Polynomial solutions of algebraic difference equations and homogeneous symmetric polynomials

Olha Shkaravska and Marko van Eekelen

Radboud University, Nijmegen and Open University of the Netherlands, Heerlen

Abstract

This article addresses the problem of computing an upper bound of the degree d of a polynomial solution P of an algebraic difference equation of the form $G\big(x\big)(P(x-\tau_1),\ldots,P(x-\tau_s)\big)+G_0(x)=0$ when such $P\in\mathbb{K}[x]$ exists and where a field $\mathbb{K}$ is of characteristic zero, $G\in\mathbb{K}[x][x_1,\ldots,x_s]$ and $G_0\in\mathbb{K}[x]$. It is known that, contrary to linear difference equations, there is no general theory for algebraic ones where G has total degree greater than 1.

It will be shown that if G is a quadratic polynomial with constant coefficients then one can construct a countable family of polynomials $f_l(u_0)$ with the following property: if a nonnegative integer number l_0 is the minimal index such that $f_{l_0}(u_0)$ is a non-zero polynomial, then either the degree d is among its roots, or $d\leq l_0$, or $d<\deg(G_0)$. Moreover, the existence of such l_0 is guaranteed if $\mathbb{K}$ is the field of real numbers, and an explicit upper bound for this case will be given. It will be shown that these results do not hold for polynomials G of degree three or greater due to a module-rank reason.

A sufficient condition for the existence of an indicial polynomial for difference equations with G of arbitrary total degree and with variate coefficients will be proven. Moreover we will give an example of the connection between Diophantine equations and algebraic difference equations with variate coefficients.

Key words: quadratic difference equation, power-sum symmetric polynomial, partition, homogeneous symmetric polynomial, Diophantine equation

1. Introduction

This article addresses the problem of determining a range of the degree d of a polynomial solution P of an algebraic difference equation (ADE for short) of the form

$$G\big(x\big)(P(x-\tau_1),\ldots,P(x-\tau_s)\big)+G_0(x)=0, \tag{1}$$

if such a solution $P\in\mathbb{K}[x]$ exists, where $G\in\mathbb{K}[x][x_1,\ldots,x_s]$ and $G_0\in\mathbb{K}[x]$. Here $\mathbb{K}$ denotes a field of characteristic zero and will be used as it throughout this article unless something else for $\mathbb{K}$ is stated explicitly. Moreover, $\tau_i\in\mathbb{K}$ are pairwise distinct.

We study in detail the case of difference equations with constant coefficients

$$G\big(P(x-\tau_1),\ldots,P(x-\tau_s)\big)+G_0(x)=0, \tag{2}$$

where $G(x_1,\ldots,x_s):=\sum\limits_{0\le i_1+\cdots+i_s\le D} a_{i_1,\ldots,i_s}x_1^{i_1}\cdots x_s^{i_s}$ is a polynomial from $\mathbb{K}[x_1,\ldots,x_s]$. The relevant notions and facts from the previous article of the authors (Shkaravska and van Eekelen, 2014) will be recapitulated in *Section 2*, and necessary statements about homogeneous symmetric polynomials will be given in *Section 3*.

It will be shown in Theorem 4 in *Section 4* that given a polynomial G of total degree $D=2$ with constant coefficients w.r.t the variables $x_1,\ldots,x_s$, and the corresponding difference equation (2), one can construct a countable family of univariate polynomials f_l with the following property: if l_0 is the minimal index such that f_{l_0} is a non-zero polynomial, then $f_{l_0}(d)=0$ or $d\le l_0$, or $d<\deg(G_0)$, where d is the degree of a polynomial solution P if such solution exists. The polynomial f_{l_0} is then an *indicial polynomial* for equation (2) similarly to an indicial polynomial defined for first-order linear difference systems in (Abramov and Barkatou, 1998). To prove Theorem 4 it is not required that the field $\mathbb{K}$ is algebraically closed and/or ordered. However, when applying Theorem 4 to instances of equation (2), it is convenient to consider the finite set of shifts $\{\tau_1,\ldots,\tau_s\}$ as a totally-ordered one so that $\tau_1<\cdots<\tau_s$. It is possible since any finite set can be totally-ordered.

In the above mentioned previous article we established the existence of a *finite* family of 6 polynomials which could be candidates for an indicial polynomial for equation (2) where $D\ge 2$. If for a given ADE all the candidates from that family were zero polynomials then the method did not give a bound for the ADE. In the present article we improve that result for quadratic equations showing that the family of the candidates in the quadratic case is countable and search can be continued until the first non-zero candidate is met. Note that the existence of a non-zero candidate f_l in the family is not considered in Theorem 4.

However, in Theorem 5 in *Section 5* we will establish the existence of a non-zero f_{l_0} for $\mathbb{K}=\mathbb{R}$ where $\mathbb{R}$ is the field of real numbers. The index l_0 is determined by the coefficients of the polynomial $G\in\mathbb{K}[x_1,\ldots,x_s]$, the shifts $\{\tau_1,\ldots,\tau_s\}$ and the degree of the polynomial $G_0\in\mathbb{K}[x]$. Therefore an upper bound of the degree d of a polynomial solution of equation (2) with $D=2$ is defined and finite.

In *Section 6* we study difference equations of degree $D\ge 2$ with polynomial coefficients of the form $a_{i_1\ldots i_s}(x)$. We will construct the polynomials $S^*_l(u_0,\mathbf{u}_l)$ representing the corresponding coefficients of x^{Dd+N-l} on the left-hand side of a given equation, where N is the maximal degree of the polynomial coefficients of the terms of degree D in the polynomial G. If one of the polynomials S^*_0, $S^*_1(u_0,0)$ or $S^*_2(u_0,0,0)$ is non-zero, then an upper bound of the solutions' degrees is defined similarly to difference equations with constant coefficients, otherwise the method does not give an answer. Moreover, it will be explained why in general, contrary to ADE with constant coefficients, the argument of homogeneous polynomials does not improve the method. Also, an example of a quadratic equation with linear coefficients, such that it has a polynomial solution of any degree, will be given as well.

If $\mathbb{K}$ has a decidable first-order theory then knowing an upper bound of the degree of a possible polynomial solution for a given ADE allows to find all its polynomial solutions or to establish their absence. This fact is stated in Lemma 15 in *Section 7* and proven by means of *undetermined coefficients*. Indeed, given an upper bound d_0 of the degree of a hypothetical solution P of equation (1), one can solve it by 1) substituting a symbolic solution $P(x) = a_{d_0}x^{d_0} + \cdots + a_1x + a_0$ with unknown coefficients a_i into it, then 2) equating to zero the symbolic coefficients of x^l on the l.h.s. of (1), which yields a nonlinear system w.r.t. a_i, and then 3) solving this system applying a decision procedure for $\mathbb{K}$. If this system is inconsistent, then the difference equation does not have a polynomial solution. A remarkable example of fields whose first-order theories admit quantifier-elimination are real closed fields with the widely-used *cylindrical algebraic decomposition* (Collins, 1998) decision procedure. Before Lemma 15 we will consider an example of reconstructing all possible polynomial solutions for an ADE given an upper bound on their degrees. Moreover, in Section 7 we will give an example illustrating the connection between Diophantine equations and algebraic difference equations with variate coefficients.

In the Appendix A we give a table with notations and definitions is given. This is followed by a bridge between previous results and the results in this paper.Then the proof of an auxiliary Lemma 17 is given. Further, to support the proofs for ADE with variate coefficients, we give a table with the expressions used in these proofs. We have used the computer algebra system (CAS) Maxima to obtain these expressions.Further, the influence of G_0 on the existence of an upper bound of the degree of a polynomial solution is considered in detail. To support the reasoning a table with necessary expressions is given and it is shown how they can be computed using Maxima.

