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THE k -CUBE IS k -REPRESENTABLE

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ABSTRACT

A graph is called k -representable if there exists a word w over the nodes of the graph, each node occurring exactly k times, such that there is an edge between two nodes x, y if and only after removing all letters distinct from x, y , from w , a word remains in which x, y alternate. We prove that if G is k -representable for $k > 1$, then the Cartesian product of G and the complete graph on n nodes is $(k + n - 1)$ -representable. As a direct consequence, the k -cube is k -representable for every $k \geq 1$.

Our main technique consists of exploring occurrence based functions that replace every i th occurrence of a symbol x in a word w by a string $h(x, i)$. The representing word we construct to achieve our main theorem is purely composed from concatenation and occurrence based functions.

1. Introduction

For a word w over an alphabet A , the two letters x and y are said to *alternate* in w if between every two x 's in w a y occurs and between every two y 's in w an x occurs. Stated otherwise: deleting all letters but x and y from w results in a word $xyxy\dots$ or $yxyx\dots$ of even or odd length.

A graph $G = (V, E)$ is *word-representable* if there is a word w over the alphabet V , such that $(x, y) \in E$ if and only if x and y alternate in w . The word w is said to *represent*, or be a *representant* of, G . A word only represents one graph, while a graph can have multiple words representing it.

A lot of work has been done on investigating which graphs are word-representable; this is a main topic of the book [6]. More recent work includes [4, 2, 8, 3]. A first basic observation is that one may restrict to *uniform* words: a word w over an alphabet A is called *uniform* if there exists a number k such that every letter in A occurs exactly k times in w . For such k , the word w is called *k -uniform*. So the basic observation states

Theorem 1. ([7]) *A graph G is representable if and only if it is representable by a k -uniform word for some $k \geq 1$.*

In this case G is called k -representable. The minimum k such that there exists a k -uniform word representing a graph G is called the *representation number* of the graph G ; for a non-word-representable graph its representation number is defined to be ∞ .

Now it is a natural question which representation numbers occur, and what are the representation numbers of particular graphs. Lots of results in this direction are given in the book [6]. A graph of particular interest is the k -cube Q_k , so an obvious question is to establish the representation number of Q_k . The nodes of the k -cube Q_k are the 2^k Boolean vectors of length k , and two such nodes are connected by an edge if and only if they differ in exactly one position. Equivalently, we can define inductively $Q_1 = K_2$ consisting of two nodes connected by an edge, and for $k > 1$ the k -cube Q_k is defined to be the Cartesian product of K_2 and $(k-1)$ -cube. The Cartesian product we present in more detail in Section 3. In this paper we answer this question in one direction: we show that the representation number of the k -cube is at most k by constructing a k -uniform word for which show that it represents the k -cube. In fact it is an instance of a more general construction: for every graph G represented by a k -uniform word, we construct a $(k+1)$ -uniform word representing the Cartesian product of G and K_2 .

After having done this we generalize this further: for K_n being the complete graph on $n > 1$ nodes, for every graph G represented by a k -uniform word, we construct a $(k+n-1)$ -uniform word representing the Cartesian product of G and K_n .

Our constructions are based *occurrence based functions*: functions h on k -uniform words in which every i th occurrence of a letter x is mapped to a fixed string $h(x, i)$. All our constructions are just concatenations of occurrence based functions.

The paper is organized as follows. First, in Section 2 we give some basic notations and preliminaries on occurrence-based functions. In Section 3 we define Cartesian products and present the main result on taking the product with K_2 . In Section 4 we discuss its consequences for cubes and prisms. Next, in Section 5 we extend our main result to the product with K_n for arbitrary n . We conclude in Section 6.

2. Preliminaries

In this section we collect some preliminaries, in particular on our notion of occurrence-based functions, and some convenient notations.

Definition 2. ([6]) *If w is a word over an alphabet A , and $B \subseteq A$, then the word w_B is defined to be obtained by removing all letters in $A \setminus B$ from w .*

So two letters x and y alternate in w if and only if $w_{\{x,y\}}$ is either $xyxy\dots$ or $yxyx\dots$, and two letters x and y alternate in a k -uniform word w if and only if $w_{\{x,y\}}$ is either $(xy)^k$ or $(yx)^k$.

We now introduce the notion of occurrence-based functions. These functions appear to be very useful in constructing representants for word-representable graphs.

Definition 3. Let V and V' be (possibly different) alphabets, and let $N_k = \{1, \dots, k\}$.

