STABILISATION BY NOISE ON THE BOUNDARY FOR A CHAFEE-INFANTE EQUATION WITH DYNAMICAL BOUNDARY CONDITIONS

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ABSTRACT. The stabilisation by noise on the boundary of the Chafee-Infante equation with dynamical boundary conditions subject to a multiplicative Itô noise is studied. In particular, we show that there exists a finite range of noise intensities that imply the exponential stability of the trivial steady state. This differs from previous works on the stabilisation by noise of parabolic PDEs, where the noise acts inside the domain and stabilisation typically occurs for an infinite range of noise intensities. To the best of our knowledge, this is the first result on the stabilisation of PDEs by boundary noise.

CONTENTS

1. INTRODUCTION

Let $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with smooth boundary ∂D and denote by $Q_T = D \times (0,T)$, $S_T = \partial D \times (0,T)$ for any $T > 0$. We consider the following stochastic Chafee-Infante equation with dynamical boundary conditions

$$\begin{cases} du + (-\Delta u + u^3 - \beta u)dt = 0 & \text{in } Q_T, \\ du + (\partial_\nu u + \lambda u)dt = \alpha u\, dW_t & \text{on } S_T, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,0) = \phi(x), & x \in \partial D, \end{cases} \tag{1}$$

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where $\beta > 0$, $\lambda > 0$ and $\alpha \in \mathbb{R}$ are constants, and ∂_ν denotes the outward normal derivative on ∂D. Moreover, W_t is a standard real-valued scalar Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$, dW_t denotes the Itô differential and $(u_0, \phi) \in L^2(D) \times L^2(\partial D)$ are given initial data.

The key feature of the model system (1) is the dynamical boundary condition in conjunction with the applied noise. There are many examples in the literature concerning partial differential equations with boundary noise which are similar to (1). Let us mention an incomplete list of references concerning the well-posedness [2, 4, 28], stability [3], or control [12, 25].

More generally, parabolic equations with dynamical boundary conditions are used to model heat conduction in solids, for example when a solid is in contact with a well-stirred fluid at its surface, see e.g. [7]. For more results on derivation and analysis of dynamical boundary conditions, we refer the interested reader to e.g., [13, 19].

The main goal of this paper is to analyse whether a *multiplicative Itô noise applied to* the boundary conditions will yield a stabilising effect for (1) compared to the deterministic problem $\alpha = 0$.

The stabilisation of PDEs by noise has been widely studied over the past decades, see e.g., [6, 9, 10, 20, 24] and the references therein, as well as the extensive review [8]. Typically, the results strongly depend on the choice of interpretation, i.e. whether the SPDE is interpreted in the sense of Itô or Stratonovich. Moreover, also the type of noise is important, where mainly perturbations by additive and multiplicative noise are considered. Most works focus on the effect of a multiplicative Itô noise, less results have been obtained for Stratonovich noise, or SPDEs with additive noise. Despite the extensive literature on the stabilisation of PDEs by noise, to the best of our knowledge, the stabilisation by a noise acting only on the boundary of a domain has not been addressed so far. This open problem and preliminary results were mentioned in [5, 8], but we are not aware of any published articles. This is the motivation for our work. As a first step in this direction we investigate the effect of a multiplicative Itô noise acting on the boundary of the domain on the Chafee-Infante equation with dynamical boundary conditions (1). More precisely, we derive sufficient conditions on the intensity of the noise $|\alpha|$, depending on β, λ and the geometry of the domain that imply exponential stability of the trivial steady state solution $u \equiv 0$.

Let us briefly describe the method and highlight the difficulties of the problem. With the use of classical Lebesgue- and Sobolev spaces $L^2(D)$, $H^1(D)$ and $H^{1/2}(\partial D)$ (see e.g. [1]), we introduce the Hilbert spaces

$$V_0 = H^1(D) \times H^{1/2}(\partial D), \qquad H = L^2(D) \times L^2(\partial D) \tag{2}$$

and

$$V = \{(u, T(u)) : u \in H^1(D)\}, \tag{3}$$

where $T(u) = u|_{\partial D}$ denotes the trace operator $T \in \mathcal{L}(H^1(D); H^{\frac{1}{2}}(\partial D))$. Then, $V \subset V_0$ is a closed vector subspace, the embedding $V \hookrightarrow H$ is dense and compact, and $V \hookrightarrow H \hookrightarrow V^*$

is a Gelfand triple, where V^* denotes the dual space of V. The bilinear form $a : V \times V \to \mathbb{R}$ is defined by

$$a(U, \Phi) = \int_D \nabla u(x) \cdot \nabla \varphi(x) dx + \lambda \int_{\partial D} u(x)|_{\partial D} \cdot \varphi|_{\partial D}(x) dS(x) \qquad \forall\, U, \Phi \in V, \tag{4}$$

where $U = (u, u|_{\partial D})$ and $\Phi = (\varphi, \varphi|_{\partial D})$. Since a is symmetric, continuous, positive and coercive (due to the Poincaré-Trace inequality (6), see Section 2), it defines a positive self-adjoint operator A with compact resolvent in H. Moreover, introducing the operators $B : L^4(D) \times L^2(\partial D) \to L^{\frac{4}{3}}(D) \times L^2(\partial D)$, $B(U) = (u^3 - \beta u, 0)$, and $C : H \to H$, $C(U) = (0, \alpha u)$, we can rewrite problem (1) in the abstract form

$$\begin{aligned} dU + (AU + B(U))dt = C(U)\, dW_t, \qquad \text{in } V^* + (L^{\frac{4}{3}}(D) \times L^2(\partial D)) \\ U(0) = (u_0, \phi) \in H. \end{aligned} \tag{5}$$

At this point, we remark that due to the "degeneracy" of the noise $C(U) = (0, \alpha u)$ which only acts on the second solution component, general results on the stabilisation by noise for abstract differential equations, e.g., see [10], are not directly applicable to (5). For the same reason, the stochastic system (1) can also not be transformed into a random PDE, which would allow to apply deterministic methods, as done, e.g. in [6] for the stochastic Chafee-Infante equation with homogeneous Dirichlet boundary conditions.

Instead, to resolve these technical issues, we shall refine the method in [10]. The following Poincaré-Trace Inequality will be essential for our analysis: For any $\theta > 0$, there exists an optimal constant $C_\theta^* > 0$ such that

$$\int_D |\nabla u(x)|^2 dx + \theta \int_{\partial D} |u(x)|^2 dS(x) \geqslant C_\theta^* \int_D |u(x)|^2 dx \qquad \forall u \in H^1(D). \tag{6}$$

The optimal constant C_θ^* is the first positive eigenvalue of the Laplacian $-\Delta$ in D with Robin boundary conditions $\partial_\nu u + \theta u = 0$ on ∂D. To our knowledge, an explicit formula for C_θ^* is unknown. We remark that even in the limit $\theta \to \infty$, the constant C_θ^* remains bounded above by, for instance, the Poincaré constant for functions $u \in H_0^1(D)$, i.e. $C_\theta^* \leqslant \lambda_1$, where λ_1 denotes the first eigenvalue of the Laplacian $-\Delta$ in D with homogeneous Dirichlet boundary conditions. In this work, in order to determine an explicit range of noise intensities for which stabilisation occurs, we derive an explicit expression for a sub-optimal constant C_θ fulfilling (6), which depends only on the dimension d and the diameter of the domain D (see Lemma 2.1).

The first main result of this paper is Theorem 3.2, which proves stabilisation by noise if the intensity $|\alpha|$ belongs to a specific and finite range. The proof uses the Poincaré-Trace inequality (6) to quantify the stabilising effect of the noisy dynamical boundary conditions to solutions of equation (1). Our approach requires that the constant β is strictly below a certain threshold. Though this restriction appears in our approach as a technical assumption, we conjecture due to the upper bound for the constant C_θ in (6) that there indeed exists a critical value β_{crit} such that problem (1) cannot be stabilised by noise on the boundary if $\beta > \beta_{\text{crit}}$.

This is in contrast to the literature on the stabilisation by noise of parabolic SPDEs, where stabilisation typically occurs for an infinite range of noise intensities, i.e. whenever $|\alpha|$ is sufficiently large. It is an interesting open problem, whether the finite range of stabilising noise intensity is a characteristic of the stabilisation by noise on the boundary, or it is due to the imposed dynamical boundary conditions, or a technical limitation of our method.

To further investigate this question, we compare the boundary noise problem (1) to the following Chafee-Infante equation with a multiplicative Itô noise acting *inside the domain* and subject to noise free dynamical boundary conditions

$$\begin{cases} du + (-\Delta u + u^3 - \beta u)dt = \alpha u\, dW_t & \text{in } Q_T, \\ du + (\partial_\nu u + \lambda u)dt = 0 & \text{in } S_T, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,0) = \phi(x), & x \in \partial D. \end{cases} \tag{7}$$

By applying a similar approach as for problem (1), we obtain again a finite range of noise intensities $|\alpha|$ that yields stabilisation of the equation. This might suggest that the dynamical boundary conditions *per se* prevent the stabilisation of the Chafee-Infante equation by noise with too large intensities (if the noise acts either on the boundary or inside the domain, but not simultaneously on both parts). Interestingly, for certain parameter regimes, our method of proof leads to a significant difference between the problems (7) and (1). Namely, in the case that $\lambda > \beta$ one can always stabilise equation (7) by noise with suitable intensity, no matter how large β is, whereas for problem (1) we can only show stabilisation for β below a critical value β_{crit}.