As a *running example* in this article the following difference equation will be used:

$$\begin{aligned}&P(x-1)P(x-1) - 3P(x-1)P(x-2)+\\&\tfrac{5}{2}P(x-2)P(x-2) - \tfrac{1}{2}P(x-2)P(x-4)+\\&\big(-P(x)\big) + 2P(x-1) - \tfrac{1}{8}P(x-2) = 0.\end{aligned} \tag{3}$$

Connection between the previous work of the authors with the work presented in this article

The work under consideration reassesses and extends the results of the earlier research (Shkaravska and van Eekelen, 2014). Let here and below $\mathbf{x}_D$ abbreviate a vector variable $(x_1, \ldots, x_D)$. Also one will use similar abbreviations: $\mathbf{u}_l$, $\mathbf{r}_d$ and $\mathbf{0}_l$ for $(u_1, \ldots, u_l)$, $(r_1, \ldots, r_d)$, and the 0-vector of length l respectively, where $r_1, \ldots, r_d$ denote the roots of a polynomial solution P. In the previous article one constructed a family of polynomials $S_l(u_0, \mathbf{u}_l)$ for equation (2), such that for $l \geq 0$ the polynomial $S_l\big(d, p_1(\mathbf{r}), \ldots, p_l(\mathbf{r})\big)$ is the coefficient of x^{Dd-l} on the left-hand side of equation (2) expanded as a symbolic polynomial in x. If $Dd-l > (D-1)d$ (that is $d > l$) and $(D-1)d \geq \deg(G_0)$ then clearly the identity $S_l\big(d, p_1(\mathbf{r}), \ldots, p_l(\mathbf{r})\big) = 0$ holds. The conditions under which $S_l(u_0, \mathbf{u}_l) = S_l(u_0, \mathbf{0}_l)$ for some $0 \leq l \leq 5$ as polynomials, were given. Therefore, under those conditions, $S_l(d, \mathbf{0}_l) = 0$ and $S_l(u_0, \mathbf{0}_l)$ could be taken as an indicial polynomial for equation (2). In general, if an indicial polynomial is not found among $S_0(u_0, \mathbf{0}_l), \ldots, S_5(u_0, \mathbf{0}_l)$ then the method does not give an answer.

In the current article it will be shown in Theorem 4 that for $D = 2$ the search for an indicial polynomial can be continued for all l until the first $S_l(u_0, \mathbf{0}_l)$ distinct from the zero polynomial is found. The improvement is done through the introduction of the $\mathbb{K}[u_0]$-modules generated by the power-sum products $p_1^{j_1}(\mathbf{x}_D) \cdots p_D^{j_D}(\mathbf{x}_D)$ with $j_1 + 2j_2 + \cdots + Dj_D = l$ being a partition of the number l, with every part at most D.

Related work

The case of quadratic difference equations with shifts $\tau_1, \dots, \tau_s$, considered in this article, is in a sense dual to the case of equations of the form $G\big(P(x), P(x-\tau)\big) = G_0(x)$ of any degree D, that is the values for degree $D = 2$ and the number of shifts τ_j are "swapped". In (Feng et al., 2008) it has been proven that if $G_0(x) \equiv 0$ and G is irreducible, then the degree of a polynomial solution is D.

In the book (Agarwal, 2000) one can find a detailed review of the known analytic and numerical methods for solving difference equations. In particular, in chapter 6 there are statements about the asymptotic and oscillating behavior of solutions of nonlinear equations. However, we believe that these results cannot be used for our purpose since the main statements in the quoted article assume rather strict preconditions, or they consider lower bounds for non-oscillating solutions (whereas our aim is to bound the degree from above). For instance, in Section 6.17 one considers a non-linear equation of a very general form into which equation (2) fits in, but Theorem 6.17.1 about the asymptotic behavior of solutions assumes in its precondition that $G(x_1, \dots, x_s) + G_0(n)$ is bounded by some linear w.r.t. x_i function of the form $\sum_{i=1}^{s} p_i(n)x_i$, which is not possible if G is a nonlinear polynomial in $x_1, \dots, x_s$ and one looks for polynomial substitutions for x_s.

The overview of the related articles about analytical methods was given in (Shkaravska and van Eekelen, 2014). To our knowledge, since that time no new results appeared, that can be used to limit the degree of a polynomial solution. Researchers are mainly interested in wave-like solutions of algebraic difference equations, see, e.g., (Lee and Lee, 2016), whereas our aim are polynomials.

Speaking about algebraic methods for difference equations one should mention the book (van der Put and Singer, 2003), devoted to Galois theory for linear difference equations. The present article might be a step towards developing a similar argument for non-linear equations.

Motivation and applications

Besides being mathematically intriguing objects, nonlinear ADEs have various applications. In particular, they appear in the time- or memory- or other resource-consumption analysis of computer programs with recursive calls. For instance, for a natural number x, equations of the form $P(x) = G\big(x, P(x-1), \dots, P(x-s)\big)$ can represent the resource consumption in the recursive step x with $P(x-1)$,..., $P(x-s)$ representing the corresponding resource consumptions on the previous steps. In general, resource-consumption analysis often yields *inequalities* of the form $P(x) \leq G\big(x, P(x-1), \dots, P(x-s)\big)$, but it will be a topic of our further research since first we want to understand better the nature of the equations.

From the practical point of view, the results discussed in this article improve polynomial resource analysis of computer programs as, for instance, studied in Shkaravska et al. (2009). There the authors consider the size of output as a polynomial function on the sizes of inputs (Tamalet et al., 2008; Shkaravska et al., 2013). In the EU Charter project, the authors developed the ResAna tool (Shkaravska et al., 2007; van Kesteren et al., 2008; Shkaravska et al., 2010; Kersten et al., 2014) that applies polynomial interpolation to generate an upper bound on Java loop iterations. The tool requires the user to input the degree of the solution. The above mentioned previous research (Shkaravska and van Eekelen, 2014) provided a partial result for that. The results of this article will make it possible to automatically obtain the degree of the polynomial in all cases for quadratic ADE with constant coefficients and a subclass of ADE with variate coefficients.

2. Recapitulation: polynomial solutions of difference equations with constant coefficients

To facilitate further reading, we recapitulate the machinery from the previous article (Shkaravska and van Eekelen, 2014) as far as it is necessary to prove the new results.

Let $G_D(x_1, \ldots, x_s) = \sum_{i_1+\ldots+i_s=D} a_{i_1,\ldots,i_s} x_1^{i_1} \cdots x_s^{i_s}$ denote the homogeneous degree D sub-polynomial of the polynomial G. We introduce a reindexation φ for its coefficients in the following way.

Definition 1. Reindexation φ is a map from the set of s-tuples $\{\mathbf{i} = (i_1, \ldots, i_s) | \sum_{j=1}^{s} i_j = D\}$ to the set $\{\tau_1, \ldots, \tau_s\}^D$ such that

$$\varphi: \ (i_1, \ldots, i_s) \mapsto (\underbrace{\tau_1, \ldots, \tau_1}_{i_1}, \underbrace{\tau_2, \ldots, \tau_2}_{i_2}, \ldots, \underbrace{\tau_s, \ldots, \tau_s}_{i_s}).$$

For instance, in equation (3) with $\tau_i = i$ for $i = 0, \ldots, 4$ one has $G_2(x_0, x_1, x_2, x_3, x_4) = x_1^2 - 3x_1x_2 + \frac{5}{2}x_2^2 - \frac{1}{2}x_2x_4$, and G_2 can be considered as a polynomial in $x_1, \ldots, x_4$. The reindexation φ is defined for the non-vanishing coefficients of G_2 in the following way:

	(i_1, i_2, i_3, i_4)	$\varphi(i_1, i_2, i_3, i_4)$	$a_{i_1 i_2 i_3 i_4} = \alpha_{\varphi(i_1,i_2,i_3,i_4)}$	
$x_1^2 = x_1x_1$	$(2, 0, 0, 0)$	$(1, 1)$	$a_{2000} = \alpha_{11} = 1$	
x_1x_2	$(1, 1, 0, 0)$	$(1, 2)$	$a_{1100} = \alpha_{12} = -3$	(4)
$x_2^2 = x_2x_2$	$(0, 2, 0, 0)$	$(2, 2)$	$a_{0200} = \alpha_{22} = \frac{5}{2}$	
x_2x_4	$(0, 1, 0, 1)$	$(2, 4)$	$a_{0101} = \alpha_{24} = -\frac{1}{2}$	

For the sake of convenience introduce the following abbreviations for the tuples of the shifts t_j and the tuples of the roots r_i.

