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On modifications of the concepts of perfect and proper equilibria

A. P. Jurg[1], I. Garcia Jurado[2], and P. E. M. Borm[1]

[1] Department of Mathematics/NICI, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands
[2] Department of Statistics and Operation Research, University of Santiago de Compostela, Spain

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Summary. Further refinements of perfect and proper Nash equilibria are introduced, where it is assumed that the players put equal probability on pure strategies with the same payoff vector. Relations with persistent equilibria and stable sets are studied.

Zusammenfassung. In dieser Arbeit werden Eigenschaften von Nash-Gleichgewichten für Situationen diskutiert, in denen die Spieler reine Strategien mit dem gleichen Ergebnisvektor mit gleichen Wahrscheinlichkeiten bewerten. Die Beziehungen zum Konzept der stabilen Mengen werden untersucht.

Key words: Bimatrix game, Nash-equilibria, perfect and proper equilibria

Schlüsselwörter: Zwei-Personen-Spiele, Nash-Gleichgewichte, perfekte Gleichgewichte

1. Introduction

In a strategic game not all Nash equilibria (Nash 1951) are reasonable outcomes. Many authors have tried to overcome this problem by considering refinements of the Nash equilibrium concept. Among them Selten (1975) defined and showed existence of perfect equilibria. These equilibria are stable against small mistakes the players could make in choosing their strategies. Myerson (1978) was able to refine the concept of perfect equilibrium by imposing a rationality restriction which says that the players make more costly mistakes with less probability. Thus proper equilibria were defined and it was shown that every game has at least one proper equilibrium.

Recently Garcia Jurado and Prada Sanchez (1990) defined equalized proper equilibria and gave a proof of existence. In this concept Myerson's definition is pushed further in the sense that more rationality is asked from the players. Besides the properness features it is assumed that each player puts equal probability on any two pure strategies that correspond to the same payoff vector, regardless of the actions of the opponents.

In this paper we study the latter assumption of rationality in case of (finite) two person games (bimatrix games). First we also define equalized perfect equilibria in an obvious way. Secondly we assign to each game a so-called "equalized game" and prove that perfect and proper equilibria for this game exactly correspond to the equalized perfect and equalized proper equilibria for the original game. This approach provides and alternative existence proof for both equilibrium concepts. Formal definitions of equalized perfect and equalized proper equilibria are given in Sect. 2. In Sect. 3 we introduce the equalized game and prove the result mentioned above concerning the perfect and proper equilibria for this game. In this section we further define iterated equalized perfect and iterated equalized proper equilibria as those equilibria that correspond to perfect and proper equilibria for the so-called iterated equalized game. Section 4 studies relations between equalized perfect and proper equilibria and persistent equilibria (Kalai and Samet 1984) and stable sets (Kohlberg and Mertens 1986). In particular, we provide a new proof for the result of Garcia Jurado (1989) that every game has a persistent equilibrium which is also an equalized proper equilibrium. The paper concludes with some remarks in Sect. 5.

Notation. Let $e_1, e_2, \ldots, e_t$ denote the unit vectors in $\mathbf{R}^t$. For $S \subset \{1, \ldots, t\}$ the vector $e^S \in \mathbf{R}^t$ is defined by

$$e_k^S = \begin{cases} 1 & \text{if } k \in S \\ 0 & \text{elsewhere} \end{cases}$$

For $x, y \in \mathbf{R}^t$ we denote $x \leq y$ $(x < y)$ if $x_k \leq y_k$ $(x_k < y_k)$ for all $k \in \{1, \ldots, t\}$. Finally, for $A \subset \mathbf{R}^t$, we denote by $\mathrm{Conv}(A)$ the convex hull of A.

2. Equalized perfect and proper equilibria

Let $A=[a_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ and $B=[b_{ij}]_{i=1}^{m}{}_{j=1}^{n}$ be real $m\times n$ matrices. The *$m\times n$ bimatrix game* (A,B) is the two-person game in strategic form with strategy spaces $\Delta_m:=\{p\in\mathbf{R}^m|p\geq 0,\sum_{i=1}^{m}p_i=1\}$ and $\Delta_n:=\{q\in\mathbf{R}^n|q\geq 0,\sum_{j=1}^{n}q_j=1\}$ and payoff functions $K:\Delta_m\times\Delta_n\to\mathbf{R}$ and $L:\Delta_m\times\Delta_n\to\mathbf{R}$ for player 1 and player 2 respectively, where $K(p,q)=pAq$ and $L(p,q)=pBq$ for all $(p,q)\in\Delta_m\times\Delta_n$. Strategies $e_i\in\Delta_m$ and $e_j\in\Delta_n$ are called *pure* and strategies in $\mathring{\Delta}_m:=\{p\in\Delta_m|p>0\}$ and $\mathring{\Delta}_n$ *completely mixed.*