We remark that the parameter conditions of our results imply the *preservation of stability*, i.e. if the trivial steady state of the deterministic problem is stable, then, its stability is preserved under stochastic perturbations by noise with sufficiently small intensity. On the other hand, we also show *stabilisation by noise*, i.e. for parameter ranges for which the zero solution of the deterministic equation is unstable, adding noise on the boundary with an appropriate intensity leads to stabilisation.

To highlight this latter case, we analyse problem (1) in one dimension in more details in the last section of the paper. Firstly, we study the linearised equation around the zero steady state of the deterministic problem, where we can derive an explicit representation for the solution by separation of variables. Then, by choosing appropriate values for the parameters $\beta > 0$ and $\lambda > 0$ we show that the zero solution of the linearised equation is unstable. Since all the eigenvalues of the stationary problem have non-zero real part, an infinite dimensional version of the Hartman-Grobman theorem implies the instability of the zero solution of the nonlinear deterministic Chafee-Infante model. Finally, applying our main results on stabilisation by noise allows to determine an explicit range of noise intensities that stabilise equation (1).

Organisation of the paper: In Section 2, we prove the crucial functional inequality (6), derive an explicit representation for the constant C_θ and analyse the stability of the zero solution for the unperturbed deterministic problem. The main result on the stabilisation by noise on the boundary is established in Section 3. In particular, we determine an explicit range for the noise intensities for (1) that lead to stabilisation. Section 4 is devoted to problem (7), where the noise acts inside the domain, and the results on stabilisation are compared to the setting with boundary noise (1). In section (5), a one-dimensional example the instability of the deterministic problem and the stabilisation by noise on the boundary is detailed. And finally Section 6 is a conclusion.

2. Preliminaries

Notations: Here and in the sequel, we denote by $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$ the norms in $L^2(D)$ and $L^2(\partial D)$, respectively. The inner products in $L^2(D)$ and $L^2(\partial D)$ are denoted by $\langle\cdot,\cdot\rangle_D$ and $\langle\cdot,\cdot\rangle_{\partial D}$, and the norm in $L^p(D)$ for $p \neq 2$ by $\|\cdot\|_{p,D}$.

2.1. A Poincaré-Trace inequality.

The following inequality and the explicit expression for the corresponding constant play an important role in our analysis.

Lemma 2.1 (A Poincaré-Trace Inequality)**.** *Let $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with smooth boundary ∂D. For every $\theta > 0$, there exists an optimal constant $C_\theta^* > 0$ such that*

$$C_\theta^* \|u\|_D^2 \leqslant \|\nabla u\|_D^2 + \theta \|u\|_{\partial D}^2 \qquad \forall u \in H^1(D), \tag{8}$$

where C_θ^* is continuous and non-increasing with respect to θ, and $C_0^* = 0$.

Let $R = \frac{1}{2}\mathrm{diam}(D)$. Then the constant C_θ defined as

$$C_\theta = \begin{cases} \theta\,(d/R - \theta)\,, & \text{if } \theta \in \left[0, \frac{d}{2R}\right), \\ \frac{d^2}{4R^2}, & \text{if } \theta \in \left[\frac{d}{2R}, \infty\right), \end{cases} \tag{9}$$

also fulfils (8).

Remark 2.1.

(i) *For the rest of this paper, we denote by C_θ^* the optimal constant in (8) while C_θ is the explicit constant defined in (9).*

(ii) *Inequality (8) is well-known in functional analysis concerning equivalent norms in $H^1(D)$, see e.g., [27]. However, its proof is usually based on a contradiction argument and thus, does not yield an explicit expression nor a quantitative estimate for C_θ^*. C_θ^* is the first eigenvalue of the Laplace operator with Robin boundary conditions $\partial_\nu u + \theta u = 0$, see e.g. [15, 18]. In one dimension, one can solve the eigenvalue problem explicitly. Obtaining the optimal constant C_θ^* in higher dimensions goes beyond the scope of this paper.*

(iii) *As θ varies in $(0,\infty)$, C_θ^* always has an upper bound $C_{\theta,\max}^*$. Indeed, choosing $u \in H_0^1(D)$ it follows from Poincaré's inequality that $C_\theta^* \leqslant \lambda_1$ for any $\theta > 0$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian $-\Delta$ with homogeneous Dirichlet boundary conditions.*

(iv) *The explicit expression for C_θ in (9) allows us to specify ranges of noise intensities, depending on α, β, d and R, that stabilise the equation.*

Proof of Lemma 2.1. Consider the eigenvalue problem for the Laplacian with Robin boundary conditions

$$\begin{cases} -\Delta u = \kappa u, & \text{in } D, \\ \partial_\nu u + \theta u = 0, & \text{on } \partial D. \end{cases}$$

By classical results from the calculus of variations, the first eigenvalue is given by

$$C_\theta^* := \inf_{u \in H^1(D), u \neq 0} \frac{\|\nabla u\|_D^2 + \theta \|u\|_{\partial D}^2}{\|u\|_D^2},$$

it is positive and the corresponding first eigenfunction $\psi_1 \in H^1(\Omega)$ is positive. Then, certainly, (8) is satisfied. The continuity of C_θ^* with respect to θ follows from [14, Theorem 1] (C_θ^* is even differentiable w.r.t θ).

We now show that the constant C_θ defined in (9) also fulfills inequality (8). Since $2R = \mathrm{diam}(D)$ we can choose $x_0 \in \mathbb{R}^d$ such that $|x - x_0| \leqslant R$ for all $x \in \partial D$, where $|\cdot|$ denotes the norm in $\mathbb{R}^d$. We consider the function $\Phi(x) = \frac{|x-x_0|^2}{2d}$. Then, $\nabla\Phi(x) = \frac{x-x_0}{d}$ and $\Delta\Phi(x) = 1$. Let $\theta \in (0, \frac{d}{R})$. Then, using integration by parts and Young's inequality it follows that

$$\begin{aligned} \int_D u^2(x)dx &= \int_D u^2(x)\Delta\Phi(x)dx \\ &= \int_{\partial D} u^2(x)\partial_\nu\Phi(x)dS(x) - 2\int_D u(x)\nabla u(x)\cdot\nabla\Phi(x)dx \\ &\leqslant \|\partial_\nu\Phi\|_{\infty,\partial D}\|u\|_{\partial D}^2 + \|\nabla\Phi\|_{\infty,D}\left(\theta\|u\|_D^2 + \frac{1}{\theta}\|\nabla u\|_D^2\right). \end{aligned}$$

Hence, we obtain

$$\left(1 - \theta\|\nabla\Phi\|_{\infty,D}\right)\|u\|_D^2 \leqslant \|\partial_\nu\Phi\|_{\infty,\partial D}\|u\|_{\partial D}^2 + \frac{\|\nabla\Phi\|_{\infty,D}}{\theta}\|\nabla u\|_D^2,$$

which implies that

$$\theta\left(\frac{1}{\|\nabla\Phi\|_{\infty,D}} - \theta\right)\|u\|_D^2 \leqslant \theta\frac{\|\partial_\nu\Phi\|_{\infty,\partial D}}{\|\nabla\Phi\|_{\infty,D}}\|u\|_{\partial D}^2 + \|\nabla u\|_D^2.$$

Now, using that

$$\|\nabla\Phi\|_{\infty,D} = \frac{R}{d} \qquad \text{and} \qquad \|\partial_\nu\Phi\|_{\infty,\partial D} = \|\nabla\Phi\cdot\nu\|_{\infty,\partial D} \leqslant \frac{R}{d}$$

we obtain

$$\theta\left(\frac{d}{R} - \theta\right)\|u\|_D^2 \leqslant \|\nabla u\|_D^2 + \theta\|u\|_{\partial D}^2.$$

As required, $C_0 = 0$ and $C_\theta := \theta(\frac{d}{R} - \theta)$ is increasing for $\theta \in \left(0, \frac{d}{2R}\right]$. However, since $\theta(\frac{d}{R} - \theta)$ is decreasing within the interval $\left(\frac{d}{2R}, \frac{d}{R}\right)$, we observe that for $\theta \geqslant \frac{d}{2R}$

$$\|\nabla u\|_D^2 + \theta\|u\|_{\partial D}^2 \geqslant \|\nabla u\|_D^2 + \frac{d}{2R}\|u\|_{\partial D}^2 \geqslant \frac{d^2}{4R^2}\|u\|_D^2 \qquad \forall u \in H^1(D),$$

and hence, we set $C_\theta = \frac{d^2}{4R^2}$ for all $\theta \in \left[\frac{d}{2R}, \infty\right)$. Then, C_θ is non-decreasing and depends continuously on θ, which completes the proof. □

2.2. **The deterministic equation.** In this section, we consider the unperturbed deterministic Chafee-Infante equation with dynamical boundary conditions,

$$\begin{cases} u_t - \Delta u + u^3 - \beta u = 0 & \text{in } Q_T, \\ u_t + \partial_\nu u + \lambda u = 0 & \text{on } S_T, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,0) = \phi(x), & x \in \partial D, \end{cases} \tag{10}$$

and formulate sufficient conditions for the exponential stability of the trivial steady state.

