A q-MICROSCOPE FOR SUPERCONGRUENCES

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Abstract. By examining asymptotic behavior of certain infinite basic (q-) hypergeometric sums at roots of unity (that is, at a 'q-microscopic' level) we prove polynomial congruences for their truncations. The latter reduce to non-trivial (super)congruences for truncated ordinary hypergeometric sums, which have been observed numerically and proven rarely. A typical example includes derivation, from a q-analogue of Ramanujan's formula

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^2}{2^{8n}3^{2n}}(8n+1) = \frac{2\sqrt{3}}{\pi},$$

of the two supercongruences

$$S(p-1) \equiv p\left(\frac{-3}{p}\right) \pmod{p^3} \quad \text{and} \quad S\left(\frac{p-1}{2}\right) \equiv p\left(\frac{-3}{p}\right) \pmod{p^3},$$

valid for all primes $p>3$, where $S(N)$ denotes the truncation of the infinite sum at the N-th place and $\left(\frac{-3}{\cdot}\right)$ stands for the quadratic character modulo 3.

1. Introduction

In our study, through several years, of Ramanujan's and Ramanujan-type formulae [29] for $1/\pi$, a lot of arithmetic mystery have been discovered along the way. A typical example on the list is the identity

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^2}{2^{8n}3^{2n}}(8n+1) = \frac{2\sqrt{3}}{\pi}. \tag{1}$$

Part of the arithmetic story, which is the main topic of the present note, is a production of Ramanujan-type supercongruences [34] (with some particular instances indicated in the earlier work [32] of Van Hamme): truncation of a Ramanujan-type infinite sum at the $(p-1)$-th place happens to be a simple expression modulo p^3 for all but finitely many primes p. In our example (1), the result reads

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}}(8k+1) \equiv p\left(\frac{-3}{p}\right) \pmod{p^3} \quad \text{for } p>3 \text{ prime}, \tag{2}$$

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where the Jacobi–Kronecker symbol $\big(\frac{-3}{\cdot}\big)$ 'replaces' the square root of 3. Another experimental observation, which seems to be true in several cases but not in general, is that truncation of the sum at the $(p-1)/2$-th place, results in a similar congruence with the same right-hand side, like

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{4k}{2k}\binom{2k}{k}^2}{2^{8k}3^{2k}}(8k+1) \equiv p\left(\frac{-3}{p}\right) \pmod{p^3} \quad \text{for } p>3 \text{ prime}, \tag{3}$$

in the example above. By noticing that the intermediate terms corresponding to k in the range $(p-1)/2 < k \le p-1$ are not necessarily 0 modulo p^3, we conclude that (2) and (3) are in fact different congruences. The experimental findings (2), (3) were implicitly discussed in [34] and later recorded in [30, Conjecture 5.6].

Development of methods [7, 20–22, 24, 25, 27, 31, 32, 34] for establishing Ramanujan-type supercongruences like (2) and (3), sometimes modulo a smaller power of p and normally on a case-by-case study, was mainly hypergeometric. It involved tricky applications of numerous hypergeometric identities and use of the algorithm of creative telescoping, namely of suitable WZ (Wilf–Zeilberger) pairs. These strategies have finally led [14, 19] to q-analogues of Ramanujan-type formulae for $1/\pi$ including

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n^2(q;q^2)_{2n}}{(q^2;q^2)_{2n}(q^6;q^6)_n^2}[8n+1] = \frac{(q^3;q^2)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty(q^6;q^6)_\infty}. \tag{4}$$

At this stage we already need to familiarize ourselves with standard hypergeometric notation. We always consider q inside the unit disc, $|q|<1$, and define

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

Then the q-Pochhammer symbol and its non-q-version are given by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{j=0}^{n-1}(a+j)$$

for non-negative integers n, so that

$$\lim_{q\to 1}\frac{(q^a;q)_n}{(1-q)^n} = (a)_n \quad \text{and} \quad \lim_{q\to 1}\frac{(q;q)_\infty(1-q)^{1-a}}{(q^a;q)_\infty} = \Gamma(a).$$

The related q-notation also includes the q-numbers and q-binomial coefficients

$$[n] = [n]_q = \frac{1-q^n}{1-q} \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}.$$

In the case of formula (4), we see that

$$\lim_{q\to 1}\frac{(q;q^2)_{2n}}{(q^2;q^2)_{2n}} = \frac{(\frac12)_{2n}}{(2n)!} = \frac{(\frac14)_n(\frac34)_n}{(\frac12)_n\,n!},$$

$$\lim_{q\to 1}\frac{(q;q^2)_n}{(q^6;q^6)_n} = \lim_{q\to 1}\frac{(q;q^2)_n}{(q^2;q^2)_n\prod_{j=1}^n(1+q^{2j}+q^{4j})} = \frac{(\frac12)_n}{n!\,3^n}$$

and

$$\lim_{q\to1}\frac{(q^3;q^6)_\infty}{(q^6;q^6)_\infty(1-q^6)^{1/2}}=\lim_{q\to1}\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty(1-q^2)^{1/2}}=\frac{1}{\Gamma(\frac12)}=\frac{1}{\sqrt{\pi}},$$

hence in the limit as $q\to1$ we obtain

$$\sum_{n=0}^\infty\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3\,9^n}\,(8n+1)=\frac{2\sqrt3}{\pi}.$$

This equality transforms into (1) after a simple manipulation of the Pochhammer symbols.

What are q-analogues good for?

It is not hard to imagine that suitable truncations of q-sums like (4) satisfy certain q-analogues of supercongruences of type (2) or (3). It is also reasonable to expect that the earlier strategies for establishing Ramanujan-type supercongruences possess suitable q-analogues. This is indeed the case, and in a series of papers [8–12] the first author uses the q-WZ machinery to produce many such examples of q-supercongruences, in particular, q-analogues for those from Van Hamme's famous list [32]. Drawbacks of this approach are short supply of wanted q-WZ pairs and lack of q-analogues of classical congruences in the required strength.

In this note we offer a different strategy for proving q-congruences. The idea rests on the fact that the asymptotic behavior of an infinite q-sum at roots of unity is determined by its truncation evaluated at the roots. This leads us to a natural extraction of the truncated sum and its evaluation modulo cyclotomic polynomials

$$\Phi_n(q)=\prod_{\substack{j=1\\ \gcd(j,n)=1}}^{n}(q-e^{2\pi ij/n})\in\mathbb{Z}[q]$$

and their products. In what follows, the congruence $A_1(q)/A_2(q)\equiv0\pmod{P(q)}$ for polynomials $A_1(q),A_2(q),P(q)\in\mathbb{Z}[q]$ is understood as $P(q)$ divides $A_1(q)$ and is coprime with $A_2(q)$; more generally, $A(q)\equiv B(q)\pmod{P(q)}$ for rational functions $A(q),B(q)\in\mathbb{Z}(q)$ means $A(q)-B(q)\equiv0\pmod{P(q)}$.

Our principal result in this direction is the following theorem observed experimentally in [19].

Theorem 1.1. *Let n be a positive integer coprime with 6. Then*

$$\sum_{k=0}^{n-1}\frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^2;q^2)_{2k}(q^6;q^6)_k^2}[8k+1]q^{2k^2}\equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right)\pmod{[n]\Phi_n(q)^2},\quad(5)$$

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^2;q^2)_{2k}(q^6;q^6)_k^2}[8k+1]q^{2k^2}\equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right)\pmod{[n]\Phi_n(q)^2}.\quad(6)$$

Clearly, the limiting case $q\to1$ for $n=p$ leads to the Ramanujan-type supercongruences (2) and (3); importantly, it also leads to more general supercongruences when choosing $n=p^s$, an arbitrary power of prime $p>3$. The significance of our proof is that it really deals with the q-hypergeometric sum (4) at a 'q-microscopic' level (that is, at roots of unity), hence it cannot be transformed into a derivation of (2) and (3) directly from (1).

Our proof of Theorem 1.1 combines two principles. One corresponds to achieving the congruences in (5) and (6) modulo $[n]$ only, and it can be easier illustrated in the following 'baby' situations also from [19].

Theorem 1.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{n-1}(-1)^k\frac{(q;q^2)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{3k^2}\equiv 0 \pmod{[n]}, \tag{7}$$

$$\sum_{k=0}^{(n-1)/2}(-1)^k\frac{(q;q^2)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{3k^2}\equiv 0 \pmod{[n]}. \tag{8}$$

Theorem 1.3. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{n-1}\frac{(q^2;q^4)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{k^2}\equiv 0 \pmod{[n]}, \tag{9}$$

$$\sum_{k=0}^{(n-1)/2}\frac{(q^2;q^4)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{k^2}\equiv 0 \pmod{[n]}. \tag{10}$$

The second principle is about getting one more parameter involved in the q-story, to opt for 'creative microscoping'.

Theorem 1.4. *Let n be a positive integer coprime with 6. Then, for any indeterminates a and q, we have modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{n-1}\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}[8k+1]q^{2k^2}\equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right), \tag{11}$$

$$\sum_{k=0}^{(n-1)/2}\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}[8k+1]q^{2k^2}\equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right). \tag{12}$$

Our exposition below is as follows. In Section 2 we prove Theorems 1.2, 1.3 and highlight some difficulties in doing so by the q-WZ method. In Section 3 we demonstrate Theorem 1.4 and show how it implies Theorem 1.1. Section 4 contains several further results on Ramanujan-type q-supercongruences and outlines of their proofs. Known congruences and q-congruences for truncated hypergeometric sums already form a broad area of research; in our final Section 5 we record several open problems and directions which will initiate further development of the method in this note and of traditional hypergeometric techniques.

We remark that asymptotic behavior of q-series at roots of unity attracts a lot of attention in recent studies of mock theta functions and so-called quantum modular forms; we limit our citations about related notion and results to [4, 33]. This gives a good indication, at least a hope, that the methods developed in those areas may shed some light on q-supercongruences and their $q\to 1$ implications.

2. Asymptotics at roots of unity

A fundamental principle for computing basic hypergeometric sums at roots of unity is encoded in the following simple observation known as the q-Lucas theorem (see [26] and [3, Proposition 2.2]).