For $p\in\Delta_m$, $C(p):=\{i\in\{1,\ldots,m\}|p_i>0\}$ denotes the *carrier* of p, $PB_2(p):=\{j\in\{1,2,\ldots,n\}|pBe_j=\max_{s\in\{1,\ldots,n\}}pBe_s\}$ represents the set of *pure best replies* of player 2 to p (in (A,B)) and $B_2(p):=\operatorname{Conv}(\{e_j\in\Delta_n|j\in PB_2(p)\})$ is the set of *best replies* of player 2 to p. For $q\in\Delta_n$, the sets $C(q)$, $PB_1(q)$ and $B_1(q)$ are defined analogously.

A strategy pair $(p,q)\in\Delta_m\times\Delta_n$ is called a *(Nash) equilibrium* for (A,B) (cf. Nash 1951) if both $p\in B_1(q)$ and $q\in B_2(p)$, or equivalently, if both $C(p)\subset PB_1(q)$ and $C(q)\subset PB_2(p)$. It was shown that the set $E(A,B)$ of all Nash equilibria for (A,B) is non-empty.

In this paper we introduce and investigate modifications of the concepts of perfect equilibria (Selten 1975) and proper equilibria (Myerson 1978). A strategy pair $(\hat{p},\hat{q})\in\Delta_m\times\Delta_n$ is called *perfect (proper)* for (A,B) if there exist sequences $\{\varepsilon^k\}_{k\in\mathbf{N}}$ of positive reals converging to zero and $\{(p^k,q^k)\}_{k\in\mathbf{N}}$ of pairs of completely mixed strategies converging to $(\hat{p},\hat{q})$ such that (p^k,q^k) is ε^k-perfect (ε^k-proper) for (A,B) for all $k\in\mathbf{N}$. Here, with $\varepsilon>0$, a pair $(p,q)\in\Delta_m\times\Delta_n$ is called *ε-perfect* for (A,B) if for all $i,r\in\{1,\ldots,m\}$ and $j,s\in\{1,\ldots,n\}$

$$\begin{cases} e_iAq<e_rAq\Rightarrow p_i\leq\varepsilon \\ pBe_j<pBe_s\Rightarrow q_j\leq\varepsilon \end{cases}$$

and *ε-proper* for (A,B) if for all $i,r\in\{1,\ldots,m\}$ and $j,s\in\{1,\ldots,n\}$

$$\begin{cases} e_iAq<e_rAq\Rightarrow p_i\leq\varepsilon p_r \\ pBe_j<pBe_s\Rightarrow q_j\leq\varepsilon q_s\,. \end{cases}$$

For the sets $PE(A,B)$ of perfect and $PR(A,B)$ of proper equilibria for (A,B) it was shown that

$$\emptyset\neq PR(A,B)\subset PE(A,B)\subset E(A,B).$$

Now we introduce the following

Definition. Let (A,B) be an $m\times n$ bimatrix game. A strategy pair $(\hat{p},\hat{q})\in\Delta_m\times\Delta_n$ is called *equalized perfect (equalized proper)* for (A,B), or shortly e-perfect (e-proper), if there exist sequences $\{\varepsilon^k\}_{k\in\mathbf{N}}$ of positive reals converging to zero and $\{(p^k,q^k)\}_{k\in\mathbf{N}}$ of pairs of completely mixed strategies converging to $(\hat{p},\hat{q})$ such that (p^k,q^k) is ε^k-perfect (ε^k-proper) and equalized for all $k\in\mathbf{N}$. Here, a pair $(p,q)\in\Delta_m\times\Delta_n$ is called *equalized* if

$$\begin{aligned} &e_iA=e_rA\Rightarrow p_i=p_r \quad (i,r\in\{1,\ldots,m\}) \\ &Be_j=Be_s\Rightarrow q_j=q_s \quad (j,s\in\{1,\ldots,n\}). \end{aligned} \tag{1}$$

From this definition it is immediately clear that e-proper implies e-perfect and that e-proper (e-perfect) implies proper (perfect). Equalized proper equilibria have already been studied in Garcia Jurado and Prada Sanchez (1990) and existence was shown. We will provide a new proof of existence in Sect. 3.

The following example shows that a proper equilibrium need not be e-perfect.