As in the introduction we first rewrite problem (10) in an abstract form. To this end let

$$H = L^2(D) \times L^2(\partial D), \qquad V_0 = H^1(D) \times H^{1/2}(\partial D)$$

be as defined in the Introduction with the norms $\|\cdot\|_H^2 = \|\cdot\|_D^2 + \|\cdot\|_{\partial D}^2$ and $\|\cdot\|_{V_0}^2 = \|\cdot\|_{H^1(D)}^2 + \|\cdot\|_{H^{\frac{1}{2}}(\partial D)}^2$ and the inner product in H be denoted by $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle_D + \langle\cdot,\cdot\rangle_{\partial D}$. Moreover, let

$$V = \{(u, T(u)) : u \in H^1(D)\} \subset V_0,$$

with the norm induced by V_0, where $T \in \mathcal{L}(H^1(D); H^{\frac{1}{2}}(\partial D))$ denotes the trace operator $T(u) = u|_{\partial D}$. Then, V is a closed vector subspace of V_0, and densely and compactly embedded into H. Identifying H with its dual we have the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$, where V^* denotes the dual of V. Let $A : V \to V^*$ be the continuous linear operator defined by the symmetric, continuous bilinear form

$$a(U, \Phi) = \int_D \nabla u(x) \cdot \nabla \varphi(x) dx + \lambda \int_{\partial D} u(x)|_{\partial D} \varphi(x)|_{\partial D} dS(x) \qquad U, \Phi \in V,$$

where $U = (u, u|_{\partial D})$, $\Phi = (\varphi, \varphi|_{\partial D})$. The Poincaré-Trace inequality (8) implies that a is coercive and hence, by the Lax-Milgram Theorem A has a bounded inverse $A^{-1} : V^* \to V$. Its restriction to H is a compact bounded operator and its inverse is the operator $A : D(A) \to H$ with domain $D(A) = \{U \in V : AU \in H\}$. Moreover, Lemma 2.1 implies that A is positive,

$$\begin{aligned} \langle AU, U\rangle = a(U,U) &= \int_D |\nabla u(x)|^2 dx + \lambda \int_{\partial D} |u(x)|_{\partial D}|^2 dS(x) \\ &\geqslant C^*_{\frac{\lambda}{2}} \|u\|_D^2 + \frac{\lambda}{2}\|u\|_{\partial D}^2 \geqslant \min\left\{\frac{\lambda}{2}, C^*_{\frac{\lambda}{2}}\right\} \|U\|_H^2 \qquad \forall U \in D(A), \end{aligned}$$

where $C^*_{\frac{\lambda}{2}}$ is the optimal constant in (8). Hence, since A is a positive, self-adjoint operator with compact resolvent, there exists an orthonormal basis $\{V_j\} \subset D(A)$ in H consisting of eigenfunctions of A with corresponding eigenvalues λ_j such that

$$0 < \lambda_j \leqslant \lambda_{j+1}, \ j \in \mathbb{N}, \qquad \lambda_j \to \infty.$$

Defining $B : L^4(D) \times L^2(\partial D) \to L^{\frac{4}{3}}(D) \times L^2(\partial D)$, $B(U) = (u^3 - \beta u, 0)$, we rewrite (10) as

$$\begin{aligned} \frac{d}{dt}U + AU + B(U) &= 0, \quad \text{ in } V^* + (L^{\frac{4}{3}}(D) \times L^2(\partial D)), \\ U(0) = U_0 &= (u_0, \phi) \in H. \end{aligned} \tag{11}$$

With an abuse of notation, in the sequel we will drop the amendment $|_{\partial D}$ in the second component of U, where the value of the function on the boundary is taken. We observe that

$$\begin{aligned} \langle AU, U\rangle &= \|\nabla u\|_D^2 + \lambda\|u\|_{\partial D}^2, & U \in D(A), \\ \langle B(U), U\rangle &= \|u\|_{4,D}^4 - \beta\|u\|_D^2, & U \in V. \end{aligned} \tag{12}$$

Theorem 2.1. *For any initial data $(u_0, \phi) \in L^2(D) \times L^2(\partial D)$ and $T > 0$, there exists a unique weak solution u of (10), and*

$$u \in C([0,T]; L^2(D)) \cap L^2([0,T]; H^1(D)) \cap L^4([0,T]; L^4(D)),$$
$$u|_{\partial D} \in C([0,T]; L^2(\partial D)) \cap L^2([0,T]; H^{1/2}(\partial D)).$$

Moreover, if λ and β are such that

$$C_\lambda^* > \beta,$$

then the zero steady state is exponentially stable. On the other hand, if

$$C_\lambda^* < \beta$$

the zero steady state is unstable.

Proof. The existence and uniqueness of solutions follows as, e.g. in [11], pp. 824–825, see also [21, Theorem 1.4] and the subsequent remark.

To show the exponential stability of the zero steady state, we compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_D^2 &= \int_D u(x)u_t(x)dx = \int_D u(x)(\Delta u(x) - u^3(x) + \beta u(x))dx \\ &= -\|\nabla u\|_D^2 + \int_{\partial D} u(x)\partial_\nu u(x)dS(x) - \|u\|_{4,D}^4 + \beta\|u\|_D^2 \\ &= -\|\nabla u\|_D^2 - \frac{1}{2}\frac{d}{dt}\|u\|_{\partial D}^2 - \lambda\|u\|_{\partial D}^2 - \|u\|_{4,D}^4 + \beta\|u\|_D^2, \end{aligned}$$

and therefore,

$$\frac{1}{2}\frac{d}{dt}(\|u\|_D^2 + \|u\|_{\partial D}^2) \leqslant -(\|\nabla u\|_D^2 + \lambda\|u\|_{\partial D}^2 - \beta\|u\|_D^2).$$

Since $C_\lambda^* > \beta$ and C_λ^* depends continuously on λ (see Lemma 2.1), there exists $\varepsilon > 0$ such that $C_{\lambda-\varepsilon}^* > \beta$. Thus, we obtain the estimate

$$\|\nabla u\|_D^2 + \lambda\|u\|_{\partial D}^2 = \varepsilon\|u\|_{\partial D}^2 + (\|\nabla u\|_D^2 + (\lambda - \varepsilon)\|u\|_{\partial D}^2) \geqslant \varepsilon\|u\|_{\partial D}^2 + C_{\lambda-\varepsilon}^*\|u\|_D^2,$$

and it follows that

$$\frac{1}{2}\frac{d}{dt}(\|u\|_D^2 + \|u\|_{\partial D}^2) \leqslant -\varepsilon\|u\|_{\partial D}^2 - (C_{\lambda-\varepsilon}^* - \beta)\|u\|_D^2 = -\delta(\|u\|_D^2 + \|u\|_{\partial D}^2),$$

where $\delta = \min\{\varepsilon, C^*_{\lambda-\varepsilon} - \beta\} > 0$. Finally, we get

$$\|u(t)\|_D^2 + \|u(t)\|_{\partial D}^2 \leqslant e^{-2\delta t}(\|u_0\|_D^2 + \|\phi\|_{\partial D}^2) \qquad \forall t \geqslant 0,$$

which proves the exponential stability of the zero steady state.

To prove instability, we use Kaplan's method. It suffices to show the instability for the linearised problem. By an infinite dimensional Hartman-Grobman theorem, see e.g. [23] or [17, Corollary 5.1.6], it then follows the instability for the nonlinear problem. Since $\beta > C^*_\lambda$ and C^*_λ depends continuously on λ, we can choose some $\theta > \lambda$ such that $\beta > C^*_\theta$. Denote by ψ_1 the positive eigenfunction corresponding to the first eigenvalue C^*_θ, i.e. $-\Delta\psi_1 = C^*_\theta \psi_1$ in D and $\partial_\nu \psi_1 + \theta\psi_1 = 0$ on ∂D. Testing the linear equation

$$\begin{cases} u_t - \Delta u - \beta u = 0, & \text{in } D, \\ u_t + \partial_\nu u + \lambda u = 0, & \text{on } \partial D \end{cases}$$

with ψ_1 and integrating by parts, we obtain

$$\begin{aligned} \frac{d}{dt}\int_D u\psi_1 dx &= \int_D \psi_1(\Delta u + \beta u)dx \\ &= -\frac{d}{dt}\int_{\partial D} u\psi_1 dS + \beta\int_D u\psi_1 dx - \lambda\int_{\partial D} u\psi_1 dS + \int_D u\Delta\psi_1 dx + \theta\int_{\partial D} u\psi_1 dS. \end{aligned}$$

By using $\Delta\psi_1 = -C^*_\theta\psi_1$, we have

$$\begin{aligned} \frac{d}{dt}\left(\int_D u\psi_1 dx + \int_{\partial D} u\psi_1 dS\right) &= (\beta - C^*_\theta)\int_D u\psi_1 dx + (\theta - \lambda)\int_{\partial D} u\psi_1 dS \\ &\geqslant \min\{\beta - C^*_\theta; \theta - \lambda\}\left(\int_D u\psi_1 dx + \int_{\partial D} u\psi_1 dS\right), \end{aligned}$$

which implies the exponential growth of $\int_D u\psi_1 dx + \int_{\partial D} u\psi_1 dS$, and consequently the instability of the zero steady state.

□

3. Stabilisation by noise on the boundary

In this section, we formulate an existence result for strong solutions of the stochastic Chafee-Infante equation (1), and investigate whether the equation can be stabilised by noise on the boundary.

As in the introduction we first rewrite problem (1) in the abstract form

$$\begin{aligned} dU + (AU + B(U))dt &= C(U)\, dW_t, \\ U(0) = U_0 &= (u_0, \phi) \in H, \end{aligned} \tag{13}$$

where the operators A and B were defined in the previous section, and $C : H \to H$ is given by $C(U) = (0, \alpha u)$. We will frequently use the identities (12) and

$$\langle C(U), U\rangle = \alpha\|u\|_{\partial D}^2, \qquad U \in H.$$

The existence and uniqueness of strong solutions of (13) follows, e.g., from a slight modification of [26, Theorem 4.1].