Notation 1. Notations $\mathbf{t}_D$ and $\mathbf{r}_d$ abbreviate the tuples $(t_1, \ldots, t_D)$ of the shifts from the set $\{\tau_1, \ldots, \tau_s\}$ and the tuples $(r_1, \ldots, r_d)$ of the roots of P respectively.

Since the further definitions and proofs are given in terms of the D-tuples $\mathbf{t}_D$ we introduce the following set T over which the D-tuples will range.

Definition 2. The set T is the image $\varphi(\{\mathbf{i} = (i_1, \ldots, i_s) | \sum_{j=1}^{s} i_j = D\})$.

For instance, in the running example with $D = 2$, $s = 4$ and $\tau_i = i$, one has $T = \{(j_1, j_2) | 1 \leq j_1 \leq j_2 \leq 4\}$. Clearly, the reindexation φ is a bijection from the set of all indices $\{\mathbf{i} = (i_1, \ldots, i_s) | \sum_{j=1}^{s} i_j = D\}$ to T since the τ_i-s are pairwise distinct.

Let a polynomial P be represented via its roots: $P(x) = a_d(x - r_1) \cdots (x - r_d)$. The product $P(x-t_1) \cdots P(x-t_D)$ is equal to $a_d^D \prod_{j=1}^{D} \prod_{i=1}^{d} (x-r_i-t_j)$. For this product we are interested in the coefficients of the highest powers of x, namely $x^{Dd}, x^{Dd-1}, \ldots, x^{(D-1)d+1}$. That is we study in detail the coefficient $\varepsilon_l(\mathbf{t}_D, \mathbf{r}_d)$ of x^{Dd-l}, where l is such that $0 \leq l \leq d-1$. This inequation appears due to the following reason: if a polynomial P of degree d solves equation (2) and $Dd - l > (D-1)d$ and $(D-1)d \geq \deg(G_0)$ for some $l \geq 0$ then the coefficients of x^{Dd-l} on the left-hand side of equation (2) must vanish. The inequation $l \leq d-1$ is equivalent to $Dd - l > (D-1)d$.

The sums $(t_j + r_i)$, where $1 \leq j \leq D$, $1 \leq i \leq d$ are the only roots of the polynomial $\prod_{j=1}^{D}\prod_{i=1}^{d}(x - r_i - t_j)$. Therefore, its coefficient $\varepsilon_l(\mathbf{t}_D, \mathbf{r}_d)$ is represented via *the elementary symmetric polynomials* $e_l(y_1, \ldots, y_{dD}) := \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq dD} y_{i_1} \cdots y_{i_l}$ and $e_0(y_1, \ldots, y_{dD}) := 1$ (Macdonald, 1979) in the standard way:

$$\varepsilon_l(\mathbf{t}_D, \mathbf{r}_d) = (-1)^l e_l(t_1 + r_1, \ldots, t_j + r_i, \ldots, t_D + r_d). \tag{5}$$

If the coefficients of x^{Dd-l} on the left-hand side of equation (2) must vanish then the roots $\mathbf{r}_d$ of $P(x)$ must satisfy the identity

$$\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} \varepsilon_l\big(\mathbf{t}_D, \mathbf{r}_d\big) = 0, \tag{6}$$

where $\alpha_{\mathbf{t}_D} = a_{i_1,\ldots,i_s}$ is the coefficient of $x_1^{i_1} \cdots x_s^{i_s}$ in G_D with $\varphi(i_1, \ldots, i_s) = \mathbf{t}_D$.

Equation (6) does not give direct information about d, since each $\varepsilon_l(\mathbf{t}_D, \mathbf{r}_d)$ depends on d implicitly: d is the dimension of $\mathbf{r}_d$. To obtain an explicit equation for d from equation (6), we employ power-sum symmetric polynomials and the Newton-Girard formulæ (Macdonald, 1979):

$$e_l(y_1, \ldots, y_m) = (1/l) \sum_{\kappa=1}^{l} (-1)^{\kappa-1} e_{l-\kappa}(y_1, \ldots, y_m) p_\kappa(y_1, \ldots, y_m), \tag{7}$$

where the power-sum symmetric polynomial $p_\kappa(x_1, \ldots, x_m)$ of degree κ is

$$p_\kappa(x_1, \ldots, x_m) = x_1^\kappa + \cdots + x_m^\kappa. \tag{8}$$

with $p_0(x_1, \ldots, x_m) = m$. As an instance we compute the values $p_\kappa(t_1, t_2)$ for $\kappa = 0, 1, 2$ and $1 \leq t_1 \leq t_2 \leq 4$ which will be used further when studying the running example given by equation (3):

	$(1,1)$	$(1,2)$	$(2,2)$	$(2,4)$
$p_0(t_1,t_2) = 2$	2	2	2	
$p_1(t_1,t_2) = t_1 + t_2$	2	3	4	6
$p_2(t_1,t_2) = t_1^2 + t_2^2$	2	5	8	20

Now we note that by the definition of power-sum polynomials and the binomial formula one has

$$p_\kappa(\ldots, t_j + r_i, \ldots) = \sum_{j=1}^{D}\sum_{i=1}^{d}(t_j + r_i)^\kappa = \sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda} p_\lambda(\mathbf{r}_d) p_{\kappa-\lambda}(\mathbf{t}_D). \tag{9}$$

Also, we introduce the following shortcuts.

Notation 2. The notations $\mathbf{u}_l$ and $\mathbf{v}_l$ abbreviate the tuples of variables $(u_1, \ldots, u_l)$ and $(v_1, \ldots, v_l)$ respectively.

Substituting the tuple $(y_1, \ldots, y_{dD})$ by the tuple $(t_1 + r_1, \ldots, t_j + r_i, \ldots, t_D + r_d)$ and using equality (9) in the Newton-Girard formulæ (7), where $m = dD$, one may see the idea behind the following construction.

Definition 3.

$$E_0\big(v_0, (), u_0, ()\big) := 1,$$
$$E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) := -(1/l) \sum_{\kappa=1}^{l} E_{l-\kappa}(v_0, \mathbf{v}_{l-\kappa}, u_0, \mathbf{u}_{l-\kappa}) (\sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda}).$$

As an instance for this definition, we consider the values of E_l for $l = 0, 1, 2$:

$$\begin{aligned} E_1() &= 1, \\ E_1(v_0, \mathbf{v}_1, u_0, \mathbf{u}_1) &= -v_1 u_0 - v_0 u_1, \\ E_2(v_0, \mathbf{v}_2, u_0, \mathbf{u}_2) &= -\tfrac{1}{2} v_2 u_0 + \tfrac{1}{2} v_1^2 u_0^2 - \tfrac{1}{2} v_0 u_2 - (v_1 - v_1 v_0 u_0) u_1 + \tfrac{1}{2} v_0^2 u_1^2. \end{aligned} \tag{10}$$

One can obtain the expression for any E_l implementing its inductive definition as a symbolic recursive function a computer algebra system. [1]

Now we introduce the notations for the tuples of the values of power-sum polynomials.

Notation 3. The notations $\mathbf{p}(\mathbf{t}_D)$ and $\mathbf{p}(\mathbf{r}_d)$ are the shortcuts denoting the tuples $\big(p_1(\mathbf{t}_D), \ldots, p_l(\mathbf{t}_D)\big)$ and $\big(p_1(\mathbf{r}_d), \ldots, p_l(\mathbf{r}_d)\big)$ respectively.

