Explicit Whittaker data for essentially tame supercuspidal representations

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Abstract

Based on the ideas of Bushnell-Henniart and Paskunas-Stevens, we construct explicit Whittaker data for an essentially tame supercuspidal representation of $GL_n(F)$, where $F$ is a non-Archimedean local field.

1 Introduction

Let $F$ be a non-Archimedean local field, $V$ be an $n$-dimensional $F$-vector space, and $G$ be the group $\text{Aut}_F(V)$ of $F$-linear automorphisms of $V$, usually regarded as $GL_n(F)$ by choosing a basis of $V$. Let $\pi$ be a supercuspidal representation of $G$. As a classical result in [GK75], we know that $\pi$ admits a unique Whittaker model. More precisely, take a tuple of Whittaker data $(N, \psi)$ consisting of a maximal unipotent subgroup $N$ of $G$ and a non-degenerate character $\psi$ of $N$, in the sense that its restriction to every simple root subgroup of $N$ is non-trivial, then we have

$$\text{Hom}_G(\pi, \text{Ind}^G_N \psi) = 1.$$ 

As another classical result in [BK93], we know that $\pi$ is isomorphic to a compactly induced representation from a finite dimensional representation $\Lambda$ of an open compact-mod-center subgroup $J$ of $G$, that is,

$$\pi \cong c\text{Ind}_J^G \Lambda.$$ 

Using Frobenius reciprocity and Mackey’s formula [Kut77], the existence and uniqueness of a Whittaker model is equivalent to the existence of a pair $(N, \psi)$ as above such that

$$\text{Hom}_{N \cap J}(\psi, \Lambda) \neq 0,$$

which is unique up to conjugation by $J$ [BH98].

The above is the starting point of [BH98] to describe an explicit Whittaker function for a supercuspidal representation. This description, together with the analogous result in [PS08], turn out to be useful in computing the epsilon factor of a certain pair of supercuspidal representations (see [PS08, Section 7]). However, as pointed out in [PS08, Intro.], the proof of [BH98, Lemma 2.10] contains a gap, so that they have to bypass the problem using a “black box” case (explained below) in [PS08].

The purpose of this paper is to construct explicit Whittaker data $(N, \psi)$ for an essentially tame supercuspidal representation $\pi$. The essential tameness condition means that, by the definition in [BH05], if the group

$$\{\chi : \text{unramified character of } F^\times \text{ such that } (\chi \circ \det) \otimes \pi \cong \pi\}$$

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has order $f$, which is necessarily a divisor of $n$, then the residual characteristic $p$ of $F$ does not divide $n/f$. We will explain the advantage of restricting to the essentially tame case at the end of this Introduction.

We summarize briefly the method of constructing our Whittaker data, following mostly [PS08, Theorem 3.3 and Section 4.2]. Let $\theta$ be the simple character of a compact subgroup $H^1$ of $G$, in the sense of [BK93, Section 3.2], underlying a chosen inducing type $\Lambda$ contained in $\pi$. Associated to $\theta$ is an element $\beta \in A = \text{End}_F(V)$ such that $E_0 = F[\beta]$ is a subfield of $A$ and is tamely ramified over $F$ in the essentially tame case. We will construct a unipotent subgroup $N$ satisfying

$$\theta|_{H^1 \cap N} = \psi_{\beta}|_{H^1 \cap N},$$

where $\psi_{\beta} : A \to \mathbb{C}$, $x \mapsto \psi_F(\text{tr}_{A/F}(\beta x))$ with $\psi_F$ being an additive character of $F$ trivial on $p_F$ but not on $\sigma_F$, together with other conditions in [PS08, Theorem 3.3].

The above unipotent subgroup $N$ is defined by a particular ordered basis $b$ of $V$ given in (4.4) below. To describe it briefly, associated to the element $\beta$ is a set $\{\beta_i\}_{i=0}^t$ of approximation elements such that $\sum_{i=0}^t \beta_i = \beta$ and, in the essentially tame case, that $E_1 = F[\sum_{j \leq \gamma} \beta_j]$ form a tower of intermediate extensions between $E_0$ and $F$. When $t = 0$, which is known as the minimal case in [BK93, (1.4.15)], we define $b$ cyclicly using the element $\beta$, similar to the one defined in [BH98, 2.1 Proposition].

In the presence of multiple approximation elements (i.e. $t > 0$), we have to define $b$ also cyclicly but in an inductive way along the approximation elements. Our $b$ will be different from the one defined in [BH98, 2.1 Proposition] at the level of the complexity of the bases, and it is unknown whether the two bases give rise to $J$-conjugate unipotent subgroups.

It is also unknown whether $\psi_{\beta}$ can be extendable to the whole $N$; however, using the matrix presentation of $\beta$ with respect to $b$, we construct an analogous element $\alpha \in A$ such that $\psi = \psi_{\alpha}$ is a non-degenerate additive character of $N$. As the main result in Theorem 5.1, we will show that

$$\psi|_{H^1 \cap N} = \psi_{\beta}|_{H^1 \cap N},$$

and the conditions in [PS08, Theorem 3.3] as well, which are sufficient to imply that (1.1) holds for our $(N, \psi)$.

Note that, using the rational canonical form of $\beta$, we can of course extend $\psi_{\beta}$ to a character of a certain maximal unipotent subgroup $N_{\beta}$ (as in [BH98, Proposition 2.1]). However, this unipotent subgroup may not be a good one for describing explicit Whittaker data for $\pi$; in particular, we do not know if (1.1) holds for $(N_{\beta}, \psi_{\beta})$ as claimed in [BH98]. As another note, the case when $[E_0 : F] = n$, i.e. $E_0$ is a maximal subfield in $A$, is the ‘black box’ case deemed by [PS08]. Hence our result also fills up the ‘black box’ of [PS08, Theorem 3.3] in the essentially tame case.

Finally, we remark that the whole development of our main result requires $E_0/F$ to be tamely ramified. As we have seen above, the tower of intermediate extensions $E_j = F[\sum_{i \geq j} \beta_i]$ between $E_0$ and $F$ allows us to define inductively the basis $b$ in (4.4), and consequently decompose the simple character $\theta$ and the compact subgroup $H^1 \cap N$ inductively to derive our main result. The author believes that harder technique is required for the general case beyond the essentially tame case, and hopes to deal with it in his future work.

### 1.1 Notations

Let $F$ be a non-Archimedean local field with an algebraic closure denoted by $\bar{F}$. Denote by $\sigma_F$ the ring of integers of $F$ and $p_F$ the maximal ideal of $\sigma_F$. The residue field $k_F = \sigma_F/p_F$ of $F$ is a finite extension of $\mathbb{F}_p$. Denote by $v_F : F \to \mathbb{Z} \cup \{\infty\}$ the valuation of $F$. 
If \( r \in \mathbb{Q} \), we denote by \( r^+ \) the smallest integer strictly greater than \( r \).

## 2 Tamely ramified extensions

The main purpose of this section is to gather some known facts concerning tamely ramified extensions. More importantly, we consider minimal elements in a tamely ramified extension, and study how they form bases with nice properties on lattice filtrations.

### 2.1 Complementary subgroup

Let \( E/F \) be a tamely ramified extension of degree \( n = n(E/F) \) and ramification degree \( e = e(E/F) \). Put \( f = f(E/F) = n/e \).