Theorem 3.1. *Let $T > 0$ and assume that the initial data $U_0 = (u_0, \phi)$ is an H-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|U_0\|_H^2 < \infty$. Then, there exists a unique strong solution of (13) in $C([0,T];H)$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\|U(t)\|_H^2\Big) < \infty \quad \text{and} \quad \mathbb{E}\Big(\int_0^T \|U(t)\|_V^2 dt\Big) < \infty.$$

As shown in Section 2.2, in the deterministic case the zero steady state is exponentially stable provided that $\beta < C_\lambda^*$, while the stability is lost for $\beta > C_\lambda^*$. We now investigate the stability of the zero solution when the system is perturbed by a multiplicative Itô noise $\alpha u\, dW_t$ on the boundary. In particular, we show that if $\beta < C_\lambda^*$ the exponential stability of the zero steady state is preserved for small noise intensities $|\alpha|$ and, if $\beta \geqslant C_\lambda^*$, the zero steady state can be stabilised by noise on the boundary for certain parameter ranges of λ and β.

Our main result is the following, it yields sufficient conditions for the exponential stability of the trivial steady state of the stochastic problem (13).

Theorem 3.2. *Let C_θ^* denote the optimal constant in Lemma 2.1 and the initial data $U_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$ be such that $\|U_0\|_H \neq 0$ P-a.s..*

If there exists a constant $\theta > 0$ such that either

$$C_\theta^* - \beta > \theta - \lambda > 0,$$

or

$$C_\theta^* > \beta \quad \text{and} \quad \lambda \geqslant \theta,$$

then, the solution of (13) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log\|U(t)\|_H^2 < 0 \qquad P\text{-a.s.}$$

for all noise intensities $|\alpha|$ such that

$$\frac{\alpha^2}{2} \in \begin{cases} [\max\{0, \theta-\lambda\}, Z_1) & \text{if } C_\theta^* - \beta > 2(\theta - \lambda), \\ (Z_1, Z_2) & \text{if } 2(\theta-\lambda) \geqslant C_\theta^* - \beta > \theta - \lambda, \end{cases}$$

where

$$Z_{1,2} := 3(C_\theta^* - \beta) + \lambda - \theta \mp 2\sqrt{2(C_\theta^* - \beta)(C_\theta^* - \beta + \lambda - \theta)}$$

with Z_1 and Z_2 are corresponding to the sign $-$ and $+$ respectively.

Proof. We will apply Itô's formula (e.g., see [26]) to $\psi(U) = \log\|U\|_H^2$, where we assume (w.l.o.g.) that $\|U\|_H^2 \neq 0$ since otherwise $\|U\|_H = 0$ which implies $u = 0$ a.e. in D and thus no further stabilisation is needed. In applying Itô's formula, we observe that the Fréchet derivatives ψ' and ψ'' can be expressed as

$$\psi'(U) = \frac{2\langle U, \cdot\rangle}{\|U\|_H^2} \quad \text{and} \quad \psi''(U)h = \frac{2\langle h, \cdot\rangle}{\|U\|_H^2} - \frac{4\langle U, h\rangle\langle U, \cdot\rangle}{\|U\|_H^4}, \qquad U, h \in H.$$

Hence, we obtain

$$\begin{aligned}\log\|U(t)\|_H^2 &= \log\|U(0)\|_H^2 - 2\int_0^t \frac{1}{\|U\|_H^2}\left(\langle AU+B(U),U\rangle - \frac{1}{2}\|C(U)\|_H^2\right)ds\\ &\quad - 2\int_0^t \frac{\langle C(U),U\rangle^2}{\|U\|_H^4}ds + 2\int_0^t \frac{\langle C(U),U\rangle}{\|U\|_H^2}dW_s\\ &= \log\|U(0)\|_H^2 - 2\int_0^t \frac{\|\nabla u\|_D^2 + \lambda\|u\|_{\partial D}^2 + \|u\|_{4,D}^4 - \beta\|u\|_D^2 - \frac{1}{2}\alpha^2\|u\|_{\partial D}^2}{\|U\|_H^2}ds\\ &\quad - 2\int_0^t \frac{\langle C(U),U\rangle^2}{\|U\|_H^4}ds + 2\int_0^t \frac{\langle C(U),U\rangle}{\|U\|_H^2}dW_s.\end{aligned} \tag{14}$$

In order to estimate the stochastic integral we apply the exponential martingale inequality (see e.g., [22, Lemma 1.1]) and obtain

$$P\left\{\omega : \sup_{0\leqslant t\leqslant T}\left[\int_0^t \frac{\langle C(U),U\rangle}{\|U\|_H^2}dW_s - \frac{\kappa}{2}\int_0^t \frac{\langle C(U),U\rangle^2}{\|U\|_H^4}ds\right] > \frac{2\log k}{\kappa}\right\} \leqslant \frac{1}{k^2}, \tag{15}$$

where $\kappa \in (0,1)$ and $k \in \mathbb{N}$. Now, the Borel-Cantelli lemma implies that there exists $k_0(\omega) > 0$ for almost all $\omega \in \Omega$ such that

$$\int_0^t \frac{\langle C(U),U\rangle}{\|U\|_H^2}dW_s \leqslant \frac{\kappa}{2}\int_0^t \frac{\langle C(U),U\rangle^2}{\|U\|_H^4}ds + \frac{2\log k}{\kappa} = \frac{\kappa}{2}\int_0^t \frac{\alpha^2\|u\|_{\partial D}^4}{\|U\|_H^4}ds + \frac{2\log k}{\kappa}$$

for all $k \geqslant k_0(\omega)$. Inserting this estimate into (14) yields

$$\begin{aligned}&\log\|U(t)\|_H^2\\ &\leqslant \log\|U(0)\|_H^2 + \frac{4\log k}{\kappa} - (2-\kappa)\int_0^t \frac{\alpha^2\|u\|_{\partial D}^4}{\|U\|_H^4}ds\\ &\quad - 2\int_0^t \frac{\|\nabla u\|_D^2 + \lambda\|u\|_{\partial D}^2 + \|u\|_{4,D}^4 - \beta\|u\|_D^2 - \frac{1}{2}\alpha^2\|u\|_{\partial D}^2}{\|U\|_H^2}ds\\ &\leqslant \log\|U(0)\|_H^2 + \frac{4\log k}{\kappa} + 2\beta t\\ &\quad - \int_0^t \frac{(2\|\nabla u\|_D^2 + 2\lambda\|u\|_{\partial D}^2 + 2\beta\|u\|_{\partial D}^2 - \alpha^2\|u\|_{\partial D}^2)\|U\|_H^2 + (2-\kappa)\alpha^2\|u\|_{\partial D}^4}{\|U\|_H^4}ds.\end{aligned} \tag{16}$$

It is now sufficient to show that there exists a range of $\alpha \in \mathbb{R}$ such that

$$(2\|\nabla u\|_D^2 + 2\lambda\|u\|_{\partial D}^2 + 2\beta\|u\|_{\partial D}^2 - \alpha^2\|u\|_{\partial D}^2)\|U\|_H^2 + (2-\kappa)\alpha^2\|u\|_{\partial D}^4 > 2\beta\|U\|_H^4. \tag{17}$$

Indeed, if (17) holds, then (16) implies that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|U(t)\|_H^2 < \limsup_{t\to\infty}\frac{1}{t}\left(\log\|U(0)\|_H^2 + \frac{4\log k}{\kappa}\right) = 0,$$

which is the statement of the theorem.

We now prove (17). By Lemma 2.1 for every $\theta > 0$ there exists an optimal constant C_θ^* such that

$$\|\nabla u\|_D^2 \geqslant C_\theta^* \|u\|_D^2 - \theta \|u\|_{\partial D}^2,$$

and hence, we obtain a lower bound for left hand side of (17)

$$\begin{aligned}
&(2\|\nabla u\|_D^2 + 2\lambda\|u\|_{\partial D}^2 + 2\beta\|u\|_{\partial D}^2 - \alpha^2\|u\|_{\partial D}^2)\|U(s)\|_H^2 + (2-\kappa)\alpha^2\|u\|_{\partial D}^4\\
&\geqslant (2C_\theta^*\|u\|_D^2 + (2\lambda + 2\beta - \alpha^2 - 2\theta)\|u\|_{\partial D}^2)(\|u\|_D^2 + \|u\|_{\partial D}^2) + (2-\kappa)\alpha^2\|u\|_{\partial D}^4\\
&= 2C_\theta^*\|u\|_D^4 + (2C_\theta + 2\lambda + 2\beta - \alpha^2 - 2\theta)\|u\|_D^2\|u\|_{\partial D}^2 + (2\lambda + 2\beta - 2\theta + (1-\kappa)\alpha^2)\|u\|_{\partial D}^4.
\end{aligned}$$

Setting $X = \|u\|_D^2$ and $Y = \|u\|_{\partial D}^2$, we see that (17) holds if we can find α and θ fulfilling

$$2C_\theta^* X^2 + (2C_\theta^* + 2\lambda + 2\beta - \alpha^2 - 2\theta)XY + (2\lambda + 2\beta - 2\theta + (1-\kappa)\alpha^2))Y^2 > 2\beta(X+Y)^2,$$

or equivalently,

$$2(C_\theta^* - \beta)X^2 + (2C_\theta^* + 2\lambda - 2\beta - \alpha^2 - 2\theta)XY + (2\lambda - 2\theta + (1-\kappa)\alpha^2)Y^2 > 0$$

for all $X, Y > 0$. Moreover, since $\kappa \in (0,1)$ was arbitrary, it suffices that

$$2(C_\theta^* - \beta)X^2 + (2C_\theta^* + 2\lambda - 2\beta - \alpha^2 - 2\theta)XY + (2\lambda - 2\theta + \alpha^2)Y^2 > 0$$

for all $X, Y > 0$, which is equivalent to the condition

$$2(C_\theta^* - \beta)Z^2 + (2C_\theta^* + 2\lambda - 2\beta - \alpha^2 - 2\theta)Z + (2\lambda - 2\theta + \alpha^2) > 0 \tag{18}$$

for all $Z > 0$. To simplify notations we introduce

$$A = C_\theta^* - \beta \quad \text{and} \quad B = \theta - \lambda,$$

and rewrite (18) as

$$2AZ^2 + (2A - 2B - \alpha^2)Z + \alpha^2 - 2B > 0 \qquad \forall Z > 0. \tag{19}$$