Using the definition of E_l and identities (5) and (7), by induction on l one can prove that the following identity holds:

$$\varepsilon_l(\mathbf{t}_D, \mathbf{r}_d) = E_l\big(D, \mathbf{p}_l(\mathbf{t}_D), d, \mathbf{p}_l(\mathbf{r}_d)\big). \tag{11}$$

That is $E_l\big(v_0, \mathbf{p}_l(\mathbf{t}_D), u_0, \mathbf{p}_l(\mathbf{r}_d)\big)$ is the coefficient of x^{Dd-l} in the product $P(x-t_1) \cdots P(x-t_D)$. Therefore using the functions E_l, one can symbolically compute $\varepsilon_l(\mathbf{t}_D, \mathbf{r}_r)$ for any $l \geq 0$. For instance, $\varepsilon_1(\mathbf{t}_D, \mathbf{r}_d) = -d\, p_1(\mathbf{t}_D) - D\, p_1(\mathbf{r})_d$.

We define the polynomials $S_l(u_0, \mathbf{u}_l)$ in the following way.

Definition 4. $S_l(u_0, \mathbf{u}_l) := \sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}_D} E_l\big(D, \mathbf{p}_l(\mathbf{t}_D), u_0, \mathbf{u}_l\big)$.

Then by equation (6) and identity (11) one proves the next lemma.

Lemma 1. If a polynomial P of degree d solves equation (2) with constant coefficients and $d > l$ for some $l \geq 0$ and $d \geq \deg(G_0)/(D-1)$ then $S_l\big(d, \mathbf{p}_l(\mathbf{r}_d)\big) = 0$.

Proof. This statement is proven as Lemma 6 in the article (Shkaravska and van Eekelen, 2014). The conditions $d > l$ and $d \geq \deg(G_0)/(D-1)$ together imply that the coefficient $S_l\big(d, \mathbf{p}_l(\mathbf{r}_d)\big)$ of x^{Dd-l} on the left-hand side of equation (2) must vanish. □

[1] We have implemented the symbolic computations for E_l in the Maxima script, available at `http://resourceanalysis.cs.ru.nl#Algebraic Difference Equations`.

The following definitions and notations will be necessary for the new results as well.

Notation 4. Let l be a nonnegative integer. Then $\mathbf{i}_l$ and $\mathbf{j}_l$ denote index tuples $(i_1, \ldots, i_l)$ and $(j_1, \ldots, j_l)$ respectively, and $\mathbf{0}_l$ denote the tuple $(0, \ldots, 0)$ of l zeros.

Definition 5. A polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$, that is $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) = \sum_{\mathbf{i}_l} A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0) u_1^{i_1} \cdots u_l^{i_l}$.

For instance, as one can see from identities (10), the values for $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ with $l = 0, 1, 2$ are

$$\begin{aligned} A_{()}()(u_0) &= 1, \\ A_{(0)}(\mathbf{v}_1)(u_0) &= -u_0 v_1, \\ A_{(00)}(\mathbf{v}_2)(u_0) &= -\tfrac{1}{2} v_2 u_0 + \tfrac{1}{2} v_1^2 u_0^2. \end{aligned} \tag{12}$$

Note that despite $A_{\mathbf{0}_l}(v_0, \mathbf{v}_l)(u_0)$ is formally a polynomial of the variable v_0 as well, it can be shown by induction on l that it does not depend on v_0, so one can use the notation $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ for it.

Definition 6. $B_{l,m}(\mathbf{v}_l)$ is the coefficient of u_0^m in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$.

For instance, $B_{0,0} = 1$, $B_{1,1} = -v_1$, $B_{2,1} = -\frac{1}{2} v_2$ and $B_{2,2} = \frac{1}{2} v_1^2$, see also the article (Shkaravska and van Eekelen, 2014) for more detail. For studying the running example we need the values of $B_{l,m}(\mathbf{v}_l)$ at $p_l(t_1, t_2)$, where $l = 0, 1, 2$:

	$(1,1)$	$(1,2)$	$(2,2)$	$(2,4)$
$B_{1,1} := -p_1(t_1, t_2)$	-2	-3	-4	-6
$B_{2,1} := -\frac{1}{2} p_2(t_1, t_2)$	-1	$-\frac{5}{2}$	-4	-10
$B_{2,2} := \frac{1}{2} p_1^2(t_1, t_2)$	2	$\frac{9}{2}$	8	18

(13)

Applying Definition 5 it is a routine to obtain the representation of $S_l(u_0, \mathbf{u}_l)$ as a polynomial in $\mathbf{u}_l$:

$$S_l(u_0, \mathbf{u}_l) = \sum_{\mathbf{i}_l} \Big(\sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}_D} A_{(i_1, \ldots, i_l)}\big(\mathbf{p}_l(\mathbf{t}_D)\big)\Big)(u_0) u_1^{i_1} \cdots u_l^{i_l}. \tag{14}$$

For the running example with $\mathbf{u}_l := \mathbf{0}_l$ one has

$$\begin{aligned} S_0(u_0) &= 1 - 3 + \tfrac{5}{2} - \tfrac{1}{2} = 0 \\ S_1(u_0, 0) &= 1 \cdot (-2) - 3 \cdot (-3) + \tfrac{5}{2} \cdot (-4) - \tfrac{1}{2} \cdot (-6) = 0 \\ S_2(u_0, 0, 0) &= \big(1 \cdot 2 - 3 \cdot \tfrac{9}{2} + \tfrac{5}{2} \cdot 8 - \tfrac{1}{2} \cdot 18\big) u_0^2 + \\ &\quad \big(1 \cdot (-1) - 3 \cdot (-\tfrac{5}{2}) + \tfrac{5}{2} \cdot (-4) - \tfrac{1}{2} \cdot (-10)\big) u_0 \\ &= -\tfrac{1}{2} u_0^2 + \tfrac{3}{2} u_0 = \tfrac{1}{2} u_0 (3 - u_0). \end{aligned} \tag{15}$$

It was proven that for all $1 \leq L \leq 5$, for all $\mathbf{i}_L \neq \mathbf{0}_L$ and for all $0 \leq m \leq l < L$ there exist polynomials $H_{\mathbf{i}_L, l, m}(v_0, u_0)$ such that

$$A_{\mathbf{i}_L}(v_0, \mathbf{v}_L)(u_0) = \sum_{l=0}^{L-1} \sum_{m=0}^{l} H_{\mathbf{i}_L,l,m}(u_0, v_0) B_{l,m}(\mathbf{v}_l)$$

that is $A_{\mathbf{i}_L}(v_0, \mathbf{v}_L)(u_0)$ is a $\mathbb{K}[u_0, v_0]$-linear combination of $B_{l,m}(\mathbf{v}_l)$ where $0 \le l \le L-1$ and $0 \le m \le l$. This allowed us to prove the main result of the previous work:

Theorem 1. Let $P(x)$ be a polynomial solution of equation (2) and let d be its degree. If the set $\{l | S_l(u_0, \mathbf{0}_l)$ is a non-zero polynomial$\}$ is not empty and, moreover, $L := \min\{l | S_l(u_0, \mathbf{0}_l)$ is a non-zero polynomial$\} \le 5$, then either $d \le L$ or $d < \deg(G_0)/(D-1)\}$, or d must be among the non-negative integer roots of $S_L(u_0, \mathbf{0}_L)$.

For the running example one has $L = 2$, the degree of a polynomial solution is either $d = 0, 1, 2$ or it solves $S_2(u_0, 0, 0) = 0$, that is $d = 3$.

The evidence that for $D = 2$ the theorem above can be refined appeared in the earlier research. The reader who is interested in a smooth transition from the old results to the new ones can find it in Subsection A.2 in the Appendix at the end of this text.