The labelling function of a word over V is defined as $H : (V^n)^* \rightarrow (V \times N_k)^*$, where the i th occurrence of each letter x is mapped to the pair (x, i) , and k satisfies the property that every symbol occurs at most k times in w . The word $H(w)$ is called the labelled version of w .

An occurrence-based function is defined as applying a string homomorphism $h : V \times N_k \rightarrow (V')^*$ to an already labelled version of a word. As a shorthand we will write $h(w)$ instead of $h(H(w))$.

Definition 4. For a k -uniform word w and a non-empty set $A \subseteq N_k = \{1, \dots, k\}$ the occurrence based function p_A is defined by $p_A(x, i) = x$ for all $i \in A$, and $p_A(x, i) = \epsilon$ for all $i \notin A$, for every symbol x .

For instance, $p_{\{i\}}(w)$ is obtained by removing all but the i th occurrence of each symbol in w ; this is called the i th permutation of a word w . Clearly, if w is k -uniform, then $p_A(w)$ is $\#A$ -uniform, and if $w_{\{x,y\}} = (xy)^k$ then $p_A(w)_{\{x,y\}} = (xy)^{\#A}$

Lemma 5. Let w be a k -uniform word representing a graph G . For some $m > 1$ let A_1, \dots, A_m be non-empty subsets of $N_k = \{1, \dots, k\}$ such that for all $j = 1, \dots, k-1$ there is $i \in \{1, \dots, m\}$ such that $\{j, j+1\} \subseteq A_i$. Then the $(\sum_{i=1}^m \#A_i)$ -uniform word $w' = p_{A_1}(w)p_{A_2}(w) \cdots p_{A_m}(w)$ also represents the graph G .

Proof. We have to prove that any two symbols x, y alternate in w if and only if they alternate in w' .

First assume they alternate in w , then $w_{\{x,y\}}$ is either $(xy)^k$ or $(yx)^k$. Assume it is $(xy)^k$, the other case is similar by swapping x and y . Then $p_{A_i}(w)_{\{x,y\}} = (xy)^{\#A_i}$ for all $i = 1, \dots, m$, so $w'_{\{x,y\}} = (xy)^{\sum_{i=1}^m \#A_i}$, by which x, y alternate in w' .

Conversely, assume that x, y alternate in w' . Then either $p_{A_i}(w)_{\{x,y\}} = (xy)^{\#A_i}$ for all $i = 1, \dots, m$, or $p_{A_i}(w)_{\{x,y\}} = (yx)^{\#A_i}$ for all $i = 1, \dots, m$; let's assume the first, the other case is similar by swapping x and y . Let $\{j, j+1\} \subseteq A_i$, then from $p_{A_i}(w)_{\{x,y\}} = (xy)^{\#A_i}$ we conclude that

- the j th x in w is left from the j th y in w ,
- the j th y in w is left from the $(j+1)$ th x in w , and
- the $(j+1)$ th x in w is left from the $(j+1)$ th y in w .

As it is assumed for all $j = 1, \dots, k-1$ there is such an A_i , we obtain this property for all $j = 1, \dots, k-1$, from which we conclude that x, y alternate in w . \square

As a direct consequence of Lemma 5 we obtain that if G is a graph represented by a k -uniform word w , then for every $1 \leq i \leq k$ the $(k+1)$ -uniform word $p_{\{i\}}(w)w$ also represents G .

Now we come to the main topic of this paper: word representations of Cartesian products of graphs.

3. Cartesian products

Definition 6. ([6]) The Cartesian product of two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is defined as $G \square H = (V_{GH}, E_{GH})$, where $V_{GH} = V_G \times V_H$ and $E_{GH} = \{((x, z), (y, x')) \mid x = y \text{ and } (z, x') \in E_H \text{ or } z = x' \text{ and } (x, y) \in E_G\}$.

It is known from [6] that if both G and H are word-representable, then $G \square H$ is word-representable. However, it is not yet clear how to find a representant of this Cartesian product directly from representations for both G and H .

In this section we will give a construction of a word representing a special case of this situation, namely $H = K_2$, the complete graph on 2 nodes.

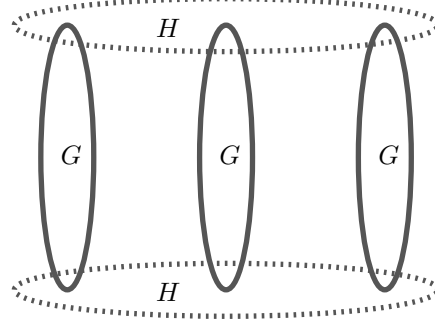


Figure 1: Cartesian product of two graphs, G and H .

