On Extensions of supersingular representations of $\mathrm{SL}_2(\mathbb{Q}_p)$.

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Abstract

In this note for $p > 5$ we calculate the dimensions of $\mathrm{Ext}^1_{\mathrm{SL}_2(\mathbb{Q}_p)}(\tau, \sigma)$ for any two irreducible supersingular representations τ and σ of $\mathrm{SL}_2(\mathbb{Q}_p)$.

1 Introduction

In this note we calculate the space of extensions of supersingular representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ for $p > 5$. The dimensions of the space of extensions between irreducible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are calculated by Paškūnas in [Paš10]. Understanding extensions between irreducible smooth representations play a crucial role in Paškūnas work on the image of Colmez Montreal functor in (see [Paš13]). We hope that these results have similar application to mod p and p-adic local Langlands correspondence for $\mathrm{SL}_2(\mathbb{Q}_p)$.

Let G be the group $\mathrm{GL}_2(\mathbb{Q}_p)$, K be the maximal compact subgroup $\mathrm{GL}_2(\mathbb{Z}_p)$ and Z be the center of G. We denote by $I(1)$ the pro-p Iwahori subgroup of G. We denote by G_S the special linear group $\mathrm{SL}_2(\mathbb{Q}_p)$. For any subgroup H of $\mathrm{GL}_2(\mathbb{Q}_p)$ we denote by H_S the subgroup $H \cap \mathrm{SL}_2(\mathbb{Q}_p)$. All representations in this note are defined over vector spaces over $\bar{\mathbb{F}}_p$. Let σ be an irreducible smooth representation of K and σ extends uniquely as a representation of KZ such that $p \in Z$ acts trivially. The Hecke algebra $\mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is isomorphic to $\bar{\mathbb{F}}_p[T]$. For any constant λ in $\overline{\mathbb{F}}_p^\times$ let μ_λ be the unramified character of Z such that $\mu_\lambda(p) = \lambda$. Let $\pi(\sigma, \mu_\lambda)$ be the representation

$$\frac{\mathrm{ind}_{KZ}^G \sigma}{T(\mathrm{ind}_{KZ}^G \sigma)} \otimes (\mu_\lambda \circ \det).$$

The representations $\pi(\sigma, \mu_\lambda)$ are irreducible (see [Bre03]) and are called **supersingular representations** in the terminology of Barthel–Livné.

Let σ_r be the representation $\mathrm{Sym}^r \bar{\mathbb{F}}_p$ of $\mathrm{GL}_2(\mathbb{F}_p)$. We consider σ_r as a representation of K by inflation. The K-socle of $\pi(\sigma_r, \mu_\lambda)$ is a direct sum of two irreducible smooth representations σ_r and σ_{p-1-r}. Let $\pi_{0,r}$ and $\pi_{1,r}$ be the G_S representations generated by $\sigma_r^{I(1)}$ and $\sigma_{p-1-r}^{I(1)}$. The representations $\pi_{0,r}$ and $\pi_{1,r}$ are irreducible supersingular representations of G_S and

$$\mathrm{res}_{G_S} \pi(\sigma_r, \mu_\lambda) \simeq \pi_{0,r} \oplus \pi_{1,r}.$$

Any irreducible supersingular representation of G_S is isomorphic to $\pi_{i,r}$ for some r such that $0 \leq r \leq p-1$ and $i \in \{0, 1\}$. Moreover the only isomorphisms between $\pi_{i,r}$ are $\pi_{0,r} \simeq \pi_{1,p-1-r}$ and $\pi_{1,r} \simeq \pi_{0,p-1-r}$ (see [Abd14]). Our main theorem on extensions of supersingular representations of G_S is:

Theorem 1.1. *Let $p \geq 5$ and $0 \leq r \leq (p-1)/2$. For any irreducible supersingular representation τ of G_S the space $\mathrm{Ext}^1_{G_S}(\tau, \pi_{i,r})$ is non-zero if and only if $\tau \simeq \pi_{j,r}$ for some $j \in \{0,1\}$. If $0 \leq r < (p-1)/2$ then $\dim_{\bar{\mathbb{F}}_p} \mathrm{Ext}^1_{G_S}(\pi_{i,r}, \pi_{j,r}) = 2$ for $i \neq j$ and $\dim_{\bar{\mathbb{F}}_p} \mathrm{Ext}^1_{G_S}(\pi_{i,r}, \pi_{i,r}) = 1$. For $r = (p-1)/2$ we have $\dim_{\bar{\mathbb{F}}_p} \mathrm{Ext}^1_{G_S}(\pi_{0,r}, \pi_{0,r}) = 3$.*

We briefly explain the method of proof. We essentially follow [Paš10]. The functor sending a smooth representation to its $I(1)_S$-invariants induces an equivalence of categories of smooth representations of G_S generated by $I(1)_S$-invariants and the module category of the pro p-Iwahori Hecke algebra (see [Koz16, Theorem 5.2]). We use the Ext spectral sequence thus obtained by this equivalence of categories to calculate $\mathrm{Ext}^1_{G_S}$. Extensions of pro p-Iwahori Hecke algebra modules are calculated from resolutions of Hecke modules due to Schneider and Ollivier. We crucially use results from work of Paškūnas [Paš10]. We first obtain lower bounds on the dimensions of $\mathrm{Ext}^1_{G_S}$ spaces using the spectral sequence and then obtain upper bounds using Paškūnas results on $\mathrm{Ext}^1_K(\sigma, \pi(\sigma, \mu_\lambda))$.