Lemma 2.1. *Let ζ be a primitive d-th root of unity and let a, b, ℓ, k be non-negative integers with $0 \le b, k \le d-1$. Then*

$$\begin{bmatrix} ad+b \\ \ell d+k \end{bmatrix}_\zeta = \binom{a}{\ell} \begin{bmatrix} b \\ k \end{bmatrix}_\zeta.$$

We recall that the congruences (7)–(10) are motivated by the following q-hypergeometric identities:

$$\sum_{k=0}^{\infty} (-1)^k \frac{(q;q^2)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{3k^2} = \frac{(q^3;q^4)_\infty(q^5;q^4)_\infty}{(-q^4;q^4)_\infty^2}, \tag{13}$$

$$\sum_{k=0}^{\infty} \frac{(q^2;q^4)_k(-q;q^2)_k^2}{(q^4;q^4)_k(-q^4;q^4)_k^2}[6k+1]q^{k^2} = \frac{(-q^2;q^4)_\infty^2}{(1-q)(-q^4;q^4)_\infty^2}, \tag{14}$$

derived in [19] with a help of the quadratic transformation [28, eq. (4.6)]. The equalities (13), (14) can be written as

$$\sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \frac{(-q;q^2)_k^2[6k+1]q^{3k^2}}{(-q;q)_k^2(-q^2;q^2)_k(-q^4;q^4)_k^2} = \frac{(q^3;q^2)_\infty}{(-q^4;q^4)_\infty^2}, \tag{15}$$

$$\sum_{k=0}^{\infty} \begin{bmatrix} 2k \\ k \end{bmatrix}_{q^2} \frac{(-q;q^2)_k^2[6k+1]q^{k^2}}{(-q^2;q^2)_k^2(-q^4;q^4)_k^2} = \frac{(-q^2;q^4)_\infty^2}{(1-q)(-q^4;q^4)_\infty^2}. \tag{16}$$

Proof of Theorem 1.2. It is immediate that (7) and (8) are true for $n=1$.

For $n>1$, let $\zeta \ne 1$ be an n-th root of unity, not necessarily primitive. This means that ζ is a primitive root of unity of odd degree $d \mid n$. For the right-hand side in (15), we clearly have

$$\frac{(q^3;q^2)_\infty}{(-q^4;q^4)_\infty^2} = \prod_{j=1}^{\infty} \frac{1-q^{2j+1}}{(1+q^{4j})^2} \to 0 \quad \text{as } q \to \zeta, \tag{17}$$

because the numerator of the product 'hits' the zero at $q=\zeta$, while the denominator does not vanish at $q=\zeta$. If $c_q(k)$ denotes the k-th term on the left-hand side in (15) (or (13)), then we write the left-hand side as

$$\sum_{\ell=0}^{\infty} c_q(\ell d) \sum_{k=0}^{d-1} \frac{c_q(\ell d+k)}{c_q(\ell d)}.$$

Observe that, for the internal terms,

$$\lim_{q\to\zeta} \frac{c_q(\ell d+k)}{c_q(\ell d)} = c_\zeta(k),$$

and also that

$$c_\zeta(k)=0 \quad \text{for } k \text{ in the range } (d-1)/2<k\le d-1, \tag{18}$$

because of the factor $(q;q^2)_k=\prod_{j=1}^k(1-q^{2j-1})$ in the numerator of $c_q(k)$ (as seen in (13)). With the help of Lemma 2.1,

$$\lim_{q\to\zeta}c_q(\ell d)=c_\zeta(\ell d)=(-1)^\ell\binom{2\ell}{\ell}\frac{(-\zeta;\zeta^2)_d^{2\ell}}{(-\zeta;\zeta)_d^{2\ell}(-\zeta^2;\zeta^2)_d^{\ell}(-\zeta^4;\zeta^4)_d^{2\ell}}=\frac{(-1)^\ell}{8^\ell}\binom{2\ell}{\ell},$$

since ζ,ζ^2,ζ^4 are all primitive d-th roots of unity, and

$$\begin{gathered}(-\zeta;\zeta)_d=(1+\zeta^d)\prod_{j=1}^{d-1}(1+\zeta^j)=2,\quad (-\zeta^2;\zeta^2)_d=(-\zeta^4;\zeta^4)_d=2,\\ (-\zeta;\zeta^2)_d=(1+\zeta)(1+\zeta^3)\cdots(1+\zeta^d)(1+\zeta^{d+2})\cdots(1+\zeta^{2d-1})=(-\zeta;\zeta)_d=2.\end{gathered} \tag{19}$$

Using the binomial theorem

$$(1-z)^{-1/2}=\sum_{\ell=0}^\infty\frac{(\frac12)_\ell}{\ell!}z^\ell=\sum_{\ell=0}^\infty\binom{2\ell}{\ell}\Big(\frac{z}{4}\Big)^\ell$$

we deduce that

$$\sum_{\ell=0}^\infty(-1)^\ell\binom{2\ell}{\ell}\frac{1}{8^\ell}=\frac{\sqrt6}{3}. \tag{20}$$

It follows that the limiting case of the equality (15) as $q\to\zeta$ assumes the form

$$\frac{\sqrt6}{3}\sum_{k=0}^{d-1}c_\zeta(k)=\frac{\sqrt6}{3}\sum_{k=0}^{(d-1)/2}c_\zeta(k)=0,$$

where the first equality follows from (18) and the second equality is implied by (17). By above it remains to notice that

$$\sum_{k=0}^{n-1}c_\zeta(k)=\sum_{\ell=0}^{n/d-1}\sum_{k=0}^{d-1}c_\zeta(\ell d+k)=\sum_{\ell=0}^{n/d-1}\frac{(-1)^\ell}{8^\ell}\binom{2\ell}{\ell}\sum_{k=0}^{d-1}c_\zeta(k)=0$$

and

$$\sum_{k=0}^{(n-1)/2}c_\zeta(k)=\sum_{\ell=0}^{(n/d-3)/2}\frac{(-1)^\ell}{8^\ell}\binom{2\ell}{\ell}\sum_{k=0}^{d-1}c_\zeta(k)+\sum_{k=0}^{(d-1)/2}c_\zeta((n-d)/2+k)=0,$$

which imply that both sums $\sum_{k=0}^{n-1}c_q(k)$ and $\sum_{k=0}^{(n-1)/2}c_q(k)$ are divisible by the cyclotomic polynomial $\Phi_d(q)$. Since this is true for any divisor $d>1$ of n, we conclude that they are divisible by

$$\prod_{\substack{d\mid n\\ d>1}}\Phi_d(q)=[n]. \qquad \square$$

Proof of Theorem 1.3. Similarly, we can prove (9) and (10). The difference is that we replace the evaluation (20) with

$$\sum_{\ell=0}^{\infty}\binom{2\ell}{\ell}\frac{(-\zeta;\zeta^2)_d^{2\ell}}{(-\zeta^2;\zeta^2)_d^{2\ell}(-\zeta^4;\zeta^4)_d^{2\ell}}=\sum_{\ell=0}^{\infty}\binom{2\ell}{\ell}\frac{1}{4^\ell}=\infty,$$

while the right-hand side in (16) is uniformly bounded as $q\to\zeta$. Indeed, the latter follows from

$$\lim_{q\to\zeta}\frac{(-q^2;q^4)_{\ell d+k}^2}{(1-q)(-q^4;q^4)_{\ell d+k}^2}=\frac{(-\zeta^2;\zeta^4)_{\ell d+k}^2}{(1-\zeta)(-\zeta^4;\zeta^4)_{\ell d+k}^2}=\frac{(-\zeta^2;\zeta^4)_k^2}{(1-\zeta)(-\zeta^4;\zeta^4)_k^2}$$

for any $\ell\ge 0$ and $0\le k<d$ (see (19)), so that the expression

$$\frac{(-q^2;q^4)_n^2}{(1-q)(-q^4;q^4)_n^2}$$

is bounded above by

$$\frac{1}{|1-\zeta|}\max_{0\le k<d}\frac{|(-\zeta^2;\zeta^4)_k|^2}{|(-\zeta^4;\zeta^4)_k|^2}+1$$

as $q\to\zeta$. Thus, we conclude that

$$\sum_{k=0}^{d-1}\begin{bmatrix}2k\\k\end{bmatrix}_{\zeta^2}\frac{(-\zeta;\zeta^2)_k^2[6k+1]_\zeta\zeta^{k^2}}{(-\zeta^2;\zeta^2)_k^2(-\zeta^4;\zeta^4)_k^2}=\sum_{k=0}^{(d-1)/2}\begin{bmatrix}2k\\k\end{bmatrix}_{\zeta^2}\frac{(-\zeta;\zeta^2)_k^2[6k+1]_\zeta\zeta^{k^2}}{(-\zeta^2;\zeta^2)_k^2(-\zeta^4;\zeta^4)_k^2}=0$$

for any (odd) divisor $d>1$ of n, and this leads — in exactly the same way as in the proof of Theorem 1.2 — to the divisibility of the truncated q-sums by $[n]$. □