3. Homogeneous symmetric polynomials

As usual, $e_l(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_l \le n} x_{i_1} \cdots x_{i_l}$ and $p_l(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^l$ are the elementary symmetric polynomial and the power-sum symmetric polynomial respectively, of degree l and of variables $x_1, \ldots, x_n$, with $e_0(x_1, \ldots, x_n) = 1$ and $p_0(x_1, \ldots, x_n) = n$.

The following statement is known as the *fundamental theorem of symmetric polynomials* (van der Waerden et al., 2003).

Theorem 2. Let $\mathbb{A}$ be a commutative ring with multiplicative identity $\mathbb{1}$. Then every symmetric polynomial $f(x_1, \ldots, x_n)$ from the subring of symmetric polynomials in $\mathbb{A}[x_1, \ldots, x_n]$ has a unique representation

$$f(x_1, \ldots, x_n) = q\big(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)\big)$$

for some polynomial $q \in \mathbb{A}[y_1, \ldots, y_n]$.

Notation 5. The field of rational numbers is denoted via $\mathbb{Q}$.

Due to the Newton-Girard identities the elementary symmetric polynomial e_l is a rational linear combination of the products of the power-sum symmetric polynomials $p_1, \ldots, p_l$. Therefore one can straightforwardly reformulate the theorem in terms of the power-sum symmetric polynomials:

Theorem 3. Let $\mathbb{A}$ be a commutative ring containing the field $\mathbb{Q}$ (e.g. $\mathbb{A} = \mathbb{L}[x]$ where $\mathbb{L}$ is a field extension of $\mathbb{Q}$). Then every symmetric polynomial f from the subring of symmetric polynomials in $\mathbb{A}[x_1, \ldots, x_n]$ has a unique representation

$$f(x_1, \ldots, x_n) = q\big(p_1(x_1, \ldots, x_n), \ldots, p_n(x_1, \ldots, x_n)\big)$$

for some polynomial $q \in \mathbb{A}[y_1, \ldots, y_n]$.

Notation 6. . Let n be a non-negative integer number. Then $\mathbf{x}_n$ denotes the tuple of variables $(x_1, \ldots, x_n)$.

Notation 7. $|\mathbf{i}_l|$ and $|\mathbf{j}_l|$ denote the sums $i_1 + 2i_2 + \cdots + li_l$ and $j_1 + 2j_2 + \cdots + lj_l$ respectively.

Notation 8. The product $p_1^{j_1}(\mathbf{x}_n) \cdots p_n^{j_n}(\mathbf{x}_n)$ is denoted via $\pi^{\mathbf{j}_n}(\mathbf{x}_n)$.

Definition 7. Let $\mathbb{A}$ be a commutative ring containing $\mathbb{Q}$. Then $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_n|=l}$ denotes the $\mathbb{A}$-module generated by the products $\pi^{\mathbf{j}_n}$ such that $j_1 + 2j_2 + \cdots + nj_n = l$.

Lemma 2. The set of all homogeneous symmetric polynomials of degree l from $\mathbb{A}[\mathbf{x}_n]$ coincides with the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_n|=l}$. [2]

Proof. First, it is easy to see that every polynomial from the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_n|=l}$ is homogeneous and symmetric since every generating polynomial $p_1^{j_1} \cdots p_n^{j_n}$ is symmetric and homogeneous of degree $j_1 + 2j_2 + \cdots + nj_n = l$ w.r.t. $\mathbf{x}_n$. We use the fact that an $\mathbb{A}$-linear combination of homogeneous (resp. symmetric) polynomials is a homogeneous (resp. symmetric) polynomial of the same degree.

Second, the opposite inclusion holds as well, that is every homogeneous symmetric polynomial of degree l from $\mathbb{A}[\mathbf{x}_n]$ belongs to the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_n|=l}$. Indeed, it follows from Theorem 3 that for any symmetric polynomial $f(\mathbf{x}_n)$ there is a polynomial $q(y_1, \ldots, y_n)$ such that $f = q(p_1, \ldots, p_n)$ that is $f(\mathbf{x}_n) = \sum\limits_{|\mathbf{j}_n| \leq l} b_{\mathbf{j}_n} p_1^{j_1}(\mathbf{x}_n) \cdots p_n^{j_n}(\mathbf{x}_n)$ with $b_{\mathbf{j}_n} \in \mathbb{A}$. Next, since f is homogeneous then for every $l_0 = j_1 + \cdots + nj_n < l$ the the part $\sum_{|\mathbf{j}_n|=l_0} b_{\mathbf{j}_n} p_1^{j_1}(\mathbf{x}_n) \cdots p_n^{j_n}(\mathbf{x}_n)$ must vanish, otherwise f would not be homogeneous. Therefore $f(\mathbf{x}_n)$ is an $\mathbb{A}$-linear combination of products $p_1^{j_1} \cdots p_n^{j_n}$ with $j_1 + \cdots + nj_n = l$ and therefore belongs to the $\mathbb{A}$-module $\langle \pi^{\mathbf{j}_n} \rangle_{|\mathbf{j}_n|=l}$. □

Lemma 3. Let $f(x, y)$ be a symmetric homogeneous polynomial in $\mathbb{R}[x, y]$ of even degree l and let (x_m, y_m) be a collection of $l/2 + 1$ points, where $0 \leq m \leq l/2$, such that $y_m \geq x_m > 0$. Moreover, let at most for one pair the equality $y_m = x_m$ may hold and let all the lines $y = \frac{y_m}{x_m} x$ be pairwise distinct.

If $f(x_m, y_m) = 0$ for all these nodes then $f(x, y)$ is the zero polynomial.

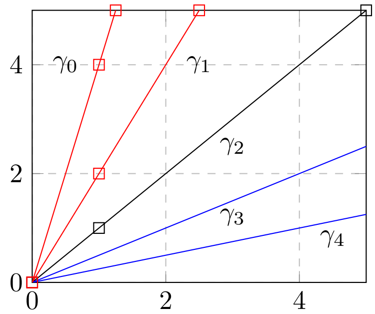
Proof. If $f(x_m, y_m) = 0$ then $f(\lambda x_m, \lambda y_m) = \lambda^l f(x_m, y_m) = 0$ due to the fact that f is homogeneous. The set $\gamma_m = \{(\lambda x_m, \lambda y_m) | \lambda \in \mathbb{R}\}$ is a parametric definition of the line defined by points $(0, 0)$ and (x_m, y_m).

Since f is symmetric, $f(x, y) = 0$ for all (x, y) that lie on the lines γ_{l-m}, where the line γ_{l-m} is symmetric to γ_m w.r.t. the line $y = x$. This implies that all together there are at least $l + 1$ lines on which the polynomial $f(x, y)$ is equal to 0.

---

[2] This statement mimics Corollary 7.7.2 from the book (Stanley, 1999). The difference is that there not symmetric polynomials but formal power series over infinite number of variables are considered, that is constructions of the form $\sum_{(\alpha_1, \alpha_2, \ldots), \alpha_1 + \alpha_2 + \cdots = n} x_1^{\alpha_1} x_2^{\alpha_2} \ldots \in \mathbb{R}[x_1, x_2, \ldots]$. For instance $p_3(x_1, x_2, \ldots, x_\ell)$ for some finite $\ell \geq 3$ is an element of the canonical generating set of the collection Λ^3 of such symmetric homogeneous functions since ℓ and (therefore degrees of the power-sums in the products of the generators) are not bounded, however $p_3(x_1, x_2)$ does not belong to the canonical generating set of $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=3}$.

Example sketch for $l = 4$, with $(x_0, y_0) = (1, 4)$, $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (1, 1)$:



As one can see, the polynomial $f(x, y)$ has at least $l + 1$ zeros in the projective space $\mathbb{P}(\mathbb{R})$ of dimension one. Therefore $f(x, y)$ is the zero polynomial. □

4. Properties of the coefficients of the u-terms

4.1. General properties of the coefficients of the u-terms if D is an arbitrary integer greater than 1.

Here we consider the properties of the polynomials $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)$ which were defined earlier in Section 2.