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2 Pro-p Iwahori Hecke algebra

Let B be the Borel subgroup consisting of invertible upper triangular matrices, U be the unipotent radical of B and T be the maximal torus consisting of diagonal matrices. We denote by $\bar{U}$ the unipotent radical of $\bar{B}$ the Borel subgroup consisting of invertible lower triangular matrices. We denote by I the standard Iwahori-subgroup of G. Let $I(1)$ be the pro-p Iwahori subgroup of G and $I(1)_S$ be the pro-p-Iwahori subgroup of G_S. We note that $I(1)_S(Z \cap I(1))$ is equal to $I(1)$. Let $\mathcal{H}$ be the pro-p Iwahori–Hecke algebra $\mathrm{End}_G(\mathrm{ind}_{I(1)_S}^{G_S} \mathrm{id})$. Let Rep_{G_S} and $\mathrm{Rep}_{G_S}^{I(1)_S}$ be the category of smooth representations of G_S and its full subcategory consisting of those smooth representations generated by $I(1)_S$-invariant vectors respectively. We denote by $\mathrm{Mod}_{\mathcal{H}}$ the category of modules over the ring $\mathcal{H}$. We have two functors

$$\mathcal{I} : \mathrm{Rep}_{G_S}^{I(1)_S} \to \mathrm{Mod}_{\mathcal{H}}$$
$$\mathcal{I}(\pi) = \pi^{I(1)_S}$$

and

$$\mathcal{T} : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Rep}_{G_S}^{I(1)_S}$$
$$\mathcal{T}(M) = M \otimes_{\mathcal{H}} \mathrm{ind}_{I(1)_S}^{G_S} \mathrm{id}\,.$$

From [Koz16, Theorem 5.2] the functors $\mathcal{T}$ and $\mathcal{I}$ are quasi-inverse to each other. Let σ and τ be any two smooth representations of G_S and σ_1 be the G_S subrepresentation of σ generated by $I(1)_S$-invariants of σ. We have

$$\mathrm{Hom}_G(\tau, \sigma) = \mathrm{Hom}_G(\tau, \sigma_1) = \mathrm{Hom}_H(\mathcal{I}(\tau), \mathcal{I}(\sigma_1)) = \mathrm{Hom}_H(\mathcal{I}(\tau), \mathcal{I}(\sigma)). \tag{1}$$

We get a Grothendieck spectral sequence with E_2^{ij} equal to $\mathrm{Ext}^i(\mathcal{I}(\tau), \mathbb{R}^j\mathcal{I}(\sigma))$ such that

$$\mathrm{Ext}^i(\mathcal{I}(\tau), \mathbb{R}^j\mathcal{I}(\sigma)) \Rightarrow \mathrm{Ext}^{i+j}_G(\tau, \sigma). \tag{2}$$

The 5-term exact sequence associated to the above spectral sequence gives the following exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\sigma)) \xrightarrow{i} \mathrm{Ext}^1_G(\tau, \sigma) \xrightarrow{\delta} \\ \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbb{R}^1\mathcal{I}(\sigma)) \to \mathrm{Ext}^2_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\sigma)) \to \mathrm{Ext}^2_G(\tau, \sigma) \tag{3}$$

for all τ such that $\tau =< G_S\tau^{I(1)} >$. In particular we apply these results when τ and σ are irreducible supersingular representations of G_S. We first recall the structure of the ring $\mathcal{H}$, its modules $M(i,r) = \pi_{i,r}^{I(1)}$ for i in $\{0,1\}$ and $0 \leq r \leq p-1$. The $\mathcal{H}$ module $M(i,r)$ is a character and we first calculate the dimensions of the spaces $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(j,s))$.