Example 1. Consider the 2×3 bimatrix game (A,B) given by

$$(A,B)=\begin{bmatrix} (3,6) & (0,6) & (5,5) \\ (0,0) & (2,0) & (5,5) \end{bmatrix}.$$

Defining $\varepsilon^k=\frac{1}{k+1}$, $q^k=\frac{k}{k+1}e_3+\frac{1}{k+1}\left(\frac{2}{5}e_1+\frac{3}{5}e_2\right)$ and $p^k=\frac{k}{k+1}e_2+\frac{1}{k+1}e_1$, it is easily checked that (p^k,q^k) is ε^k-proper for all $k>1$.

Hence, $(e_2,e_3)\in PR(A,B)$. However, (e_2,e_3) is not e-perfect because all equalized and ε-perfect strategy pairs $(p,q)\in\Delta_2\times\Delta_3$ must satisfy $q_1=q_2$, which implies that $e_2Aq<e_1Aq$ and so $p_2\leq\varepsilon p_1$. In fact the unique e-perfect (e-proper) equilibrium for (A,B) is $\left(e_1,\frac{1}{2}e_1+\frac{1}{2}e_2\right)$.

3. Equalized games

In this section we assign to each bimatrix game a so-called equalized bimatrix game. It turns out that perfect and proper equilibria for the equalized game exactly correspond to the equalized perfect and equalized proper equilibria for the original game.

Let (A,B) be an $m\times n$ bimatrix game. Two pure strategies $e_i,e_r\in\Delta_m$ are called *payoff equivalent* for player 1 (in (A,B)) if $e_iA=e_rA$ (cf. (1)). The equivalence classes of payoff equivalent pure strategies induce a partition $\{M_1,M_2,\ldots,M_{\bar{m}}\}$ of the set $\{1,2,\ldots,m\}$. Similarly one defines payoff equivalent pure strategies for player 2 and a partition $\{N_1,N_2,\ldots,N_{\bar{n}}\}$ of $\{1,2,\ldots,n\}$. Furthermore, the *$\bar{m}\times\bar{n}$ equalized game* $(\bar{A},\bar{B})$ corresponding to (A,B) is defined by $\bar{A}=[\bar{a}_{\mu\nu}]_{\mu=1}^{\bar{m}}{}_{\nu=1}^{\bar{n}}$ and $\bar{B}=[\bar{b}_{\mu\nu}]_{\mu=1}^{\bar{m}}{}_{\nu=1}^{\bar{n}}$, where

$$\bar{a}_{\mu\nu}:=\frac{e^{M_\mu}}{|M_\mu|}A\frac{e^{N_\nu}}{|N_\nu|} \quad\text{and}\quad \bar{b}_{\mu\nu}:=\frac{e^{M_\mu}}{|M_\mu|}B\frac{e^{N_\nu}}{|N_\nu|} \tag{2}$$

for all $\mu\in\{1,\ldots,\bar{m}\}$ and $\nu\in\{1,\ldots,\bar{n}\}$. Note that the pure strategy $e_\mu\in\Delta_{\bar{m}}$ of the equalized game $(\bar{A},\bar{B})$ corresponds to the barycentre of the equivalence class of pure strategies corresponding to M_μ.

Let the mapping $f: \Delta_m \times \Delta_n \to \Delta_{\bar{m}} \times \Delta_{\bar{n}}$ be defined by

$$f(p,q) = \left(\left(\sum_{i \in M_\mu} p_i \right)_{\mu \in \{1,\ldots,\bar{m}\}}, \left(\sum_{j \in N_\nu} q_j \right)_{\nu \in \{1,\ldots,\bar{n}\}} \right) \quad (3)$$

for all $(p,q) \in \Delta_m \times \Delta_n$. Obviously f is continuous. Further we introduce the sets $\bar{\Delta}_m \subset \Delta_m$ and $\bar{\Delta}_n \subset \Delta_n$ by

$$\bar{\Delta}_m := \mathrm{Conv}\left(\left\{ \frac{e^{M_\mu}}{|M_\mu|} \right\}_{\mu \in \{1,\ldots,\bar{m}\}} \right) \quad \text{and}$$

$$\bar{\Delta}_n := \mathrm{Conv}\left(\left\{ \frac{e^{N_\nu}}{|N_\nu|} \right\}_{\nu \in \{1,\ldots,\bar{n}\}} \right) \quad (4)$$

such that all e-perfect equilibria for (A,B) are contained in $\bar{\Delta}_m \times \bar{\Delta}_n$. It is clear that the restriction $\bar{f}$ of f to $\bar{\Delta}_m \times \bar{\Delta}_n$ is a bijection and has a continuous inverse $\bar{f}^{-1}$.