Throughout we fix a chosen uniformizer \( \varpi_F \) of \( F \), and let \( \mu_F \) be the group of roots of unity with order coprime to \( p \). We also fix \( \varpi_E \) and let \( \mu_E \) similarly, and assume that \( \varpi^e_E \varpi_F \in \mu_E \).

We define the complementary subgroup \( C_E \) of \( E \times \) to be the subgroup generated by \( \varpi_E \) and \( \mu_E \). It can be shown that \( C_E \) depends only on the choice of \( \varpi_F \). Moreover, if \( K/F \) is an intermediate field extension in \( E \), then \( C_K \subseteq C_E \). We denote by \( C_{\text{tame}}^F \) to be the union of all \( C_E \), with \( E \) ranges over all tamely ramified extensions of \( F \).

Let \( r \in \mathbb{R} \). If \( r \) is a positive integer, let \( U_F^r \) be the \( r \)-th unit group \( 1 + \mathfrak{p}_F^r \). In general, we write \( U_F^r = U_F^{\lceil r \rceil} \) where \( \lceil r \rceil \) is the smallest integer \( \geq r \), and write \( U_F^{r^+} = U_F^{\lceil r \rceil^+} \) where \( \lceil r \rceil^+ \) is the smallest integer \( > r \). We let \( U_E^r \) similarly. Any element \( b \in E \times \) can be uniquely decomposed as \( cu \) where \( c \in C_E \) and \( u \in U_1^E \). We call \( c \) the first term of \( b \).

### 2.2 Minimality

At the beginning of this subsection, we only require \( E/F \) to be a finite separable extension of degree \( n \). Later we will require \( E/F \) to be moreover tamely ramified.

Let \( E = F[\alpha] \) for some \( \alpha \in E \). Denote \( e = e(E/F) \) and \( v = v_E(\alpha) \). From [KM88, Proposition 1.5], we say that \( \alpha \) is minimal over \( F \) if it satisfies

(I) \( \gcd(v, e) = 1 \), and

(II) any one of the following conditions.

(a) \( \mathfrak{o}_F[\beta] = \mathfrak{o}_K \), where \( K/F \) is the maximal unramified extension in \( E/F \) and \( \beta = N_{E/K}(\alpha)/\varpi_F^v \).

(b) The elements \( \{x_j\}_{j=1}^n \), where \( x_j = \alpha^j/\varpi_F^v \), form an \( \mathfrak{o}_F \)-basis of \( \mathfrak{o}_E \). In particular we have \( \mathfrak{o}_E = \bigoplus_{j=0}^{[E:F]-1} \mathfrak{o}_Fx_j \).

(c) \( k_E = k_F[\gamma + \mathfrak{p}_E] \), where \( \gamma = x_e = \alpha^e/\varpi_F^v \) ([BK93, (1.4.15)]).

By [KM88, Proposition 1.5], given (I), the three conditions in (II) are equivalent.

To incorporate the construction of simple character, we recall another equivalent minimality condition from [BK93]. Let \( V \) be a finite dimensional \( E \)-vector space. We first regard \( V \) is an \( F \)-vector space and denote \( A = \text{End}_F(V) \). Let \( \mathfrak{A} \) be an hereditary \( \mathfrak{o}_F \)-order in \( A \), with Jacobson radical \( \mathfrak{P} \) and which is normalized by \( E^\times \). Let \( v_\mathfrak{A} \) be the valuation on \( A \) associated
with \( \mathfrak{A} \). Let \( B \) be the centralizer of \( E \) in \( A \), and denote \( \mathfrak{B} = \mathfrak{A} \cap B \). Recall from [BK93, 1.4] the \( \mathfrak{o}_F \)-lattice
\[
\mathfrak{N}_k(\alpha, \mathfrak{A}) = \{ x \in \mathfrak{A} : \alpha x - x \alpha \in \mathfrak{P}^k \}
\]
and the critical exponent
\[
k_0(\alpha, \mathfrak{A}) = \max\{ k \in \mathbb{Z} : \mathfrak{N}_k(\alpha, \mathfrak{A}) \not\subset \mathfrak{B} + \mathfrak{P} \}.
\]
By convention, if \( \alpha \in F \), we put \( k_0(\alpha, \mathfrak{A}) = \infty \). In fact, by [BK93, (1.4.13)(ii)], the definition is independent of the vector space \( V \) and the hereditary order \( \mathfrak{A} \). One important property is [BK93, (1.4.15)],
\[
v_\mathfrak{A}(\alpha) \leq k_0(\alpha, \mathfrak{A}),
\]
with equality holds if and only if \( \alpha \) is minimal over \( F \).

**Proposition 2.1.** An element \( \alpha \in \bar{F}^\times \) with finite order modulo \( F^\times \) coprime to \( p \) is minimal over \( F \). In particular, any element in \( C_{F_{\text{ tame}}}^\mathfrak{A} \) is minimal over \( F \).

**Proof.** The second statement is direct from the first, so we focus on proving the first statement. The idea of the proof comes from [Rei91, Lemma 2.8]. We take \( \alpha \in C_{F_{\text{ tame}}}^\mathfrak{A} \) and assume \( c \notin F \), otherwise the result is trivial. We will use condition (2.1) and, since the condition does not depend on the choice of the vector space \( V \), we can assume \( V = \mathfrak{V} \), and denote \( A, \mathfrak{A} \) and \( \mathfrak{P} \) as above, so that \( B = E \) and \( \mathfrak{B} = \mathfrak{o}_E \). The statement can be proved if we can show that
\[
\mathfrak{N}_k(\alpha, \mathfrak{A}) = \mathfrak{o}_E + \mathfrak{P}^{k-v_\mathfrak{A}(\alpha)}
\]
for all \( k \in \mathbb{Z} \). Let \( \tau(x) = \alpha x - x \alpha \) for all \( x \in \mathfrak{A} \), which is an \( F \)-algebra automorphism of \( A \). We hence take \( x \in \mathfrak{A} \) such that \( \tau(x) - x \in \mathfrak{P}^{k-v_\mathfrak{A}(\alpha)} \). If \( m \) is the order of \( c \) in \( \bar{F}^\times / F^\times \), which indeed lies in \( U_F \), we define
\[
s : A \to A, \ x \mapsto \frac{1}{m} \sum_{i=0}^{m-1} \tau^i(x),
\]
which is an \( F \)-linear projection onto \( E \), and so \( s(x) \in \mathfrak{o}_E \) as \( m \in o_E^\times \). The relation
\[
x = s(x) - \sum_{i=0}^{m-1} \sum_{j=1}^{\tau^j-1} (\tau^i(x) - x)
\]
implies that \( x \in \mathfrak{o}_E + \mathfrak{P}^{k-v_\mathfrak{A}(\alpha)} \). The converse inclusion is straightforward. \( \square \)

**Corollary 2.2.** Suppose that the field \( E = F[\alpha] \), for some \( \alpha \in E \), is tamely ramified over \( F \). Then \( \alpha \) is minimal if and only if the first term of \( \alpha \) also generates \( E \) over \( F \).

**Proof.** We write \( \alpha = au \) for some \( u \in U_E \). It is straightforward to see that \( \alpha \) satisfies minimality conditions (I) and (IIc) if and only if \( a \) does for the same field \( E \). \( \square \)

We provide a useful calculation of the critical exponent of an element generating a tamely ramified field extension.