With the summands in Theorems 1.2 and 1.3, we can associate the q-WZ pairs

$$F(n,k)=(-1)^{n+k}\frac{[6n-2k+1](q;q^2)_{n-k}(-q;q^2)_{n-k}(-q;q^2)_{n+k}}{(q^4;q^4)_n(-q^4;q^4)_n(-q^4;q^4)_{n-k}},$$
$$G(n,k)=\frac{(-1)^{n+k}(q;q^2)_{n-k}(-q;q^2)_{n-k}(-q;q^2)_{n+k-1}}{(1-q)(q^4;q^4)_{n-1}(-q^4;q^4)_{n-1}(-q^4;q^4)_{n-k}},$$

and

$$\begin{aligned}\widetilde{F}(n,k)&=\frac{q^{(n-k)^2}[6n-2k+1](q^2;q^4)_n(-q;q^2)_{n-k}(-q;q^2)_{n+k}}{(q^4;q^4)_n(-q^4;q^4)_n(-q^4;q^4)_{n-k}(-q^2;q^4)_k},\\ \widetilde{G}(n,k)&=\frac{q^{(n-k)^2}(q^2;q^4)_n(-q;q^2)_{n-k}(-q;q^2)_{n+k-1}}{(1-q)(q^4;q^4)_{n-1}(-q^4;q^4)_{n-1}(-q^4;q^4)_{n-k}(-q^2;q^4)_k},\end{aligned}\tag{21}$$

respectively, where the convention $1/(q^4;q^4)_m=0$ for negative integers m is applied (and justified by the extended definition

$$(a;q)_{-n}=\frac{(-1)^na^nq^{n(n+1)/2}}{(q/a;q)_n}\quad\text{for } n=1,2,\dots$$

of the q-Pochhammer symbol). However, we do not see a way to use the pairs for proving the required congruences. Let us illustrate the difficulty in the example of the q-WZ pair (21), which satisfies

$$\widetilde{F}(n,k-1)-\widetilde{F}(n,k)=\widetilde{G}(n+1,k)-\widetilde{G}(n,k),$$

and the related congruence (9). For an odd integer $m>1$, summing the last equality over $n=0,1,\dots,(m-1)/2$ we obtain, via telescoping,

$$\sum_{n=0}^{(m-1)/2} F(n,k-1) - \sum_{n=0}^{(m-1)/2} F(n,k) = G\Big(\frac{m+1}{2},k\Big) \equiv 0 \pmod{[m]}$$

for any integer k, so that

$$\sum_{n=0}^{(m-1)/2} F(n,0) \equiv \sum_{n=0}^{(m-1)/2} F(n,1) \equiv \cdots \equiv \sum_{n=0}^{(m-1)/2} F(n,k) \pmod{[m]}.$$

At the same time, there seems to be no natural choice of k, for which

$$\sum_{n=0}^{(m-1)/2} F(n,k) \equiv 0 \pmod{[m]}$$

follows. The same obstacles occur for the remaining congruences in Theorems 1.2 and 1.3.

3. q-Supercongruences of Ramanujan type

We recall that the q-analogue (4) of Ramanujan's formula (1) is established in [19] on the basis of

$$\begin{aligned}\sum_{k=0}^{\infty} &\frac{(1-acq^{4k})(a;q)_k(q/a;q)_k(ac;q)_{2k}}{(1-ac)(cq^3;q^3)_k(a^2cq^2;q^3)_k(q;q)_{2k}}\, q^{k^2} \\ &= \frac{(acq^2;q^3)_\infty(acq^3;q^3)_\infty(aq;q^3)_\infty(q^2/a;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty(a^2cq^2;q^3)_\infty(cq^3;q^3)_\infty}. \end{aligned} \tag{22}$$

Replace q with q^2, take $c=q/a$ and then aq for a:

$$\begin{aligned}\sum_{k=0}^{\infty} &[8k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}\, q^{2k^2} \\ &= \frac{(q^5;q^6)_\infty(q^7;q^6)_\infty(aq^3;q^6)_\infty(q^3/a;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty(aq^6;q^6)_\infty(q^6/a;q^6)_\infty}. \end{aligned} \tag{23}$$

Observe that the truncated q-sums in Theorem 1.4 correspond to the left-hand side of (23). Furthermore,

$$\frac{(q^5;q^6)_\infty(q^7;q^6)_\infty}{(q^2;q^6)_\infty(q^4;q^6)_\infty} = \frac{(q;q^2)_\infty(q^6;q^6)_\infty}{(1-q)\,(q^2;q^2)_\infty(q^3;q^6)_\infty}. \tag{24}$$

Lemma 3.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{(q^{1-n};q^2)_k(q^{1+n};q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{6-n};q^6)_k(q^{6+n};q^6)_k}[8k+1]q^{2k^2} = q^{-(n-1)/2}[n]\Big(\frac{-3}{n}\Big). \tag{25}$$

Proof. We substitute $a = q^n$ into (23). Then the left-hand side of (23) terminates at $k = (n-1)/2$; therefore, it is exactly the left-hand side of (25), while the right-hand side of (23) becomes $q^{-(n-1)/2}[n]$ if $n \equiv 1 \pmod 3$, $-q^{-(n-1)/2}[n]$ if $n \equiv 2 \pmod 3$, and 0 if $3 \mid n$. $\square$

Lemma 3.2. *Let n be an integer with $n > 1$ and $(n, 6) = 1$. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{1 - q^{1-n+8k}}{1 - q^{1-n}} \frac{(aq;q^2)_k (q/a;q^2)_k (q^{1-n};q^2)_{2k}}{(q^2;q^2)_{2k} (aq^{6-n};q^6)_k (q^{6-n}/a;q^6)_k} q^{2k^2} = 0.$$

Proof. We specialise (22) by taking q^2 for q, then $c = q^{1-n}/a$ and replacing a with aq. The right-hand side of the resulting identity vanishes, because of the factors $(q^{5-n};q^6)_\infty$ and $(q^{7-n};q^6)_\infty$ in the numerator, while the left-hand side terminates at $k = (n-1)/2$ (in fact, even earlier, at $k = \lfloor n/4 \rfloor$), since the summand involves $(q^{1-n};q^2)_{2k}$. $\square$

Proof of Theorem 1.4. Let $\zeta \neq 1$ be a primitive d-th root of unity, where $d \mid n$ and $n > 1$ is coprime with 6. Denote by

$$c_q(k) = [8k+1] \frac{(aq;q^2)_k (q/a;q^2)_k (q;q^2)_{2k}}{(q^2;q^2)_{2k} (aq^6;q^6)_k (q^6/a;q^6)_k} q^{2k^2}$$

the k-th term of the sum (23). Use (24) to write the identity in (23) in the form

$$\sum_{\ell=0}^{\infty} c_q(\ell d) \sum_{k=0}^{d-1} \frac{c_q(\ell d + k)}{c_q(\ell d)} = \frac{(q;q^2)_\infty (q^6;q^6)_\infty (aq^3;q^6)_\infty (q^3/a;q^6)_\infty}{(1-q)\,(q^2;q^2)_\infty (q^3;q^6)_\infty (aq^6;q^6)_\infty (q^6/a;q^6)_\infty}. \tag{26}$$

Consider the limit as $q \to \zeta$ radially. On the left-hand side we get

$$\lim_{q \to \zeta} \frac{c_q(\ell d + k)}{c_q(\ell d)} = \frac{c_\zeta(\ell d + k)}{c_\zeta(\ell d)} = c_\zeta(k)$$

and, since $(q;q^2)_{2k}/(q^2;q^2)_{2k} = {4k \brack 2k}_{q^2}/(-q;q)_{4k}$, by Lemma 2.1 and the reduction formulas (19),

$$\lim_{q \to \zeta} c_q(\ell d) = \frac{1}{2^{4\ell}} \binom{4\ell}{2\ell} \frac{(a\zeta;\zeta^2)_{\ell d} (\zeta/a;\zeta^2)_{\ell d}}{(a\zeta^6;\zeta^6)_{\ell d} (\zeta^6/a;\zeta^6)_{\ell d}} = \frac{1}{2^{4\ell}} \binom{4\ell}{2\ell}.$$

By Stirling's approximation, $\binom{4\ell}{2\ell} \sim 2^{4\ell}/\sqrt{2\pi\ell}$ as $\ell \to \infty$, hence

$$\sum_{\ell=0}^{\infty} \frac{1}{2^{4\ell}} \binom{4\ell}{2\ell} = \infty.$$

For the right-hand side of (26),

$$\begin{aligned} &\lim_{q \to \zeta} \frac{(q;q^2)_{\ell d+k} (q^6;q^6)_{\ell d+k} (aq^3;q^6)_{\ell d+k} (q^3/a;q^6)_{\ell d+k}}{(1-q)\,(q^2;q^2)_{\ell d+k} (q^3;q^6)_{\ell d+k} (aq^6;q^6)_{\ell d+k} (q^6/a;q^6)_{\ell d+k}} \\ &\quad = \frac{(\zeta;\zeta^2)_k (\zeta^6;\zeta^6)_k (a\zeta^3;\zeta^6)_k (\zeta^3/a;\zeta^6)_k}{(1-\zeta)\,(\zeta^2;\zeta^2)_k (\zeta^3;\zeta^6)_k (a\zeta^6;\zeta^6)_k (\zeta^6/a;\zeta^6)_k} \end{aligned}$$

for any $\ell \ge 0$ and $0 \le k < d$. Therefore, the expression on the right-hand side of (26) is bounded above by

$$\max_{0\le k<d} \frac{|(\zeta;\zeta^2)_k(\zeta^6;\zeta^6)_k(a\zeta^3;\zeta^6)_k(\zeta^3/a;\zeta^6)_k|}{|(1-\zeta)\,(\zeta^2;\zeta^2)_k(\zeta^3;\zeta^6)_k(a\zeta^6;\zeta^6)_k(\zeta^6/a;\zeta^6)_k|}+1$$

as $q\to\zeta$. As in the proof of Theorem 1.3 we conclude that

$$\sum_{k=0}^{d-1} c_\zeta(k)=0, \tag{27}$$

while Lemma 3.2 applied for d in place of n after the substitution $q=\zeta$ results in

$$\sum_{k=0}^{(d-1)/2} c_\zeta(k)=0. \tag{28}$$

These equalities imply

$$\sum_{k=0}^{n-1} c_\zeta(k)=\sum_{k=0}^{(n-1)/2} c_\zeta(k)=0$$

for any d-th primitive root of unity ζ with $d\mid n$ and $d>1$, so that

$$\sum_{k=0}^{m}\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}[8k+1]q^{2k^2}\equiv 0\equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right)\pmod{[n]} \tag{29}$$

for both $m=n-1$ and $m=(n-1)/2$. Furthermore, it follows from Lemma 3.1 that

$$\sum_{k=0}^{m}\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}[8k+1]q^{2k^2}=q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right)$$

when $a=q^n$ or $a=q^{-n}$, again for both $m=n-1$ and $m=(n-1)/2$. This implies that the congruences

$$\sum_{k=0}^{m}\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_{2k}}{(q^2;q^2)_{2k}(aq^6;q^6)_k(q^6/a;q^6)_k}[8k+1]q^{2k^2}\equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right)$$

hold modulo $1-aq^n$ and $a-q^n$. Since $[n]$, $1-aq^n$ and $a-q^n$ are relatively prime polynomials, we obtain (11) and (12). □

Notice that our proofs of equations (27) and (28) are based on two different arguments — somewhat less uniform compared to the earlier derivation of similar equalities in the proofs of Theorems 1.2 and 1.3. We also stress on the fact that the congruences (29) obtained are valid for *any* a, including $a=1$:

$$\sum_{k=0}^{m}\frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^2;q^2)_{2k}(q^6;q^6)_k^2}[8k+1]q^{2k^2}\equiv 0\equiv q^{-(n-1)/2}[n]\left(\frac{-3}{n}\right)\pmod{[n]} \tag{30}$$

hold for $m=n-1$ and $m=(n-1)/2$.

