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Asymptotically safe $f(R)$-gravity coupled to matter II: Global solutions

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ABSTRACT

Ultraviolet fixed point functions of the functional renormalisation group equation for $f(R)$-gravity coupled to matter fields are discussed. The metric is split via the exponential parameterisation into a background metric and a fluctuating part, the former is chosen to be the one of a four-sphere. Also when scalar, fermion and vector fields are included global quadratic solutions exist as in the pure gravity case for discrete sets of values for some endomorphism parameters defining the coarse-graining scheme. The asymptotic, large-curvature behaviour of the fixed point functions is analysed for generic values of these parameters. Examples for global numerical solutions are provided. A special focus is given to the question whether matter fields might destabilise the ultraviolet fixed point function. Similar to a previous analysis of a polynomial, small-curvature approximation to the fixed point functions different classes for such functions are found.

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1. Introduction

Searching for a viable theory of quantum gravity is one of the most important open problems in theoretical physics. Many different approaches try to elucidate it from various perspectives. In this letter, the asymptotic safety scenario for quantum gravity [1] will be employed based on a specific ansatz for the effective action: an investigation of $f(R)$-gravity minimally coupled to an arbitrary number of scalar, Dirac, and vector fields is discussed with a special focus on the study of global fixed functions, the generalisation of non-Gaußian fixed point (NGFP). Recently, the flow equation used herein has been derived within the functional renormalisation group (FRG) for the effective average action, and this equation has been solved to obtain the respective NGFP function in a polynomial, small-curvature approximation [2], see also ref. [3]. These solutions provide the foundation of the here reported study. Results for the NGFP function for the pure gravity case within the employed version of the flow equation have been given recently in refs. [4,5]. Note that such solutions of NGFP functions but for different truncations of the flow equation have been obtained in refs. [6–20]. Hereby the characteristics of the solutions for these functions differ significantly depending on the technical aspects of the respective work. Given the fixed functions' importance for the asymptotic safety scenario this requires further understanding. In the following it will be studied whether coupling matter might give an important hint to resolve ambiguities.

Coupling matter to gravity within the asymptotic safety scenario has a long history, see refs. [18,21–47], however, with mixed results. In ref. [2] comprehensible estimates have been provided which gravity-matter systems may give rise to NGFPs suitable for rendering the theory asymptotically safe. In this reference the flow equation has been derived within a seven-parameter family of non-trivial endomorphisms in the regularisation procedure. Herein this freedom will be exploited to show the existence of global quadratic solutions. In ref. [2] it was also shown that for vanishing endomorphisms gravity coupled to the matter content of the standard model of particle physics (and also many beyond the standard model extensions) exhibit a NGFP whose properties are strikingly similar to the case of pure gravity: there are two UV-relevant directions, and the position and critical exponents converge rapidly when higher powers of the scalar curvature beyond the quadratic ones are included. Building on this result numerical solutions will be obtained for a global fixed function for the pure gravity as well as for the gravity-matter system with standard model matter content. Hereby the discussion of the singular points of the flow equation and the asymptotic behaviour of the solution for large scalar curvatures turns out to be the crucial element.

Based on generic features of the employed flow equation one can show for global solutions a property already visible at the level of the polynomial approximations, namely that addition of fermions, stabilise an existing NGFP if the coarse graining operator is chosen as the Laplacian. As the Standard Model is dominated by fermions therefore a NGFP function for $f(R)$-gravity coupled to the Standard Model matter content exists in this case.

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This letter is organised as follows: in sect. 2 the derivation of the flow equation is briefly reviewed. In sect. 3 the properties of the flow equation in four dimensions are discussed. The existence of global quadratic solutions is shown by constructing two explicit examples. Furthermore, the asymptotic behaviour is analysed, and numerical solutions for two selected cases are presented. In sect. 4 the results are summarised and conclusions are provided.

2. RG equation for gravity and matter in $f(R)$ truncation

To make this letter self-contained the derivation of the flow equation given in ref. [2] is briefly reviewed. It is based on the form of the FRG equations given in refs. [48,49], adapted to the case of gravity [1],

$$\partial_t \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right]. \tag{1}$$

Here, k is the RG scale and $t = \ln(k/k_0)$ the RG "time" with k_0 being an arbitrary reference scale. Γ_k denotes the effective average action and $\Gamma_k^{(2)}$ its second variation with respect to the fluctuation fields. $\mathcal{R}_k$ is a regulator introduced such that solving the flow equation effectively integrates out quantum fluctuations with momenta above the scale k.

Throughout this work for the gravitational part of the effective average action a $f(R)$ truncation on d-spheres as background will be employed. To be explicit, the corresponding total effective action reads

$$\Gamma_k = \Gamma_k^{\text{grav}} + \Gamma_k^{\text{matter}}\,, \tag{2}$$

where Γ_k^{grav} is the gravitational part of the effective average action and Γ_k^{matter} contains the matter fields. The gravitational part of the action is assumed to be given by

$$\Gamma_k^{\text{grav}} = \int d^d x \sqrt{g}\, f_k(R) \;+ \Gamma_k^{\text{gf}} + \Gamma_k^{\text{gh}}\,, \tag{3}$$

where $f_k(R)$ is an arbitrary, scale-dependent function of the Ricci scalar R, and the action is supplemented by suitable gauge fixing and ghost terms. This sector is taken to be identical to the one studied in ref. [5]. The matter sector is assumed to consist of N_S scalar fields, N_D Dirac fermions, and N_V Abelian gauge fields. The latter ones are fixed to Feynman gauge, and thus on curved backgrounds the related ghosts are included. Matter self-interactions as well as the RG scale dependence of the matter wave-function renormalisations are neglected.