**Proposition 2.3.** Suppose that \( \beta \in A \) such that \( E = F[\beta] \) is a tamely ramified extension of \( F \), and \( \mathfrak{A} \) is an \( \mathfrak{o}_F \)-hereditary order normalized by \( E^\times \). Take \( c \in C_E \) and denote \( \gamma = \beta - c \). If \( k_0(\gamma, \mathfrak{A}) < v_\mathfrak{A}(c) \), then
\[
k_0(\beta, \mathfrak{A}) = \begin{cases} k_0(\gamma, \mathfrak{A}) & \text{if } c \in F[\gamma], \\ v_\mathfrak{A}(c) & \text{otherwise.} \end{cases}
\]

**Proof.** This can be derived from [BK93, (2.2.8)]. \( \square \)
2.3 Some special properties

We consider a more general situation. Suppose that $V$ is an $n$-dimensional $F$-vector space containing an $\mathfrak{O}_F$-lattice chain $L$. We call an $F$-basis $\{y_i\}_{i=1}^n$ of $V$ an $\mathfrak{O}_F$-basis of $L$, in the sense of [BK93, (1.1.7)] if it is an $\mathfrak{O}_F$-basis of $L(r)$ for some $r \in \mathbb{Z}$, and there exists $a(j, r) = 0, \ldots, e-1$ such that

$$L(r) = \bigoplus_j p_F^{a(j, r)}x_j \quad \text{for all } r \in \mathbb{Z}. $$

We may arrange the order so that $a(j, r) \leq a(j + 1, r)$.

For example, if $V$ is a field extension $E = F[a]$ as in the last section, then the set $\{x_j\}_{j=1}^n$ in the minimality condition (IIb) is an $\mathfrak{O}_F$-basis of $\{p_E^{j}\}_{r \in \mathbb{Z}}$. Indeed, suppose that $\{y_j\}$ is an ordered set equal to $\{x_j\}$ as a set but with order re-arranged such that $v_E(y_j) = i$ if $j = fi + k$ with $0 \leq i < e$ and $1 \leq k \leq f$, then we have

$$p_E^i = \bigoplus_{i=t}^{e-1} \bigoplus_{k=1}^{f} p_F^{fj+i+k} \oplus \bigoplus_{i=0}^{t-1} \bigoplus_{k=1}^{f} p_F^{fj+i+k}$$

(2.2)

if $r = se + t$ for all $s \in \mathbb{Z}$ and $t = 0, \ldots, e-1$. Indeed we always have an inclusion $\subseteq$ for all $r \in \mathbb{Z}$, and we just have to show the equality for $r = 0, \ldots, e-1$ by periodicity. We of course have the equality for $r = 0$ and $r = e$. We then obtain the equality for other $r$ by counting the $k_F$-dimensions of successive quotients of the two sides of (2.2).

For later constructing Whittaker data, we consider a more general property. Denote by $v_L : V \to \mathbb{Z} \cup \{\infty\}$ the associated valuation of $L$. Let $\{u_j\}_{j=1}^n$ be an $\mathfrak{O}_F$-basis of $L$ satisfying the following condition:

\[ (*) \text{ For every } u = \sum_j a_ju_j \in V \text{ for some } a_j \in F, \text{ we have } v_L(a_ju_j) \geq v_L(u) \text{ for all } j. \]

For example, the basis of $E$ in the minimality condition (IIb) satisfies the above condition, by observing from (2.2).

This condition leads to the following simple useful result.

**Proposition 2.4.** Suppose further that $v_L(u_i) \geq v_L(u_j)$ if $i \leq j$. For every $u = \sum_i a_iu_i \in V$, if $v_L(u) > v_L(u_i)$ for some $i$, then $a_j \in \mathbb{p}_F$ for all $j \geq i$.

**Proof.** It is just because $v_L(a_ju_j) \geq v_L(u) > v_L(u_i) \geq v_L(u_j)$, where the first inequality comes from condition (*) above. \hfill \Box

3 Essentially tame supercuspidal representation

In this section, we recall the construction of essentially tame supercuspidal representations of $G$ using admissible characters.

3.1 Structure of admissible characters

Given a character $\xi$ of $F^\times$, the level of $\xi$ is the smallest integer $r = r_F(\xi)$ such that $\xi|_{U_F^{r+1}}$ is trivial. We call $\xi$ tamely ramified if $r = 0$. 


Suppose that $E/F$ is tamely ramified and $\xi$ is an admissible character of $E^\times$ over $F$ in the sense of [How77], which means that for some intermediate subfield $K$ between $E$ and $F$,

- if $\xi$ factors through $N_{E/K}$, then $E = K$, and
- if $\xi|_{U_E}$ factors through $N_{E/K}$, then $E/K$ is unramified.

It is known that an admissible character $\xi$ admits a factorization

$$\xi = \xi_{-1}(\xi_0 \circ N_{E_0/E}) \cdots (\xi_t \circ N_{E_t/E})(\xi_{t+1} \circ N_{E/F}),$$

(3.1)

with notations specified as follows.

- We have a decreasing sequence of fields

$$E = E_{-1} \supseteq E_0 \supseteq E_1 \supseteq \cdots \supseteq E_t \supseteq E_{t+1} = F,$$

and each $\xi_i$ is a character of $E_i^\times$. This tower is uniquely determined by $\xi$.
- Let $r_i$ be the level of $\xi_i \circ N_{E_i/E}$, then $r = r_{t+1}$ is the level of $\xi$. We assume that $\xi_{t+1}$ is trivial if $r_{t+1} = r_t$. We call the increasing sequence of integers $r_0 < \cdots < r_t$ the jumps of $\xi$, which are uniquely determined by $\xi$. For later computation, we put $r_{-1} = 0$.
- If $E_0 = E$, then we replace $(\xi_0 \circ N_{E_0/E_0})\xi_{-1}$ by $\xi_0$. If $E_0 \subsetneq E$, then we assume that $\xi_{-1}$ is tamely ramified and $E/E_0$ is unramified.

We put $\xi_{-1} = \Xi_0 \circ N_{E_0/E}$, where $\Xi_0 = \xi_0(\xi_1 \circ N_{E_0/E_1}) \cdots (\xi_t \circ N_{E_0/E_t})(\xi_{t+1} \circ N_{E_0/F})$. Note that the jumps $\{r_i\}$ depend only on $\Xi_0 \circ N_{E_0/E_0}|_{U_E}$, and are invariant under the Galois action on $\xi$.

We fix an additive character of $F$, which is assumed to be trivial on $p_F$ but not on $\mathfrak{o}_F$. For any tamely ramified extension $K/F$, we write $\psi_K = \psi_F \circ \text{tr}_{K/F}$.

We recall several results from [Moy86, Section 2.2]. For $i = 0, \ldots, t+1$, suppose that $s_i$ is the level of $\xi_i$, which means that $s_i \epsilon(E/E_i) = r_i$, then there is $\beta_i \in p_{E_i}^{s_i} - p_{E_i}^{-s_i}$ such that

$$\xi_i(1 + x) = \psi_F(\beta_i x)$$

for all $x \in p_{E_i}^{r_i/2^+}$.