Since $aX^2 + bX + c > 0 \ \forall X > 0$ if and only if

$$a > 0, \quad c \geqslant 0 \quad \text{and} \quad b \geqslant -2\sqrt{ac},$$

(19) is equivalent to the following set of conditions

$$2A > 0, \tag{20}$$

$$\alpha^2 - 2B \geqslant 0, \tag{21}$$

$$2A - 2B - \alpha^2 > -2\sqrt{2A(\alpha^2 - 2B)}. \tag{22}$$

- If $A > 2B$ we distinguish two different cases.
  - If $\alpha^2 < 2A - 2B$ then (22) is automatically satisfied. Hence, together with (21) we obtain the following admissible range for α

$$B \leqslant \frac{\alpha^2}{2} < A - B.$$

– If $\alpha^2 \geqslant 2A - 2B$, then taking the square of both sides of (22) implies that

$$\alpha^4 - 4(3A - B)\alpha^2 + 4(A + B)^2 < 0. \tag{23}$$

A necessary condition for the existence of a solution α is, that the discriminant is positive, i.e.

$$(-2(3A - B))^2 - 4(A + B)^2 = 32A(A - B) > 0,$$

which is true due to $A > 2B$ and (20). Hence, we solve (23) and get

$$Z_1 < \frac{\alpha^2}{2} < Z_2,$$

where

$$Z_{1,2} := 3A - B \mp 2\sqrt{2A(A - B)}. \tag{24}$$

It is easy to check that $B < Z_1 < A - B$ and $Z_2 > A - B$. Therefore, in the case $A > 2B$ the admissible range of noise intensities determined by (20)–(22) is

$$B \leqslant \frac{\alpha^2}{2} < Z_2.$$

- If $A \leqslant 2B$, then (21) implies that $\alpha^2 \geqslant 2B \geqslant A + (A - 2B) = 2(A - B)$. Moreover, taking the square of both sides of (22) we again obtain inequality (23). Necessary for the existence of a solution α is that $A > B$, and hence, it follows that

$$Z_1 < \frac{\alpha^2}{2} < Z_2,$$

with $Z_{1,2}$ defined in (24).

In conclusion, from (17) we derive the following admissible ranges for the noise intensity $|\alpha|$:

- If $\quad A > 2B \quad$ then $\quad B \leqslant \dfrac{\alpha^2}{2} < Z_2.$
- If $\quad 2B \geqslant A > B \quad$ then $\quad Z_1 < \dfrac{\alpha^2}{2} < Z_2.$

This concludes the proof of Theorem 3.2. □

Theorem 3.2 provides sufficient conditions for the stabilisation by noise on the boundary. Whether for a given domain D and parameters β and λ a suitable θ exists crucially depends on the optimal constant C_θ^*. However, since C_θ^* is not explicitly known, these conditions are hard to verify. We remark that Theorem 3.2 remains valid if C_θ^* is replaced by another constant C_θ satisfying (8). Hence, in the following we use the expression for the constant C_θ in Lemma 2.1 to derive explicit conditions for the exponential stability of the zero solution that are determined by α, β, d and the diameter of the domain D. We recall that in Lemma 2.1 for every $\theta \geqslant 0$ the constant C_θ is given by

$$C_\theta = \begin{cases} \theta\left(d/R - \theta\right) & \text{if } \theta \in \left[0, \frac{d}{2R}\right), \\ \frac{d^2}{4R^2} & \text{if } \theta \in \left[\frac{d}{2R}, \infty\right), \end{cases}$$

where $R > 0$ is such that $2R = \mathrm{diam}(D)$. Note that $C_\theta \leqslant d^2/4R^2$ for all $\theta \geqslant 0$.

Theorem 3.3.

(a) *Persistence of stability: If $\beta < C_\lambda$, the zero steady state is exponentially stable for all noise intensities $|\alpha|$ such that*

$$\frac{\alpha^2}{2} \in \left[0, (3+2\sqrt{2})(C_\lambda - \beta)\right).$$

(b) *Stabilisation by noise: In case $\beta \geqslant C_\lambda$, if*

$$0 < \lambda < \frac{1}{2}(d/R - 1) \quad \text{and} \quad \beta < \frac{1}{4}(d/R-1)^2 + \lambda, \tag{25}$$

then there exists θ such that

$$C_\theta - \beta > \theta - \lambda > 0, \tag{26}$$

and consequently, the zero solution of (1) is exponentially stable for all noise intensities $|\alpha|$ such that

$$\frac{\alpha^2}{2} \in \begin{cases} [\theta - \lambda, Z_2) & \text{if } C_\theta - \beta > 2(\theta - \lambda), \\ (Z_1, Z_2) & \text{if } 2(\theta - \lambda) \geqslant C_\theta - \beta, \end{cases}$$

where $Z_{1,2}$ are defined in (24) with C_θ in place of C_θ^*.

Remark 3.1. *For the condition (25) to hold it is necessary that $d > R$, which means that we can only show stabilisation if the diameter of the domain is not too large in comparison to the dimension.*

Proof. For (a) we can choose $\theta = \lambda$ in Theorem 3.2, and replace C_θ^* by C_θ to obtain the desired range of α.

From Theorem 3.2, again with C_θ in place of C_θ^*, we know that if (26) is satisfied then the equation can be stabilised. To ensure that the set of parameters λ and β satisfying (25) is not empty, we first observe that (25) implies that $\beta < \frac{d^2}{4R^2}$. Moreover, we need that

$$\frac{1}{4}(d/R-1)^2 + \lambda > C_\lambda = \lambda(d/R - \lambda),$$

since $\lambda < \frac{1}{2}(d/R-1) < d/2R$. But this condition is equivalent to $(\lambda - \frac{1}{2}(d/R-1))^2 > 0$, which is obviously true due to (25).

Our goal now is to show that the hypotheses (25) imply the existence of some $\theta \leqslant d/2R$ satisfying (26). Indeed, since $\theta \leqslant d/2R$, we use that $C_\theta = \theta(d/R - \theta)$ by Lemma 2.1 and rewrite the condition (26) as

$$\begin{cases} d/2R \geqslant \theta > \lambda, \\ \theta^2 + (1 - d/R)\theta + \beta - \lambda < 0. \end{cases}$$

This system is equivalent to

$$d/2R \geqslant \theta > \lambda \quad \text{and} \quad \frac{d/R - 1 - \sqrt{\Psi}}{2} < \theta < \frac{d/R - 1 + \sqrt{\Psi}}{2},$$

where

$$\Psi := (1 - d/R)^2 - 4(\beta - \lambda) > 0$$

due to the condition on β in (25). It is obvious that

$$\frac{d}{2R} > \frac{d/R - 1 - \sqrt{\Psi}}{2}.$$

Moreover, by (25) it also follows that

$$\lambda < \frac{d/R - 1 + \sqrt{\Psi}}{2}.$$

Therefore, (26) is satisfied for all θ within the interval

$$\max\left\{\lambda; \frac{d/R - 1 + \sqrt{\Psi}}{2}\right\} < \theta < \min\left\{\frac{d}{2R}; \frac{d/R - 1 + \sqrt{\Psi}}{2}\right\},$$

which proves part (b). □

In Section 5 we consider a concrete one-dimensional example for the stabilisation by noise in (b), where the trivial solution of the deterministic problem is unstable, but adding noise on the boundary with appropriate intensity leads to stabilisation.

Discussion.

- The results of Theorem 3.3 can be interpreted as follows:
  - If $\beta < C_\lambda \leqslant C_\lambda^*$, the zero steady state of the deterministic equation is exponentially stable due to Theorem 2.1. Theorem 3.3 shows that the stability is preserved for the stochastic problem, if the intensity of the noise is not too large; more precisely, for $\frac{\alpha^2}{2} \in [0, (3 + 2\sqrt{2})(C_\lambda - \beta))$.
  - If $\beta \geqslant C_\lambda$, the zero solution of the unperturbed deterministic problem might be unstable, while for $\beta > C_\lambda^*$ the zero solution is unstable by Theorem 2.1. In this case, Theorem 3.3 shows that for sufficiently small domains and the parameter ranges (25), adding noise with suitable intensity implies stability. In fact, by choosing $\theta > 0$ such that (26) holds, the zero solution of the stochastic problem is exponentially stable if the intensity of the noise is within the stated range. In this case, the lower bound for the noise intensity is strictly positive.
- These results are essentially different from related results in the literature on the stabilisation of parabolic PDEs by noise. In case of homogeneous Dirichlet boundary conditions and a multiplicative Itô noise $\alpha u\, dW_t$ acting inside the domain, the noise typically helps to stabilise the system. More precisely, the zero solution is exponentially stable for all sufficiently large intensities $|\alpha|$ (see e.g., [6, 9, 20, 10]).

  With the current technique, we are able to prove the stabilisation by noise on the boundary only for a finite range of α. Whether it is always possible to stabilise equation (1), or whether a noise with higher intensity destroys the stability of the zero steady state remain interesting open questions.