Proof of Theorem 1.1. The denominators of (11) and (12) related to a are the factors $(aq^6;q^6)_{n-1}(q^6/a;q^6)_{n-1}$ and $(aq^6;q^6)_{(n-1)/2}(q^6/a;q^6)_{(n-1)/2}$, respectively; their limits as $a\to 1$ are relatively prime to $\Phi_n(q)$, since n is coprime with 6. On the other hand, the limit of $(1-aq^n)(a-q^n)$ as $a\to 1$ has the factor $\Phi_n(q)^2$. Thus, letting $a\to 1$ in (11) and (12), we see that (5) and (6) are true modulo $\Phi_n(q)^3$. At the same time, from (30) they are also valid modulo $[n]$. This completes the proof of (5) and (6). □

4. More q-supercongruences

Throughout this section, m always stands for $n-1$ or $(n-1)/2$. We shall give generalizations of some known q-supercongruences and also confirm some conjectures on q-analogues of Ramanujan-type supercongruences in [8–12].

4.1. **Two congruences of Van Hamme.** We start with the following q-supercongruence from [8, Theorem 1.2]:

$$\sum_{k=0}^{(n-1)/2}(-1)^k q^{k^2}[4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}\equiv q^{(n-1)^2/4}[n](-1)^{(n-1)/2}\pmod{[n]\Phi_n(q)^2} \qquad (31)$$

for odd n, which is a q-analogue of the (B.2) supercongruence of Van Hamme [32]. Along the same lines as Theorem 1.4, we have the following generalization of (31).

Theorem 4.1. *Let n be a positive odd integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{m}(-1)^k q^{k^2}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k}\equiv q^{(n-1)^2/4}[n](-1)^{(n-1)/2}. \qquad (32)$$

As we have seen in the proof of Theorem 1.1, the modulus $[n](1-aq^n)(a-q^n)$ transforms to contain the factor $\Phi_n(q)^3$ as $a\to 1$ and from the (sketch of) proof below the congruence (32) is true for $a=1$ modulo $[n]$. Therefore, the congruence (32) reduces to (31) when $m=(n-1)/2$ but it also confirms [8, Conjecture 5.1] when $m=n-1$, as $a\to 1$.

Sketch of proof. The terminating case of the sum of the very-well-poised ${}_6\phi_5$ series,

$$\sum_{k=0}^{\infty}\frac{(1-aq^{2k})(a;q)_k(b;q)_k(c;q)_k(d;q)_k}{(1-a)(q;q)_k(aq/b;q)_k(aq/c;q)_k(aq/d;q)_k}\left(\frac{aq}{bcd}\right)^k$$
$$=\frac{(aq;q)_\infty(aq/bc;q)_\infty(aq/bd;q)_\infty(aq/cd;q)_\infty}{(aq/b;q)_\infty(aq/c;q)_\infty(aq/d;q)_\infty(aq/bcd;q)_\infty}, \qquad (33)$$

reads

$$\sum_{k=0}^{N}\frac{(1-aq^{2k})(a;q)_k(b;q)_k(c;q)_k(q^{-N};q)_k}{(1-a)(q;q)_k(aq/b;q)_k(aq/c;q)_k(aq^{N+1};q)_k}\left(\frac{aq^{N+1}}{bc}\right)^k=\frac{(aq;q)_N(aq/bc;q)_N}{(aq/b;q)_N(aq/c;q)_N} \qquad (34)$$

(see [5, Appendix, eqs. (II.20) and (II.21)]). Letting $N \to \infty$, $q \to q^2$, $a = q$ in (34), then $b = aq$ and $c = q/a$ we obtain

$$\sum_{k=0}^{\infty}(-1)^k q^{k^2}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k} = \frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(aq^2;q^2)_\infty(q^2/a;q^2)_\infty}.$$

Considering the limits of both sides as $q \to \zeta$, a d-th root of unity for each $d \mid n$, we see that (32) holds modulo $[n]$, while letting $a = q^n$ we conclude that (32) holds modulo $(1-aq^n)(a-q^n)$. □

In [16], the first author and Wang obtained a q-analogue of [22, Theorem 1.1 with $r = 1$]: for odd n,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv q^{(1-n)/2}[n] + \frac{(n^2-1)(1-q)^2}{24}\,q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3},$$

which modulo $[n]\Phi_n(q)^2$ corresponds to the following q-analogue of the (C.2) supercongruence of Van Hamme [32]:

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv q^{(1-n)/2}[n] \pmod{[n]\Phi_n(q)^2}. \tag{35}$$

We have the following two-parameter common generalization of (32) (corresponding to $c \to 0$) and (35) (corresponding to $c = 1$).

Theorem 4.2. *Let n be a positive odd integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{m}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q/c;q^2)_k(q;q^2)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(cq^2;q^2)_k(q^2;q^2)_k}\,c^k \equiv \frac{(c/q)^{(n-1)/2}(q^2/c;q^2)_{(n-1)/2}}{(cq^2;q^2)_{(n-1)/2}}\,[n]. \tag{36}$$

Sketch of proof. Taking $q \to q^2$, $a = q$ in (33), then $c = aq$ and $d = q/a$ we obtain

$$\sum_{k=0}^{\infty}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(b;q^2)_k(q;q^2)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^3/b;q^2)_k(q^2;q^2)_k}\left(\frac{q}{b}\right)^k$$
$$= \frac{(q^3;q^2)_\infty(q;q^2)_\infty(q^2/ab;q^2)_\infty(aq^2/b;q^2)_\infty}{(aq^2;q^2)_\infty(q^2/a;q^2)_\infty(q^3/b;q^2)_\infty(q/b;q^2)_\infty}.$$

For a root of unity ζ of odd degree $d \mid n$, the limit of the right-hand side is 0 as $q \to \zeta$, because of the presence of the factor $(q;q^2)_\infty$. Letting q tend to ζ on the left-hand side results in

$$\sum_{\ell=0}^{\infty}\frac{(b;\zeta^2)_{\ell d}}{b^{\ell d}(\zeta^3/b;\zeta^2)_{\ell d}}\binom{2\ell}{\ell}\frac{1}{4^\ell}\sum_{k=0}^{d-1}[4k+1]_\zeta\frac{(a\zeta;\zeta^2)_k(\zeta/a;\zeta^2)_k(b;\zeta^2)_k(\zeta;\zeta^2)_k}{(a\zeta^2;\zeta^2)_k(\zeta^2/a;\zeta^2)_k(\zeta^3/b;\zeta^2)_k(\zeta^2;\zeta^2)_k}\left(\frac{\zeta}{b}\right)^k$$
$$= \sum_{\ell=0}^{\infty}\binom{2\ell}{\ell}\frac{1}{4^\ell}\sum_{k=0}^{(d-1)/2}[4k+1]_\zeta\frac{(a\zeta;\zeta^2)_k(\zeta/a;\zeta^2)_k(b;\zeta^2)_k(\zeta;\zeta^2)_k}{(a\zeta^2;\zeta^2)_k(\zeta^2/a;\zeta^2)_k(\zeta^3/b;\zeta^2)_k(\zeta^2;\zeta^2)_k}\left(\frac{\zeta}{b}\right)^k$$

implying

$$\sum_{k=0}^{m}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(b;q^2)_k(q;q^2)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^3/b;q^2)_k(q^2;q^2)_k}\left(\frac{q}{b}\right)^k \equiv 0 \pmod{[n]}$$

for any $b \neq 0$, in particular, for $b = q/c$.

Finally, the congruence (36) modulo $1-aq^n$ and $a-q^n$ follows from the summation

$$\sum_{k=0}^{m}[4k+1]\frac{(q^{1-n};q^2)_k(q^{1+n};q^2)_k(q;q^2)_k(q/c;q^2)_k}{(q^{2-n};q^2)_k(q^{2+n};q^2)_k(q^2;q^2)_k(cq^2;q^2)_k}\,c^k = \frac{(c/q)^{(n-1)/2}(q^2/c;q^2)_{(n-1)/2}}{(cq^2;q^2)_{(n-1)/2}}\,[n],$$

which is the specialization $q \to q^2$, $a = q$, $c = q^{1+n}$, $N = (n-1)/2$ and $b = q/c$ of (34). □

4.2. **Another two congruences from Van Hamme's list.** The following q-supercongruence conjectured in [10, eqs. (1.4) and (1.5)] is a partial q-analogue of the (J.2) supercongruence of Van Hamme [32]:

$$\sum_{k=0}^{m} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3} \equiv (-q)^{(1-n)/2}[n] \pmod{[n]\Phi_n(q)^2}. \tag{37}$$

It is established modulo $[n]\Phi_n(q)$ in [10, Theorem 1.3]. Here we confirm (37) by showing the following more general form, which is also a generalization of Theorem 1.3.