This $\beta_i$ can be chosen mod $p_{E}^{(-r_i/2)^+}$, and is usually regarded as in $p_{E}^{r_i} - p_{E}^{-r_i}$ and chosen mod $p_{E}^{(-r_i/2)^+}$. This element $\beta_i$ depends on the choice of $\xi_i$, but its ‘first term’ $c_i \in C_E$ is uniquely determined by $\xi$. Moreover, for $i = 0, \ldots, t$, each character $\xi_i$ is generic over $E_{i+1}$, in the sense that

$$E_{i+1}[c_i] = E_i.$$

(3.3)

In particular, (3.3) implies that

$$\gcd(s_i, \epsilon(E_i/E_{i+1})) = 1.$$  

(4.4)

We write

$$\beta = \beta(\xi) = \beta_0 + \cdots + \beta_{t+1}.$$  

(3.5)

Note that $v_E(\beta) = -r$, the level of $\xi$. When $r = 0$, i.e., $\xi$ is tamely ramified, then all $\xi_i$, with $i = 0, \ldots, t+1$, are trivial, and we take $\beta = 0$.

**Proposition 3.1.**  

(i) $E_i = F[\beta_{t+1} + \cdots + \beta_i]$.

(ii) Each $\beta_i \in E_i$ is minimal over $E_{i+1}$.
Proof. (i) is because we have a decreasing sequence (3.2) of field extensions, while (ii) is from (3.3) and Corollary 2.2.

For future computation, we define an extra element $\beta_{-1} = c_{-1}$ to be a primitive root of unity in $\mu_E$ when $E \neq E_0$, and put $\beta_{-1} = c_{-1} = 0$ when $E = E_0$.

### 3.2 Construction of simple characters

We now identify $E$ as an $n$-dimensional vector space $V$ and hence obtain an embedding $E \hookrightarrow A$. We then define

$$\mathfrak{A} = \{X \in \text{End}_F(E) : Xp_E^k \subseteq p_E^k \text{ for all } k \in \mathbb{N}\}$$

and

$$\mathfrak{P}_{\mathfrak{A}}^j = \{X \in \text{End}_F(E) : Xp_E^{k+j} \subseteq p_E^k \text{ for all } k \in \mathbb{N}\},$$

which are respectively an hereditary order in $A$ and its $j$th radicals. We also extend the definition such that, for $r \in \mathbb{R}$, we have $\mathfrak{P}_{\mathfrak{A}}^r = \mathfrak{P}_{\mathfrak{A}}^{[r]}$ and $\mathfrak{P}_{\mathfrak{A}}^{r+} = \mathfrak{P}_{\mathfrak{A}}^{[r]+}$. We then define the following subgroups in $G$,

$$U_{\mathfrak{A}} = \mathfrak{A}^\times \text{ and } U_{\mathfrak{A}}^j = 1 + \mathfrak{P}_{\mathfrak{A}}^j,$$

and similarly $U_{\mathfrak{A}}^r$ and $U_{\mathfrak{A}}^{r+}$ for $r \in \mathbb{R}$. Finally, we define $\mathfrak{P}_{\mathfrak{A}}^i$, $\mathfrak{P}_{\mathfrak{A}}^i$, and $\mathfrak{P}_{\mathfrak{A}}^{i+}$ the centralizers of $E_i$ in $\mathfrak{A}$, $\mathfrak{P}_{\mathfrak{A}}^i$, and and $\mathfrak{P}_{\mathfrak{A}}^{i+}$ respectively, and define the subgroups $U_{\mathfrak{A}}^i$, $U_{\mathfrak{A}}^i$, and $U_{\mathfrak{A}}^{i+}$ in $U_{\mathfrak{A}}$ the centralizers of $E_i^x$ in $U_{\mathfrak{A}}^i$, $U_{\mathfrak{A}}^i$, and $U_{\mathfrak{A}}^{i+}$ respectively.

The inducing type $\pi$ is a finite-dimensional irreducible representation $A$ of the compact mod-center subgroup

$$J = J_\xi = E^x U_{\mathfrak{A}}^0 U_{\mathfrak{A}}^{r_0/2} \cdots U_{\mathfrak{A}}^{r_{t-1}/2} U_{\mathfrak{A}}^{r_t/2}$$

whose restriction onto the compact subgroup

$$H^1 = H^1 = U_{\mathfrak{A}}^1 U_{\mathfrak{A}}^{r_0/2+} \cdots U_{\mathfrak{A}}^{r_{t-1}/2+} U_{\mathfrak{A}}^{r_t/2+}$$

is a direct sum of a unique simple character $\theta = \theta_{\xi}$, in the sense of [BK98], whose restriction onto $U_{\mathfrak{A}}$ coincides with $\xi|_{U_{\mathfrak{A}}^1}$. Like $\xi$, this simple character also admits a factorization

$$\theta = \theta_0 \theta_1 \cdots \theta_{t+1}$$

such that

$$\theta_i|_{U_{\mathfrak{A}}^0 U_{\mathfrak{A}}^{r_0/2+} \cdots U_{\mathfrak{A}}^{r_{i-1}/2+}} = \xi_i \circ \text{det}_{B_i/E_i} \text{ and } \theta_i|_{U_{\mathfrak{A}}^{r_i/2+} \cdots U_{\mathfrak{A}}^{r_{t+1}/2+}} = \psi_F \circ \text{tr}_{A/F}(\beta_i^*).$$

(Note that $\mathfrak{A}_{i+1} = \mathfrak{A}$.) It is well-defined since on the intersection $U_{\mathfrak{A}}^{r_t/2+}$ the characters are equal. Note that when $r = 0$, we take $\mathfrak{A} = M(E_F)$ with $H^1 = U_{\mathfrak{A}}^1$, and $\theta$ is the trivial character of $H^1$.

**Proposition 3.2.** The assignment $\xi|_{U_{\mathfrak{A}}^1} \mapsto \theta$ is well-defined, i.e., it is independent of the factorization (3.1).

Proof. The verifying arguments are quite routine, so we only provide a brief idea as follows. Before we begin, in order to reduce the load of notations, we denote the restriction of any character $\phi$ of some $E_i^x$ to $U_{E_i}^1$, just by $\phi$ instead of $\phi|_{U_{E_i}^1}$, and similarly if we replace $E$ by other fields.

First of all, remember that the jumps $\{r_i\}$ and the intermediate subfields $\{E_i\}$ in (3.2) are uniquely determined by $\xi$. Suppose we have another factorization of $\xi$ whose factors are $\{\xi_i\}_{i=0}^{t+1}$, then we can inductively deduce that, for $i = 0, \ldots, t+1$,

$$\xi_{i-1}^{-1} \xi_i \phi_{i-1} = \phi_i \circ N_{E_i/E_{i+1}} \quad (3.6)$$
for some characters \( \phi_i \) of \( U^1_{E_i+1} \) whose level \( t_i \) is less than \( s_i = r_i/\epsilon(E/E_i) \) because of (3.4).

We remark that here we take \( \phi_{-1} \) and \( \phi_{t+1} \) to be trivial. In the additive level, suppose that
\[
\phi_i|_{U^r_{E_i+1}} = \psi_{E_{i+1}}(\gamma_i),
\] (3.7)
then (3.6) becomes, for \( i = 0, \ldots, t + 1, \)
\[
\beta_i' + \gamma_{i-1} - \beta_i = \gamma_i
\] (3.8)
for some element \( \gamma_i \in E_{i+1} \), and we take \( \gamma_{-1} = \gamma_{t+1} = 0 \).