With respect to Nash equilibria we now can formulate

Theorem 3.1. *Let (A,B) be an $m \times n$ bimatrix game and let $f: \Delta_m \times \Delta_n \to \Delta_{\bar{m}} \times \Delta_{\bar{n}}$ be defined as in (3). Let $(p,q) \in \bar{\Delta}_m \times \bar{\Delta}_n$.*

Then $(p,q) \in E(A,B)$ if and only if $f(p,q) \in E(\bar{A},\bar{B})$.

Proof. Let $p = \sum_{\mu=1}^{\bar{m}} c_\mu \dfrac{e^{M_\mu}}{|M_\mu|}$ and $q = \sum_{\nu=1}^{\bar{n}} d_\nu \dfrac{e^{N_\nu}}{|N_\nu|}$.

Define $(x,y) \in \Delta_{\bar{m}} \times \Delta_{\bar{n}}$ by $(x,y) := f(p,q)$. Let $\mu \in \{1,\ldots,\bar{m}\}$ be fixed. Then

$$\begin{aligned} e_\mu \bar{A} y &= \sum_{\nu=1}^{\bar{n}} \frac{e^{M_\mu}}{|M_\mu|} A \left(\frac{e^{N_\nu}}{|N_\nu|} \sum_{j \in N_\nu} q_j \right) \\ &= \sum_{\nu=1}^{\bar{n}} \frac{e^{M_\mu}}{|M_\mu|} A \left(d_\nu \frac{e^{N_\nu}}{|N_\nu|} \right) \\ &= \frac{e^{M_\mu}}{|M_\mu|} Aq = e_i A q \quad \text{for all } i \in M_\mu. \end{aligned} \quad (5)$$

It follows that $\mu \in PB_1(y)$ in $(\bar{A},\bar{B})$ if and only if $M_\mu \subset PB_1(q)$ in (A,B). Furthermore, since $x_\mu = \sum_{i \in M_\mu} p_i = c_\mu$ we have that $\mu \in C(x)$ if and only if $M_\mu \subset C(p)$.

This proves that $C(p) \subset PB_1(q)$ in (A,B) if and only if $C(x) \subset PB_1(y)$ in $(\bar{A},\bar{B})$. Similarly one shows that $C(q) \subset PB_2(p)$ in (A,B) if and only if $C(y) \subset PB_2(x)$ in $(\bar{A},\bar{B})$. □

With respect to e-perfect and e-proper equilibria we have

Theorem 3.2. *Let (A,B) be an $m \times n$ bimatrix game. Let $f: \Delta_m \times \Delta_n \to \Delta_{\bar{m}} \times \Delta_{\bar{n}}$ be defined as in (3). Let $(p,q) \in \bar{\Delta}_m \times \bar{\Delta}_n$. Then*

(i) (p,q) is e-perfect for (A,B) if and only if $f(p,q)$ is perfect for $(\bar{A},\bar{B})$

(ii) (p,q) is e-proper for (A,B) if and only if $f(p,q)$ is proper for $(\bar{A},\bar{B})$.

Proof. We only prove (ii). First we demonstrate the "only if"-part. Let (p,q) be e-proper for (A,B). Then we can find sequences $\{\varepsilon^k\}_{k \in \mathbf{N}}$ of positive reals converging to zero and $\{(p^k,q^k)\}_{k \in \mathbf{N}}$ of pairs of completely mixed strategies converging to (p,q) such that (p^k,q^k) is ε^k-proper and equalized for all $k \in \mathbf{N}$. Defining $(x,y) := f(p,q)$ and $(x^k,y^k) := f(p^k,q^k)$ for all $k \in \mathbf{N}$, it follows that $(x^k,y^k) \in \mathring{\Delta}_{\bar{m}} \times \mathring{\Delta}_{\bar{n}}$ for all $k \in \mathbf{N}$ and, by continuity of f, (x^k,y^k) converges to (x,y). Let, for all $k \in \mathbf{N}$,

$$\delta^k := \varepsilon^k \cdot \max\{ \max_{\mu \in \{1,\ldots,\bar{m}\}} |M_\mu|, \max_{\nu \in \{1,\ldots,\bar{n}\}} |N_\nu| \}. \quad (6)$$