Theorem 4.3. *Let n be a positive odd integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{m} q^{k^2}[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k} \equiv (-q)^{(1-n)/2}[n].$$

Sketch of proof. Our derivation in [19] of the related q-analogue of a formula for $1/\pi$ uses the formula [28, eq. (4.6)]:

$$\sum_{k=0}^{\infty}\frac{(a;q)_k(1-aq^{3k})(d;q)_k(q/d;q)_k(b;q^2)_k}{(q^2;q^2)_k(1-a)(aq^2/d;q^2)_k(adq;q^2)_k(aq/b;q)_k}\frac{a^k q^{\binom{k+1}{2}}}{b^k}$$
$$= \frac{(aq;q^2)_\infty(aq^2;q^2)_\infty(adq/b;q^2)_\infty(aq^2/bd;q^2)_\infty}{(aq/b;q^2)_\infty(aq^2/b;q^2)_\infty(aq^2/d;q^2)_\infty(adq;q^2)_\infty}. \tag{38}$$

Letting $q \to q^2$ and taking $a = q$, $d = aq$ and $b = q^2$, we are led to

$$\sum_{k=0}^{\infty} q^{k^2}[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q^2;q^4)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k} = \frac{(aq^2;q^4)_\infty(q^2/a;q^4)_\infty}{(1-q)(aq^4;q^4)_\infty(q^4/a;q^4)_\infty}.$$

The rest is similar to the proof of Theorem 1.4. □

We complement the result by the following complete q-analogue of Van Hamme's supercongruence (J.2) (see [10, Conjecture 1.1]), which remains open:

$$\sum_{k=0}^{(n-1)/2} q^{k^2}[6k+1]\frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^4;q^4)_k^3}$$
$$\equiv (-q)^{(1-n)/2}[n] + \frac{(n^2-1)(1-q)^2}{24}(-q)^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}$$

for odd n.

Similarly, we have the following q-analogue of the (L.2) supercongruence of Van Hamme [32]:

$$\sum_{k=0}^{m}(-1)^k[6k+1]\frac{(q;q^2)_k^3}{(q^4;q^4)_k^3} \equiv (-q)^{-(n-1)(n+5)/8}[n] \pmod{[n]\Phi_n(q)^2}, \tag{39}$$

which is conjectured in [9, Conjecture 1.1] and proved in [9, Theorem 1.2] for special cases. Here we are able to confirm (39) in the full generality as a consequence of the following result, which is also a generalization of Theorem 1.2.

Theorem 4.4. *Let n be a positive odd integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{m}(-1)^k[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k} \equiv (-q)^{-(n-1)(n+5)/8}[n].$$

Sketch of proof. Replacing q by q^{-1}, we see that the desired congruence is equivalent to

$$\sum_{k=0}^{m}(-1)^k[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k q^{3k^2}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k} \equiv (-q)^{(n-1)(n-3)/8}[n].$$

Letting $q\to q^2$ and $b\to\infty$ in (38), then taking $a=q$ and $d=aq$ we obtain

$$\sum_{k=0}^{\infty}(-1)^k[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k q^{3k^2}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k} = \frac{(q^3;q^4)_\infty(q^5;q^4)_\infty}{(aq^4;q^4)_\infty(q^4/a;q^4)_\infty}.$$

The remaining argument is as before. □

We also have a common generalization of Theorems 4.3 and 4.4 as follows.

Theorem 4.5. *Let $n\equiv r \pmod 4$ be a positive odd integer, where $r=\pm1$. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{m}[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k(b;q^4)_k q^{k^2+2k}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k(q^3/b;q^2)_k b^k}$$
$$\equiv \frac{(q^rb;q^4)_{(n-r)/4}}{(q^{4+r}/b;q^4)_{(n-r)/4}}\, b^{-(n-r)/4}(-q)^{(1-r)/2}[n].$$

Sketch of proof. This follows along the lines of our previous proofs, for the specialization $q \to q^2$, $a = q$ and $d = aq$ of the quadratic summation (38):

$$\sum_{k=0}^{\infty}[6k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k(b;q^4)_k q^{k^2+2k}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k(q^3/b;q^2)_k b^k}$$
$$= \frac{(q^3;q^4)_\infty(q^5;q^4)_\infty(aq^4/b;q^2)_\infty(q^4/ab;q^2)_\infty}{(q^3/b;q^2)_\infty(q^5/b;q^2)_\infty(aq^4;q^2)_\infty(q^4/a;q^2)_\infty}. \qquad \square$$

4.3. **'Divergent' congruences.** The first author obtained in [12] the following q-analogues of two 'divergent' Ramanujan-type supercongruences of Guillera and the second author [7]:

$$\sum_{k=0}^{m}[3k+1]\frac{(q;q^2)_k^3 q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k} \equiv q^{(1-n)/2}[n] \pmod{[n]\Phi_n(q)^2}, \tag{40}$$

$$\sum_{k=0}^{n-1}(-1)^k[3k+1]\frac{(q;q^2)_k^3}{(q;q)_k^3} \equiv q^{(n-1)^2/4}[n](-1)^{(n-1)/2} \pmod{[n]\Phi_n(q)^2}. \tag{41}$$

For both cases, the corresponding infinite hypergeometric sums diverge. Observing their connection to Rahman's quadratic transformation [28, eq. (3.12)] (also recorded in [5, eq. (3.8.13)]) we have arrived numerically at the following three-parameter common generalization of (40) and (41).

Conjecture 4.6. *Let n be a positive odd integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$\sum_{k=0}^{m}[3k+1]\,\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k(q/b;q)_k(q/c;q)_k(bc;q)_k\,q^k}{(aq;q)_k(q/a;q)_k(q;q)_k(bq^2;q^2)_k(cq^2;q^2)_k(q^3/bc;q^2)_k}$$
$$\equiv \frac{(bcq;q^2)_{(n-1)/2}(q^2/b;q^2)_{(n-1)/2}(q^2/c;q^2)_{(n-1)/2}}{(q^3/bc;q^2)_{(n-1)/2}(bq^2;q^2)_{(n-1)/2}(cq^2;q^2)_{(n-1)/2}}\,[n]. \tag{42}$$

Note that the infinite sum for the left-hand side of (42) is the specialization $a = q$ of the left-hand side in [5, eq. (3.8.13)], where one further sets $d = aq$ and replaces b, c with $q/b, q/c$, respectively:

$$\sum_{k=0}^{\infty}[3k+1]\,\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k(q/b;q)_k(q/c;q)_k(bc;q)_k\,q^k}{(aq;q)_k(q/a;q)_k(q;q)_k(bq^2;q^2)_k(cq^2;q^2)_k(q^3/bc;q^2)_k}$$
$$= \frac{(q^2/b;q^2)_\infty(q^2/c;q^2)_\infty(bcq;q^2)_\infty}{(1-q)\,(bq^2;q^2)_\infty(cq^2;q^2)_\infty(q^3/bc;q^2)_\infty}\sum_{k=0}^{\infty}\frac{(q/b;q^2)_k(q/c;q^2)_k(bc;q^2)_k\,q^{2k}}{(q^2;q^2)_k(aq^2;q^2)_k(q^2/a;q^2)_k}. \tag{43}$$

The a-parametric versions of the congruences (40) and (41) are obtained from (42) by setting $b \to 0$ followed by $c = 1$ and $c \to 0$, respectively. We cannot establish this numerical observation in its entirety but we can settle two particular cases.

Theorem 4.7. *For n a positive odd integer, the congruence (42) is valid modulo $(1-aq^n)(a-q^n)$.*

Sketch of proof. For convenience, we will use here the standard notation

$$ {}_{r+3}W_{r+2}(a_0;a_1,a_2,\dots,a_r;q,z)=\sum_{k=0}^{\infty}\frac{(1-a_0q^{2k})\,(a_0;q)_k(a_1;q)_k\cdots(a_r;q)_k\,z^k}{(1-a_0)\,(q;q)_k(qa_0/a_1;q)_k\cdots(qa_0/a_r;q)_k} $$

for very-well-poised (basic) hypergeometric series.

Take $a=q^{1+2N}$. Then the transformation [5, eq. (3.8.14)] applies, in which the parameters a, b, c and f are replaced with our q/b, q/c, b and q, respectively:

$$ \begin{aligned} &\sum_{k=0}^{N}[3k+1]\,\frac{(q^{2+2N};q^2)_k(q^{-2N};q^2)_k(q;q^2)_k(q/b;q)_k(q/c;q)_k(bc;q)_k\,q^k}{(q^{2+2N};q)_k(q^{-2N};q)_k(q;q)_k(bq^2;q^2)_k(cq^2;q^2)_k(q^3/bc;q^2)_k}\\ &\qquad=[2N+1]\,\frac{(bcq;q^2)_N(q^2/bc;q^2)_N}{(bcq^2;q^2)_N(q^3/bc;q^2)_N}\,{}_{10}W_9(bc;q,b,c,bc,qbc,q^{2N+2},q^{-2N};q^2,q^2)\\ &\qquad=[2N+1]\,\frac{(bcq;q^2)_N(q^2/bc;q^2)_N}{(bcq^2;q^2)_N(q^3/bc;q^2)_N}\,{}_{8}W_7(bc;b,c,bc,q^{2N+2},q^{-2N};q^2,q^2) \end{aligned} $$

(this is summable by Jackson's q-analogue of Dougall's ${}_7F_6$ sum [5, eq. (II.22)])

$$ \begin{aligned} &\qquad=[2N+1]\,\frac{(bcq;q^2)_N(q^2/bc;q^2)_N}{(bcq^2;q^2)_N(q^3/bc;q^2)_N}\,\frac{(bcq^2;q^2)_N(q^2/b;q^2)_N(q^2/c;q^2)_N}{(bq^2;q^2)_N(cq^2;q^2)_N(q^2/bc;q^2)_N}\\ &\qquad=[2N+1]\,\frac{(bcq;q^2)_N(q^2/b;q^2)_N(q^2/c;q^2)_N}{(q^3/bc;q^2)_N(bq^2;q^2)_N(cq^2;q^2)_N}. \end{aligned} $$

This establishes (42) simultaneously modulo $a-q^n$ and $1-aq^n$ for $n=2N+1$. ☐

Theorem 4.8. *Let n be a positive odd integer. Then, modulo $[n](1-aq^n)(a-q^n)$,*

$$ \begin{aligned} \sum_{k=0}^{m}[3k+1]\,&\frac{(aq;q^2)_k(q/a;q^2)_k(b;q)_k(q/b;q)_k(q;q^2)_k\,q^k}{(aq;q)_k(q/a;q)_k(bq^2;q^2)_k(q^3/b;q^2)_k(q^2;q^2)_k}\\ &\equiv\frac{(bq;q^2)_{(n-1)/2}(q^2/b;q^2)_{(n-1)/2}}{(bq^2;q^2)_{(n-1)/2}(q^3/b;q^2)_{(n-1)/2}}[n]. \end{aligned} $$

This confirms Conjecture 4.6 when $c=1$.