Now we consider the restriction of \( \theta \) to \( U^{r_i/2+}_{A_i+1} \), on which each factor \( \theta_j \) is equal to
\[
\psi_{A/F} \circ \text{tr}_{A/F}(\beta_j) \text{ if } j \leq i, \quad \text{and} \quad \xi_j \circ \text{det}_{B_j/E_j} = (\xi_j \circ N_{E_i/E_j}) \circ \text{det}_{B_i/E_i} \text{ if } j > i.
\]
Similar results apply to each factor \( \theta'_j \) of \( \theta' \). We then apply (3.6) and (3.8) to obtain
\[
\theta(\theta')^{-1}|_{U^{r_i/2+}_{A_i+1}} = \phi_i \circ \text{det}_{B_{i+1}/E_{i+1}} \cdot \psi_{A/F}(\gamma_i)^{-1}
\]
which is just trivial because of (3.7). Therefore, we have \( \theta = \theta' \).

Given \( \beta \) as in (3.5), we form a stratum \([A, r, 0, \beta]\), in the sense of [BK93, (1.5)]. Note that we have taken \( \beta = 0 \) when \( \xi \) is tamely ramified, in which case the associated stratum is null.

**Proposition 3.3.** (i) If the level \( r \) of \( \xi \) is positive, then the stratum \([A, r, 0, \beta]\) is simple, with a sequence of approximation strata \([A, r, r_i, \gamma_i]\) where
\[
\gamma_i = \sum_{j=1}^{t+1} \beta_j,
\]
and each with a derived stratum \([A_i, r_i, r_i - 1, c_i]\), all in the sense of [BK93, (2.4.2)].

(ii) \( \theta \in \mathcal{C}(A, 0, \beta) \), the set of simple characters in the sense of [BK93, (3.2.3)].

**Proof.** We first prove (i), which is to show that the sequence \([A, n, r_i, \gamma_i]\) satisfies the conditions in [BK93, (2.4.1)]. In fact, many of the arguments are routine, mostly following from constructions. One technical part is [BK93, (2.4.1)(iv)], where we have to show that
\[
k_0(\gamma_i, A) = -r_i \quad \text{for each } i = 0, \ldots, t.
\]
We first decompose \( \beta \) term by term as \( \sum_{i=1}^r a_i \) with \( a_i \in C_E \) and \( v_E(a_i) = -i \). Hence \( \beta_i = \sum_{j=r_i+1}^r a_j \) and \( a_{r_i} = c_i \). We now apply induction, assuming that \( k_0(\gamma_{i+1}, A) = -r_{i+1} \), which is less than \( v_E(c_i) \). By the second case of Proposition 2.3, we have \( k_0(c_i + \gamma_i + 1, A) = -r_i \).

Now notice that all \( a_k \) with \( k = r_i, \ldots, r_i+1 \), lies in \( E_i = F[\sum_{l=r_i}^k a_l + \gamma_i + 1] \). In particular \( \sum_{l=r_i}^{r_i+1} a_l + \gamma_i + 1 = \gamma_i \), and so by the first case of Proposition 2.3 \( k_0(\gamma_i, A) = -r_i \).

Once (i) is established, (ii) can be checked just by the definition in [BK93, (3.2.3)]. The case for \( \theta \) being trivial is just by convention, so we move on to the positive level case. By induction along the approximation sequence in (i), it suffices to show that for each \( i = 0, \ldots, t + 1 \), we have
\[
\Theta_i := \theta_i \cdots \theta_{i+1} \in \mathcal{C}(A, r_i-1/2+/+1, \gamma_i).
\]
Now the subgroup \( H^{r_{i-1}+1/2+} \) is \( U_{A_i+1}^{r_{i-1}+1/2+} \). For each \( j \geq i \), the factor \( \theta_j|_{H^{r_{i-1}+1/2+}} \) is equal to
\[
\xi_j \circ \text{det}_{B_j/E_j}|_{U_{A_i+1}^{r_{i-1}+1/2+} \cdots U_{A_{i+1}}^{r_{i-2}+1/2+} \cdots} \cdot \psi_{A/F}(\beta_j)^{-1}|_{U_{A_i+1}^{r_{i-1}+1/2+} \cdots U_{A_{i+1}}^{r_{i-2}+1/2+} \cdots}
\]
We hence check the conditions in [BK93, (3.2.3)] for the character \( \Theta_i \).
(a) We have $\Theta_i|_{U_{\mathfrak{a}^i}^{r_i/2+}} = (\xi_{i+1} \circ N_{E_i/E_{i+1}}) \cdots (\xi_{t+1} \circ N_{E_t/E_t}) \circ \det_{E_i/E_i}$.

(b) The compact subgroup $H^{r_i-1/2+}$ is clearly normalized by

$$\mathfrak{H}(\mathfrak{a}_i) = \{ x \in B_i^x : x^{-1} \mathfrak{a}_i x = \mathfrak{a}_i \},$$

and so are the characters $\xi_j \circ \det_{B_j/E_j}$ and $\psi_{F} \circ \text{tr}_{A/F} (\beta_j^*)$ for $j \geq i$.

(c) We have $H^{r_i/2+} = U_{\mathfrak{a}_i^{r_i/2+}} \cdots U_{\mathfrak{a}_{t+1}^{r_{t+1}/2+}}$, on which the factor $\theta_i$ is equal to $\psi_{F} \circ \text{tr}_{A/F} (\beta_j^*)$, and $\Theta_{i+1} \in C(\mathfrak{a}, r_{i/2+}, \gamma_{i+1})$ by induction assumption.

We show very briefly that our $\theta$ agrees with the one in [BH05, Section 2.3]. We will not go into detail as it incurs heavy definitions and notations from transfers [BK93], endo-classes [BH96, Section 7], and tame liftings [BH96, Section 9], but only refer to the references as given.

Suppose that $\xi$ is an admissible character of $E \times$, with an associated stratum $[\mathfrak{a}, r, 0, \beta]$ as constructed in the previous section, and $\theta \in C(\mathfrak{a}, 0, \beta)$ is a simple character of a compact subgroup $H^1$ of $G$. Recall from [BH05, Section 2.3] that, if we write $\xi|_{U_0} = \Xi \circ N_{E_0/E_0}$ for some character $\Xi_0$ of $U_0$, and denote the endo-classes $E_{F}(\theta)$ and $E_{F}(\Xi_0)$ of $\theta$ and $\Xi_0$ respectively, then a specific simple character $\theta_0$ is characterized by the condition that $E_{F}(\Xi_0)$ is an $E_0/F$-lift of $E_{F}(\theta_0)$.

Our simple character $\theta$ constructed above satisfies this condition, because of the relation $\theta|_{U_0} = \xi|_{U_0}$

which is exactly the relation in [BH96, (9.2)] that defines the tame lifting of a simple character. Hence $\theta_0 = \theta$.

We hence follow [BH05, Section 2.3] and define an extended maximal simple type $\Lambda = \Lambda_{\xi}$ of $J$ containing $\theta = \theta_{\xi}$. We then put $\pi = \pi_{\xi} := \text{cclnd}_{E}^{F} \Lambda$.

**Proposition 3.4.** (i) The representation $\pi$ is irreducible, supercuspidal, and essentially tame. Moreover, any such a representation arises from the above construction.

(ii) We have $f(\pi) = n/e(E_0/F)$.