Notation 9. $\mathbb{K}[u_0]$ stays for the ring of polynomials of the variable u_0 with coefficients in $\mathbb{K}$.

Let D be a non-negative integer. Similarly to Notation 3, let $\mathbf{p}_l(\mathbf{x}_D)$ denote the l-tuple of the power-sum symmetric polynomials $\big(p_1(\mathbf{x}_D), \ldots, p_l(\mathbf{x}_D)\big)$.

Lemma 4. The polynomial $A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0) \in \mathbb{K}[\mathbf{x}_D][u_0] \cong \mathbb{K}[u_0][\mathbf{x}_D]$ is a homogeneous polynomial of $\mathbf{x}_D$ of degree $l - |\mathbf{i}_l|$.

Proof. The proof follows by induction on l. For the base case $l = 0$ one has $A_{0,()} = 1$ and the statement of the lemma is obvious.

For the induction step for $l \geq 1$ we fix some $\mathbf{i}_l = (i_1, \ldots, i_l)$. The term of the monomial $u_1^{i_1} \cdots u_l^{i_l}$ in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ is obtained by the definition of E_l inductively as the sum

- of all the terms of E_{l-k} containing the monomials of the form
$$u_1^{i_1} \cdots u_{\lambda-1}^{i_{\lambda-1}} u_\lambda^{i_\lambda - 1} u_{\lambda+1}^{i_{\lambda+1}} \cdots u_l^{i_l},$$
multiplied by $\binom{k}{\lambda} u_\lambda v_{k-\lambda}$, if $i_\lambda \geq 1$,
- of all the terms of E_{l-k} containing the monomials of the form $u_1^{i_1} \cdots u_l^{i_l}$ multiplied by $\binom{k}{0} u_0 v_k$.

Formally this fact is expressed as follows. Let l_0 denote the maximal l' such that $i_{l'} \neq 0$ i.e. $\mathbf{i}_l = (i_1, \ldots, i_{l'}, \ldots, i_l) = (i_1, \ldots, i_{l_0}, 0, \ldots, 0)$. Since by the definition the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$, is the coefficient of the monomial $u_1^{i_1} \cdots u_l^{i_l}$ in $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$, one has

$$
\begin{aligned}
&A_{(i_1,\dots,i_{l_0},0,\dots,0)}\big(D,\mathbf{p}_l(\mathbf{x}_D)\big)(u_0) = (-1/l)\cdot\Big(\\
&\sum_{1\le k\le l,\, l-k\ge l_0}\ \sum_{\lambda, i_\lambda>0} A_{(i_1,\dots,i_{\lambda-1},(i_\lambda-1),i_{\lambda+1},\dots,i_{l-k})}\big(D,\mathbf{p}_{l-k}(\mathbf{x}_D)\big)(u_0)\binom{k}{\lambda}p_{k-\lambda}(\mathbf{x}_D)+\\
&\sum_{1\le k\le l, l-k\ge l_0} A_{(i_1,\dots,i_{l-k})}\big(D,\mathbf{p}_l(\mathbf{x}_D)\big)(u_0)u_0p_k(\mathbf{x}_D)\Big).
\end{aligned}
\tag{16}
$$

By the induction assumption the polynomial

$$
A_{(i_1,\dots,i_{\lambda-1},(i_\lambda-1),i_{\lambda+1},\dots,i_{l-k})}\big(D,\mathbf{p}_l(\mathbf{x}_D)\big)(u_0)
$$

from the first sum is homogeneous in $\mathbf{x}_D$ of degree $\deg_1 := (l-k)-\big(i_1+2i_2+\cdots+(\lambda-1)i_{\lambda-1}+\lambda(i_\lambda-1)+(\lambda+1)i_{\lambda+1}+\cdots+(l-k)i_{l-k}\big)$. Obviously, its product with $p_{k-\lambda}(\mathbf{x}_D)$ is homogeneous of degree $\deg_1+(k-\lambda) = l-(i_1+2i_2+\cdots+li_l)$. Indeed,

$$
\begin{aligned}
&(l-k)-\big(i_1+\cdots+(\lambda-1)i_{\lambda-1}+\lambda(i_\lambda-1)+(\lambda+1)i_{\lambda+1}+\cdots+(l-k)i_{l-k}\big)+(k-\lambda) =\\
&l-\big(i_1+\cdots+(\lambda-1)i_{\lambda-1}+\lambda(i_\lambda-1)+\lambda+(\lambda+1)i_{\lambda+1}+\cdots+(l-k)i_{l-k}\big) =\\
&l-\big(i_1+\cdots+(\lambda-1)i_{\lambda-1}+\lambda i_\lambda+(\lambda+1)i_{\lambda+1}+\cdots+(l-k)i_{l-k}+\\
&(l-k+1)\cdot 0+\cdots+l\cdot 0\big) =\\
&l-(i_1+\cdots+li_l).
\end{aligned}
$$

using the fact that $i_{l'}=0$ for all $l'>l-k\ge l_0$.This implies that every summand in the first sum is homogeneous of degree $l-|\mathbf{i}_l|$.

Similarly, by the induction assumption the polynomial $A_{\mathbf{i}_{l-k}}\big(D,p_l(\mathbf{x}_D)\big)(u_0)$ from the second sum is homogeneous in $\mathbf{x}_D$ of degree $\deg_2 := (l-k)-\big(i_1+\cdots+(l-k)i_{l-k}\big)$. Therefore its product with p_k is homogeneous of degree $\deg_2+k = l-|\mathbf{i}_l|$. Indeed,

$$
\begin{aligned}
&(l-k)-\big(i_1+\cdots+(l-k)i_{l-k}\big)+k =\\
&l-\big(i_1+\cdots+(l-k)i_{l-k}+(l-k+1)\cdot 0+\cdots+l\cdot 0\big) =\\
&l-(i_1+\cdots+li_l).
\end{aligned}
$$

This implies that every summand in the second sum is homogeneous of degree $l-|\mathbf{i}_l|$ as well. Therefore $A_{\mathbf{i}_l}\big(D,p_l(\mathbf{x}_D)\big)(u_0)$ is a homogeneous polynomial of degree $l-|\mathbf{i}_l|$ in $\mathbb{K}[u_0][\mathbf{x}_D]$. □

Strictly speaking the proof above does not show that $A_{\mathbf{i}_l}\big(D,p_l(\mathbf{x}_D)\big)(u_0)$ is a non-zero polynomial, but this is not relevant for the way Lemma 4 will be used.

From Lemma 4 and Lemma 2 one immediately obtains the following result.

Lemma 5. The polynomial $A_{\mathbf{i}_l}\big(D,\mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ belongs to the $\mathbb{K}[u_0]$-module $\langle\pi^{\mathbf{j}_D}\rangle_{|\mathbf{j}_D|=l-|\mathbf{i}_l|}$; that is it is a $\mathbb{K}[u_0]$-linear combination of the products of the form $p_1^{j_1}\cdots p_D^{j_D}$ where $|\mathbf{j}_D| = l-|\mathbf{i}_l|$.

Lemma 6. For $D \geq 3$ the family $\{B_{l,m}\big(\mathbf{p}_l(\mathbf{x}_D)\big)\}_{m=1}^{l}$ does not contain a generator set of the module $\langle\pi^{\mathbf{j}_D}\rangle_{|\mathbf{j}_D|=l}$.

Proof. Since $\mathbb{K}[u_0]$ is a commutative ring one can apply rank reasons for the $\mathbb{K}[u_0]$-module as one would apply dimension reasons for a linear space over a field because for a commutative ring $\mathbb{A}$ an isomorphism $\mathbb{A}^m \cong \mathbb{A}^n$ implies $m = n$, see e.g. (Dummit and Foote, 2003), Exercise 2 of Section 10.3. This means that the size of any generator set of the $\mathbb{K}[u_0]$-module $\langle\pi^{\mathbf{j}_D}\rangle_{|\mathbf{j}_D|=l}$ must be exactly the same as the size of its "canonical generator" set, which is the collection of the products $\pi^{\mathbf{j}_D}$ where $|\mathbf{j}_D| = l$.