We are finished if we can show that (x^k,y^k) is δ^k-proper for $(\bar{A},\bar{B})$ for all $k \in \mathbf{N}$. Let $\mu, \varrho \in \{1,\ldots,\bar{m}\}$ be such that $e_\mu \bar{A} y^k < e_\varrho \bar{A} y^k$. Then $e_i A q^k < e_r A q^k$ for all $i \in M_\mu$ and $r \in M_\varrho$ (cf. (5)). Hence, $p_i^k \leq \varepsilon^k p_r^k$ for all $i \in M_\mu$ and $r \in M_\varrho$. Consequently,

$$x_\mu^k = \sum_{i \in M_\mu} p_i^k \leq \varepsilon^k |M_\mu| \sum_{r \in M_\varrho} p_r^k = \varepsilon^k |M_\mu| x_\varrho^k \leq \delta^k x_\varrho^k.$$

Similarly one finds that $x^k \bar{B} e_\nu < x^k \bar{B} e_\sigma$ implies that $y_\nu^k \leq \delta^k y_\sigma^k$ for all $k \in \mathbf{N}$ and $\nu, \sigma \in \{1,\ldots,\bar{n}\}$.

Secondly we prove the "if"-part. Let $(x,y) := f(p,q)$ be proper for $(\bar{A},\bar{B})$. Let $\{\varepsilon^k\}_{k \in \mathbf{N}}$ and $\{(x^k,y^k)\}_{k \in \mathbf{N}} \subset \mathring{\Delta}_{\bar{m}} \times \mathring{\Delta}_{\bar{n}}$ be sequences as required for the properness of (x,y). Defining $(p^k,q^k) := \bar{f}^{-1}(x^k,y^k) \in \mathring{\Delta}_m \times \mathring{\Delta}_n$, the continuity of $\bar{f}^{-1}$ implies that (p^k,q^k) converges to $\bar{f}^{-1}(x,y) = (p,q)$.

Let δ^k be defined as in (6). By definition (p^k,q^k) is equalized, so it suffices to show that (p^k,q^k) is δ^k-proper for all $k \in \mathbf{N}$.

Let $i, r \in \{1,\ldots,m\}$ be such that $e_i A q^k e_r A q^k$. For $\mu, \varrho \in \{1,\ldots,\bar{m}\}$ with $i \in M_\mu$ and $r \in M_\varrho$, (5) implies that $e_\mu \bar{A} y^k < e_\varrho \bar{A} y^k$ so $x_\mu^k \leq \varepsilon^k x_\varrho^k$. Consequently,

$$\begin{aligned} p_i^k &= \frac{x_\mu^k}{|M_\mu|} \leq \frac{\varepsilon^k x_\varrho^k}{|M_\mu|} = \frac{\varepsilon^k}{|M_\mu|} |M_\varrho| \frac{x_\varrho^k}{|M_\varrho|} \\ &\leq \varepsilon^k \cdot |M_\varrho| p_r^k \leq \delta^k p_r^k. \end{aligned}$$

Similarly one finds that $p^k B e_j < p^k B e_s$ implies that $q_j^k \leq \delta^k q_s^k$ for all $k \in \mathbf{N}$ and $j, s \in \{1,\ldots,n\}$. □

The existence of proper equilibria immediately implies the following

Corollary. *Every bimatrix game has at least one e-proper equilibrium.*

These results are illustrated in

Example 2. Consider the 2×3 bimatrix game (A,B) of Example 1.

Then $\bar{m} = \bar{n} = 2$, $M_1 = \{1\}$, $M_2 = \{2\}$, $N_1 = \{1,2\}$ and $N_2 = \{3\}$. The equalized game $(\bar{A},\bar{B})$ is given by

$$(\bar{A},\bar{B}) = \begin{bmatrix} (1\frac{1}{2}, 6) & (5,5) \\ (1,0) & (5,5) \end{bmatrix}$$

It is easily checked that the unique perfect (proper) equilibrium for $(\bar{A},\bar{B})$ is (e_1,e_1). Hence, $\bar{f}^{-1}(e_1,e_1) = (e_1, \frac{1}{2}e_1 + \frac{1}{2}e_2) \in \Delta_2 \times \Delta_3$ is the unique e-perfect (e-proper) equilibrium for (A,B).

In Example 1 it was shown that $(e_2,e_3) \in PR(A,B)$. By Theorem 3.1 we have $f(e_2,e_3) = (e_2,e_2) \in E(\bar{A},\bar{B})$. However, $(e_2,e_2) \notin PR(\bar{A},\bar{B})$.

The following example indicates that the concepts of e-perfectness and e-properness can again be refined.