- The conditions in Theorem 3.3 essentially depend on the constant C_θ defined in Lemma 2.1, and thus, implicitly depend on the geometry of the domain D. These conditions are obviously not optimal since $C_\theta \leqslant C_\theta^*$. Nevertheless, Lemma 2.1 yields an explicit expression for the constant C_θ which allows to specify parameter ranges for λ and β for which stabilisation by noise can be shown. These ranges increase with decreasing diameter of the domain, or in other words, it is easier to stabilise by boundary noise as the domain gets smaller.
- If $\beta \geqslant C^*_{\theta,\max}$ (see Remark 2.1 (iii)), the hypotheses of Theorem 3.2 are never satisfied, and our method of proof does not apply. However, due to the existence of an upper bound for C_θ^*, we conjecture that there is indeed a threshold value β_{crit} so that (1) cannot be stabilised by noise on the boundary if $\beta \geqslant \beta_{\text{crit}}$.

  Conjecture. *There exists $\beta_{\text{crit}} > 0$ such that if $\beta \geqslant \beta_{\text{crit}}$, the zero solution of (14) is unstable for any $\lambda > 0$ and $\alpha \in \mathbb{R}$.*

4. Stabilisation by noise inside the domain

We now analyse whether the results on stabilisation change if the noise acts inside the domain for noise free dynamical boundary conditions. Using the same notations as in the previous section we consider the following stochastic Itô problem

$$\begin{cases} du + (-\Delta u + u^3 - \beta u)dt = \alpha u\, dW_t & \text{in } Q_T, \\ du + (\partial_\nu u + \lambda u)dt = 0 & \text{on } S_T, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,0) = \phi(x), & x \in \partial D. \end{cases} \tag{27}$$

Similarly to (13), it can be rewritten in the abstract form

$$\begin{cases} dU + (AU + B(U))dt = \widetilde{C}(U)dW_t, \\ U(0) = U_0 = (u_0, \phi) \in H, \end{cases} \tag{28}$$

where A and B were defined in the beginning of Section 3. The stochastic perturbation $\widetilde{C} : H \to H$ is given by $\widetilde{C}(U) = (\alpha u, 0)$, i.e., it acts on the first component of the solution, and different from (3) we now have

$$\langle \widetilde{C}(U), U\rangle = \alpha \|u\|_D^2, \qquad \forall U \in H.$$

The existence of strong solutions follows again from [26, Theorem 4.1].

Theorem 4.1. *Let $T > 0$ and assume that the initial data $U_0 = (u_0, \phi)$ is an H-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|U_0\|_H^2 < \infty$. Then, there exists a unique strong solution of (28) in $C([0,T]; H)$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|U(t)\|_H^2\Big) < \infty \quad \textit{and} \quad \mathbb{E}\Big(\int_0^T \|U(t)\|_V^2 dt\Big) < \infty.$$

Theorem 4.2. *Let C_θ^* denote the optimal constant in Lemma 2.1 and the initial data $U_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$ be such that $\|U_0\|_H \neq 0$ P-a.s..*

Assume that there exists $\theta > 0$ such that

$$\lambda - \theta > \beta - C_\theta^* > 0, \tag{29}$$

or

$$\lambda > \theta \quad \text{and} \quad C_\theta^* \geqslant \beta.$$

Then, the solution $U(t)$ of (28) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log \|U(t)\|_H^2 < 0 \qquad P\text{-a.s.}$$

for all noise intensities $|\alpha|$ such that

$$\frac{\alpha^2}{2} \in \begin{cases} [\max\{0, \beta - C_\theta^*\}, T_2) & \text{if} \quad \lambda - \theta > 2(\beta - C_\theta^*), \\ (T_1, T_2) & \text{if} \quad 2(\beta - C_\theta^*) \geqslant \lambda - \theta > \beta - C_\theta^*, \end{cases}$$

where

$$T_{1,2} := C_\theta^* - \beta + 3(\lambda - \theta) \mp 2\sqrt{2(\lambda - \theta)(\lambda - \theta + C_\theta^* - \beta)}.$$

Proof. Following the same arguments as in the proof of Theorem 3.2, by applying the Itô formula to $\log \|U(t)\|_H^2$ and using the exponential martingale inequality we obtain

$$\begin{aligned} &\log \|U(t)\|_H^2 \\ &\leqslant \log \|U(0)\|_H^2 + 4\frac{\log k}{\kappa} - \int_0^t \frac{2\langle AU + B(U), U\rangle - \|\widetilde{C}(U)\|_H^2}{\|U\|_H^2} ds - (2-\kappa)\int_0^t \frac{\langle \widetilde{C}(U), U\rangle^2}{\|U\|_H^4} ds \\ &\leqslant \log \|U(0)\|_H^2 + 4\frac{\log k}{\kappa} \\ &\quad + \int_0^t \frac{2\|U\|_H^2(-\|\nabla u\|_D^2 - \lambda\|u\|_{\partial D}^2 + \beta\|u\|_D^2) + \alpha^2\|u\|_D^2\|U\|_H^2 - (2-\kappa)\alpha^2\|u\|_D^4}{\|U\|_H^4} ds. \end{aligned}$$

Now, we need that

$$2\|U\|_H^2(-\|\nabla u\|_D^2 - \lambda\|u\|_{\partial D}^2 + \beta\|u\|_D^2) + \alpha^2\|u\|_D^2\|U\|_H^2 - (2-\kappa)\alpha^2\|u\|_D^4 < 0,$$

where $\kappa \in (0,1)$. Since $\kappa \in (0,1)$ is arbitrary, and using the functional inequality $\|\nabla u\|_D^2 + \theta\|u\|_{\partial D}^2 \geqslant C_\theta^*\|u\|_D^2$ in Lemma 2.1, it is sufficient that

$$[2(C_\theta^* - \beta) + \alpha^2]\|u\|_D^4 + [2(C_\theta^* - \beta + \lambda - \theta) - \alpha^2]\|u\|_D^2\|u\|_{\partial D}^2 + 2(\lambda - \theta)\|u\|_{\partial D}^4 > 0. \tag{30}$$

To shorten notations we set $A = C_\theta^* - \beta$ and $B = \lambda - \theta$. As in the proof of Theorem 3.2 we conclude that (30) is equivalent to the set of conditions

$$\begin{aligned} B &> 0, \\ \alpha^2 + 2A &\geqslant 0, \\ 2(A + B) - \alpha^2 &> -2\sqrt{2B(\alpha^2 + 2A)}. \end{aligned}$$

We solve this system of inequalities analog (with $(-B, A)$ instead of (A, B)) as in the proof of Proposition 3.2, in fact, by replacing the constants we obtain the following admissible ranges of noise intensities:

- If $\quad B > -2A \quad$ then $\quad \max\{0, -A\} \leqslant \frac{\alpha^2}{2} < T_2.$
- If $\quad -2A \geqslant B > -A \quad$ then $\quad T_1 \leqslant \frac{\alpha^2}{2} < T_2.$

Here,

$$T_{1,2} := A + 3B \mp 2\sqrt{2B(A+B)}.$$

This completes the proof of Theorem 4.2. □

As in the previous section we now deduce from Theorem 4.2 some explicit sufficient conditions for the persistence of stability and stabilization by noise for problem (27), using the explicit expression for the constant C_θ in Lemma 2.1. We remark again that Theorem 4.2 is valid for any constant fulfilling inequality (8) in place of C_θ^*.

Theorem 4.3. *Let C_θ denote the explicit constant in Lemma 2.1, i.e. $C_\theta = \theta(d/R - \theta)$ for $\theta \leqslant d/2R$ and $C_\theta = d^2/(4R^2)$ for $\theta \geqslant d/2R$.*

(a) *Persistence of stability: If $\beta < C_\lambda$, then the trivial solution of (27) is exponentially stable for all noise intensities $|\alpha|$ such that*

$$\frac{\alpha^2}{2} \in [0, 8(\lambda - \widehat{\theta})),$$

where $\widehat{\theta} < \lambda$ is the unique constant such that $\widehat{\theta} + C_{\widehat{\theta}} = \beta + \lambda$.

(b) *Stabilisation by noise I: If $\beta < \lambda$, then problem (27) can be stabilised by noise. More precisely, a range of noise intensities that stabilise the equation is given by*

$$\frac{\alpha^2}{2} \in \begin{cases} [\beta, T_2) & \text{if } \ 2\beta < \lambda, \\ (T_1, T_2) & \text{if } \ \beta < \lambda \leqslant 2\beta, \end{cases}$$

where

$$T_{1,2} = 3\lambda - \beta \mp 2\sqrt{2\lambda(\lambda - \beta)}.$$

(c) *Stabilisation by noise II: In case $\beta > \max\{\lambda, C_\lambda\}$, if*

$$\beta \leqslant \frac{1}{4}(d/R - 1)^2 + \lambda, \tag{31}$$

and $d > R$, then there exists $\theta > 0$ such that

$$\lambda - \theta > \beta - C_\theta \geqslant 0, \tag{32}$$

and consequently, the zero solution of (27) is exponentially stable for all noise intensities $|\alpha|$ such that

$$\frac{\alpha^2}{2} \in \begin{cases} [\beta - C_\theta, T_2) & \text{if } \ \lambda - \theta > 2(\beta - C_\theta), \\ (T_1, T_2) & \text{if } \ 2(\beta - C_\theta) \geqslant \lambda - \theta > \beta - C_\theta, \end{cases}$$

where

$$T_{1,2} := C_\theta - \beta + 3(\lambda-\theta) \mp 2\sqrt{2(\lambda-\theta)(\lambda-\theta+C_\theta-\beta)}.$$

Proof. To prove part (a) we observe that $T_1 = 0$ in Theorem 4.2 if and only if $C_\theta+\theta = \lambda+\beta$. Moreover, if $\beta < C_\lambda$, then $\lambda+\beta = \lambda + C_\lambda - \varepsilon$ for some $\varepsilon > 0$, and since $\theta + C_\theta$ is strictly increasing, there exists a unique $\widehat{\theta} < \lambda$ such that $\widehat{\theta} + C_{\widehat{\theta}} = \lambda + C_\lambda - \varepsilon = \beta+\lambda$. In this case, we obtain $T_2 = 8(\lambda - \widehat{\theta})$ in Theorem 4.2, which implies the stated range of noise intensities.