Let T_S^0 and T_S^1 be the maximal compact subgroup of T_S and its maximal pro-p-subgroup. We denote by s_0, s_1 and θ the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -p^{-1} \\ p & 0 \end{pmatrix}$ and $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ respectively. Let $N(T_S)$ be the normaliser of the torus. The extended Weyl group $W = \theta^{\mathbb{Z}} \coprod s_0\theta^{\mathbb{Z}}$ sits into an exact sequence of the form

$$0 \to \Omega := \frac{T_S^0}{T_S^1} \to \tilde{W} := \frac{N(T_S)}{T_S^1} \to W = \frac{N(T_S)}{T_S^0} \to 0.$$

The length function l on W, given by $l(\theta^i) = |2i|$ and $l(s_0\theta^i) = |1-2i|$, extends to a function on $\tilde{W}$ such that $l(\Omega) = 0$. Let T_w be the element $\mathrm{Char}_{I(1)wI(1)}$ for all $w \in \tilde{W}$. We denote by e_1 the element $\sum_{w\in\Omega} T_w$. The functions T_w span $\mathcal{H}$ and the relations in $\mathcal{H}$ are given by

$$T_wT_v = T_{wv} \text{ whenever } l(v) + l(w) = l(vw),$$
$$T_{s_i}^2 = -e_1T_{s_i}.$$

The pro-p-Iwahori Hecke algebra is generated by $T_wT_{s_i}$ for w in Ω. For any character χ of Ω let e_χ be the element $\sum_{w\in\Omega} \chi^{-1}(w)T_w$. Let γ be a W_0 orbit of the characters χ and e_γ be the element $\sum_{\chi\in\gamma} e_\chi$. The elements $\{e_\gamma; \gamma \in \hat{\Omega}/W_0\}$ are central idempotents in the ring $\mathcal{H}$ and we have

$$\mathcal{H} = \bigoplus_{\hat{\Omega}/W_0} \mathcal{H}e_\gamma. \tag{4}$$

For the group G_S, we know that $\mathcal{H}$ is the affine Hecke algebra. The characters of affine Hecke algebra are described in a simple manner we recall this for G_S. Let I be a subset of $\{s_0, s_1\}$ and W_I be the subgroup of W generated by elements of I and $W_\emptyset$ is trivial group. The characters of $\mathcal{H}$ are parametrised by pairs (λ, I) where λ is a character of Ω and $I \subset S_\lambda$. For such a pair (λ, I) the character $\chi_{\lambda,I}$ associated to it is given by

$$\chi_{\lambda,I}(T_{wt}) = 0 \text{ for all } w \in W\backslash W_I \text{ and for all } t \in \Omega, \tag{5}$$
$$\chi_{\lambda,I}(T_{wt}) = \lambda(t)(-1)^{l(w)} \text{ for all } w \in W_I \text{ and for all } t \in \Omega. \tag{6}$$

If λ is nontrivial then we have $\chi_{\lambda,\emptyset}(T_t) = \lambda(t)$, for all $t \in \Omega$ and $\chi_{\lambda,\emptyset}(T_{wt}) = 0$ for all $w \neq \mathrm{id}$ and $t \in \Omega$.

We denote by $\chi_{r,\emptyset}$ the character $\chi_{x\mapsto x^r,\emptyset}$. From the above description we get that $M(0,r) = \chi_{r,\emptyset}$ and $M(1,r) = \chi_{p-1-r,\emptyset}$ for $r \notin \{0, p-1\}$. If $r \in \{0, p-1\}$ then [OS16, Proposition 3.9] says that $\chi_{\mathrm{id},\emptyset}$ and $\chi_{\mathrm{id},S}$ are not supersingular characters. This shows that $M(i,r)$ is either $\chi_{\mathrm{id},I}$ or $\chi_{\mathrm{id},J}$, for $r \in \{0, p-1\}$, where $I = \{s_0\}$ and $J = \{s_1\}$. Since the element T_{s_0} belongs to pro-p Iwahori–Hecke algebra of G and using the presentation in [BP12, Corollary 6.4] we obtain that $M(1,0)=\chi_{\mathrm{id},I}$ and $M(0,0)=\chi_{\mathrm{id},J}$. Similarly $M(1,p-1)$ is given by the character $\chi_{\mathrm{id},J}$ and $M(0,p-1)$ is given by the character $\chi_{\mathrm{id},I}$. Let $0\le r,s\le (p-1)/2$ then (4) shows that

$$\mathrm{Ext}^1_{\mathcal{H}}(M(i,r),M(j,s))=0 \tag{7}$$

for $r\neq s$.