In a next step one splits the metric into a background and a fluctuating (quantum) part. In the following the exponential split

$$g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho{}_\nu \tag{4}$$

is used thereby avoiding any signature change even for large fluctuating fields. A detailed derivation of the gravitational part of the RG equation can be found in refs. [4,5], see also refs. [2,3].

Some of the global solutions for fixed functions presented here can only be obtained if additional endomorphisms in the regulator functions appearing in the RG equations are introduced. To this end, a set of parameters $\alpha^{G,M}_{S,D,V,T}$, with the subscript labelling the type of field and the superscript the gravity or matter sector, are introduced: the regulators are chosen to depend on $\Box^{G,M}_{S,D,V,T} := \Delta - \alpha^{G,M}_{S,D,V,T}\bar{R}$ where Δ is the Laplace operator, and $\bar{R}$ is the positive curvature scalar of the background sphere. The labelling of subscripts refer to: S scalar, D Dirac, V transverse vector, and T transverse traceless symmetric tensor field.

In ref. [2] all the traces appearing in the RG equation have been done by explicitly summing over the eigenvalues of the Laplacian on the sphere in the different spin channels, for the respective expressions for these eigenvalues see, *e.g.*, ref. [50]. Using a Litim-type regulator [51,52]

$$\begin{aligned} R_k(z) &= (k^2 - z)\theta(k^2 - z)\,, \\ \partial_t R_k(z) &= 2k^2\theta(k^2 - z)\,, \end{aligned} \tag{5}$$

these sums are all finite. Subsequently, an additional smoothing operation (namely averaging over two sums performed on the upper and lower limit of the resulting "staircase" function) are employed as part of the regularisation, see ref. [2] for more details.

Two widely used choices for the coarse graining operators termed "Type I" and "Type II" (see [53] for detailed definitions and a discussion of this typology) are given by the following choice of endomorphism parameters,

$$\begin{aligned} \text{Type I:}\quad & \alpha_T^G = \alpha_S^G = \alpha_V^G = \\ & = \alpha_D^M = \alpha_{V_1}^M = \alpha_{V_2}^M = \alpha_S^M = 0\,, \end{aligned} \tag{6a}$$

$$\begin{aligned} \text{Type II:}\quad & \alpha_T^G = -\tfrac{2}{d(d-1)}\,,\ \alpha_S^G = \tfrac{1}{d-1}\,,\ \alpha_V^G = \tfrac{1}{d}\,, \\ & \alpha_D^M = -\tfrac{1}{4}\,,\ \alpha_{V_1}^M = -\tfrac{1}{d}\,,\ \alpha_{V_2}^M = \alpha_S^M = 0\,. \end{aligned} \tag{6b}$$

The RG equation takes the form of a partial differential equation for the scale-dependent function $f_k(\bar{R})$. As usual it is advantageous to formulate it in dimensionless variables $r = \bar{R}/k^2$ and $\varphi(r) = f(\bar{R})/k^d$. This leads to a separation of the "classical" scale dependence of $f(\bar{R})$ from the quantum one which reads in four dimensions:

$$\begin{aligned} \partial_t \Gamma_k &= \int d^4x\,\sqrt{g}\,\partial_t f(\bar{R}) \\ &= V_4\, k^4\left(\partial_t\varphi(r) + 4\varphi(r) - 2r\varphi'(r)\right), \end{aligned} \tag{7}$$

where $V_4 = 384\pi^2/\bar{R}^2$ is the volume of the 4-sphere. The flow equation is then given by:

$$\begin{aligned} \dot{\varphi} + 4\varphi - 2r\varphi' = \mathcal{T}^{\text{TT}} + \mathcal{T}^{\text{ghost}} + \mathcal{T}^{\text{sinv}} \\ + \mathcal{T}^{\text{scalar}} + \mathcal{T}^{\text{Dirac}} + \mathcal{T}^{\text{vector}}\,, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mathcal{T}^{\text{TT}} = {}& \frac{5}{2(4\pi)^2}\,\frac{1}{1+(\alpha_T^G+\frac{1}{6})\,r}\left(1+\left(\alpha_T^G-\tfrac{1}{6}\right)r\right) \\ & \times\left(1+\left(\alpha_T^G-\tfrac{1}{12}\right)r\right) \\ & + \frac{5}{12(4\pi)^2}\,\frac{\dot{\varphi}'+2\varphi'-2r\varphi''}{\varphi'}\left(1+\left(\alpha_T^G-\tfrac{2}{3}\right)r\right) \\ & \times\left(1+\left(\alpha_T^G-\tfrac{1}{6}\right)r\right), \end{aligned} \tag{9a}$$

$$\begin{aligned} \mathcal{T}^{\text{sinv}} = {}& \frac{1}{2(4\pi)^2}\,\frac{\varphi''}{\left(1+\left(\alpha_S^G-\frac{1}{3}\right)r\right)\varphi''+\frac{1}{3}\varphi'} \\ & \times\left(1+\left(\alpha_S^G-\tfrac{1}{2}\right)r\right)\left(1+\left(\alpha_S^G+\tfrac{11}{12}\right)r\right) \\ & + \frac{1}{12(4\pi)^2}\,\frac{\dot{\varphi}''-2r\varphi'''}{\left(1+\left(\alpha_S^G-\frac{1}{3}\right)r\right)\varphi''+\frac{1}{3}\varphi'} \\ & \times\left(1+\left(\alpha_S^G+\tfrac{3}{2}\right)r\right)\left(1+\left(\alpha_S^G-\tfrac{1}{3}\right)r\right) \\ & \times\left(1+\left(\alpha_S^G-\tfrac{5}{6}\right)r\right), \end{aligned} \tag{9b}$$

