)^2$:

$$\begin{aligned}\delta_S^*(\overline{Q}LS) &= \delta_S^*(\{(\phi_1,\phi_2) \mid \forall W \in \mathcal{F}(A \times US).\, \phi_1(W) \leq \phi_2(W)\}) \\ &= \{(t_1,t_2) \mid \forall W \in \mathcal{F}(A \times US).\, \bigwedge_{(a,x)\in W} \bigvee_{\phi \in t_1(a)} \phi(x) \leq \bigwedge_{(a,x)\in W} \bigvee_{\phi\in t_2(a)} \phi(x)\}\,.\end{aligned}$$

Further, $\overline{B}\,\overline{Q}S = \{(t_1,t_2) \mid \forall a \in A.\forall \phi_1 \in t_1(a).\,\exists \phi_2 \in t_2(a).\,\forall x \in S.\,\phi_1(x) \leq \phi_2(x)\}$. For adequacy, we need to prove $\overline{B}\,\overline{Q}S \leq \delta_S^*(\overline{Q}LS)$; but this is trivial, given the above computations. For expressivity, let $(t_1,t_2) \in \delta_S^*(\overline{Q}LS)$. We need to show that $(t_1,t_2) \in \overline{B}\,\overline{Q}S$. Suppose, towards a contradiction, that $(t_1,t_2) \notin \overline{B}\,\overline{Q}S$, i.e., there exist $a \in A$ and $\phi_1 \in t_1(a)$ such that for all $\phi_2 \in t_2(a)$, there is $x \in S$ with $\phi_1(x) = 1$ and $\phi_2(x) = 0$. We choose such an element x_{ϕ_2} for every $\phi_2 \in t_2(a)$. Note that the collection $\{x_{\phi_2} \mid \phi_2 \in t_2(a)\}$ is *finite* – here we make use of the image-finiteness captured by the functor B. Now, consider the conjunction $\psi = \bigwedge_{\phi_2\in t_2(a)} x_{\phi_2} \in S$. Using that ϕ_1 is a homomorphism, we have $\phi_1(\psi) = \phi_1(\bigwedge_{\phi_2\in t_2(a)} x_{\phi_2}) = \bigwedge_{\phi_2\in t_2(a)} \phi_1(x_{\phi_2}) = 1$, and consequently $\bigvee_{\phi\in t_1(a)} \phi(\psi) = 1$. We also have $\bigvee_{\phi\in t_2(a)} \phi(\psi) = \bigvee_{\phi\in t_2(a)} \bigwedge_{\phi_2\in t_2(a)} \phi(x_{\phi_2}) = 0$ since $\phi_2(x_{\phi_2}) = 0$ for every $\phi_2 \in t_2(a)$. Finally, to arrive at a contradiction, let $W = \{(a,\psi)\}$. Since $(t_1,t_2) \in \delta_S^*(\overline{Q}LS)$ this implies $\bigvee_{\phi\in t_1(a)} \phi(\psi) \leq \bigvee_{\phi\in t_2(a)} \phi(\psi)$, which is in contradiction with the above. It is easy to check that $(\overline{B}, B)$ is a fibration map (cf. [16]). Hence, we conclude expressivity from Theorem 19.

5 Future work

We proposed suitable notions of expressivity and adequacy, connecting coinductive predicates in a fibration to coalgebraic modal logic in a contravariant adjunction. Further, we gave sufficient conditions on the one-step semantics that guarantee expressivity and adequacy, and showed how to put these methods to work in concrete examples.

There are several avenues for future work. First, an intriguing question is whether the characterisation of behavioural metrics in [24, 37] can be covered in the setting of this paper, as well as logics for other distances such as the (abstract, coalgebraic) Wasserstein distance. Those behavioural metrics are already framed in a fibrational setting [4, 34, 2, 23]. While all our examples are for coalgebras in Set, the fibrational framework allows different base categories, which might be useful to treat, e.g., behavioural metrics for continuous probabilistic systems [35].

A further natural question is whether we can automatically *derive* logics for a given predicate. As mentioned in the introduction, there are various tools to find expressive logics for behavioural equivalence. But extending this to the current general setting is non-trivial. Finally, we note that our expressivity result requires the relevant lifting defining the coinductive predicate to be a fibration map, which in particular implies weak pullback preservation for the canonical relation lifting. This is natural, since the latter captures bisimilarity, while logics capture coalgebraic behavioural equivalence. However, it remains an interesting question whether we can use different liftings to obtain expressivity for behavioural equivalence; perhaps based on the lifting in [21], techniques related to Λ-bisimulations [11, 1, 10] or the lax relation lifting from [28].

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