2.1 Resolutions of Hecke modules

In order to calculate extensions between the characters $M(i,r)$, we use resolutions constructed by Schneider and Ollivier for $\mathcal{H}$. Let $\mathfrak{X}$ be the Bruhat–Tits tree of G_S and let $A(T_S)$ be the standard apartment associated to T_S. We fix an edge E and vertices v_0 and v_1 contained in E such that the G_S-stabiliser of v_0 is K_S. For any facet F of $\mathfrak{X}$ we denote by $\mathbf{G}_F$ the $\mathbb{Z}_p$-group scheme with generic fibre $\mathbf{SL}_2$ and $\mathbf{G}_F(\mathbb{Z}_p)$ is the G-stabiliser of F. We denote by I_F the subgroup of $\mathbf{G}_F(\mathbb{Z}_p)$ whose elements under mod- $\mathfrak{p}$ reduction of $\mathbf{G}_F(\mathbb{Z}_p)$ belong to the $\mathbb{F}_p$-points of the unipotent radical of $\mathbf{G}_F\times\mathbb{F}_p$. We denote by $\mathcal{H}_F$ the finite subalgebra of $\mathcal{H}$ defined as

$$\mathcal{H}_F:=\mathrm{End}_{\mathbf{G}_F(\mathbb{Z}_p)}(\mathrm{ind}_{I_F}^{\mathbf{G}_F(\mathbb{Z}_p)}(\mathrm{id})).$$

In particular $\mathcal{H}_E$ is a semi-simple algebra.

For any $\mathcal{H}$-module $\mathfrak{m}$ the construction of Schneider and Ollivier [OS14, Theorem 3.12, (6.4)] gives us a $(\mathcal{H},\mathcal{H})$-exact resolution

$$0\to\mathcal{H}\otimes_{\mathcal{H}_E}\mathfrak{m}\xrightarrow{\delta_1}(\mathcal{H}\otimes_{\mathcal{H}_{v_0}}\mathfrak{m})\oplus(\mathcal{H}\otimes_{\mathcal{H}_{v_1}}\mathfrak{m})\xrightarrow{\delta_0}\mathfrak{m}\to 0. \tag{8}$$

Using the resolution (8) and the observation that $\mathcal{H}_E$ is semi-simple for $p\neq 2$ we get that

$$0\to\mathrm{Hom}_{\mathcal{H}}(\mathfrak{m},\mathfrak{n})\to\bigoplus_{v_0,v_1}\mathrm{Hom}_{\mathcal{H}_{v_i}}(\mathfrak{m},\mathfrak{n})\to\mathrm{Hom}_{\mathcal{H}_E}(\mathfrak{m},\mathfrak{n})\xrightarrow{\delta}\mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m},\mathfrak{n})\to\bigoplus_{v_0,v_1}\mathrm{Ext}^1_{\mathcal{H}_{v_i}}(\mathfrak{m},\mathfrak{n})\to 0 \tag{9}$$

Note that we have an isomorphism of algebras

$$\mathcal{H}_{v_0}\simeq\mathcal{H}_{v_1}\simeq\mathrm{End}_{\mathrm{SL}_2(\mathbb{F}_p)}(\mathrm{ind}_{N(\mathbb{F}_p)}^{\mathrm{SL}_2(\mathbb{F}_p)}\mathrm{id}).$$

The above isomorphism is not a canonical isomorphism. Let K_0 and K_1 be the compact open subgroups $K\cap G_S$ and $K^{\Pi}\cap G_S$ respectively.

2.2 Extensions of supersingular modules over pro-p Iwahori–Hecke algebra.

The Hecke algebra $\mathcal{H}_{v_i}$ is isomorphic to $\mathrm{End}_{K_i}(\mathrm{ind}_{I(1)}^{K_i}\mathrm{id})$. The Hecke algebra $\mathcal{H}_{v_i}$ is generated by T_t and T_{s_i} for $t\in\Omega$. The relations among them are given by

$$\begin{aligned}&T_{t_1}T_{t_2}=T_{t_1t_2},\\&T_tT_{s_i}=T_{ts_i}=T_{s_it^{-1}}=T_{s_i}T_{t^{-1}},\\&T_{s_i}^2=-e_1T_{s_i}\end{aligned}$$

where $e_1=\sum_{t\in\Omega}T_t$.

Lemma 2.1. *Let $0 < r \leq (p-1)/2$ the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(j,s))$ is non-zero if and only if $i \neq j$ and has dimension 2 when $i = j$. If $r = (p-1)/2$ then the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ has dimension 2.*

Proof. Since $r \neq 0$ the characters $M(0,r)$ and $M(1,r)$ are isomorphic to $\chi_{r,\emptyset}$ and $\chi_{p-1-r,\emptyset}$ respectively (see (5)). Let E_c be a 2-dimensional $\bar{\mathbb{F}}_p$ module $\bar{\mathbb{F}}_p e_1 \oplus \bar{\mathbb{F}}_p e_2$ and $\bar{\mathbb{F}}_p[\Omega]$ acts on E by $T_t e_0 = t^r e_0$ and $T_t e_1 = t^{p-1-r} e_1$. We set $T_{s_i} e_0 = 0$ and $T_{s_i} e_1 = c e_0$ for some $c \neq 0$. This makes E a $\mathcal{H}_{v_i}$ module and is a non-trivial extension

$$0 \to \chi_{r,\emptyset} \to E \to \chi_{p-1-r,\emptyset} \to 0.$$

Let E be a $\mathcal{H}_{v_i}$-extension of $W := \chi_{s,\emptyset}$ by $V := \chi_{r,\emptyset}$ i.e, we have an exact sequence