The set $\{B_{l,m}\big(\mathbf{p}_l(\mathbf{x}_D)\big)\}_{m=1}^{l}$ contains l non-zero polynomials, whereas the rank of $\langle\pi^{\mathbf{j}_D}\rangle_{|\mathbf{j}_D|=l}$ is the number $part_D(l)$ of the partitions $(1^{j_1}, 2^{j_2}, \ldots, D^{j_D})$ of l such that $j_1 + 2j_2 + \cdots + Dj_D = l$. For $D = 2$ this number is $part_2(l) = \lfloor l/2 \rfloor + 1$. However for $D = 3$ this is $part_3(l)$ which is the nearest to $(l+3)^2/12$ integer number (Stanley, 1997). It is a routine to check (e.g. by induction on $l \geq 6$, and direct calculations for $l = 0, \ldots, 5$) that this number, which is an increasing function of l, may be less or equal to l only for $l \leq 5$, and otherwise the rank of $\langle\pi^{\mathbf{j}_D}\rangle_{|\mathbf{j}_D|=l}$ exceeds the number of polynomials in $B_{l,m}\big(\mathbf{p}_l(\mathbf{x}_D)\big)$. In particular, for $l = 6$ one has that $(l+3)^2/12 = 81/12 = 6,75$ with the nearest integer number equal to 7.

In general, $part_D(l)$ is bounded from below by a polynomial of degree $D - 1$, see (Stanley, 1997) and a similar argument holds for any $D \geq 3$. [3] $\square$

4.2. Specific properties of the coefficients of the u-terms when D is 2

Everywhere in this sub-section it is assumed that $D = 2$ and the power-sum polynomials are bivariate of the form $p_l(x_1, x_2)$.

In Lemma 8 below we will show that for any $l \geq 1$ and any $0 \leq k \leq \lfloor l/2 \rfloor$ the polynomial $B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big)$ is a rational linear combination of the products p_1^l, $p_1^{l-2}p_2$, $p_1^{l-4}p_2^2$,..., $p_1^{l-2k}p_2^k$ and moreover the coefficient of $p_1^{l-2k}p_2^k$ in this combination does not vanish. This will allow us to express for any l the generators $p_1^{j_1}p_2^{j_2}$ of the module $\langle\pi^{\mathbf{j}_2}\rangle_{|\mathbf{j}_2|=l}$ as linear combinations of the polynomials $\{B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big)\}$. This fact will be used to prove Lemma 10 which states that $\{B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big)\}$ is a generator set of the $\mathbb{K}[u_0]$-module $\langle\pi^{\mathbf{j}_2}\rangle_{|\mathbf{j}_2|=l}$.

Since the polynomials $B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big)$ are homogeneous polynomials of degree l in the variables x_1 and x_2, they are linear combinations of the products $p_1^{j_1}(x_1, x_2)p_1^{j_2}(x_1, x_2)$ by Lemma 2. However to prove Lemma 8 we must know more about these linear combinations. For this we need the following auxiliary statement.

Lemma 7. Let $D = 2$, $l \geq 0$ and $0 \leq k \leq \lfloor l/2 \rfloor$. Then the maximum possible degree of the power-sum $p_2(x_1, x_2)$ in the linear combination representing $B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_2)\big)$ in the module $\langle\pi^{\mathbf{j}_2}\rangle_{|\mathbf{j}_2|=l}$ is k. Moreover for $l \geq 1$, given the recurrent formula (see Lemma 17 in the Appendix)

---

[3] In item 10 under Corollary 1.4 of Stanley's book it is shown that the number of partitions $part'_k(n)$ of n into k parts is the same as the number of partitions $part_k(n)$ where the largest part is at most k. In Example 4.4.2 of the book it shown that $part'_k(n)$ is a *quasipolynomial* of degree $k - 1$ with the minimal period equal to the least common multiple N of $1, \ldots, k$, that is there are N polynomials f_i of degree at most $k - 1$ such that $part'_k(l) = f_i(l)$ once $l \equiv i \mod N$.

$$B_{l,l-k}\big(\mathbf{p}_l(\mathbf{x}_D)\big) = -1/l \sum_{h=1}^{k+1} B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(\mathbf{x}_D)\big)p_h(\mathbf{x}_D) \tag{17}$$

this maximum k is achieved for $h = 1, 2$ if $0 < k < l/2$, for $h = 1$ if $k = 0$, and for $h = 2$ if $k = l/2$.

Proof. The proof is done by induction on l. We begin with the base case where $l = 0, 1, 2$.

When $l = 0$ one has $B_{0,0} = 1$ and $k = 0$. The maximal power of $p_2(x_1, x_2)$ in the representation of $B_{0,0}$ is obviously $0 = k$.

Now, let $l = 1$. Then for $k = 0$ one has $B_{l,l-k}\big(\mathbf{p}_l(x_1, x_2)\big) = B_{1,1}\big(\mathbf{p}_1(x_1, x_2)\big) = -p_1(x_1, x_2)$. The maximal power of $p_2(x_1, x_2)$ here is $0 = k$ and it is achieved for $h = 1$ with the only summand $B_{0,0}p_1(x_1, x_2)$.

Further, for $l = 1$ the index $k = 1$ does not satisfy $k \le \lfloor \frac{l}{2} \rfloor$ so $B_{1,0} = 0$ is out of consideration.

Let $l = 2$. For $k = 0$ one has $B_{2,2}\big(\mathbf{p}_2(x_1, x_2)\big) = \frac{1}{2}p_1^2(x_1, x_2)$. The maximal power of $p_2(x_1, x_2)$ here is $0 = k$ and it is achieved for $h = 1$ with the only summand $B_{1,1}p_1(x_1, x_2) = -p_1(x_1, x_2)p_1(x_1, x_2)$.

For $l = 2$ and $k = 1 = l/2$ one has $B_{2,1}\big(\mathbf{p}_2(x_1, x_2)\big) = -\frac{1}{2}p_2(x_1, x_2)$. There are two summands corresponding to $h = 1$ and $h = 2$ in the recurrent formula: $B_{1,0}p_1(x_1, x_2)$ and $B_{0,0}p_2(x_1, x_2)$ respectively. Recall that by Lemma 17 one has $B_{l,0}(\mathbf{v}_l) = 0$ since $l > 0$. Therefore the first summand vanishes and the maximum of the degree of p_2 in the representation of $B_{2,1}\big(\mathbf{p}_2(x_1, x_2)\big)$ is $1 = k$ and it is achieved for $h = 2$.

Now we continue with the induction step. We fix $l \ge 3$ and $h \ge 1$ and consider the product $B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(x_1, x_2)\big)p_h(x_1, x_2)$ in detail.

By Lemma 2 the product $B_{l-h}p_h$ expands into a $\mathbb{Q}$-linear combination of all the possible terms of the form $p_1^{j_1'}p_2^{j_2'}p_1^{j_1''}p_2^{j_2''}$ where the first sub-term $p_1^{j_1'}p_2^{j_2'}$ comes from the expansion of $B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(x_1, x_2)\big) = B_{l-h,(l-h)-(k-h+1)}\big(\mathbf{p}_{l-h}(x_1, x_2)\big)$ and the second sub-term $p_1^{j_1''}p_2^{j_2''}$ comes from the expansion of p_h.

Later we will analyse all three cases: $0 < k < l/2$, $k = 0$ and $k = l/2$. Before doing that we consider now the technical situation when $0 \le k - h + 1 \le \lfloor (l - h)/2 \rfloor$, which, firstly, falls under the induction assumption, and, secondly, as we will check later, covers the cases $0 \le k < l/2$ with $1 \le h \le k + 1$ and $k = l/2$ with $h \ge 2$.