$$\mathcal{T}^{\text{ghost}} = -\frac{1}{48(4\pi)^2}\frac{1}{1+(\alpha_V^G-\frac{1}{4})r} \times \Big(72+18r(1+8\alpha_V^G) - r^2(19-18\alpha_V^G-72(\alpha_V^G)^2)\Big), \tag{9c}$$

and

$$\mathcal{T}^{\text{scalar}} = \frac{N_S}{2(4\pi)^2}\frac{1}{1+\alpha_S^M r}\left(1+\left(\alpha_S^M+\frac{1}{4}\right)r\right) \times\left(1+\left(\alpha_S^M+\frac{1}{6}\right)r\right), \tag{10a}$$

$$\mathcal{T}^{\text{Dirac}} = -\frac{2N_D}{(4\pi)^2}\left(1+\left(\alpha_D^M+\frac{1}{6}\right)r\right), \tag{10b}$$

$$\mathcal{T}^{\text{vector}} = \frac{N_V}{2(4\pi)^2}\left(\frac{3}{1+\left(\alpha_{V_1}^M+\frac{1}{4}\right)r}\left(1+\left(\alpha_{V_1}^M+\frac{1}{6}\right)r\right)\times\left(1+\left(\alpha_{V_1}^M+\frac{1}{12}\right)r\right) - \frac{1}{1+\alpha_{V_2}^M r}\left(1+(\alpha_{V_2}^M+\frac{1}{2})r\right)\times\left(1+(\alpha_{V_2}^M-\frac{1}{12})r\right)\right). \tag{10c}$$

The first line of eq. (8) stems from the gravitational sector and depends correspondingly on the endomorphism parameters $\alpha^G_{S,V,T}$. The second line originates from the matter part: the contributions from the transverse vector and scalar ghost fields are proportional to N_V. In addition there are the ones of the Dirac and the scalar fields.

The common factor $1/(4\pi)^2$ can be removed from the coefficients defined in eqs. (9) and (10) by a suitable rescaling of $\varphi(r)$, and this is assumed in the following (*cf., e.g.*, eq. (12) below).

3. Flow equation and fixed functions in four dimensions

3.1. Discussion of the flow equation

The coefficient $\mathcal{T}^{\text{Dirac}}$ shows a peculiarity, it is only linear in the curvature. During derivation it is also a ratio of a quadratic numerator and a linear denominator like the other coefficients, however, for the chosen regularisation procedure (and only for this one amongst the ones used in ref. [2]) this denominator cancels against one of the two numerator terms to yield $\mathcal{T}^{\text{Dirac}} \propto 2(1+(\alpha_D+1/6)r)$.

Although simpler than the other terms the coefficient for the Dirac fields already displays a qualitative difference when changing the related endomorphism parameter. The allowed interval for α_D^M is $-1/4 \leq \alpha_D^M \leq 0$ (see the corresponding discussion in refs. [2,3]) with the lower end corresponding to a type-II- and the upper end to a type-I-regulator. It is plain that therefore the sign of the linear term depends on this parameter, and this will qualitatively change how fermions contribute to the flow equation. This property will be important when discussing the solutions for fixed functions.

Solving the non-linear partial differential equation (8) for flows of the function $\varphi(r)$ is an extremely complicated task. The necessary first step in such an analysis is calculating its fixed functions, the generalisation of fixed points. Those are the solutions of the ordinary differential equation obtained by setting $\partial_t\varphi(r)=0$ and thus $\partial_t\varphi'(r)=0=\partial_t\varphi''(r)$ in eq. (8). To distinguish them from the general scale-dependent function $\varphi(r)$ a fixed function will be denoted as usual by $\varphi^\star(r)$ in the following.

In its normal form $\varphi^{\star\prime\prime\prime}=\ldots$ the flow equation has the following singularities: first, from the term proportional to $\varphi^{\star\prime\prime\prime}$ in (9b)

$$r_1^{\text{sing}} = -\frac{1}{\alpha_S^G+\frac{3}{2}},\quad r_2^{\text{sing}}=0, \quad r_3^{\text{sing}} = -\frac{1}{\alpha_S^G-\frac{5}{6}},\quad r_4^{\text{sing}} = -\frac{1}{\alpha_S^G-\frac{1}{3}}, \tag{11}$$

and, second, from the denominators in the expressions (9) and (10). Hereby the extrema of $\varphi^\star(r)$ via the second term in (9a) are moving singularities. As seen below one can arrange the parameters and the solutions such that the moving singularities are cancelled against the numerators. Note that the singularity at vanishing curvature, $r_2^{\text{sing}}=0$, reflects the non-smooth transition from a sphere to a flat space.

In the pure gravitational sector global fixed functions which are polynomials of quadratic order have been found and described in ref. [5]. For them, the third derivative vanishes and thus the second summand of the expression (9b) does not contribute. In all other non-trivial solutions this term (which stems from the conformal mode) determines the structure of the differential equation in the normal form because in this and only this term a third-order derivative, *i.e.*, $\varphi'''(r)$, appears.