$$0 \to V \to E \xrightarrow{f} W \to 0.$$

There exists a $\bar{\mathbb{F}}_p[\Omega]$-equivariant section $s : W \to E$ of f. Let V' be the image of this section. Now $E = V \oplus V'$. The action of T_{s_i} is trivial on V and observe that $f(T_{s_i}(V')) = T_{s_i}(f(V')) = 0$. **If E is nontrivial then** $T_{s_i}(V') = V$. This implies that $r + s = p - 1$ and hence E is isomorphic to E_c for some $c \neq 0$. This shows that the space of $\mathcal{H}_{v_i}$ extensions of W by V is one dimensional if $r + s = p - 1$ and zero otherwise. Now consider the exact sequence (9) when $\mathfrak{m}$ is $M(i,r)$ and $\mathfrak{n}$ is $M(j,r)$. For $i = j$ the map δ in zero (9) hence the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is trivial. When $i \neq j$ the Hom spaces in (9) are all trivial. This shows that the dimension of the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is 2 from our calculations. ∎

Lemma 2.2. *The space of extensions $\mathrm{Ext}^1_{\mathcal{H}}(M(i,0), M(j,0))$ is trivial for $i = j$ and has dimension 1 for $i \neq j$.*

Proof. The algebra $e_1 \mathcal{H}_{v_i}$ is semi-simple algebra and hence we get that

$$\mathrm{Ext}^i_{\mathcal{H}_{v_i}}(\chi_{\mathrm{id},S}, \chi_{\mathrm{id},S'}) = 0 \tag{10}$$

for all $i > 0$ and for subsets S and S' of $\{s_0, s_1\}$. Now consider the exact sequence (9) when $\mathfrak{m}$ is $M(i,r)$ and $\mathfrak{n}$ is $M(j,r)$. For $i = j$ the map δ in (9) is zero hence the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is trivial. When $i \neq j$ the first two Hom spaces in (9) are trivial. The space $\mathrm{Hom}_{\mathcal{H}_E}(\mathfrak{m}, \mathfrak{n})$ has dimension one. This shows that the dimension of the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is 1 for $i \neq j$. ∎

3 The Hecke module $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$.

Paškūnas calculated the cohomology groups $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$ and we now recall his results. Let $\tilde{\pi}_r$ be the supersingular representation $\pi(\sigma_r, \mu_1)$ of G. Recall that the K-socle of $\tilde{\pi}_r$ is isomorphic to $\sigma_r \oplus \sigma_{p-1-r}$ and the space of $I(1)$ invariants has a basis $(\mathbf{v}_0, \mathbf{v}_1)$ where $\mathbf{v}_0$ and $\mathbf{v}_1$ belong to $\sigma_r^{N_p}$ and $\sigma_{p-1-r}^{N_p}$ respectively. Let I^+ and I^- be the groups $I \cap U$ and $I \cap \bar{U}$ respectively. Consider the spaces

$$M_0 :=< I^+ \theta^n \mathbf{v}_1;\ n \geq 0 > \text{ and } M_1 :=< I^+ \theta^n \mathbf{v}_2;\ n \geq 0 >$$

and let Π be the matrix $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ which normalizes I and $I(1)$. We denote by π_0 and π_1 the spaces $M_0 + \Pi M_1$ and $M_1 + \Pi M_0$. Let G^0 be subgroup of G consisting of elements with integral discriminant. Let G^+ be the group ZG^0. We denote by Z_1 the group $I(1) \cap Z$.

Proposition 3.1 (Paškūnas). *The spaces σ_0 and σ_1 are G^+ stable. The space $\tilde{\pi}_r$ is the direct sum of the representations π_1 and π_0 as G^+ representations and hence $\pi_{i,r}$ is isomorphic to π_i as G_S representations for $i \in \{0,1\}$. If r be an integer such that $0 < r < (p-1)/2$ then the Hecke module $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$ is isomorphic to $\mathcal{I}(\pi_{i,r}) \oplus \mathcal{I}(\pi_{i,r})$. In the Iwahori case (i.e, $r = 0$) the Hecke module $\mathbb{R}\mathcal{I}(\pi_{0,0}) \oplus \mathbb{R}\mathcal{I}(\pi_{1,0})$ is isomorphic to $\mathcal{I}(\pi_{0,0})^{\oplus 2} \oplus \mathcal{I}(\pi_{0,0})^{\oplus 2}$.*

Proof. The first part follows from [Paš10, Corollary 6.5]. The second part follows from [Paš10, Proposition 10.5, Theorem 10.7 and equation (49)]. □

Corollary 3.2. *Let τ be an irreducible supersingular representation of G_S. If the space of extensions $\mathrm{Ext}^1_{G_S}(\tau, \pi_{i,r})$ is non-trivial then $\tau \simeq \pi_{j,r}$ for some $j \in \{0,1\}$.*