When looking at the Cartesian product of a graph G with the complete graph on $n > 1$ nodes, K_n , the resulting graph consists of n copies of G , in which moreover any two nodes corresponding to the same node in G are connected by an edge. It is also easily verified that the complete graph on n nodes is represented by the 1-uniform word $w = 12 \dots n$. In fact, the complete graphs are the only graphs of representation number 1, see [5].

The complete graph K_2 just consists of two nodes connected by a single edge; the nodes of $G \square K_2$ are denoted by x_1, x_2 for x running over the nodes of G ; two nodes x_i, y_j are connected by an edge in $G \square K_2$ if and only if

- $i = j$ and (x, y) is an edge in G , or
- $i \neq j$ and $x = y$.

Write V_1 for the set of nodes x_1 and V_2 for the set of nodes x_2 , so $V_1 \cup V_2$ is the set of nodes of $G \square K_2$.

Theorem 7. Let G be a k -representable graph for $k > 1$ and let w be a k -representant of G . Then the graph $G \square K_2$ is $(k+1)$ -representable with representant $w' = f(w)g(w)$

for the occurrence based functions f, g defined by

$$f(x, i) = \begin{cases} x_1 & \text{if } i = 1 \\ x_2x_1 & \text{if } 1 < i \leq k \end{cases} \quad g(x, i) = \begin{cases} x_2 & \text{if } i = 1 \\ x_1x_2 & \text{if } i = 2 \\ \epsilon & \text{if } 2 < i \leq k \end{cases}$$

Proof. For every x the word $f(w)$ contains k copies of x_1 and $k - 1$ copies of x_2 , and the word $g(w)$ contains 1 copy of x_1 and 2 copies of x_2 , so indeed w' is $(k+1)$ -uniform.

We have to prove that x_i, y_j alternate in w' for $x_i \neq y_j$ if and only if (x_i, y_j) is an edge in $G \square K_2$, for $i, j = 1, 2$, that is

- if $x \neq y$ and $i = j$ then x_i, y_j alternate in w' if and only if x, y alternate in w ,
- if $x = y$ and $i \neq j$ then x_i, y_j do alternate in w' , and
- if $x \neq y$ and $i \neq j$ then x_i, y_j do not alternate in w' .

We do this by considering all these cases.

Let $x \neq y$ and $i = j = 1$. Observe that $f(w)_{V_1} = w_1$ and $g(w)_{V_1} = p_2(w_1)$, in which w_1 is a copy of w in which every symbol is indexed by 1. Now x_1, y_1 alternate in w' if and only if they alternate in $w'_{V_1} = f(w)_{V_1}g(w)_{V_1} = w_1p_2(w_1)$, and by Lemma 5 for $A_1 = N_k$ and $A_2 = \{2\}$, with $N_k = \{1, \dots, k\}$, this holds if and only if x, y alternate in w , which we had to prove.

Let $x \neq y$ and $i = j = 2$. Observe that $f(w)_{V_2} = p_{N_k \setminus \{1\}}(w_2)$ and $g(w)_{V_2} = p_{\{1,2\}}(w_2)$, in which w_2 is a copy of w in which every symbol is indexed by 2. Now x_2, y_2 alternate in w' if and only if they alternate in $w'_{V_2} = f(w)_{V_2}g(w)_{V_2} = p_{N_k \setminus \{1\}}(w_2)p_{\{1,2\}}(w_2)$, and by Lemma 5 for $A_1 = N_k \setminus \{1\}$ and $A_2 = \{1, 2\}$ this holds if and only if x, y alternate in w , which we had to prove.

Let $x = y$ and $i \neq j$, say $i = 1, j = 2$. Then x_1, x_2 alternate in w' since $w'_{\{x_1, x_2\}} = x_1(x_2x_1)^{k-1}x_2x_1x_2 = (x_1x_2)^{k+1}$, which we had to prove.

Now, let $x \neq y$ and $i \neq j$, say $i = 1, j = 2$.

If $w_{\{x,y\}} = (xy)^k$ then $f(w)_{\{x_1, y_2\}}$ starts by x_1x_1 , so x_1, y_2 do not alternate in $w' = f(w)g(w)$.

If $w_{\{x,y\}} = (yx)^k$ then $g(w)_{\{x_1, y_2\}} = y_2y_2x_1$, so x_1, y_2 do not alternate in $w' = f(w)g(w)$.