Sketch of proof. In view of Theorem 4.7, we only need to verify the required congruence modulo $[n]$. Take $c=q^{1+2N}$ in (43) for N a positive integer, so that the q-Saalschütz theorem [5, eq. (II.12)] applies to the right-hand side:

$$ \begin{aligned} &\sum_{k=0}^{2N}[3k+1]\,\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k(q/b;q)_k(q^{-2N};q)_k(bq^{1+2N};q)_k\,q^k}{(aq;q)_k(q/a;q)_k(q;q)_k(bq^2;q^2)_k(q^{3+2N};q^2)_k(q^{2-2N}/b;q^2)_k}\\ &\quad=\frac{(q^{1-2N};q^2)_\infty(q^2/b;q^2)_\infty(bq^{2+2N};q^2)_\infty}{(1-q)\,(q^{3+2N};q^2)_\infty(q^{2+2N}/b;q^2)_\infty(bq^2;q^2)_\infty}\,\frac{(abq;q^2)_N(aq^{1-2N}/b;q^2)_N}{(aq^2;q^2)_N(aq^{-2N};q^2)_N}\\ &\quad=\frac{(-1)^N(q/b)^Nq^{-N^2}(q;q^2)_N^2(q^2/b;q^2)_N(abq;q^2)_N(bq/a;q^2)_N}{(1-q)\,(bq^2;q^2)_N(aq^2;q^2)_N(q^2/a;q^2)_N}. \end{aligned} $$

Now for $d>1$ odd take a primitive d-th root of unity ζ, then $M>0$ odd and specialize N above to be $(dM-1)/2$. The limit of the right-hand side as $q\to\zeta$ is equal to 0, because of the factor $(q;q^2)_N$. The limit of the left-hand side is

$$\sum_{\ell=0}^{M-1}\frac{1}{2^\ell}\binom{2\ell}{\ell}\sum_{k=0}^{d-1}[3k+1]\,\frac{(aq;q^2)_k(q/a;q^2)_k(b;q)_k(q/b;q)_k(q;q^2)_k\,q^k}{(aq;q)_k(q/a;q)_k(bq^2;q^2)_k(q^3/b;q^2)_k(q^2;q^2)_k}\bigg|_{q=\zeta},$$

where we use that $\zeta^{-2N}=\zeta^{1-dM}=\zeta$, so that we conclude with the congruence

$$\sum_{k=0}^{m}[3k+1]\,\frac{(aq;q^2)_k(q/a;q^2)_k(b;q)_k(q/b;q)_k(q;q^2)_k\,q^k}{(aq;q)_k(q/a;q)_k(bq^2;q^2)_k(q^3/b;q^2)_k(q^2;q^2)_k}\equiv 0 \pmod{[n]}$$

for odd n. □

Although Theorem 4.8 implies the a-parametric version of (40), there is a stronger version of the latter congruence (see [12, Conjecture 7.1]) which remains open: if n is odd then

$$\sum_{k=0}^{n-1}[3k+1]\frac{(q;q^2)_k^3q^{-\binom{k+1}{2}}}{(q;q)_k^2(q^2;q^2)_k}$$
$$\equiv q^{(1-n)/2}[n]+\frac{(n^2-1)(1-q)^2}{24}\,q^{(1-n)/2}[n]^3 \pmod{[n]\Phi_n(q)^3}.$$

Some other specializations of Theorem 4.8 are interesting by themselves. For example, the choice $q\to q^2$, $b=q$ and $a\to 1$ leads us to

$$\sum_{k=0}^{(n-1)/2}[3k+1]_{q^2}\,\frac{(q;q^2)_k^2(q^2;q^4)_k^3\,q^{2k}}{(q^2;q^2)_k^2(q^4;q^4)_k(q^5;q^4)_k^2}\equiv 0 \pmod{\Phi_n(q)^3}$$

for a positive integer $n\equiv 3 \pmod 4$. Notice the equivalence of the congruences for $m=(n-1)/2$ and $m=n-1$ in this special case. This, in turn, implies that for a prime p congruent to 3 modulo 4 we have

$$\sum_{k=0}^{(p-1)/2}(3k+1)\,\frac{(\frac12)_k^5}{(1)_k^3(\frac54)_k^2}\equiv 0 \pmod{p^3}.$$

4.4. **Generalized Van Hamme's congruences.** In [11, Theorem 1.3], a uniform version of q-analogues of the (B.2), (E.2) and (F.2) supercongruences of Van Hamme are given. The following result provides a generalization of the q-supercongruence that depends on an additional parameter a.

Theorem 4.9. *Let d be a positive integer and let r be an integer with $\gcd(r,d)=1$. Then, for any positive integer $n\equiv r \pmod d$ such that $n+d-nd\le r\le n$, we have*

$$\sum_{k=0}^{M}(-1)^kq^{d\binom{k+1}{2}-rk}[2dk+r]\frac{(aq^r;q^d)_k(q^r/a;q^d)_k(q^r;q^d)_k}{(aq^d;q^d)_k(q^d/a;q^d)_k(q^d;q^d)_k}$$
$$\equiv q^{(n-r)(n-d+r)/(2d)}[n](-1)^{(n-r)/d} \pmod{[n](1-aq^n)(a-q^n)}, \quad (44)$$

where $M=(n-r)/d$ or $M=n-1$.

Note that the $a\to 1$ and $M=n-1$ case of (44) partially confirms [11, Conjecture 5.1]. In particular, if p is a prime with $p^s\equiv 1 \pmod d$, then

$$\sum_{k=0}^{p^s-1}(-1)^k(2dk+1)\frac{(\frac{1}{d})_k^3}{k!^3}\equiv p^s(-1)^{(p^s-1)/d}\pmod{p^{s+2}}.$$

Sketch of proof. Letting $N\to\infty$, $q\to q^d$, $a=q^r$ in (34), followed by $b=aq^r$ and $c=q^r/a$, we obtain

$$\sum_{k=0}^{\infty}(-1)^k q^{d\binom{k+1}{2}-rk}[2dk+r]\frac{(aq^r;q^d)_k(q^r/a;q^d)_k(q^r;q^d)_k}{(aq^d;q^d)_k(q^d/a;q^d)_k(q^d;q^d)_k}=[r]\,\frac{(q^{d-r};q^d)_\infty(q^{d+r};q^d)_\infty}{(aq^d;q^d)_\infty(q^d/a;q^d)_\infty}. \tag{45}$$

Let ζ be an e-th primitive root of unity with $e\mid n$. By the hypothesis of the theorem, we see that $\gcd(n,d)=1$ and so $\gcd(e,d)=1$. This means that there is one and only one number divisible by e in the arithmetic progression $r,r+d,\dots,r+(e-1)d$. Denote this number by $r+ud=ve$. Then by L'Hôpital's rule we see that

$$\begin{aligned}\lim_{q\to\zeta}\frac{(q^r;q^d)_{\ell e+k}}{(q^d;q^d)_{\ell e+k}}&=\frac{v\,(v+d)\cdots(v+(\ell-1)d)}{d\cdot 2d\,\cdots\,\ell d}\,\lim_{q\to\zeta}\frac{(q^r;q^d)_k}{(q^d;q^d)_k}\\&=\binom{v/d+\ell-1}{\ell}\lim_{q\to\zeta}\frac{(q^r;q^d)_k}{(q^d;q^d)_k}\end{aligned}$$

for $\ell\ge 0$ and $0\le k<e$. It is clear that $v\ne d$. Since

$$\sum_{\ell=0}^{\infty}(-1)^\ell\binom{v/d+\ell-1}{\ell}=\begin{cases}2^{-v/d}\ne 0 & \text{if } v<d,\\ \infty & \text{if } v>d,\end{cases}$$

the proof of (44) modulo $[n]$ follows the lines of the proofs of Theorems 1.2 and 1.3.

Finally, the congruence (44) modulo $1-aq^n$ and $a-q^n$ follows from setting $a=q^{-n}$ in (45):

$$\begin{aligned}\sum_{k=0}^{M}&(-1)^k q^{d\binom{k+1}{2}-rk}[2dk+r]\frac{(q^{r-n};q^d)_k(q^{r+n};q^d)_k(q^r;q^d)_k}{(q^{d-n};q^d)_k(q^{d+n};q^d)_k(q^d;q^d)_k}\\&=[r]\,\frac{(q^{d-r};q^d)_\infty(q^{d+r};q^d)_\infty}{(q^{d-n};q^d)_\infty(q^{d+n};q^d)_\infty}=[r]\,\frac{(q^{d+r};q^d)_{(n-r)/d}}{(q^{d-n};q^d)_{(n-r)/d}}\\&=(-1)^{(n-r)/d}q^{(n-r)(n-d+r)/(2d)}[r]\,\frac{(q^{d+r};q^d)_{(n-r)/d}}{(q^r;q^d)_{(n-r)/d}}\\&=q^{(n-r)(n-d+r)/(2d)}[n](-1)^{(n-r)/d}.\end{aligned}$$

Note that the conditions $n\ge r$ and $n\equiv r \pmod d$ imply that the left-hand side terminates at $k=(n-r)/d$, while the hypothesis $n+d-nd\le r$ means that $(n-r)/d\le n-1$. □

Using the above basic hypergeometric series identity, we can also prove the following generalization of [11, Theorem 1.5].

Theorem 4.10. *Let d be a positive integer and let r be an integer with $\gcd(r,d)=1$. Then, for any positive integer $n\equiv -r \pmod d$ such that $d-n\le r\le (d-1)n$, we have*

$$\sum_{k=0}^{M}(-1)^k q^{d\binom{k+1}{2}-rk}[2dk+r]\frac{(aq^r;q^d)_k(q^r/a;q^d)_k(q^r;q^d)_k}{(aq^d;q^d)_k(q^d/a;q^d)_k(q^d;q^d)_k}$$
$$\equiv q^{(nd-n-r)(nd-n-d+r)/(2d)}[(d-1)n](-1)^{((d-1)n-r)/d} \pmod{[n](1-aq^n)(a-q^n)}, \tag{46}$$

where $M=((d-1)n-r)/d$ or $M=n-1$.