Proof. This follows from (3), (7) and Proposition 3.1. □

Corollary 3.3. *Let $0 < r < (p-1)/2$ and $i \neq j$ then the dimensions of the space $\mathrm{Ext}^1_{G_S}(\pi_{i,r}, \pi_{j,r})$ is 2.*

Proof. Observe that for $0 < r < (p-1)/2$ the modules $M(i,r)$ and $M(j,s)$ are not isomorphic. Now using the exact sequence (3) and Proposition 3.1 we get that

$$\mathrm{Ext}^1_{G_S}(\pi_{i,r}, \pi_{j,r}) \simeq \mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(j,r)).$$

The corollary follows from the Lemma 2.1. □

Remark 3.4. *The results of Corollary 3.3 remain valid for $r = 0$ but we prove this later. It is interesting to note that for $0 < r < (p-1)/2$ and $i \neq j$ any extension E of $\pi_{i,r}$ by $\pi_{j,r}$ for $i \neq j$ is generated by its $I(1)_S$ invariants, i.e, $E = < G_S E^{I(1)_S} >$.*

4 Calculation of degree one self extensions.

Let us first consider the case when $0 < r \leq (p-1)/2$. In order to determine the dimensions of $\mathrm{Ext}^1(\pi_{i,r}, \pi_{i,r})$ we first show that the map

$$\mathrm{Ext}^1_{G_S}(\pi_{i,r}, \pi_{i,r}) \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\pi_{i,r}), \mathbb{R}^1\mathcal{I}(\pi_{i,r}))) \tag{11}$$

is non-zero. Explicitly the above map takes an extension E, with $0 \to \pi_{i,r} \to E \to \pi_{i,r} \to 0$, to the delta map in the associated long exact sequence, given by: $\mathcal{I}(\pi_{i,r}) \xrightarrow{\delta_E} \mathbb{R}^1\mathcal{I}(\pi_{i,r})$. Note that the dimension of $E^{I(1)}$ is one if and only if $\delta_E \neq 0$.

Lemma 4.1. *For $0 < r \leq (p-1)/2$ then map (11) is non-zero.*

Proof. For $0 < r \leq (p-1)/2$ there exists a self extension E of $\tilde{\pi}_r$ such that the map $\mathcal{I}(\tilde{\pi}_r) \xrightarrow{\delta_E} \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r)$ is non-zero. We fix an extension E such that $\delta_E \neq 0$. Since δ_E is a Hecke-equivariant map and $\mathcal{I}(\tilde{\pi}_r)$ is an irreducible Hecke-module of dimension 2 we get that the inclusion map of $\mathcal{I}(\tilde{\pi}_r)$ in $\mathcal{I}(E)$ is an isomorphism i.e, $\dim E^{I(1)} = 2$. Now consider the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\pi}_r & \longrightarrow & E_1 & \longrightarrow & \pi_{i,r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\pi}_r & \longrightarrow & E & \longrightarrow & \tilde{\pi}_r & \longrightarrow & 0. \end{array} \tag{12}$$

The long exact sequences in $I(1)$-group cohomology attached to (12) gives us:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}(\tilde{\pi}_r) & \xrightarrow{f} & \mathcal{I}(E_1) & \longrightarrow & \mathcal{I}(\pi_{i,r}) & \xrightarrow{\delta_2} & \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}(\tilde{\pi}_r) & \longrightarrow & \mathcal{I}(E) & \longrightarrow & \mathcal{I}(\tilde{\pi}_r) & \xrightarrow[\delta_1]{} & \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r). \end{array}$$

Since the dimension of $\mathcal{I}(E)$ is 2 we get that δ_1 is injective and hence the map δ_2 is non-zero. The dimension of the space $\mathcal{I}(\pi_{i,r})$ is one hence f is an isomorphism. This shows that the space $\mathcal{I}(E_1)$ has dimension 2. For $r = (p-1)/2$ the representations $\pi_{1,r} \simeq \pi_{0,r}$. We assume without loss of generality $\operatorname{img}\delta_2$ is contained in $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$. For any r such that $0 < r \leq (p-1)/2$ consider the pushout of $\tilde{\pi}_r$ by $\pi_{i,r}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{i,r} & \longrightarrow & E_2 & \longrightarrow & \pi_{i,r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\pi}_r & \longrightarrow & E_1 & \longrightarrow & \pi_{i,r} & \longrightarrow & 0 \end{array} \tag{13}$$