(iii) The isomorphism class of $\pi$ depends only on the orbit of $(E/F, \xi)$ under Galois conjugation. (This orbit is called an admissible pair in [BH05].)

**Proof.** All statements can be deduced by [BH05, Proposition 2.3 and Theorem 2.3]

4 A special choice of ordered basis

In this section, we will define an ordered basis for constructing our explicit Whittaker data. We first provide the desired properties of our basis; then in the next section, we will describe this basis explicitly.

9
4.1 Properties of a special choice of basis

We again suppose that $E/F$ is a tamely ramified extension of degree $n$, and is identified with an $F$-vector space $V$, together with a lattice chain $L$ in $V$ identified with $\{p_E^r\}_{r \in \mathbb{Z}}$. For each $i = 0, \ldots, t+1$, we require an $E_i$-decomposition

$$V = \bigoplus_{j=1}^{[E:E_i]} W^i_j$$

such that the following conditions hold.

(i) Each $W^i_j$ is 1-dimensional over $E_i$.

(ii) For each $j = 1, \ldots, [E : E_i]$, we have

$$W^i_j = \bigoplus_{k=1}^{[E_i : E_{i+1}]} W^{i+1}_{j-[E_i : E_{i+1}]+k}.$$  \hfill (4.2)

(iii) For each $r \in \mathbb{Z}$, we have

$$L(r) = \bigoplus_{j=1}^{[E:E_i]} L(r) \cap W^i_j,$$

which means that the decomposition (4.1) conforms with $L$ over $E_i$, in the sense of [BK93, (7.1.1(i))] or [BH96, (10.5)].

We will define, for each $i = 0, \ldots, t+1$, an ordered $E_i$-basis $\mathfrak{b} = \{x_j\}_{j=1}^n$ of $V \cong E$.

$$x_1 = 1 \text{ and } x_{j+1} = c_ix_j$$

for $i = -1, \ldots, t$, if $j$ is a multiple of $[E_{i+1} : F]$ but not a multiple of $[E_i : F]$. Note that

- $v_E(x_j) \leq v_E(x_k)$ for all $j > k$, with equality if and only if $E \neq E_0$ and $k$ is a multiple of $[E_0 : F]$ and $j = k + 1$, in which case $x_{k+1} = c_{-1}x_k$.
- If $\beta \notin F$, then $v_E(x_j) < 0$ for all $j > 1$.
- $\beta_i \in \bigoplus_{k=1}^{[E_i : F]} Fx_k$ for each $i = -1, \ldots, t+1$. 

\hfill (4.4)
We can also define this basis inductively as follows. Let
\[ b^{-1} = \{1, c_{-1}, c_{-1}^2, \ldots, c_{-1}^{[E:E_0]-1}\}. \]

This is an ordered cyclic \( E_0 \)-basis of \( E \). For \( i = 0, \ldots, t + 1 \), we define
\[ c(e_i) = \{1, c_i, c_i^2, \ldots, c_i^{[E:E_{i+1}]-1}\} \]
and, if \( b^{i-1} \) is ordered as \( \{z_1, \ldots, z_{[E:E_i]}\} \), define
\[ b_j^i = z_j c_i^{(j-1)([E:E_{i+1}]-1)} c(e_i) \]
for \( j = 1, \ldots, [E:E_i] \). Each \( b_j^i \) is an \( E_{i+1} \)-basis of an \( E_i \)-vector space, and
\[ b^i = b_1^i \sqcup \cdots \sqcup b_{[E:E_i]}^i, \]
is an \( E_{i+1} \)-basis of \( E \). Finally, we have \( b = b' \).

We hence define, for \( j = 1, \ldots, [E:E_i] \),
\[ W_j^i = \text{span}_F b_j^i, \]
which is an \( E_i \)-vector space of dimension 1, and define \( W_j^i \), for \( j = 1, \ldots, [E:E_i] \), inductively by (4.2). Conditions (i) and (ii) in the previous section are clearly satisfied. We then take a suitable \( a(r,j) \) such that
\[ p_E^r = \sum_{j=1}^n a(r,j) (x_j / \varpi_F)^{[v_E(x_j)]/e_F} \]
for each \( r \in \mathbb{Z} \). If \( L(r) \) is identified with \( p_E^r \), then condition (iii) is satisfied by inductively using (2.2). In particular, the condition (*) in Section 2.3 is satisfied.

**Proposition 4.1.** (i) The matrix presentation \( \beta_{j,k} \), where \( j, k = 1, \ldots, n \), of \( \beta \) with respect to \( b \) takes the form
\[ \beta_{j,k} \in \begin{cases} 1 + p_F & \text{if } j - k = 1 \text{ and } k \text{ is not a multiple of } [E_0 : F], \\ p_F & \text{if } j - k > 1 \text{ or } (j - k = 1 \text{ and } k \text{ is a multiple of } [E_0 : F]). \end{cases} \]

(ii) When \( E \neq E_0 \), the matrix presentation \( \beta_{-1,j,k} \) of \( \beta_{-1} \) with respect to \( b \) takes the form
\[ \beta_{-1,j,k} \in \begin{cases} 1 + p_F & \text{if } j - k = 1 \text{ and } k \text{ is a multiple of } [E_0 : F], \\ p_F & \text{if } j - k > 1 \text{ or } (j - k = 1 \text{ and } k \text{ is not a multiple of } [E_0 : F]). \end{cases} \]

(iii) In the matrix presentation of \( H^1 \) with respect to \( b \), the entries in the strictly upper triangle belong to \( a_F \).

(We remark that, in cases (i) and (ii), we do not need to study the \((j,k)\)-entries with \( j \leq k \).)

**Proof.** For each \( k = 1, \ldots, n \), we will determine where the entries of the \( k \)th column of \( \beta \) belong with respect to \( b \). Let \( i = i(k) \) be the index such that \( k \) is a multiple of \([E_{i+1} : F]\) but not a multiple of \([E_i : F]\), then \( x_{k+1} = c_i x_k \) by construction. If \( E = E_0 \), we want to show that the product
\[ \beta x_k = \sum_{i=0}^{t+1} \beta_i x_k \]
lies in

\[ \bigoplus_{l=1}^{k} Fx_l + x_{k+1} + \bigoplus_{l=k+1}^{n} p_F x_l. \]

First of all, we have

\[ \beta_{t+1} x_k + \cdots + \beta_{i+1} x_k \in \bigoplus_{l=1}^{k} Fx_l, \]  

(4.6)

because if we write \( x_k = c_{t+1}^{m_{t+1}} \cdots c_{i+1}^{m_{i+1}} c_i^m \) for some integers \( m_{t+1}, \ldots, m_i \geq 0 \), then we see that \( E_i x_k \in \bigoplus_{l=1}^{k} Fx_l \), in particular (4.6) holds. We then show that

\[ \beta_i x_k + \beta_{i-1} x_k + \cdots + \beta_0 x_k \in x_{k+1} + \bigoplus_{l=k+1}^{n} p_F x_l. \]

(4.7)

Consider

\[ \beta_i x_k = c_i u x_k = x_{k+1} + w x_k \]

for some \( u = 1 + w \in U_E^1 \), and observe that

\[ v_E(w x_k) > v_E(x_k) > v_E(x_{k+1}). \]