In the remaining case x, y do not alternate in w , so $w_{\{x,y\}}$ contains either xx or yy . If it is xx , or it is yy and $w_{\{x,y\}}$ does not start in yy , then $f(w)_{\{x_1, y_2\}}$ contains x_1x_1 . Otherwise $w_{\{x,y\}}$ starts in yy , but then $g(w)_{\{x_1, y_2\}} = y_2y_2x_1$. In all cases we conclude that x_1, y_2 do not alternate in $w' = f(w)g(w)$, concluding the proof. \square

4. Cubes and Prisms

Theorem 7 has a couple of implications. In particular, it implies that the k -cube Q_k is k -representable, as is stated in the following theorem.

Theorem 8. *For every $k \geq 1$, the k -cube Q_k is k -representable.*

Proof. The proof is by induction on k . For $k = 1$ we observe that $Q_1 = K_2$, being 1-representable by the word $w = 12$.

For $k = 2$ we observe that Q_2 is the 4-cycle being 2-representable by the word $w = 31421324$.

For the induction step for $k > 2$, we use Theorem 7 giving a k -uniform representant for Q_k from a $(k - 1)$ -uniform representant of Q_{k-1} . \square

It was already known from [7] that every prism is 3-representable and that the 3-prism is not 2-representable. Theorem 7 also implies that every prism is 3-representable, as a prism is the Cartesian product of a cycle-graph and K_2 and cycle-graphs are 2-representable ([6]). From the fact that the 3-prism is not 2-representable and Theorem 7 we can prove the following.

Theorem 9. *The Cartesian product $K_n \square K_2$ has representation number n for $n = 1, 2, 3$, and representation number 3 for all $n > 3$.*

Proof. $K_1 \square K_2$ is equal to K_2 having representation number 1.

$K_2 \square K_2$ is the 4-cycle, which is known to have representation number 2 ([6]).

$K_3 \square K_2$ is equal to the 3-prism, which is not 2-representable, but it is 3-representable ([6]). If $n > 3$ then $K_n \square K_2$ contains the 3-prism as induced subgraph and thus cannot be 2-representable. Theorem 7 gives a 3-representation because K_n has a 2-representation $12 \cdots n12 \cdots n$, so $K_n \square K_2$ has representation number 3. \square

In particular, this theorem shows that the requirement $k > 1$ in Theorem 7 is essential: for $k = 1$ the claim of Theorem 7 does not hold since it would yield a non-existent 2-representation of $K_n \square K_2$ for $n > 2$.

5. Extension to K_n

The ideas used in Theorem 7 can be applied to prove the following generalization.

The nodes of $G \square K_n$ are denoted by x_1, x_2, \dots, x_n for x running over the nodes of G ; two nodes x_i, y_j are connected by an edge in $G \square K_2$ if and only if

- $i = j$ and (x, y) is an edge in G , or
- $i \neq j$ and $x = y$.

Write V_i for the set of nodes x_i , so $V_1 \cup V_2 \cup \dots \cup V_n$ is the set of nodes of $G \square K_n$.

Theorem 10. *Let G be a k -representable graph for $k > 1$ and let w be a k -representant of G . Then the graph $G \square K_n$ is $(k + n - 1)$ -representable with representant $w' = f_n(w)f_{n-1}(w) \cdots f_1(w)$ for the occurrence based functions f_i defined by*

$$f_1(x, i) = \begin{cases} x_1 & \text{if } i = 1 \\ x_n x_{n-1} \dots x_1 & \text{if } 1 < i \leq k \end{cases}$$

and

$$f_j(x, i) = \begin{cases} x_j & \text{if } i = 1 \\ x_{j-1} \dots x_1 x_n \dots x_j & \text{if } i = 2 \\ \epsilon & \text{if } 2 < i \leq k \end{cases}$$

for $j = 2, \dots, n$.

Proof. For every x the word $f_1(w)$ contains k copies of x_1 and $k - 1$ copies of x_i for $i > 2$, and the words $f_j(w)$ contain 2 copies of x_j and 1 copy of x_i for $i \neq j$. So for every i , x_i occurs either $k + (n - 1)$ times if $i = 1$, or $(k - 1) + 2 + (n - 2) = k + (n - 1)$ times if $i \neq 1$. So w' is $(k + (n - 1))$ -uniform.

We have to prove that x_i, y_j alternate in w' for $x_i \neq y_j$ if and only if (x_i, y_j) is an edge in $G \square K_n$, for $i, j = 1, 2, \dots, n$, more precisely:

- if $x \neq y$ and $i = j$ then x_i, y_j alternate in w' if and only if x, y alternate in w ,
- if $x = y$ and $i \neq j$ then x_i, y_j do alternate in w' , and
- if $x \neq y$ and $i \neq j$ then x_i, y_j do not alternate in w' .

We do this by considering all cases.