Part (b): If $\beta < \lambda$ we can choose θ in Theorem 4.2 arbitrarily small such that (29) holds, since $C_0^* = 0$ and C_θ^* depends continuously on θ. Moreover, we observe that

$$T_{1,2}|_{\theta=0} = 3\lambda - \beta \mp \sqrt{2\lambda(\lambda-\beta)},$$

and the statement is a direct consequence of Theorem 4.2.

For Part (c), we first observe by direct computation that

$$\max\{\lambda, C_\lambda\} < \frac{1}{4}(d/R-1)^2 + \lambda,$$

thus, the set of parameters β satisfying $\max\{\lambda, C_\lambda\} < \beta < (1/4)(d/R-1)^2+\lambda$ is not empty. We now prove that (31) implies that there exists $\theta \leqslant d/2R$ such that (32) holds. The claim then follows from Theorem 4.2. If $\theta \leqslant d/2R$, then $C_\theta = \theta(d/R-\theta)$, and condition (32) is equivalent to

$$\begin{cases} \theta < \min\{d/2R, \lambda\}, \\ \theta^2 - (d/R-1)\theta + \beta - \lambda \leqslant 0. \end{cases} \tag{33}$$

Moreover, the second inequality in (33) is equivalently to

$$\frac{d/R - 1 - \sqrt{\Phi}}{2} \leqslant \theta \leqslant \frac{d/R-1+\sqrt{\Phi}}{2},$$

where

$$\Phi := (d/R-1)^2 - 4(\beta-\lambda) \geqslant 0$$

thanks to (31). Note that we also need $R > d$ for the two bounds for θ to be positive. It now suffices to check that

$$\frac{d/R-1-\sqrt{\Phi}}{2} < \min\{d/2R, \lambda\}.$$

The inequality $\frac{d/R-1-\sqrt{\Phi}}{2} < d/2R$ is obviously satisfied. On the other hand, we observe that

$$\frac{d/R-1-\sqrt{\Phi}}{2} < \lambda \quad \Longleftrightarrow \quad d/R - 1 - 2\lambda < \sqrt{\Phi},$$

which certainly holds if $\lambda > \frac{1}{2}(d/R-1)$. If $\lambda \leqslant \frac{1}{2}(d/R-1)$ taking the square of both sides of the inequality leads to $\beta > \lambda(d/R-\lambda)$, which is satisfied since we have $\beta > \max\{\lambda; C_\lambda\}$.

In conclusion, under the condition $\max\{\lambda, C_\lambda\} < \beta < \frac{1}{4}(d/R-1)^2 + \lambda$, there exists $0 < \theta \leqslant d/2R$ such that (32) holds. Therefore, applying Theorem 4.2 we obtain Part (c). □

Discussion.

- The results in Theorem 4.3 can be interpreted as follows:
  - If $\beta < C_\lambda \leqslant C_\lambda^*$ the trivial steady state of the deterministic equation is exponentially stable by Theorem 2.1. Theorem 4.3 proves that this stability is preserved for the perturbed problem (27) if the noise intensity $|\alpha|$ is such that $\alpha^2/2 \in [0, 8(\lambda - \widehat{\theta}\,))$.
  - If $\beta \geqslant C_\lambda$, the zero solution of the deterministic equation might be unstable. In particular, if $\beta > C_\lambda^*$, then the zero steady state is unstable by Theorem 2.1. Theorem 4.3 shows that it is possible to stabilise the equation by noise inside the domain if either (by part (b)) $\beta < \lambda$ or (by part (c))
$$\max\{\lambda; C_\lambda\} < \beta \leqslant \frac{1}{4}(d/R - 1)^2 + \lambda.$$
- Remarkably different from the case of noise acting on the boundary (1), where we conjecture that there exists a critical value β_{crit} such that the equation cannot be stabilised if $\beta \geqslant \beta_{\text{crit}}$, Theorem 4.3 implies that if $\lambda > \beta$ then, no matter how large β is, there always exists a range of noise intensities that stabilise the equation (27).
- In Theorem 4.3, as well as in Theorem 3.3, we obtain a finite range of noise intensities that stabilise the equation. This indicates that the dynamical boundary conditions cause difficulties when trying to stabilise the deterministic problem by noise (acting either inside the domain or on the boundary) with too high intensities.

5. The case of one dimension

As shown in Section 2.2, the zero solution of the deterministic equation (10) is exponentially stable if $C_\lambda^* > \beta$, while instability occurs when $C_\lambda^* < \beta$. However, since C_λ^* is not explicit, the latter case is not easy to verify in practice. In this section, we analyse system (10) in one dimension. Firstly, we consider the linearised equation around zero, which can be solved explicitly by using separation of variables. For a particular choice of parameters, it follows that the trivial solution of the linearised problem is unstable. Since all eigenvalues of the stationary problem have non-zero real part, we can apply an infinite dimensional version of the Hartman-Grobman theorem, see e.g. [23] or [17, Corollary 5.1.6], to conclude the instability of the nonlinear problem (10). Finally, we apply Theorem 3.3 to prove that the equation can be stabilised by noise on the boundary and determine a concrete range of noise intensities that imply the exponential stability of the zero steady state.

In the one-dimensional domain $D = (0, 1)$ we consider the following linearisation of (10)

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) - \beta u(x,t) = 0, & (x,t) \in (0,1) \times (0,\infty), \\ u_t(0,t) - u_x(0,t) + \lambda u(0,t) = 0, & t \in (0,\infty), \\ u_t(1,t) + u_x(1,t) + \lambda u(1,t) = 0, & t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in (0,1), \\ u(0,0) = \phi_1, \quad u(1,0) = \phi_2, \end{cases} \tag{34}$$

where $u_0 \in L^2(0,1)$ and $\phi_1, \phi_2 \in \mathbb{R}$. Defining $\widehat{u}(x,t) = e^{-\beta t} u(x,t)$ we observe that $\widehat{u}$ satisfies the system

$$\begin{cases} \widehat{u}_t - \widehat{u}_{xx} = 0, & (x,t) \in (0,1) \times (0,\infty), \\ \widehat{u}_t(0,t) - \widehat{u}_x(0,t) + (\beta+\lambda)\widehat{u}(0,t) = 0, & t \in (0,\infty), \\ \widehat{u}_t(1,t) + \widehat{u}_x(1,t) + (\beta+\lambda)\widehat{u}(1,t) = 0, & t \in (0,\infty), \\ \widehat{u}(x,0) = u_0(x), & x \in (0,1), \\ \widehat{u}(0,0) = \phi_1, \quad \widehat{u}(1,0) = \phi_2. \end{cases} \tag{35}$$

We will solve (35) explicitly using separation of variables. We are looking for solutions of the form

$$\widehat{u}(x,t) = X(x)T(t),$$

and inserting this ansatz into (35) leads to

$$\frac{T'}{T} = \frac{X''}{X} = k$$

for some constant $k \in \mathbb{R}$. This immediately implies that

$$T(t) = c_0 e^{kt},$$

for some constant $c_0 \neq 0$. To determine X, we distinguish three cases.

Case 1: $k > 0$. From $X'' = kX$ we obtain

$$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x},$$

for some constants $c_1, c_2 \in \mathbb{R}$. Consequently,

$$\widehat{u}(x,t) = c_3 e^{kt}\left(e^{\sqrt{k}x} + c_4 e^{-\sqrt{k}x}\right),$$

for some constants $c_3, c_4 \in \mathbb{R}$, where $c_3 \neq 0$. The dynamical boundary conditions in (34) now lead to

$$(\beta + k + \lambda + \sqrt{k})c_2 = \sqrt{k} - \beta - k - \lambda,$$
$$(\beta + k + \sqrt{k} + \lambda)e^{\sqrt{k}} = c_2(\sqrt{k} - \beta - k - \lambda)e^{-\sqrt{k}},$$

and it follows that

$$(\beta + k + \lambda + \sqrt{k})^2 e^{\sqrt{k}} = (\beta + k + \lambda - \sqrt{k})^2 e^{-\sqrt{k}}.$$

This is impossible since β, λ and k are positive.

Case 2: $k = 0$. In this case we obtain

$$T(t) = c_0 \quad \text{and} \quad X(x) = c_1 x + c_2,$$

for some constants $c_0, c_1, c_2 \in \mathbb{R}$. Using again the boundary conditions it follows that

$$(\beta + \lambda)c_2 = c_1,$$
$$\beta(c_1 + c_2) + c_1 + \lambda(c_1 + c_2) = 0,$$

which implies that $c_1 = c_2 = 0$, and consequently, $u \equiv 0$.