We also have

\[ v_E(\beta_j x_k) = v_E(c_j x_k) > v_E(c_i x_k) > v_E(x_{k+1}) \]

for all \( j < i \). Therefore, the coefficients of \( x_i \), for \( l \geq k + 1 \), of all \( w x_k, \beta_0 x_k, \ldots, \beta_{i-1} x_k \) lie in \( p_F \) by Proposition 2.4, and (4.7) holds. When \( E \neq E_0 \), the proof is similar, except that when \( k \) is a multiple of \( [E_0 : F] \), we have \( x_{k+1} = c_1 x_k \) which is not a summand of \( \beta \), and so we only have

\[ \beta x_k \in \bigoplus_{l=1}^{k} Fx_l \oplus \bigoplus_{l=k+1}^{n} p_F x_l. \]

For (ii), the arguments are similar to above. If \( k \) is a multiple of \( [E_0 : F] \), then \( \beta_{i-1} x_k = x_{k+1} \). Otherwise, we have \( v_E(\beta_{i-1} x_k) = v(x_k) > v(x_{k+1}) \), and so \( \beta_{i-1} x_k \in \bigoplus_{l=1}^{k} Fx_l \oplus \bigoplus_{l=k+1}^{n} p_F x_l. \)

For (iii), note from (4.5) that \( \{ x_j/w_E(x_j)/x_j \}_{j=1}^{n} \) is an \( \mathfrak{o}_F \)-basis for the lattice \( \mathcal{L}(r) = p_E^r \) of \( V = E \), which is used to define \( H^1 \). With this basis, the entries of \( H^1 \) are in \( \mathfrak{o}_F \). If we use the basis \( \mathfrak{b} = \{ x_j \}_{j=1}^{n} \) instead, then \((j,k)\)-entry is multiplied by \( w_E(x_j)/x_j \). In the upper triangle consisting of \((j,k)\)-entries where \( j < k \), we have \( v_F(x_j) > v_F(x_k) \), and so the \((j,k)\)-entry with respect to \( \mathfrak{b} \) is still in \( \mathfrak{o}_F \).

\[ \Box \]

**Remark 4.2.** Note that the elements \( c_i \in C_E \), and so the basis \( \mathfrak{b} \) are uniquely determined by \( \xi \), which means that they are independent of the factorization (3.1). Another factorization of \( \xi \) yields another set of elements \( \{ \beta_i \} \) and so another \( \beta \), but the matrix presentation of that \( \beta \) takes the same form as in the proposition.

\[ \Box \]

Given an element \( \alpha \in A \), we denote a map

\[ \psi_\alpha : A \to \mathbb{C}, x \mapsto \psi_F \circ \text{tr}_{A/F}(\alpha(x - 1)) \]

\[ \psi_{\beta+\beta_{-1}} |_{N \cap H^1} (x) = \psi_F(\sum_{j=1}^{n-1} x_{j+1,j}), \text{ where } x_{j,k} \text{ is the } (j,k)\text{-entry of the matrix presentation of } x \in A \text{ with respect to } \mathfrak{b}. \]

**Corollary 4.3.**

**Proof.** With respect to the basis \( \mathfrak{b} \), it is easy to see that the entries of \( \beta + \beta_{-1} \) in the lower sub-diagonal belong to \( 1 + p_F \), and those underneath belong to \( p_F \). Also, \( N \) is defined by this ordered basis, and the entries of \( N \cap H^1 \) in the strictly upper triangle belong to \( \mathfrak{o}_F \). Since \( \psi_F \) is trivial on \( p_F \) but not on \( \mathfrak{o}_F \), we have the desired result.

\[ \Box \]

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4.3 Some Iwahori factorizations

We first work on a general situation. Let $E/F$ be a finite extension and $V$ be a finite dimensional $E$-vector space, also regarded as an $F$-vector space. Denote $A = \text{End}_F(V)$ and let $\mathfrak{A}$ be an hereditary $\sigma_F$-order in $A$ defined by an $\sigma_E$-lattice $\mathcal{L}$ in $V$. Let $B$ be the centralizer of $E$ in $A$, and denote $\mathfrak{B} = \mathfrak{A} \cap B$.

Let $\mathcal{F}$ be an $F$-flag in $V$ refining an $E$-flag $\mathcal{F}_E$, i.e., $\mathcal{F} \supseteq \mathcal{F}_E$ by regarding $\mathcal{F}_E$ as an $F$-flag. Moreover, suppose that the $E$-decomposition $\oplus_i W_i$ of $V$ defined by $\mathcal{F}_E$ conforms with $\mathcal{L}$ over $E$, in the sense that $\mathcal{L}(r) = \oplus_i (\mathcal{L}(r) \cap W_i)$ for all $r \in \mathbb{Z}$ (condition (iii) in Section 4.1).

Let $N_\mathcal{F}$ be the unipotent subgroup in $G$ defined by the flag $\mathcal{F}$, $N_\mathcal{F}^E$ be its opposite, and $M_\mathcal{F}$ be the subgroup of $G$ stabilizing $\mathcal{F}$. Also, let $N_\mathcal{F}_E$, $N_\mathcal{F}^E$, and $M_\mathcal{F}_E$ similarly using the flag $\mathcal{F}_E$.

**Proposition 4.4.** (i) For each positive integer $k$, the subgroups $U^k_{\mathfrak{A}}$ and $U^k_{\mathfrak{B}}$ admit an Iwahori decomposition

$$U^k_{\mathfrak{A}} = (U^k_{\mathfrak{A}} \cap N_{\mathcal{F}_E})(U^k_{\mathfrak{A}} \cap M_{\mathcal{F}})(U^k_{\mathfrak{A}} \cap N_{\mathcal{F}_E})$$

and similarly for $U^k_{\mathfrak{B}}$.

(ii) For positive integers $k_1 < k_2$, we have

$$(U^k_{\mathfrak{A}} U^{k_2}_{\mathfrak{A}}) \cap N_\mathcal{F} = (U^k_{\mathfrak{A}} \cap N_{\mathcal{F}_E})(U^{k_2}_{\mathfrak{A}} \cap N_{\mathcal{F}_E}).$$

**Proof.** (i) is given by [BH96, (10.4)] and noting that, if the decomposition conforms with $\mathcal{L}$ over $E$, it also conforms with $\mathcal{L}$ over $F$. For (ii), we first prove the ‘maximal’ case, i.e., when $V$ is 1-dimensional over $E$, in which case $N_{\mathcal{F}_E}$ is trivial, and the right hand side is $U^2_{\mathfrak{A}} \cap N_\mathcal{F}$. This is equal to the left hand side by [BS09, Lemma A.5 Appendix]. If $E$ is not maximal in $A$, we can follow the idea in [BS09, Corollary A.6 Appendix]. By (i), we have

$$(U^k_{\mathfrak{A}} U^{k_2}_{\mathfrak{A}}) \cap P_{\mathcal{F}_E} = (U^k_{\mathfrak{A}} \cap N_{\mathcal{F}_E})(U^{k_2}_{\mathfrak{A}} \cap N_{\mathcal{F}_E})(U^k_{\mathfrak{A}} \cap M_{\mathcal{F}})(U^{k_2}_{\mathfrak{A}} \cap M_{\mathcal{F}})$$

and note that $N_{\mathcal{F}} \subseteq P_{\mathcal{F}_E}$, so

$$U^k_{\mathfrak{A}} U^{k_2}_{\mathfrak{A}} \cap N_{\mathcal{F}} = (U^k_{\mathfrak{A}} \cap N_{\mathcal{F}_E})(U^{k_2}_{\mathfrak{A}} \cap N_{\mathcal{F}_E})((U^k_{\mathfrak{A}} \cap M_{\mathcal{F}})(U^{k_2}_{\mathfrak{A}} \cap M_{\mathcal{F}})) \cap N_{\mathcal{F}}.$$ 

The last bracket is a maximal case for the Levi subgroup $M_{\mathcal{F}_E}$, which is equal to $U^2_{\mathfrak{A}} \cap M_{\mathcal{F}_E} \cap N_{\mathcal{F}}$. Since $(U^{k_2}_{\mathfrak{A}} \cap N_{\mathcal{F}_E})(U^k_{\mathfrak{A}} \cap M_{\mathcal{F}_E} \cap N_{\mathcal{F}}) = U^2_{\mathfrak{A}} \cap N_{\mathcal{F}}$, we have the desired result.