3.2. Global solutions for fixed functions

3.2.1. Global quadratic solutions

As already mentioned above, global solutions of quadratic order,

$$\varphi^\star(r) = \frac{1}{(4\pi)^2}\,(g_0^\star+g_1^\star r+g_2^\star r^2) \tag{12}$$

are special. Hereby, $g_1^\star$ needs to assume a negative value. To understand why one requires this for a polynomial Ansatz one writes the action such that the Einstein–Hilbert action in standard notation is contained,

$$f(R) = \frac{\Lambda_k}{8\pi G_k} - \frac{R}{16\pi G_k} + \mathcal{O}(R^2), \tag{13}$$

which allows one to identify

$$\Lambda_k = -\frac{g_0}{2g_1}k^2 \qquad \text{and} \qquad G_k = -\frac{\pi}{k^2 g_1}, \tag{14}$$

respectively. Thus, a positive value for Newton's constant requires a negative value of g_1. As the RG flow for Newton's constant cannot cross the zero, and therefore its UV and IR value possess the same sign, also its fixed point value must be positive for an acceptable solution, and thus $g_1^\star<0$.

The fact that the constant term in the polynomial expansion, g_0, does not appear on the right hand side of the flow equation (8) allows for a simple estimate how it is changed by the presence of matter fields. This in turn permits to estimate the influence of matter onto the cosmological constant within the present setting: $g_0^{complete} \approx g_0^{gravity} + \frac{1}{4}N_V + \frac{1}{8}N_S - \frac{1}{2}N_D$, for more details see [2]. Furthermore, the related critical exponent (which is as usual defined as the negative of the eigenvalue of the stability matrix of the linearised flow equation) is always $\Theta_0=4$.

Without matter fields, *i.e.*, for $N_S=N_D=N_V=0$, five different solutions for a globally quadratic fixed function have been identified in ref. [5]. In case $\varphi^\star(r)$ is a polynomial the differential equation determining it can be written as

$$\frac{\mathcal{P}_{num}(r)}{\mathcal{P}_{den}(r)} = 0, \tag{15}$$

i.e., as the requirement that the ratio of two polynomials vanish. This can be solved in two steps: first, solve for $\mathcal{P}_{num}(r)=0$, and second, keep only those solutions where all roots of $\mathcal{P}_{den}(r)$ (*i.e.*,

Table 1
Quadratic solutions for the fixed function with different matter content derived from the pure gravity solution (16).

(N_S, N_D, N_V)	α_S^G	α_V^G	α_T	α_S^M	α_D^M	α_{V1}^M	$g_0^\star$	$g_1^\star$	$g_2^\star$	r_{min}	Θ_0, Θ_1
(0,0,0)	.0389	.3189	−.0978	–	–	–	.6800	−1.179	.4505	1.308	4, 2.02
(0,45/2,0)	.0389	.3189	−.0978	–	0	–	−10.57	−4.929	1.884	1.308	4, 1.98
(4,45/2,0)	−.0819	.0389	−.3778	−.2111	0	–	−9.970	−5.078	1.382	1.837	4, 2.35
(4,45/2,12)	−.0190	.1603	−.2563	−.0897	0	−.3397	−6.702	−8.630	1.825	2.364	4, 2.36

Table 2
Quadratic solution for the fixed function with different matter content derived from the pure gravity solution (17).

(N_S, N_D, N_V)	α_S^G	α_V^G	α_T	α_S^M	α_D	α_{V1}^M	$g_0^\star$	$g_1^\star$	$g_2^\star$	r_{min}	Θ_0, Θ_1
(0,0,0)	−.0638	−.1472	−.5638	–	–	–	1.236	−1.074	0.2134	2.518	≈ 16, 4
(0,45/2,0)	−.0638	−.1472	−.5638	–	0	–	−10.01	−4.824	0.9581	2.518	4, 2.98
(4,45/2,0)	−.0554	−.1388	−.5550	−.3855	0	–	−9.415	−4.637	0.9014	2.572	4, 3.05
(4,45/2,12)	−.0308	−.1140	−.5308	−.3644	0	−.6143	−5.813	−9.368	1.705	2.746	4, 2.8

the potential singularities of this equation) coincide with roots of the numerator.

In the case of a quadratic Ansatz for the fixed function, $\mathcal{P}_{num}(r)$ is a fifth-order polynomial,[1] and its six coefficients can be determined by a discrete set of values for $g_0^\star$, $g_1^\star$, $g_2^\star$, α_T^G, α_V^G and α_S^G, see ref. [5]. Quite surprisingly, in all five solutions found in this reference the potential singularities given by the zeros of the denominator are cancelled by the numerator. On the other hand, for two of these five solutions the eigenperturbations lead to a differential equation with four instead of three fixed singularities, and therefore such eigenperturbations cannot exist globally. For another of these five solutions $\alpha_T = (11+\sqrt{265})/54 \approx 0.505 > 2/3$, *i.e.*, the inequality for a positive argument of the regulator function is violated. This leaves us with two solutions, and the corresponding values for the parameters are given in the respective first lines of Tables 1 and 2. The exact values of these parameters are, respectively,

$$\begin{aligned} \alpha_S^G &= \tfrac{5\sqrt{265}-73}{216}, & \alpha_V^G &= \tfrac{67-2\sqrt{265}}{108}, \\ \alpha_T^G &= \tfrac{11-\sqrt{265}}{54}, & g_0^\star &= \tfrac{49+\sqrt{265}}{96}, \\ g_1^\star &= -\tfrac{4141+121\sqrt{265}}{5184}, & g_2^\star &= \tfrac{67795+3583\sqrt{265}}{279936}, \end{aligned} \tag{16}$$

or

$$\begin{aligned} &\alpha_S^G = -\tfrac{3}{47}, \quad \alpha_V^G = -\tfrac{83}{564}, \quad \alpha_T^G = -\tfrac{53}{94}, \\ &g_0^\star = \tfrac{89}{72}, \quad g_1^\star = -\tfrac{101}{94}, \quad g_2^\star = \tfrac{1414}{6627}. \end{aligned} \tag{17}$$