Example 3. Consider the 2×3 bimatrix game (A,B) given by

$$(A,B) = \begin{bmatrix} (0,1) & (1,-1) & (2,0) \\ (0,-1) & (1,1) & (2,0) \end{bmatrix}$$

The 1×3 equalized game $(\bar{A},\bar{B})$ is given by

$$(\bar{A},\bar{B}) = [(0,0) \quad (1,0) \quad (2,0)].$$

Note that in $(\bar{A},\bar{B})$ player 2 has three equivalent pure strategies. So one might consider the 1×1 equalized game $(\bar{\bar{A}},\bar{\bar{B}})$ corresponding to $(\bar{A},\bar{B})$ given by

$$(\bar{\bar{A}},\bar{\bar{B}}) = [(1,0)].$$

Obviously $(\bar{\bar{A}},\bar{\bar{B}})$ has only one (perfect and proper) equilibrium which corresponds to the e-proper equilibrium $(\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3)$ for (A,B). However, using Theorem 3.2, one finds that $(\frac{1}{2}e_1 + \frac{1}{2}e_2, q)$ is e-proper for (A,B) for all $q \in \Delta_3$.

Example 3 motivates the following definitions.

Let BG denote the set of all (finite) bimatrix games. Let the mapping $g: BG \to BG$ be defined by $g(A,B) := (\bar{A},\bar{B})$ for all $(A,B) \in BG$. It is clear that for every $(A,B) \in BG$ there exists a smallest integer t such that

$$g^t(A,B) = g^{t+1}(A,B).$$

The game $g^t(A,B)$ will be called the *iterated equalized game* corresponding to (A,B). Strategy pairs in the original game (A,B) will be called *iterated e-perfect* (*iterated e-proper*) if they correspond to perfect (proper) equilibria for the iterated equalized game.

By definition, the existence of these equilibrium concepts is guaranteed and, clearly, iterated e-perfect (iterated e-proper) implies e-perfect (e-proper). However, Example 3 shows that the converse of the last statement need not hold.

4. Relations with persistent equilibria and stable sets

Kalai and Samet (1984) introduced persistent equilibria and showed that each bimatrix game has at least one persistent equilibrium which is also proper. In this section this result will be strenghtened by showing that there exists at least one persistent equilibrium which is also e-proper.

First we recall the definitions of absorbing and persistent retracts, which are of the root of the definition of a persistent equilibrium. Let (A,B) be an $m\times n$ bimatrix game. A convex and closed set $R = R_1 \times R_2 \subset \Delta_m \times \Delta_n$ is called an *absorbing retract* for (A,B) if there exists an open neighbourhood V of R such that for all $(p,q) \in V$ there exists a pair $(\hat{p},\hat{q}) \in R$ with $\hat{p} \in B_1(q)$ and $\hat{q} \in B_2(p)$. An absorbing retract which does not properly contain another absorbing retract is called a *persistent retract.* An equilibrium for (A,B) which is contained in a persistent retract is called a *persistent equilibrium.*

Using selection retracts Kalai and Samet showed

Lemma 4.1 (Kalai and Samet 1984).

(i) Every absorbing retract contains a persistent retract.
(ii) Every persistent retract contains a proper equilibrium.

For our purposes we also need

Lemma 4.2. *Let (A,B) be an $m\times n$ bimatrix game and let $f: \Delta_m \times \Delta_n \to \Delta_{\bar{m}} \times \Delta_{\bar{n}}$ be defined as in (3). Let $R \subset \bar{\Delta}_m \times \bar{\Delta}_n$ be an absorbing retract for (A,B). Then $f(R)$ is an absorbing retract for $(\bar{A},\bar{B})$.*

Proof. It is easily checked that $f(R) = T_1 \times T_2$ for two convex and closed sets $T_1 \subset \Delta_{\bar{m}}$ and $T_2 \subset \Delta_{\bar{n}}$. Since R is an absorbing retract for (A,B) there exists an open neighbourhood $V \subset \Delta_m \times \Delta_n$, $V \supset R$, such that for all $(p,q) \in V$ there exists a pair $(\hat{p},\hat{q}) \in R$ with $\hat{p} \in B_1(q)$ and $\hat{q} \in B_2(p)$.

Define $\bar{V} := V \cap (\bar{\Delta}_m \times \bar{\Delta}_n)$. Then $f(\bar{V}) = \bar{f}(\bar{V})$ is an open set in $\Delta_{\bar{m}} \times \Delta_{\bar{n}}$ because $\bar{f}^{-1}$ is continuous. Since $\bar{V} \supset R$ we have $f(\bar{V}) \supset f(R)$. Let $(x,y) \in f(\bar{V})$. Defining $(p,q) := \bar{f}^{-1}(x,y) \in \bar{V}$, there exists a pair $(\hat{p},\hat{q}) \in R$ such that $\hat{p} \in B_1(q)$ and $\hat{q} \in B_2(p)$ in (A,B).