Case 3: $k < 0$. We set $k = -\mu^2$, where $\mu \in (0, \infty)$. From $X'' = -\mu^2 X$ we obtain

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x),$$

for some constants $c_1, c_2 \in \mathbb{R}$, and thus,

$$\widehat{u}(x,t) = c_0 e^{-\mu^2 t}(c_1 \cos(\mu x) + c_2 \sin(\mu x)).$$

Using the dynamical boundary conditions it follows that

$$\begin{aligned} (\beta + \lambda - \mu^2)c_1 - \mu c_2 = 0, \\ [(\beta + \lambda - \mu^2)\cos\mu - \mu \sin\mu]c_1 + [\mu\cos\mu + (\beta + \lambda - \mu^2)\sin\mu]c_2 = 0. \end{aligned}$$

For the existence of a nontrivial solution (c_1, c_2) it is necessary that

$$\det \begin{bmatrix} (\beta + \lambda - \mu^2) & -\mu \\ (\beta + \lambda - \mu^2)\cos\mu - \mu\sin\mu & \mu\cos\mu + (\beta + \lambda - \mu^2)\sin\mu \end{bmatrix} = 0.$$

Hence, if $\cos\mu = 0$, i.e. $\mu = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{N}$, then $\mu^2 = (\beta + \lambda - \mu^2)^2$, which yields two positive roots $\mu = \pm\frac{1}{2} + \sqrt{\frac{1}{4} + \beta + \lambda} > 0$, which require $\beta + \lambda = -\frac{1}{4} + (\pm\frac{1}{2} + \frac{\pi}{2} + k\pi)^2$ for some $k \in \mathbb{N}$. Otherwise, if $\cos\mu \neq 0$ then

$$\tan\mu = \frac{2\mu^3 - 2(\beta + \lambda)\mu}{\mu^4 - [2(\beta + \lambda) + 1]\mu^2 + (\beta + \lambda)^2}. \tag{36}$$

It is easy to see that this equation possesses countably infinitely many positive solutions

$$0 < \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n \leqslant \ldots, \qquad \lim_{n\to\infty} \mu_n = \infty.$$

In conclusion, the solution of (35) can be written as

$$\begin{aligned} \widehat{u}(x,t) &= \sum_{n=1}^{\infty} c_{0,n} e^{-\mu_n^2 t} \left(c_{1,n}\cos(\mu_n x) + c_{2,n}\sin(\mu_n x)\right) \\ &= \sum_{n=1}^{\infty} c_n e^{-\mu_n^2 t} \left(\cos(\mu_n x) + \frac{\beta + \lambda - \mu_n^2}{\mu_n}\sin(\mu_n x)\right), \end{aligned}$$

where the coefficients c_n are determined by the initial data. Finally, changing back to $u(x,t)$ we obtain the explicit solution to (35)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\langle U_0, \varphi_n \rangle}{\|\varphi_n\|_H^2} e^{(\beta - \mu_n^2)t} \varphi_n(x),$$

where $U_0 = (u_0, (\phi_1, \phi_2))$ and

$$\varphi_n(x) = \cos(\mu_n x) + \frac{\beta + \lambda - \mu_n^2}{\mu_n}\sin(\mu_n x).$$

From this, we observe that the zero solution of (34) is unstable if the parameters β and λ are such that

$$\beta > \mu_1^2. \tag{37}$$

If $\beta + \lambda \ll 1$, we can use an asymptotic analysis to determine an explicit condition for instability.

Proposition 5.1 (Instability for small $\beta + \lambda$)**.** *There exists $\varepsilon > 0$ small enough such that, if $\beta + \lambda \leqslant \varepsilon$ and $\beta > 2\lambda$, then the zero steady state of (34) is unstable.*

Proof. For simplicity we set $b = \beta + \lambda$. Note that when $b \to 0$ the smallest positive solution of (36) also converges to 0. Therefore, for small b, μ_1 is expected to be close to 0. In a general asymptotic expansion $\mu^2 = \gamma b^\sigma$ of (36), direct computations verify that only the exponent $\sigma = 1$ leads to a significant degeneration. Thus, by using the ansatz $\mu^2 = \gamma b$ and a Taylor expansion for the function $\tan \mu = \mu + \mu^3/3 + O(\mu^5)$ we obtain from (36)

$$1 + \frac{\gamma b}{3} + O(b^2) = \frac{2(\gamma - 1)}{b(\gamma - 1)^2 - \gamma}. \tag{38}$$

By expanding $\gamma = \gamma_0 + \gamma_1 b + O(b^2)$, the left hand side of (38) is $1 + \gamma_0 b/3 + O(b^2)$. Let $f(b)$ denote the right hand side of (38), then a Taylor expansion yields

$$f(b) = f(0) + f'(0)b + O(b^2) = \frac{2(1-\gamma_0)}{\gamma_0} - \frac{2\gamma_1 + 2(\gamma_0 - 1)^3}{\gamma_0} b + O(b^2).$$

By identifying the zero and the first order terms of b in (38), it follows that

$$1 = \frac{2(1-\gamma_0)}{\gamma_0} \quad \text{and} \quad \frac{\gamma_0}{3} = -\frac{2\gamma_1 + 2(\gamma_0 - 1)^3}{\gamma_0}$$

which implies that $\gamma_0 = 2/3$ and $\gamma_1 = -1/27$. Therefore, asymptotically we have

$$\mu_1^2 = \frac{2}{3}b - \frac{1}{27}b^2 + O(b^3) = \frac{2}{3}(\beta + \lambda) - \frac{1}{27}(\beta + \lambda)^2 + O((\beta + \lambda)^3).$$

It follows that, when $\beta + \lambda$ is small enough, we have $\mu_1^2 < \frac{2}{3}(\beta + \lambda) < \beta$ since $\beta > 2\lambda$. This confirms the instability of the zero solution of (34) due to (37). □

In the next proposition we formulate a concrete example where the deterministic equation is unstable and by applying Theorem 3.3 we derive an explicit range of noise intensities on the boundary which stabilise the equation.

Proposition 5.2. *We consider the equation (34) with $\beta = 0.02$, $\lambda = 0.001$. Then, the zero solution is unstable, but can be stabilised by a multiplicative Itô noise on the boundary.*

In particular, the zero steady state of the stochastic problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + u^3(x,t) - \beta u(x,t) = 0, & (x,t) \in (0,1) \times (0,\infty), \\ u_t(0,t) - u_x(0,t) + \lambda u(0,t) = \alpha u(0,t) dW_t, & t \in (0,\infty), \\ u_t(1,t) + u_x(1,t) + \lambda u(1,t) = \alpha u(1,t) dW_t, & t \in (0,\infty), \\ u(x,0) = u_0(x), & x \in (0,1), \\ u(0,0) = \phi_1, \quad u(1,0) = \phi_2 \end{cases} \tag{39}$$

is exponentially stable for all α such that

$$0.0556 < \alpha^2 < 6.274. \tag{40}$$

Proof. Thanks to Proposition 5.1, the zero solution of the linearised equation for (39) is unstable. Since all the eigenvalues of the operator A are positive, we can apply an infinite dimensional version of the Hartman-Grobman theorem, see e.g. [23] or [17, Corollary 5.1.6], to conclude that the zero solution of the nonlinear problem (39) is also unstable.

To prove the second statement of the proposition, we apply Theorem 3.3. We need to find $\theta > 0$ such that

$$C_\theta - \beta > \theta - \lambda > 0.$$

Since $D = (0,1)$ we have $R = 1/2$ and $d/2R = 1$, and consequently, by Lemma 2.1,

$$C_\theta = \begin{cases} \theta(2-\theta) & \theta < 1 \\ 1 & \theta \geqslant 1. \end{cases}$$

This leads to the conditions

$$\theta > \lambda = 0.001,$$
$$0 > \theta^2 - \theta + 0.019,$$

and solving the inequality we obtain

$$0.0194 < \theta < 0.9806.$$

By Theorem 3.3 for any θ within this range, the zero solution of equation (39) is exponentially stable if the noise intensity is such that $Z_1 < \alpha^2/2 < Z_2$, where

$$Z_{1,2}(\theta) = -3\theta^2 + 5\theta - 0.059 \mp 2\sqrt{2(\theta^2 - 2\theta + 0.02)(\theta^2 - \theta + 0.019)}.$$

We observe that

$$\min_{\theta \in (0.0194, 0.9806)} \{Z_1(\theta)\} = 0.0278 \qquad \text{and} \qquad \max_{\theta \in (0.0194, 0.9806)} \{Z_2(\theta)\} = 3.137,$$

and thus, obtain the desired range of intensities (40). □

6. Conclusion

This paper studies the stabilisation effect of noise on the boundary for a Chafee-Infante equation with dynamical boundary conditions. Under certain conditions of the domain and the parameters, we show that with suitable boundary noise one can stabilise the trivial stationary state, which is unstable in the deterministic case, i.e. without noise. The main tools are the refinements of the ideas in [10] and the functional inequality: for each $\theta > 0$ there exists an optimal constant $C_\theta^* > 0$ such that

$$\int_D |\nabla u(x)|^2 dx + \theta \int_D |u(x)|^2 dS(x) \geqslant C_\theta^* \int_D |u(x)|^2 dx \quad \text{for all } u \in H^1(D).$$

A main difference to related results in the literature on stabilisation by noise (within a domain) is the fact that we prove a *finite amplitude range of stabilising noise*. It is however an open question whether such results are due to technical limitation or the nature of stabilisation by boundary noise. However, Theorem 4.3 seems to indicate that the finite range of stabilisation might be due to the nature of dynamical boundary conditions.

Up to the best of our knowledge, this work proves the first results concerning the question of stabilisation for partial differential equations using boundary noise. The Chafee-Infante equation and the dynamical boundary condition are chosen as a specific example to present the main ideas. As future research, we expect that the results of this paper generalise e.g. to Dirichlet or Neumann boundary conditions.

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