The self extension E_2 of $\pi_{i,r}$ is non-split and the induced map δ_{E_2} is non-zero. To see this consider the long exact sequence in cohomology attached to (13):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}(\pi_{i,r}) & \xrightarrow{g} & \mathcal{I}(E_2) & \longrightarrow & \mathcal{I}(\pi_{i,r}) & \xrightarrow{\delta_3} & \mathbb{R}^1\mathcal{I}(\pi_{i,r}) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \mathbb{R}^1\mathcal{I}(p_2) \\ 0 & \longrightarrow & \mathcal{I}(\tilde{\pi}_r) & \xrightarrow{\simeq} & \mathcal{I}(E_1) & \xrightarrow{0} & \mathcal{I}(\pi_{i,r}) & \xrightarrow{\delta_2} & \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r) \end{array}$$

Note that $\mathbb{R}^1\mathcal{I}(\tilde{\pi}_r)$ is isomorphic to $\mathbb{R}^1\mathcal{I}(\pi_{0,r}) \oplus \mathbb{R}^1\mathcal{I}(\pi_{1,r})$ and $\mathbb{R}^1\mathcal{I}(p_2)$ is the projection map. This shows that $\mathbb{R}^1\mathcal{I}(p_2)\delta_2 \neq 0$ and hence $\delta_3 \neq 0$ using which we get that g is an isomorphism. This shows that E_2 is a non-split self-extension of $\pi_{i,r}$ by $\pi_{i,r}$. ☐

Corollary 4.2. *For any integer r such that $0 < r < (p-1)/2$ we have $\dim_{\mathbb{F}_p} \operatorname{Ext}^1_G(\pi_{i,r}, \pi_{i,r}) \geq 1$.*

Theorem 4.3. *Let $p \geq 5$ and $0 \leq r < (p-1)/2$ then the dimension of $\operatorname{Ext}^1_{G_S}(\pi_{i,r}, \pi_{i,r})$ is 1 and dimension of $\operatorname{Ext}^1_{G_S}(\pi_{i,r}, \pi_{j,r})$ is 2 for $i \neq j$. For $r = (p-1)/2$ the dimension of $\operatorname{Ext}^1_{G_S}(\pi_{0,r}, \pi_{0,r})$ is 3.*

Proof. The subgroup $G_S Z$ is an index 2 subgroup of G and id and Π are two double coset representatives for $K \backslash G / G_S Z$. We note that $K \cap G_S$ and $K^{\Pi} \cap G_S$ are representatives for the two distinct classes of maximal compact subgroups of G_S and we denote them by K_1 and K_2 respectively. Let σ'_r be the representation σ_r^{Π} of K^{Π}. Using Mackey-decomposition we get that

$$\operatorname{res}_{G_S} \operatorname{ind}_{KZ}^G \sigma_r = \operatorname{ind}_{K^{\Pi} \cap G_S}^{G_S} \sigma_r^{\Pi} \oplus \operatorname{ind}_{K \cap G_S}^{G_S} \sigma_r = \operatorname{ind}_{K_1}^{G_S} \sigma_r \oplus \operatorname{ind}_{K_2}^{G_S} \sigma'_r. \tag{14}$$

using the long exact sequence of Ext groups for the exact sequence,

$$0 \to \operatorname{ind}_{ZK}^G \sigma_r \xrightarrow{T} \operatorname{ind}_{ZK}^G \to \tilde{\pi}_r \to 0$$

we get that an exact sequence

$$\mathrm{Hom}_G(\mathrm{ind}_{ZK}^G \sigma_r, \tilde{\pi}_r) \to \mathrm{Ext}^1_G(\tilde{\pi}_r, \tilde{\pi}_r) \to \mathrm{Ext}^1_G(\mathrm{ind}_{ZK}^G \sigma_r, \tilde{\pi}_r) \xrightarrow{T} \mathrm{Ext}^1_G(\mathrm{ind}_{ZK}^G \sigma_r, \tilde{\pi}_r). \tag{15}$$

Now using (14) the exact sequence (15) becomes

$$0 \to \mathrm{Hom}_{K_1}(\sigma_r, \tilde{\pi}_r) \oplus \mathrm{Hom}_{K_2}(\sigma'_r, \tilde{\pi}_r) \to \mathrm{Ext}^1_G(\tilde{\pi}_r, \tilde{\pi}_r) \to \mathrm{Ext}^1_{K_1}(\sigma_r, \tilde{\pi}_r) \oplus \mathrm{Ext}^1_{K_2}(\sigma'_r, \tilde{\pi}_r). \tag{16}$$

The groups K_1 is contained in K/Z_1. For all $i \geq 0$ we note that

$$\mathrm{Ext}^i_{K_1}(\sigma_r, \tilde{\pi}_r) \simeq \mathrm{Ext}^i_{K/Z_1}(\mathrm{ind}_{K_1}^{K/Z_1}(\sigma_r), \tilde{\pi}_r) \simeq \bigoplus_{0 \leq a < p-1} \mathrm{Ext}^i_{K/Z_1}(\sigma_r \otimes \det^a, \tilde{\pi}_r).$$