With $(\hat{x},\hat{y}) := f(\hat{p},\hat{q}) \in f(R)$ this implies (cf. (5)) that $\hat{x} \in B_1(y)$ and $\hat{y} \in B_2(x)$ in $(\bar{A},\bar{B})$. Hence, $f(R)$ is an absorbing retract for $(\bar{A},\bar{B})$. □

Now we can provide an new proof for the following theorem of Garcia Jurado (1989).

Theorem 4.3. *There is a persistent retract which contains an e-proper equilibrium.*

Proof. Let (A,B) be an $m\times n$ bimatrix game. Let $R := \bar{\Delta}_m \times \bar{\Delta}_n$. Clearly R is an absorbing retract for (A,B). Using Lemma 4.1(i), R contains a persistent retract P for (A,B) and so, by Lemma 4.2, $f(P)$ is an absorbing retract for the equalized game $(\bar{A},\bar{B})$. Lemma 4.1 implies that $f(P)$ contains a proper equilibrium (x,y) for $(\bar{A},\bar{B})$. Since

$P \subset \bar{\Delta}_m \times \bar{\Delta}_n$ we have that $\bar{f}^{-1}(x,y) \in P$ and, by Theorem 3.2, $\bar{f}^{-1}(x,y)$ is an e-proper equilibrium for (A,B). □

Corollary. *Each bimatrix game has a persistent equilibrium which is also e-proper.*

The following example shows there need not be a persistent retract which contains an iterated e-proper equilibrium.

Example 4. Reconsider the 2×3 bimatrix game (A,B) of Example 3. The unique iterated e-proper equilibrium for this game is $\left(\frac{1}{2} e_1 + \frac{1}{2} e_2, \frac{1}{3} e_1 + \frac{1}{3} e_2 + \frac{1}{2} e_3\right)$, whereas the persistent retracts for (A,B) are the sets

$$\{(p, e_2)\} \text{ for all } p \in \text{Conv}\left(\left\{e_2, \tfrac{1}{2} e_1 + \tfrac{1}{2} e_2\right\}\right) \setminus \left\{\tfrac{1}{2} e_1 + \tfrac{1}{2} e_2\right\}$$

$$\{(p, e_1)\} \text{ for all } p \in \text{Conv}\left(\left\{e_1, \tfrac{1}{2} e_1 + \tfrac{1}{2} e_2\right\}\right) \setminus \left\{\tfrac{1}{2} e_1 + \tfrac{1}{2} e_2\right\}$$

and

$$\left\{\tfrac{1}{2} e_1 + \tfrac{1}{2} e_2\right\} \times \text{Conv}(\{e_1, e_2\}).$$

This can be seen as follows. Let $R_1 \times R_2 \subset \Delta_2 \times \Delta_3$ be a persistent retract for (A,B). Since $B_1(q) = \Delta_2$ for all $q \in \Delta_3$ "minimality" of $R_1 \times R_2$ implies that $|R_1| = 1$. Let $p \in \Delta_2$ be such that $R_1 = \{p\}$.

Suppose $p \in \text{Conv}\left(\left\{e_2, \frac{1}{2} e_1 + \frac{1}{2} e_2\right\}\right)$ and $p \neq \frac{1}{2} e_1 + \frac{1}{2} e_2$. It is easily seen that for any (small) neighbourhood $U \ni p$ it holds that $B_2(p') = \{e_2\}$ for all $p' \in U$. Hence, $e_2 \in R_2$. Since $\{(p, e_2)\}$ is absorbing, it then follows that $R_2 = \{e_2\}$. A similar argument can be applied in case $p \in \text{Conv}\left(\left\{e_1, \frac{1}{2} e_1 + \frac{1}{2} e_2\right\}\right)$ and $p \neq \frac{1}{2} e_1 + \frac{1}{2} e_2$.

Now suppose $p = \frac{1}{2} e_1 + \frac{1}{2} e_2$. Note that any open neighbourhood $U \ni p$ contains strategies $p^1 \in \text{Conv}(\{e_1, p\})$ and $p^2 \in \text{Conv}(\{e_2, p\})$ with $p^1 \neq p$ and $p^2 \neq p$. Since $B_2(p^1) = \{e_1\}$ and $B_2(p^2) = \{e_2\}$, it follows that $\{e_1, e_2\} \subset R_2$. Convexity then implies $\text{Conv}\{e_1, e_2\} \subset R_2$. Clearly, $\{p\} \times \text{Conv}\{e_1, e_2\}$ is absorbing, so $\text{Conv}\{e_1, e_2\} = R_2$.