As this will be important below we also give the value for the minimum of the fixed functions

$$r_{min} = -\frac{g_1^\star}{2g_2^\star} = \begin{cases} \frac{3}{20}(25-\sqrt{265}) \approx 1.3082, \\ \frac{141}{56} \approx 2.5179. \end{cases} \tag{18}$$

As a matter of fact, for a global quadratic solution one can rewrite the equation for the fixed function such that the parameters $g_i^\star$ appear only in the ratio $r_{min} = -g_1^\star/2g_2^\star$ on the left hand side because then some of the expressions in eqs. (9a) and (9b) simplify:

$$\frac{\varphi' - r\varphi''}{\varphi'} = \frac{r_{min}}{r_{min} - r} \tag{19}$$

and

$$\frac{\varphi''}{\left(1+(\alpha_S^G - \frac{1}{3})\,r\right)\varphi'' + \frac{1}{3}\varphi'} = \frac{1}{1+\alpha_S^G r - r_{min}/3}. \tag{20}$$

For the solution (16) one of the zeros of second summand of (9a) occurs exactly at r_{min} and thus the potential singularity is cancelled. The singularity in the scalar term occurs at negative values of r and is thus of no concern. For the solution (17) the potential pole due to the scalar term appears also exactly at r_{min}, and the same is true for the first term in (9a) and the term (9c). With these values of endomorphism parameters, for the pure gravity case, the four terms of the left hand side conspire to yield

$$\frac{1}{(4\pi)^2}\left(\frac{89}{18} - \frac{101}{47}r\right) \tag{21}$$

which, of course, solves then the equation for the fixed function for the parameters $g_0^\star$ and $g_1^\star$ given in eq. (17).

The usefulness of the above considerations becomes immediately clear when adding fermions, *i.e.*, when adding

$$\frac{-2N_D}{(4\pi)^2}\left(1+(\alpha_D^M + \frac{1}{6})\,r\right). \tag{22}$$

A global quadratic solution can be now easily obtained by keeping the ratio $g_1^\star/g_2^\star$ and thus r_{min} fixed. One simply keeps the values of the endomorphism parameters in the gravity sector and substitutes

$$\begin{aligned} g_0^\star &\to g_0^\star - \frac{N_D}{2} \\ g_1^\star &\to g_1^\star - (\alpha_D^M + \frac{1}{6})N_D \\ &= g_1^\star + \begin{cases} \frac{1}{12}N_D & \text{type II reg.} \\ -\frac{1}{6}N_D & \text{type I reg.} \end{cases} \\ g_2^\star &\to g_2^\star + \frac{1}{2r_{min}}(\alpha_D^M + \frac{1}{6})N_D \\ &= g_2^\star - \frac{1}{2r_{min}} \begin{cases} \frac{1}{12}N_D & \text{type II reg.} \\ -\frac{1}{6}N_D & \text{type I reg.} \end{cases} \end{aligned} \tag{23}$$

This proves to be always possible independent of whether the coefficient of the linear term is negative as, *e.g.*, for the type I regulator, or positive as, *e.g.*, for the type II regulator. However, in the latter case the value of $g_1^\star$ will change sign, and thus the solution becomes unphysical.

If one uses now the type-II regulator for the fermions there will be a critical value of N_D where $g_1^\star$ becomes positive, for the solution (16) this value is $N_D = 14.1$ whereas for the solution (17) it

[1] The l.h.s. of the flow equation (8) is for the Ansatz (12) not a polynomial of order $N = 2$ as naïvely expected because the term proportional to r^2 cancels: $4\varphi(r) - 2r\varphi'(r) = 4g_0 + 2rg_1$.

is $N_D = 12.9$. If these values are exceeded the minimum turns to a maximum (but stays at the same location) and the values of $g_1^\star$ and $g_2^\star$ change sign. Therefore, if a type-II regulator is used for fermions one can add only a finite number of them and keep a physically meaningful solution in agreement with the results obtained already in the polynomial approximation [2].

Adding now scalar and/or vector fields it turns out that one cannot fix the parameters $\alpha_S^M = \alpha_{V2}^M = 0$ and $\alpha_{V1}^M = -1/4$, *i.e.*, to their respective type-II values. Although then no new singularities arise in the matter sector one can easily convince oneself that one obtains then for the numerator polynomial the degree six, and thus seven equations for six variables because the expressions $\mathcal{T}^{\text{scalar}}$ and $\mathcal{T}^{\text{vector}}$ in (10) are of quadratic order. A similar situation arises, namely eight equations for seven variables etc., if one fixes only one or two of the three parameters to the respective type-II value. Basically the same remark applies for fixing to type-I values.

Exploring the possibility of adjusting the parameters α_S^M and $\alpha_{V1,2}^M$ to keep a global quadratic solution one notes first that adding fermions is always straightforward by applying the rule (23). For finding the endomorphism parameters which lead to a quadratic solution it proves to be easier to add scalar then vector fields. To obtain a solution with the standard model field content the following strategy has been used: first, add 45/2 Dirac fields (according to standard model matter content) with type-I regulator by applying (23) to the solution (16) and verify this numerically. Second, on the top of this four scalar fields are added and the corresponding parameter α_S^M is determined. From there on one increases N_V in small steps until the standard model value 12 was reached. The results for pure gravity, gravity plus fermions, gravity plus fermions and scalars as well as for gravity plus standard model matter content are displayed in Tables 1 and 2. In all cases one obtains $\alpha_S^M = \alpha_{V2}^M$.