The spaces $\mathrm{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \tilde{\pi}_r)$ can be calculated from the work of Paškūnas. We recall his calculations as needed. There exists a G smooth representation Ω such that $\mathrm{res}_K \Omega$ is an injective envelope of $\mathrm{Soc}_K(\tilde{\pi}_r)$ in the category of smooth representations of K. In particular we get that $\tilde{\pi}_r$ is contained in Ω. The restriction $\mathrm{res}_K \Omega$ is isomorphic to $\mathrm{inj}\sigma_r \oplus \mathrm{inj}\sigma_{p-1-r}$. Now $\mathrm{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \tilde{\pi}_r)$ is isomorphic to $\mathrm{Hom}_{K/Z_1}(\sigma_r \otimes \det^a, \Omega/\tilde{\pi}_r)$.

We now use the notations from [Paš10, Notations, Section 9]. We make one modification. Paškūnas uses the notation χ for the character

$$\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} \mapsto (\lambda)^r (\lambda\mu)^a$$

for all $\lambda, \mu \in \mathbb{F}_p^\times$ and $[\ ]$ is the Teichmuller lift. For convenience we use the notation $\chi_{a,r}$ instead of χ. The idempotent e_χ in [Paš10, Section 9] will be denoted $e_{\chi_{a,r}}$. The space $\mathrm{Hom}_{K_1}(\sigma_r \otimes \det^a, \Omega/\tilde{\pi}_r)$ is the same as

$$\ker(\mathcal{I}(\Omega/\tilde{\pi}_r)e_{\chi_{r,a}} \xrightarrow{T_{n_s}} \mathcal{I}(\Omega/\tilde{\pi}_r)e_{\chi^s_{r,a}}) \tag{17}$$

and from [Paš10, Proposition 10.10] has dimension less than or equal to 2. For $0 \leq r \leq (p-1)/2$ the space $\mathrm{Hom}_{K/Z_1}(\sigma_r \otimes \det^a, \tilde{\pi}_r)$ is non-zero if and only if $a = 0$ and has dimension 1 if $r < (p-1)/2$ and 2 otherwise. Using (17) for $0 \leq r < (p-1)/2$ the space $\mathrm{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \tilde{\pi}_r)$ is non-zero if and only if $a = 0$ and has dimension at most 2 (see [Paš10, Proposition 10.10] for $0 < r < (p-1)/2$ and [Paš13, Corollary 6.13 and Corollary 6.16] for $r = 0$). When $r = (p-1)/2$ the space $\mathrm{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \tilde{\pi}_r)$ is non-zero for $a = 0$ and $a = (p-1)/2$ and in each of these cases the dimension of the space $\mathrm{Ext}^1_{K/Z_1}(\sigma_r \otimes \det^a, \tilde{\pi}_r)$ is less than or equal to 2.

Now using exact sequence (16) the space $\mathrm{Ext}^1_{G_S}(\tilde{\pi}_r, \tilde{\pi}_r)$ has dimension less than or equal to 6 for $0 \leq r < (p-1)/2$ and its dimension is less than or equal to 12 if $r = (p-1)/2$. For $r \neq 0$ using this upper bound and the lower bounds from Corollary 4.2 and Corollary 3.3 we deduce the theorem in this case. When $r = 0$ Paškūnas showed that (see [Paš10, Proposition 6.15]) the dimension of $\mathrm{Ext}^1_{G^+/Z}(\pi_{i,0}, \pi_{j,0})$ is 2 when $i \neq j$ and 1 otherwise. Since $G_S/\{\pm 1\}$ has index a factor of 2 in G^+/Z and $G_S \cap Z$ acts trivially on $\pi_{i,0}$ we get that

$$\mathrm{Ext}^1_{G^+/Z}(\pi_{i,0}, \pi_{j,0}) \hookrightarrow \mathrm{Ext}^1_{G_S/\{\pm 1\}}(\pi_{i,0}, \pi_{j,0}) = \mathrm{Ext}^1_{G_S}(\pi_{i,0}, \pi_{j,0}). \tag{18}$$

From our upper bounds the inclusions (18) are strict and hence we prove the theorem. □

Corollary 4.4. *The Hecke module $\mathbb{R}^1\mathcal{I}(\pi_{i,0})$ is isomorphic to the module $\mathcal{I}(\pi_{i,0}) \oplus \mathcal{I}(\pi_{j,0})$ for $i \neq j$.*

Proof. From the Theorem 4.3, exact sequence (3) and (10) we get that dimension of the space $\mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\pi_{0,0}), \mathbb{R}^1\mathcal{I}(\pi_{0,0}))$ is 1. Using the Proposition 3.1 we get that

$$\mathbb{R}^1\mathcal{I}(\pi_{i,0}) \simeq \mathcal{I}(\pi_{i,0}) \oplus \mathcal{I}(\pi_{j,0}).$$

□

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