We now focus on relations between e-perfect and e-proper equilibria and *stable sets* à la Kohlberg and Mertens (1986).

Jurg et al. (1992) showed that each persistent retract contains a stable set. However, a generalization of Theorem 4.3 towards stable sets does not hold. Moreover, although each stable set consists of perfect equilibria only, Example 5 shows that there need not be a stable set that contains an e-perfect equilibrium. At this point it would carry us too far to give the precise definition of stability. Therefore, we cannot provide a detailed analysis of the game in Example 5. However, we like to note that the various assertions in Example 5 can be easily verified by means of the results in Borm (1990) on $2 \times n$ bimatrix games.

Example 5. Consider the 2×5 bimatrix game (A,B) given by

$$(A,B) = \begin{bmatrix} (1,1) & (1,3) & (0,4) & (6,2) & (0,2) \\ (0,3) & (0,1) & (0,-2) & (0,2) & (4,2) \end{bmatrix}.$$

It follows that $E(A,B) = T_1 \cup T_2$, where

$$T_1 = \left\{\tfrac{1}{2} e_1 + \tfrac{1}{2} e_2\right\} \times \text{Conv}\left\{\tfrac{4}{5} e_1 + \tfrac{1}{5} e_5, \tfrac{4}{5} e_2 + \tfrac{1}{5} e_5, \tfrac{2}{5} e_4 + \tfrac{3}{5} e_5\right\}$$

and

$$T_2 = \text{Conv}\left\{\tfrac{3}{4} e_1 + \tfrac{1}{4} e_2, e_1\right\} \times \{e_3\}.$$

Further, all stable sets consist of only one point and, with $(p,q) \in \Delta_2 \times \Delta_5$, $\{(p,q)\}$ is stable if and only if $(p,q) \in T_1$. However, since

$$(\bar{A},\bar{B}) = \begin{bmatrix} (1,1) & (1,3) & (0,4) & (3,2) \\ (0,3) & (0,1) & (0,-2) & (2,2) \end{bmatrix}$$

the set of equalized perfect equilibria for (A,B) is T_2.

5. Concluding remarks

(i) First of all we like to note that all definitions and results of the previous sections can directly be extended to the n-person case ($n \geq 3$).
(ii) It can be shown that the sets of e-perfect and iterated e-perfect equilibria for bimatrix games both are the finite union of polytopes. This immediately follows from Theorem 3.2(i), the definition of iterated e-perfect and the fact that the set of perfect equilibria for a bimatrix game is the finite union of polytopes (cf. Borm et al. 1988).
(iii) Borm (1990) provides a geometric-combinatorial approach (GC-approach) to determine e.g. perfect and proper equilibria for $2 \times n$ bimatrix games. Hence, using Theorem 3.2, the GC-approach can also be applied to determine e-perfect and e-proper equilibria for $2 \times n$ bimatrix games.

In the present paper the GC-approach was also used for determining persistent retracts and stable sets in the Examples 4 and 5.

References

Borm PEM (1990) On perfectness concepts for bimatrix games. Report 9013, Dept. of Mathematics, University of Nijmegen
Borm PEM, Jansen MJM, Potters JAM, Tijs SH (1988) On the structure of the set of undominated equilibria in bimatrix games. Report 8819, Dept. of Mathematics, University of Nijmegen

Damme EEC van (1987) Stability and perfection of Nash equilibria. Springer, Berlin Heidelberg New York

Garcia Jurado I (1989) Nuevos equilibrios estables ante tendencias at error en juegos no cooperativos. PhD Dissertation

Garcia Jurado I, Prada Sanchez JM (1990) A remark on Myerson's concept of proper equilibrium. Int J Game Theory 19:11–16

Jurg AP, Jansen MJM, Borm PEM (1992) On minimal strictly perfect sets. Games and Economic Behavior (to appear)

Kalai E, Samet D (1984) Persistent equilibria in strategic games. Int J Game Theory 13:129–144

Kohlberg E, Mertens J-F (1986) On strategic stability of equilibria. Econometrica 54:1003–1037

Myerson RB (1978) Refinements of the Nash equilibrium point concept. Int J Game Theory 7:73–80

Nash JF (1951) Non cooperative games. Ann Math 54:286–295

Selten R (1975) Reexamination of the perfectness concept for equilibrium points in extensive games. Int J Game Theory 4:25–55