It has to be emphasised that a solution with a positive value for Newton's constant could be found because the type-I value $\alpha_D^M = 0$ was used for the fermionic term. The stabilising effect of the type-I regulated fermions is very much needed. *E.g.*, the solution with no fermions at all but four scalars and twelve vectors which follows from the ones given in Table 1 possesses a negative value for Newton's constant. For the solution (17) one obtains an interesting effect of the fermions for the critical exponents: the pure gravity solution has a large critical exponent which we estimate to be around 16. Adding now type-I regulated fermions brings this one down to three (which also restores the order such that the critical exponent 4 related to the cosmological constant is the largest one). Adding scalars on top of gravity and fermions slightly increases the values of the second critical exponent but has overall not much effect. The same can be said about the vector fields. All other critical exponents $\Theta_{2,3,\dots}$ are always negative, respectively, possess a negative real part.

In summary, two solutions have been found with endomorphism parameters adjusted such that a global quadratic solution exist for matter up to the Standard Model matter content. This worked because a type-I regulator for the fermions has been used. At least for the type of solutions discussed here type-II regulated fermions quite efficiently lead to a change of sign of Newton's constant and thus outside the class of physically accepted solutions. It should, however, be emphasised that the (non-)existence of physical solutions for this very specific type of global solutions do not provide any argument in favour or against type-I and/or type-II regulators.

3.3. *Asymptotic behaviour for large curvature*

Studying the asymptotic behaviour for large curvature r serves within this investigation two purposes. On the one hand, this knowledge will be employed when numerically solving for a fixed function. On the other hand, it will allow to identify a destabilising influence of matter fields without actually searching for a numerical solution.

As shown below the possible leading asymptotic behaviour for $r \gg 1$ is either $\propto r^2$ or $\propto r^2 \ln r$ (*cf.*, ref. [9]) depending on the values of the endomorphism parameters. The left hand side of eq. (8) at the NGFP reduces to a constant plus linear term if $\varphi^\star(r)$ is a quadratic function due to a cancellation (see footnote 1), and it becomes a quadratic polynomial if a term proportional to $r^2 \ln r$ is added.

Quite obviously cancellations in differences between terms play a significant role. Therefore the most straightforward way to proceed is to infer the large curvature behaviour term by term. In this respect the simplest term is $\mathcal{T}^{\text{Dirac}}$ (10b). It is, in the presence of a non-vanishing quadratic term on the left hand side of the flow equation, subleading because it is a linear function in r. As the scalar matter term $\mathcal{T}^{\text{scalar}}$ and the contribution from the gauge ghost behave identical they can be discussed together. One clearly sees a qualitative difference for $\alpha_S^M \neq 0$, resp., $\alpha_{V2}^M \neq 0$ for which the asymptotic behaviour of the corresponding terms is linear, versus for vanishing parameter (which includes type-I and type-II coarse graining) for which the asymptotic behaviour is quadratic. In the first case these two terms provide singularities at $r = -1/\alpha_S^M$ and $r = -1/\alpha_{V2}^M$, respectively. In the latter case one has, of course, no singularities.

As for the transverse vector matter fields one has to distinguish between the type-II case $\alpha_{V1}^M = -1/4$ for which there is no singularity but a quadratic contribution, and all other cases with a singularity at $r = -1/(\alpha_{V1}^M + 1/4)$ and a leading linear asymptotic behaviour. A completely analogous discussion applies to the gravitational ghost term $\mathcal{T}^{\text{ghost}}$ with the only difference that the type-II corresponds to $\alpha_V^G = +1/4$, and in a similar way to the first line in (9a) (type II corresponds to $\alpha_T^G = -1/6$).

Last but not least, in order to obtain a global solution the moving singularities in the second line of (9a) and in both expressions in (9b) need to cancel against the numerators (Frobenius method). Even if this is arranged then these three terms have leading quadratic behaviour. However, there is one way to avoid this: if $\varphi^\star \propto r^2$ then $\varphi^{\star\prime\prime\prime} \to 0$ for $r \to \infty$, and the remaining two terms can be tuned to cancel.

To summarise this discussion, especially with respect to the impact of matter on the asymptotic behaviour, one notes that for type-II coarse-graining the generic leading behaviour on the right hand side of the flow equation is quadratic. Noting that the solution of the differential equation $4\varphi(r) - 2r\varphi'(r) = cr^2$ is $\varphi = \frac{1}{2}cr^2 \ln r + \mathcal{O}(r^2)$ this implies that the leading behaviour is then $\varphi^\star \propto r^2 \ln r$ for $r \gg 1$. As an advantage one has that then matter does not introduce any new singularities, *i.e.*, the same counting of conditions with respect to the solubility and the number of solutions for this non-linear differential equation applies. For generic endomorphism parameters the leading asymptotic behaviour of the matter contributions is linear, and thus will not qualitatively change the leading asymptotic behaviour of the solution in the pure gravity case. On the other hand, one introduces (even if one sets $\alpha_S^M = \alpha_{V2}^M$ right away) one or two new singularities which will make without fine-tuning (*e.g.*, to push them to values of r in which one is not interested, foremost to negative values) the differential equation only locally solvable. Note that for the transverse vectors and for the Dirac fermions type-I endomorphism parameters behave for this purpose alike general values.