Let $x \neq y$ and $i = j = 1$. Observe that $f_1(w)_{V_1} = w_1$ and $f_i(w)_{V_1} = p_2(w_1)$ for all $i > 1$, in which w_1 is a copy of w in which every symbol is indexed by 1. Now x_1, y_1 alternate in w' if and only if they alternate in $w'_{V_1} = f_n(w)_{V_1} f_{n-1}(w)_{V_1} \dots f_1(w)_{V_1} = (p_2(w_1))^{n-1} w_1$, and by Lemma 5 for $A_1 = A_2 = \dots = A_{n-1} = \{2\}$ and $A_n = N_k$, with $N_k = \{1, \dots, k\}$, this holds if and only if x, y alternate in w , which we had to prove.

Let $x \neq y$ and $i = j \geq 2$. Observe that $f_1(w)_{V_i} = p_{N_k \setminus \{1\}}(w_i)$ and $f_l(w)_{V_i} = p_{\{1, 2\}}(w_i)$ for all $l > 1$, in which w_i is a copy of w in which every symbol is indexed by i . Now x_i, y_i alternate in w' if and only if they alternate in $w'_{V_i} = f_n(w)_{V_i} f_{n-1}(w)_{V_2} \dots f_1(w)_{V_i} = (p_{\{1, 2\}}(w_i))^{n-1} p_{N_k \setminus \{1\}}(w_i)$, and by Lemma 5 for $A_1 = A_2 = \dots = A_{n-1} = \{1, 2\}$ and $A_n = N_k \setminus \{1\}$ this holds if and only if x, y alternate in w , which we had to prove.

Let $x = y$, $i = 1$ and $i < j$. Then x_1, x_j alternate in w' since $w'_{\{x_1, x_j\}} = (x_j x_1)^{n-j} (x_j x_1 x_j) (x_1 x_j)^{j-2} (x_1 (x_j x_1)^{k-1}) = (x_j x_1)^{k+(n-1)}$, which we had to prove.

Let $x = y$, $1 \neq i < j$. Then x_i, x_j alternate in w' since $w'_{\{x_i, x_j\}} = (x_j x_i)^{n-j} (x_j x_i x_j) (x_i x_j)^{j-i-1} (x_i x_j x_i) (x_j x_i)^{i-2} (x_j x_i)^{k-1} = (x_j x_i)^{k+(n-1)}$, which we had to prove.

It remains to consider $x \neq y$ and $i \neq j$; without loss of generality assume $i < j$. We continue by case analysis on the shape of $w_{\{x, y\}}$.

If $w_{\{x, y\}} = (xy)^k$ and $i = 1$, then $f_1(w)_{\{x_1, y_j\}}$ starts by $x_1 x_1$, so x_1, y_j do not alternate in w' .

If $w_{\{x, y\}} = (xy)^k$ and $i > 1$, then $f_i(w)_{\{x_i, y_j\}} = x_i x_i y_j$, so x_i, y_j do not alternate in w' .

If $w_{\{x, y\}} = (yx)^k$ then $f_j(w)_{\{x_i, y_j\}} = y_j y_j x_i$, so x_i, y_j do not alternate in w' .

In the remaining case x, y do not alternate in w , so $w_{\{x, y\}}$ contains either xx or yy . If it is xx , or it is yy and $w_{\{x, y\}}$ does not start in yy , then $f_1(w)_{\{x_1, y_j\}}$ contains

x_1x_1 for all $j > 1$ and $f_i(w)_{\{x_i, y_j\}} = x_ix_iy_j$ for all $1 \neq i < j$. Otherwise $w_{\{x, y\}}$ starts in yy , but then $f_j(w)_{\{x_i, y_j\}} = y_jy_jx_i$ for all $1 \neq i < j$. In all cases we conclude that x_i, y_j do not alternate in w' , concluding the proof. \square

6. Conclusions

We introduced occurrence based functions as a building block when given a word representing a graph, to construct a word representing a modified graph based on the given word. Exploiting the key lemma (Lemma 5) we succeeded in doing so for taking the Cartesian product with K_n . Doing this only for the Cartesian product with K_2 , as a consequence we have a construction for a k -uniform word representing the k -cube Q_k . Hence, the representation number of Q_k is at most k . It remains open whether the representation number of Q_k is exactly k , for that one should prove that for $m < k$ no m -uniform word exists representing Q_k .

When considering our approach combining occurrence based functions and concatenation, we see that several (but not all) constructions in [6] giving word representations of particular graphs could be presented in the same format. It would make sense to investigate which building blocks are used in which constructions.

Our Theorem 10 is expected to have further generalizations, as will be elaborated in the master thesis [1] of the first author.

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