Note that the $a\to 1$ and $M=n-1$ case of (46) confirms [11, Conjecture 5.2]. In particular, if p is a prime satisfying $p^s\equiv -1 \pmod d$, then

$$\sum_{k=0}^{p^s-1}(-1)^k(2dk+1)\frac{(\frac{1}{d})_k^3}{k!^3}\equiv (d-1)p^s(-1)^{((d-1)p^s-1)/d} \pmod{p^{s+2}}.$$

4.5. **A strange congruence.** In [12, Conjecture 7.2], the following strange conjecture was proposed: for any positive integer n with $n\equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3}q^{k(n^2-2nk-n-2)/4}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{47}$$

Note that $k(n^2-2nk-n-2)/4$ is a two-variable polynomial of degree 3. Congruences of this form are very rare. We now give a related parametric result.

Theorem 4.11. *Let $n\equiv 1 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(q;q^2)_k q^{(n-1)k/2}}{(aq^2;q^2)_k(q^2/a;q^2)_k(q^2;q^2)_k}\equiv 0 \pmod{\Phi_n(q)}. \tag{48}$$

It is easy to see that the term $q^{(n-1)k/2}$ in (48) can be replaced by $q^{k(n^2-2nk-n-2)/4}$. However, we cannot replace the term $q^{k(n^2-2nk-n-2)/4}$ in (47) by $q^{(n-1)k/2}$.

Sketch of proof. Set $q\to q^4$, $a=q^2$, $d=q^3$ in (33), then take $b=aq^2$ and $c=q^2/a$:

$$\sum_{k=0}^{\infty}[4k+1]_{q^2}\frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k q^{-k}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k}$$
$$=\frac{(q^2;q^4)_\infty(q^6;q^4)_\infty(aq;q^4)_\infty(q/a;q^4)_\infty}{(1-q^{-1})(q^3;q^4)_\infty^2(aq^4;q^4)_\infty(q^4/a;q^4)_\infty}.$$

Now choose any primitive n-th root of unity $\zeta\ne 1$ and consider the limit of both sides of the equality as $q\to\zeta$. The right-hand side clearly tends to 0, because of the presence of $(q^2;q^4)_\infty$; the factor $(q^3;q^4)_\infty^2$ in the denominator does not interfere, since $\zeta^{3+4j} \ne 1$ when $j = 0, 1, 2, \dots$ for the root of unity of degree $n \equiv 1 \pmod 4$. The standard analysis of the left-hand side leads us to

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{\zeta^2} \frac{(a\zeta^2;\zeta^4)_k(\zeta^2/a;\zeta^4)_k(\zeta^2;\zeta^4)_k\zeta^{-k}}{(a\zeta^4;\zeta^4)_k(\zeta^4/a;\zeta^4)_k(\zeta^4;\zeta^4)_k} = 0.$$

Noticing that $\zeta^{-k} = \zeta^{(n-1)k}$ for $k = 0, 1, \dots, (n-1)/2$, we have

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2} \frac{(aq^2;q^4)_k(q^2/a;q^4)_k(q^2;q^4)_k q^{(n-1)k}}{(aq^4;q^4)_k(q^4/a;q^4)_k(q^4;q^4)_k} \equiv 0 \pmod{\Phi_n(q)}. \tag{49}$$

The left-hand side here remains the same if we replace q with $-q$, therefore the congruence (49) takes place modulo $\Phi_n(-q)$ as well, hence modulo $\Phi_n(q^2) = \Phi_n(q)\Phi_n(-q)$ since n is odd. Thus, changing q^2 with q we arrive at the congruence (48). □

4.6. **A congruence from the q-Dixon sum.** As we have seen, truncating known basic hypergeometric series identities usually leads to new q-congruences, or to 'natural' candidates for q-analogues of those coming from non-q-settings. Here is another example.

Theorem 4.12. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{m} \frac{(1+aq^{4k+1})\,(a^2q^2;q^4)_k(bq^2;q^4)_k(cq^2;q^4)_k}{(1+aq)\,(a^2q^4/b;q^4)_k(a^2q^4/c;q^4)_k(q^4;q^4)_k}\left(\frac{aq}{bc}\right)^k \equiv 0 \pmod{(1-a^2q^{2n})}; \tag{50}$$

in particular,

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})\,(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)\Phi_n(-q)}. \tag{51}$$

Proof. Taking $q \to q^4$, $a \to a^2q^2$, $b \to bq^2$ and $c \to cq^2$ in the q-Dixon sum [5, eq. (II.13)] we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} &\frac{(1+aq^{4k+1})\,(a^2q^2;q^4)_k(bq^2;q^4)_k(cq^2;q^4)_k}{(1+aq)\,(a^2q^4/b;q^4)_k(a^2q^4/c;q^4)_k(q^4;q^4)_k}\left(\frac{aq}{bc}\right)^k \\ &= \frac{(a^2q^6;q^4)_\infty(aq^3/b;q^4)_\infty(aq^3/c;q^4)_\infty(a^2q^2/bc;q^4)_\infty}{(a^2q^4/b;q^4)_\infty(a^2q^4/c;q^4)_\infty(aq^5;q^4)_\infty(aq/bc;q^4)_\infty}.\end{aligned} \tag{52}$$

Since $n \equiv 3 \pmod 4$, putting $a = \pm q^{-n}$ in (52) we see that the left-hand side terminates, while the right-hand side vanishes. This proves (50). Letting $a, b, c \to 1$ in (50) we are led to (51). □

We now provide a conjectural refinement of (51), which is a new q-analogue of the (H.2) supercongruence of Van Hamme [32] for $p \equiv 3 \pmod 4$ (corresponding to $q \to 1$). It is also a partial q-analogue of the (B.2) supercongruence of Van Hamme (corresponding to $q \to -1$).

Conjecture 4.13. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{4k+1})(q^2;q^4)_k^3}{(1+q)\,(q^4;q^4)_k^3} q^k \equiv 0 \pmod{\Phi_n(q)^2\Phi_n(-q)}.$$

4.7. **A congruence from Andrews' q-analogue of Gauss' ${}_2F_1(-1)$ sum.** It is proved in [17, eq. (2.6)] that, for p a prime of the form $4\ell+3$,

$$\sum_{k=0}^{p-1} \frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k} = \sum_{k=0}^{p-1} \frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k^2(-q^2;q^2)_k} \equiv 0 \pmod{[p]^2}.$$

We now give a two-parameter extension of this congruence.

Theorem 4.14. *Let $n \equiv 3 \pmod 4$ be a positive integer. Then*

$$\sum_{k=0}^{m} \frac{(aq;q^2)_k(bq;q^2)_k q^{2k}}{(q^2;q^2)_k(abq^4;q^4)_k} \equiv 0 \pmod{(1-aq^n)(1-bq^n)}; \tag{53}$$

in particular,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 q^{2k}}{(q^2;q^2)_k(q^4;q^4)_k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{54}$$

Proof. Making the substitutions $q \to q^2$, $a \to aq$ and $b \to bq$ in Andrews' q-analogue of Gauss' ${}_2F_1(-1)$ sum (see [1,2] or [5, Appendix (II.11)]), we obtain

$$\sum_{k=0}^{\infty} \frac{(aq;q^2)_k(bq;q^2)_k q^{k^2+k}}{(q^2;q^2)_k(abq^4;q^4)_k} = \frac{(aq^3;q^4)_\infty(bq^3;q^4)_\infty}{(q^2;q^4)_\infty(abq^4;q^4)_\infty}, \tag{55}$$

Since $n \equiv 3 \pmod 4$, taking $a = q^{-n}$ or $b = q^{-n}$ in (55) we see that the left-hand side terminates, while the right-hand side vanishes. This proves that

$$\sum_{k=0}^{m} \frac{(aq;q^2)_k(bq;q^2)_k q^{k^2+k}}{(q^2;q^2)_k(abq^4;q^4)_k} \equiv 0 \pmod{(1-aq^n)(1-bq^n)},$$

which after the rearrangement $a \to a^{-1}$, $b \to b^{-1}$ and $q \to q^{-1}$ becomes the congruence (53). Letting $a \to 1$ and $b \to 1$ in (53) we arrive at (54). □

Motivated by [17, Theorem 2.5] we observe the following generalization of Theorem 4.14.

Conjecture 4.15. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{m} \frac{(aq;q^2)_k(bq;q^2)_k(x;q^2)_k q^{2k}}{(q^2;q^2)_k(abq^4;q^4)_k}$$
$$\equiv (-1)^{(n-1)/2} \sum_{k=0}^{m} \frac{(aq;q^2)_k(bq;q^2)_k(-x;q^2)_k q^{2k}}{(q^2;q^2)_k(abq^4;q^4)_k} \pmod{(1-aq^n)(1-bq^n)}.$$

When $x = 0$ and $n \equiv 3 \pmod 4$ this indeed reduces to Theorem 4.14. Inspired by [17, Conjecture 7.3], we believe that the following further generalization is true as well.

Conjecture 4.16. *Let d, n and r be positive integers with $r<d$, $\gcd(d,n)=1$, and n odd. Then, modulo $(1-aq^{n\langle r/n\rangle_d})(1-bq^{n\langle (d-r)/n\rangle_d})$,*

$$\sum_{k=0}^{n-1}\frac{(aq^r;q^d)_k(bq^{d-r};q^d)_k(x;q^d)_kq^{dk}}{(q^d;q^d)_k(abq^{2d};q^{2d})_k}\equiv(-1)^{\langle -r/d\rangle_n}\sum_{k=0}^{n-1}\frac{(aq^r;q^d)_k(bq^{d-r};q^d)_k(-x;q^d)_kq^{dk}}{(q^d;q^d)_k(abq^{2d};q^{2d})_k},$$

where $\langle z\rangle_s$ denotes the least non-negative residue of z modulo s.