For the scalars and the gauge ghosts the type-I and type-II endomorphism parameters coincide, $\alpha_S^M = \alpha_{V2}^M = 0$. Therefore, the "dangers" of type-II apply for the related two terms also for type-I coarse-graining. At this point, it is interesting to note that,

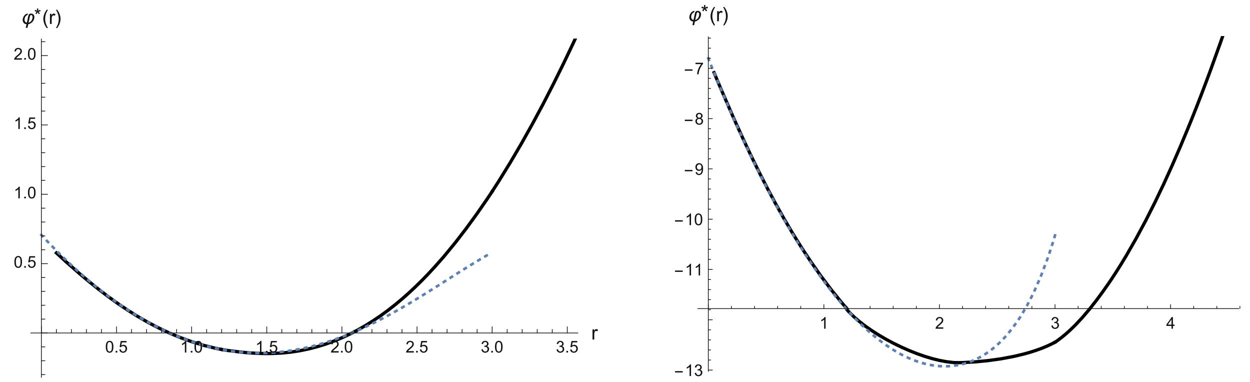

Fig. 1. Displayed are the fixed functions (black lines) for the case of pure gravity (left panel) and with standard model matter content (right panel). The respective polynomial approximations of order 14 using $r = 0$ as an expansion point [2] are shown as dashed lines for comparison.

had one employed an interpolation scheme based on the Euler–MacLaurin formula, the scalar term had simplified very much alike the fermionic one does in the here used averaging interpolation [2,3]:

$$\mathcal{T}^{\text{scalar}} = \frac{N_S}{2}\frac{1}{(4\pi)^2}(1 + (\alpha_S^M + \tfrac{1}{3})r)\,. \tag{24}$$

With this behaviour the contributions of the scalars would be as easily and semi-analytically taken into account as the ones for the fermions here.

3.4. Numerical solutions for global fixed functions

In this section two examples for a numerical solution will be presented, one for pure gravity and one for standard model matter content. Given the fact that type-II coarse graining with standard model matter content will lead to physically unacceptable solutions one may want to employ as coarse graining operator only the Laplacian (type-I). However, then already in the pure gravity case the flow equation will not possess a solution for all positive curvatures r.

The flow equation is a third order equation, and it is only then not over-constraint if there are a most three singularities [9]. Therefore, if the solution had no extremum, and there were no moving singularity one can allow for positive r at most three fixed singularities. However, the physical condition of a positive Newton constant and thus a negative g_1 implies that $\varphi(r)$ decreases at small values of the curvature. On the other hand, at large curvatures the function $\varphi(r)$ should assume a positive value to make the functional integral well-defined which is achieved by $\varphi(r) \to +\infty$ for $r \to \infty$. Consequently, $\varphi^\star(r)$ must possess at least one minimum, and one can allow for at most two fixed singularities. However, for type-I one has four additional fixed singularities at $r_2^{\text{sing}} = 0$, $r_3^{\text{sing}} = 6/5$, $r_4^{\text{sing}} = 3$ and $r_{\text{ghost}}^{\text{sing}} = 4$, where the last one originates from the ghost term $\mathcal{T}^{\text{ghost}}$. Searching for solutions for strictly positive curvature one does not require a condition at $r_2^{\text{sing}} = 0$. The ghost singularity we move to negative values of r by choosing $\alpha_V^G = 1/2$ which is well within the allowed range of parameters [2,5]. This is then the least modification of the flow equation as compared to the one in type-I coarse graining which allows for a numerical solution.

Analysing the flow equation for large r one can infer the behaviour

$$\varphi^\star(r) \propto (2\ln r - 1)\, r^2 + \mathcal{O}\left(\frac{r^2}{\ln r},\; r\,(\ln r)^2\right), \tag{25}$$

which fulfils the orders $r^2(\ln r)^3$ and $r^2(\ln r)^2$ simultaneously. However, this asymptotic behaviour only becomes reasonably precise at extremely large values of r and is thus only of limited use in the numerics.

To obtain numerical solutions a multi-shooting method will be employed. As for the pure gravity case: to this end a minimum at the zero of the second summand in $\mathcal{T}^{\text{TT}}$ (9a) at $r = 3/2$ is enforced. Shooting to the left one can then construct the solution left and right from the singularity by matching it at $r_3^{\text{sing}} \pm 10^{-4} = 6/5 \pm 10^{-4}$ such that a singularity of the third derivative is avoided. The result is displayed in the left panel of Fig. 1.

When adding the standard matter model content ($N_S = 4$, $N_D = 22.5$ and $N_V = 12$ and type-I coarse graining) a polynomial approximation is used as an Ansatz in the differential equation for the fixed function to estimate at which position the minimum of $\varphi^\star(r)$ has to be located. This estimate is then iteratively improved by repeating the analogous procedure as in the gravity case.

The result is displayed in the right panel of Fig. 1. First, one observes clearly the absence of any structure, the global fixed functions are very close to parabolas, *i.e.*, all their features can be captured a quadratic expression with only three coefficients. Second, the agreement with the polynomial approximation extends until $r \approx 2$. For the pure gravity case this implies that the position of the minimum coincides with the one of the polynomial approximation within numerical accuracy. For the matter-gravity system the minimum is slightly shifted from $r = 2.05$ to $r = 2.15$.