Now we return to the situation when $E/F$ is tamely ramified, and we have a tower of intermediate subfields (3.2) coming from an admissible character.

**Corollary 4.5.** Given a sequence of flags as in (4.3), then

$$H^1 \cap N_{\mathcal{F}} = (U^1_{\mathfrak{A}} \cap N_{\mathcal{F}_E}) \cdots (U^{r_{i-1}/2+}_{\mathfrak{B}} \cap N_{\mathcal{F}_E})(U^{r_{i}/2+}_{\mathfrak{A}} \cap N_{\mathcal{F}}).$$

**Proof.** We apply Proposition 4.4(ii) inductively. First regard $U^1_{\mathfrak{B}} \cdots U^{r_{i-1}/2+}_{\mathfrak{B}}$ as a subgroup of $U^1_{\mathfrak{A}}$, and so

$$H^1 \cap N_{\mathcal{F}} = (U^1_{\mathfrak{B}} \cdots U^{r_{i-1}/2+}_{\mathfrak{B}} \cap N_{\mathcal{F}_E})(U^{r_{i}/2+}_{\mathfrak{A}} \cap N_{\mathcal{F}}).$$

We can therefore apply induction on $U^1_{\mathfrak{A}} \cdots U^{r_{i-1}/2+}_{\mathfrak{B}} \cap N_{\mathcal{F}_E}$, and obtain the desired result.

**Proposition 4.6.** $\theta|_{N \cap H^1} = \psi_\beta|_{N \cap H^1}$.

**Proof.** For each $i$, we already know that $\theta_i$ is equal to $\psi_\beta$ on $(U^{r_{i}/2+}_{\mathfrak{B}} \cap N_{\mathcal{F}_E}) \cdots (U^{r_{i}/2+}_{\mathfrak{A}} \cap N_{\mathcal{F}})$ from its construction. It suffices to show that $\theta_i$ on $(U^1_{\mathfrak{A}} \cap N_{\mathcal{F}_E}) \cdots (U^{r_{i-1}/2+}_{\mathfrak{B}} \cap N_{\mathcal{F}_E})$, which is $\xi_i \circ \det_{B/E_i}$, is also equal to $\psi_\beta$. Indeed, on all $N_{\mathcal{F}_E}$, for $j \leq i$, the character $\det_{B/E_i}$ is trivial, while $\psi_\beta$ is also trivial since $\beta_i \in M_{\mathcal{F}_E} \subset M_{\mathcal{F}_E}$. \qed
5 The main result

Let \( \pi \) be an essentially tame supercuspidal representation compactly induced by an extended maximal type \((J, \Lambda)\) which contains a simple character \( \theta \in \mathcal{C}(\mathfrak{A}, 0, \beta) \) associated to \( \xi \), and \( N = N_F \) be the maximal unipotent subgroup defined by the \( F \)-flag

\[
F = \{ V_j \}_{j=1}^n, \quad \text{where} \quad V_j = \bigoplus_{k=1}^j F x_k,
\]

where \( \{ x_j \}_{j=1}^n \) is the ordered basis constructed in (4.4).

Let

- \( \alpha_0 \) be an element in \( \text{Mat}_n(F) \) whose matrix representation \((\alpha_0)_{j,k}\) with respect to \( b \) is 1 if \( j - k = 1 \) but is 0 if \( k \) is a multiple of \([E_0 : F]\), and is 0 if \( j - k > 1 \) (and can be anything if \( i \leq j \)),
- \( \alpha_{-1} \) be an element in \( \text{Mat}_n(F) \) whose matrix representation \((\alpha_{-1})_{j,k}\) with respect to \( b \) is 0 if \( j - k = 1 \) but is 1 if \( k \) is a multiple of \([E_0 : F]\), and is 0 if \( j - k > 1 \) (and can be anything if \( i \leq j \)), and
- \( \alpha = \alpha_{-1} + \alpha_0 \).

Hence, with the notation defined in [PS08], we have

\[
\alpha \in \mathcal{X}_F^+ := \{ x \in A : xV_i \subset V_{i+1} \text{ and } xV_i \not\subset V_i \text{ for all } i \},
\]

and so by [PS08, Lemma 1.2] \( \psi_\alpha \) defines a non-degenerate character of \( N \). (Note that \( \psi_\beta \) may not extend to a character of the whole \( N \).)

**Theorem 5.1.** \( \text{Hom}_{N \cap J}(\psi_\alpha, \Lambda) \neq 0 \).

**Proof.** We show that the condition at the beginning of [PS08, Section 4.2] is satisfied, then the result is given by [PS08, Corollary 4.13]. Hence it suffices to show that \((F, \psi_{\alpha_0}, \alpha_{-1})\) satisfies the conditions (i)-(iv) in [PS08, Theorem 3.3].

(i) This condition is just \( F_{E_0} \subset F \) in our notation, which is true by construction.

(ii) In Proposition 4.6, we showed that \( \theta|_{N \cap H^1} = \psi_\beta|_{N \cap H^1} \). Now with the matrix representation of \( \beta \) and elements in \( H^1 \) in Proposition 4.1(i), we know that \( \psi_\beta|_{N \cap H^1} = \psi_{\alpha_0}|_{N \cap H^1} \).

(iii) If \( E = E_0 \), then \( N_{F_{E_0}} \) is trivial and the results is clearly satisfied. If \( E \neq E_0 \), then \( \psi_{\alpha_0} \) on \( N_{F_{E_0}} \) is trivial since the matrix entry \((\alpha_0)_{k,k+1}\) with respect to \( b \) is 0 when \( k \) is a multiple of \([E_0 : F]\).

(iv) The maximal unipotent subgroup \( N_{F_{E_0}} \cap U_{E_0}/U_{E_0}^1 \) of \( U_{E_0}/U_{E_0}^1 \) is defined by the cyclic basis \( \{ 1, \beta_{E_0}^{\pm1}, \ldots, \beta_{E_0}^{- [E_0 : F]} \} \), where each \( x \) is \( x + p_{E_0} \in U_{E_0}/U_{E_0}^1 \) for \( x \in U_{E_0} \). The character \( \psi_{\beta_{E_0}} \) clearly defines a non-degenerate character, by arguments similar to [BH98, 2.1]. This character is equal to \( \psi_{\alpha_{-1}} \) by Proposition 4.1(ii).

\( \square \)
References