5. Concluding remarks and open problems

Since Ramanujan's formula (1) has a WZ proof [6], it is natural to ask whether there is a q-WZ proof of its q-analogue (4). If this is the case, then the corresponding q-WZ pair will possibly lead to another proof of Theorem 1.1.

The equality (1) motivates considering different families of congruences, like

$$\sum_{k=0}^{n}(8k+1)\binom{4k}{2k}\binom{2k}{k}^2 2^{8(n-k)}3^{2(n-k)}\equiv 0 \pmod{\binom{2n}{n}}, \tag{56}$$

$$\sum_{k=0}^{n}(8k+1)\binom{4k}{2k}\binom{2k}{k}^2 2^{8(n-k)}3^{2(n-k)}\equiv 0 \pmod{\binom{3n}{n}}, \tag{57}$$

$$\sum_{k=0}^{n}(8k+1)\binom{4k}{2k}\binom{2k}{k}^2 2^{8(n-k)}3^{2(n-k)}\equiv 0 \pmod{\binom{4n}{n}}, \tag{58}$$

$$\sum_{k=0}^{n}(8k+1)\binom{4k}{2k}\binom{2k}{k}^2 2^{8(n-k)}3^{2(n-k)}\equiv 0 \pmod{\binom{4n}{2n}}, \tag{59}$$

which we observe numerically, and whose proofs can be accessible to the WZ method. In view of the congruence

$$\sum_{k=0}^{n}(-1)^kq^{k^2}[4k+1]{\begin{bmatrix}2k\\k\end{bmatrix}}^3\frac{(-q;q)_n^6}{(-q;q)_k^6}\equiv 0 \pmod{(1+q^n)^2[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}}.$$

established in [8, Theorem 1.4] by the q-WZ method (see [13,15] for some other congruences related to q-binomial coefficients), we hypothesize the truth of the following q-analogues of (56) and (57).

Conjecture 5.1. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}\begin{bmatrix}4k\\2k\end{bmatrix}{\begin{bmatrix}2k\\k\end{bmatrix}}^2\frac{(-q;q)_n^4(-q;q)_{2n}^2(q^2;q^2)_k^2(q^6;q^6)_n^2}{(-q;q)_k^4(-q;q)_{2k}^2(q^2;q^2)_n^2(q^6;q^6)_k^2}[8k+1]q^{2k^2}\equiv 0 \pmod{\begin{bmatrix}2n\\n\end{bmatrix}},$$

$$\sum_{k=0}^{n}\begin{bmatrix}4k\\2k\end{bmatrix}{\begin{bmatrix}2k\\k\end{bmatrix}}^2\frac{(-q;q)_n^4(-q;q)_{2n}^2(q^2;q^2)_k^2(q^6;q^6)_n^2}{(-q;q)_k^4(-q;q)_{2k}^2(q^2;q^2)_n^2(q^6;q^6)_k^2}[8k+1]q^{2k^2}\equiv 0 \pmod{\begin{bmatrix}3n\\n\end{bmatrix}}.$$

The expression on the left-hand sides is clearly a polynomial in q, and it can also be written as

$$\frac{(-q;q)_n^4(-q;q)_{2n}^2(q^6;q^6)_n^2}{(q^2;q^2)_n^2}\sum_{k=0}^{n}\frac{(q;q^2)_k^2(q;q^2)_{2k}}{(q^2;q^2)_{2k}(q^6;q^6)_k^2}[8k+1]q^{2k^2}.$$

However, similar natural q-analogues of (58) and (59) do not hold in general.

As somewhat complementary to Theorem 4.2, we have the following collection of parametric congruences.

Conjecture 5.2. *Let d and n be positive integers with $n \equiv -1 \pmod{d}$. Then*

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(aq;q^d)_k(q/a;q^d)_k(bq;q^d)_k(q/b;q^d)_k}{(aq^d;q^d)_k(q^d/a;q^d)_k(bq^d;q^d)_k(q^d/b;q^d)_k}q^{(d-2)k} \equiv 0 \pmod{[n]}$$

and, for $d \neq 2$,

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(aq;q^d)_k(q/a;q^d)_k(q;q^d)_k^2}{(aq^d;q^d)_k(q^d/a;q^d)_k(q^d;q^d)_k^2}q^{(d-2)k} \equiv 0 \pmod{[n]\Phi_n(q)}.$$

Furthermore, for the particular case $d=2$, we also have a 'shorter' congruence

$$\sum_{k=0}^{(n-1)/2}[4k+1]\frac{(aq;q^2)_k(q/a;q^2)_k(bq;q^2)_k(q/b;q^2)_k}{(aq^2;q^2)_k(q^2/a;q^2)_k(bq^2;q^2)_k(q^2/b;q^2)_k} \equiv 0 \pmod{[n]}.$$

Because of

$$\sum_{k=0}^{n-1}[2k+1]q^{-k} = [n]^2q^{1-n},$$

Conjecture 5.2 is trivially true for $d=1$. The special case $d=2$ and $b=1$ of the conjecture is seen to be covered by Theorem 4.2.

Conjecture 5.3. *Let d and n be positive integers with $d \geq 3$ and $n \equiv -1 \pmod{d}$. Then*

$$\sum_{k=0}^{n-1}\frac{(a_1q;q^d)_k(a_2q;q^d)_k\cdots(a_dq;q^d)_kq^{dk}}{(a_1q^d;q^d)_k(a_2q^d;q^d)_k\cdots(a_dq^d;q^d)_k} \equiv 0 \pmod{\Phi_n(q)}$$

and

$$\sum_{k=0}^{n-1}\frac{(q;q^d)_k^dq^{dk}}{(q^d;q^d)_k^d} \equiv 0 \pmod{\Phi_n(q)^2}.$$

The congruences in Conjecture 5.3 do not hold in general when $d=2$. The conjecture comes with the following companion.

Conjecture 5.4. *Let d and n be positive integers with $d \geq 2$ and $n \equiv 1 \pmod{d}$. Then*

$$\sum_{k=0}^{n-1}\frac{(a_1/q;q^d)_k(a_2/q;q^d)_k\cdots(a_d/q;q^d)_kq^{dk}}{(a_1q^d;q^d)_k(a_2q^d;q^d)_k\cdots(a_dq^d;q^d)_k} \equiv 0 \pmod{\Phi_n(q)} \tag{60}$$

and

$$\sum_{k=0}^{n-1}\frac{(q^{-1};q^d)_k^dq^{dk}}{(q^d;q^d)_k^d} \equiv 0 \pmod{\Phi_n(q)^2}.$$

If $d=2$, then the congruence (60) *further holds modulo $[n]$.*

Another related entry of Conjecture 5.3 for $d=4$ is as follows.

Conjecture 5.5. *Let* $n \equiv 3 \pmod 4$ *be a positive integer. Then*

$$\sum_{k=0}^{n-1} \frac{(aq;q^4)_k (q/a;q^4)_k (q^2;q^4)_k q^{4k}}{(aq^4;q^4)_k (q^4/a;q^4)_k (q^4;q^4)_k} \equiv 0 \pmod{\Phi_n(q)},$$

$$\sum_{k=0}^{n-1} \frac{(q;q^4)_k^2 (q^2;q^4)_k q^{4k}}{(q^4;q^4)_k^3} \equiv 0 \pmod{\Phi_n(q)^2}.$$

The first author and Zeng [18, Corollary 1.2] give a q-analogue of the (H.2) supercongruence of Van Hamme [32]. In particular, they prove that

$$\sum_{k=0}^{(p-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k q^{2k}}{(q^2;q^2)_k^2 (q^4;q^4)_k} \equiv 0 \pmod{[p]^2} \quad \text{for any prime } p \equiv 3 \pmod 4.$$

We now provide a related a-parametric version of the congruence.

Conjecture 5.6. *Let* $n \equiv 3 \pmod 4$ *be a positive integer. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{(aq;q^2)_k (q/a;q^2)_k (q^2;q^4)_k q^{2k}}{(aq^2;q^2)_k (q^2/a;q^2)_k (q^4;q^4)_k} \equiv 0 \pmod{\Phi_n(q)}.$$

More generally, motivated by [18, Theorem 1.3], we believe that the following is true.

Conjecture 5.7. *Let* d*,* n *and* r *be positive integers with* $\gcd(d,n)=1$ *and* n *odd. If the least non-negative residue of* $-r/d$ *modulo* n *is odd, then*

$$\sum_{k=0}^{(n-1)/2} \frac{(aq^r;q^d)_k (q^{d-r}/a;q^d)_k (q^d;q^{2d})_k q^{dk}}{(aq^d;q^d)_k (q^d/a;q^d)_k (q^{2d};q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)}.$$

There are other classes of (super)congruences, in which truncated hypergeometric sums are compared with coefficients of modular forms. One notable example, again from Van Hamme's list [32, (M.2)], is the supercongruence

$$\sum_{k=0}^{p-1} \frac{(\frac12)_k^4}{k!^4} \equiv \sum_{k=0}^{(p-1)/2} \frac{(\frac12)_k^4}{k!^4} \equiv \gamma_p \pmod{p^3}$$

for primes $p>2$, where the right-hand side represents the p-th coefficient in the q-expansion $q\,(q^2;q^2)_\infty^4 (q^4;q^4)_\infty^4 = \sum_{n=1}^\infty \gamma_n q^n$ (of a modular form). The supercongruence was settled by T. Kilbourn [21] using p-adic methods. An obstacle to producing a suitable q-analogue is related to the coefficients γ_n (which already originate from a q-expansion!). However, the machinery of hypergeometric motives, in particular, a method due to B. Dwork, allows one to reduce the proof of the (M.2) supercongruence to verifying the congruences

$$S(p^{s+1}-1) \equiv S(p^s-1)S(p-1) \pmod{p^3} \tag{61}$$

for $s = 1$ and 2 (see [23, Section 2.1]), where $S(N)$ denotes the truncation of the hypergeometric sum

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^4}{n!^4}$$

at the N-th place. So far we could not figure out a q-analogue of the 'simpler' supercongruence (61), though we expect that the method in this note is adaptable to these settings as well.

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