In Fig. 2 the functions $\varphi^\star(r)/r^2$ are plotted for two reasons. First, in this way one can check the asymptotic behaviour for large r. Amazingly, the logarithmic dependence of eq. (25) cannot be seen. Even up to very large values of r the extracted leading term is proportional to r^2 within the numerical accuracy. Second, plotting the fixed functions in this way a comparison to the results of ref. [18] is straightforwardly possible. In this investigation fixed functions with a sphere as background have been calculated, however, using a linear split of the metric and a vertex expansion. As argued in this reference the ratio $\varphi^\star(r)/r^2$ is the effective background potential at the fixed point, and a minimum of this function signals a solution of the background equation of motion. The authors of ref. [18] found now in the pure gravity case a background potential without an extremum whereas after adding the Standard Model matter content they observed a minimum at $r \approx 0.1$. From the present work, see Fig. 2, one obtains, on the contrary, a minimum for the pure gravity case and the absence of an extremum with Standard Model matter. In addition, in the pure gravity case the minimum is at a rather large value of the curvature, $r \approx 1.2$.

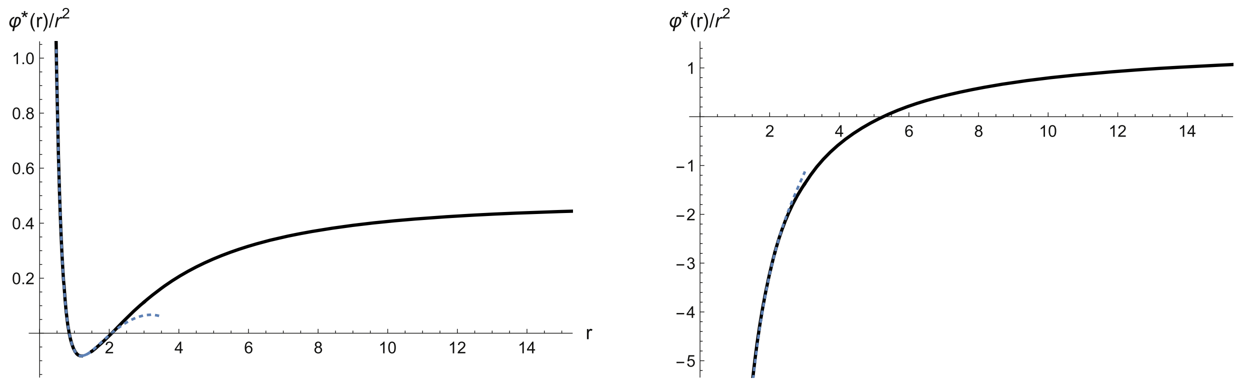

Fig. 2. Displayed are the effective potentials $\varphi^\star(r)/r^2$ (black lines) for the case of pure gravity (left panel) and with standard model matter content (right panel). The respective polynomial approximations of order 14 using $r = 0$ as an expansion point [2] are shown as dashed lines for comparison.

4. Summary and conclusions

In this letter a study of global fixed functions in the context of the asymptotic safety scenario for $f(R)$-gravity coupled to matter has been presented. For some well-chosen sets of coarse-graining parameters global quadratic solution exists. Although these choices can hardly be motivated by physics they explain a remarkable behaviour of the numerically obtained global fixed functions: for all studied cases the deviations from a global quadratic form are tiny. Given this situation one might even speculate that differences to a quite simple form of the global fixed functions might be only due to the employed truncation. In the present investigation only two relevant directions have been found, one of them is directly related to the constant term, *i.e.*, the cosmological constant. The other one is with only very small contributions from higher-order terms a linear combination of a linear and a quadratic term.

The presented investigation emphasises once more the question whether a change of a coarse-graining operator by a non-trivial endomorphism parameter still leads to consider the same theory, or whether the ultraviolet completions of such quantum gravity models are qualitatively different. Searching for global solutions for the fixed functions provided further evidence for the conclusion of ref. [2] based on polynomial approximations, namely, that the NGFPs seen in gravity-matter systems belong to (at least) two different classes. Such a situation deserves certainly further investigations.

When comparing the here obtained numerical solutions for the fixed functions with those of ref. [18] a clear difference can be noted. The vertex expansion used there is certainly a more sophisticated truncation of the flow equation than the single-metric background approach used in this work. On the other hand, the exponential split of the metric employed here might be from a conceptual point of view superior to the linear split, *cf.*, also the recent study [20] where (using also the same gauge as in the presented calculation) a two-parameter family of parameterisations of the split of the metric has been applied to the pure gravity case. Depending on these parameters two different classes for the fixed functions have been found. One might now further investigate whether such differences persist when matter is included, and how the assignment of these different classes relate to the different classes found in ref. [2] and here.

With respect to an investigation of the truncation dependence of the NGFP functions one may build on the recent work of ref. [54]: there a flow equation has been constructed retaining the consistency of the fluctuation field and background field equations of motion even for finite RG scales. Within the background approximation this leads to a modified flow equation containing some additional terms. It is certainly worthwhile to study whether including these terms brings the results for the fixed functions closer to the ones found in a vertex expansion for background and fluctuating fields.

Last but not least, the results for the fixed functions obtained here verify further what one has seen in practically all investigations of the NGFP function for $f(R)$-gravity without and with matter: if a solution for such a function can be found it is very close to a polynomial of only quadratic order.

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