Let $0 \le k - h + 1 \le \lfloor (l - h)/2 \rfloor$. Then one can apply the induction assumption: there is a term containing the product $p_1^{(l-h)-2(k-h+1)}p_2^{k-h+1}$ in the expansion of

$$B_{l-h,(l-h)-(k-h+1)}\big(\mathbf{p}_{l-h}(x_1, x_2)\big),$$

which has the maximal degree $k - h + 1$ of p_2. Fix this maximal j_2' and the corresponding j_1':

$$\begin{aligned} j_2' &= k - h + 1, \\ j_1' &= (l - h) - 2(k - h + 1). \end{aligned} \tag{18}$$

By Lemma 2 in the expansion of p_h one has $j_1'' + 2j_2'' = h$. The highest possible degree of p_2 in the expansion of p_h is therefore

$$\begin{aligned} j_2'' &= \lfloor h/2 \rfloor, \text{ and} \\ j_1'' &= h - 2\lfloor h/2 \rfloor. \end{aligned} \tag{19}$$

The corresponding product $p_1^{(l-h)-2(k-h+1)} p_2^{k-h+1} p_1^{h-2\lfloor h/2 \rfloor} p_2^{\lfloor h/2 \rfloor}$ yields the highest possible degree of p_2 in the expansion of the product

$$B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(x_1, x_2)\big) p_h(x_1, x_2). \tag{20}$$

Let the highest possible degree of $p_2(x_1, x_2)$ in this expansion is denoted via $g_{l,k}(h)$. As we have just shown, $g_{l,k}(h) = k - h + 1 + \lfloor h/2 \rfloor$. It is easy to see that it is a non-strictly decreasing function of h. Therefore its maximum is achieved for $h = 1$ with $g_{l,k}(1) = k - 1 + 1 + 0 = k$. For $k \geq 1$ this maximum is achieved for $h = 2$ as well, with $g_{l,k}(2) = k - 2 + 1 + 1 = k$. Checking at $h = 3$ yields $g_{l,k}(3) = k - 3 + 1 + 1 = k - 1$.

We have proven the technical statement about the maximal degree of the $p_2(x_1, x_2)$ in the product $B_{l-h,l-k-1}\big(\mathbf{p}_{l-h}(x_1, x_2)\big) p_h(x_1, x_2)$ when $0 \leq k - h + 1 \leq \lfloor (l - h)/2 \rfloor$. This degree is equal to k and achieved for $h = 1$ and also for $h = 2$ if $k > 0$. Now we can apply this fact to consider three cases from the statement of the lemma.

First, let $0 < k < l/2$. If l is odd then $k < l/2$ implies $k \leq (l - 1)/2$. This implies $k-h+1 \leq (l-1)/2-h+1 = (l-1-2h+2)/2 = (l+1-h-h)/2 \leq (l-h)/2$ for $h \geq 1$. Since $k-h+1$ is integer then it is less or equal to the nearest to $(l-h)/2$ integer number, that is $k-h+1 \leq \lfloor (l-h)/2 \rfloor$. If l is even then $k < l/2$ implies $k \leq (l-2)/2$ and therefore $k-h+1 \leq (l-2)/2-h+1 = (l-2-2h+2)/2 = (l-h-h)/2 \leq (l-h-1)/2$ for $h \geq 1$. If $l-h$ is odd then $(l-h-1)/2 = \lfloor (l-h)/2 \rfloor$ and therefore $k-h+1 \leq \lfloor (l-h)/2 \rfloor$. If $l-h$ is even then $(l-h-1)/2 < (l-h)/2 = \lfloor (l-h)/2 \rfloor$ and therefore $k-h+1 < \lfloor (l-h)/2 \rfloor$.

Therefore $k < l/2$ implies $k-h+1 \leq \lfloor (l-h)/2 \rfloor$ and one uses the technical statement above to obtain $g_{l,k}(h) = k - h + 1 - \lfloor h/2 \rfloor$, with the maximum value of this function equal to k achieved for $h = 1, 2$.

Second, let $k = 0$. Then h can take only one value $h = 1$, and moreover $0 \leq k-h+1 = 0 \leq \lfloor (l-h)/2 \rfloor$ holds, so one can apply reasoning for the technical case above, excluding the part about the function $g_{l,k}(h) = k - h + 1 + \lfloor h/2 \rfloor$ in $h = 2$. In the case of $k = 0$ the function $g_{l,k}(h)$ is defined only in $h = 1$ and $g_{l,k}(1) = 0$ is its maximum.

Third, let $k = l/2$ for even l. Let first $h \geq 2$. This means that $k-h+1 = l/2-h+1 = (l-2h+2)/2 = (l+2-h-h)/2 \leq (l-h)/2$ since $h \geq 2$. Since $k-h+1$ is integer then it is less or equal to the nearest to $(l-h)/2$ integer number, that is $k-h+1 \leq \lfloor (l-h)/2 \rfloor$. Therefore, in this case $g_{l,k}(h) = k-h+1-\lfloor h/2 \rfloor$, with the maximum value of this function equal to k achieved for $h = 2$. Now, let, second, $h = 1$. We will see now that $g_{l,k}(1) = k-1$ that is the function $g_{l,k}(h)$ does not reach its maximum value k on $h = 1$. Indeed, the polynomial $B_{l-h,l-k-1}\big(p_1(x_1, x_2), p_2(x_1, x_2)\big)$ is of degree $l-h$ and the maximal possible degree of the occurrences of $p_2(x_1, x_2)$ in it is $\lfloor (l-h)/2 \rfloor$. Next, the maximal degree of the occurrences of p_2 in the product $B_{l-h,l-k-1}\big(p_1(x_1, x_2), p_2(x_1, x_2)\big) p_h(x_1, x_2)$ is therefore $\lfloor (l - h)/2 \rfloor + \lfloor h/2 \rfloor$, which for $h = 1$ is equal to $\lfloor (l - 1)/2 \rfloor + \lfloor 1/2 \rfloor = k - 1 + 0 = k - 1$, since $k = l/2$ and l is even.

The lemma is proven. □

Now we can prove Lemma 8.

Lemma 8. For $D = 2$, integer numbers l, k and j such that $l \geq 0$ and $0 \leq k \leq \lfloor l/2 \rfloor$ and $0 \leq j \leq k$, there exist rational numbers $b_{l,k,j}$ such that

$$B_{l,l-k}\big(\mathbf{p}_l(x_1,x_2)\big) = \sum_{j=0}^{k} b_{l,k,j} p_1^{l-2j}(x_1,x_2) p_2^j(x_1,x_2). \tag{21}$$

and, moreover, $b_{l,k,k} \neq 0$ has sign $(-1)^{l-k}$.

Proof. The fact that $B_{l,l-k}\big(\mathbf{p}_l(x_1,x_2)\big)$ is presented as the linear combination of the form (21) follows directly from Lemma 7. We have to show that the coefficient $b^k_{l,l-k}$ of the product $p_1^{l-2k}p_2^k$, with the highest power k of p_2 in this expansion does not vanish and has the sign as stated in the conclusion of the lemma. This is proven by the induction on l.

We start the proof with the base cases where $l = 0, 1, 2$. For $B_{0,0} = 1$ one has $l = 0, k = 0$ and $b_{0,0,0} = 1 = (-1)^{0-0}$. Now, let $l = 1$. Then for $k = 0$ one has $B_{l,l-k}\big(\mathbf{p}_l(x_1,x_2)\big) = B_{1,1}\big(\mathbf{p}_1(x_1,x_2)\big) = -p_1(x_1,x_2)$, and $b_{1,0,0} = (-1)^1 = (-1)^{l-k}$. Further, for $l = 1$ the index $k = 1$ does not satisfy $k \leq \lfloor \frac{l}{2} \rfloor$ so $B_{1,0} = 0$ is out of consideration. Let $l = 2$. For $B_{2,2}\big(\mathbf{p}_2(x_1,x_2)\big) = \frac{1}{2}p_1^2(x_1,x_2)$ one has $k = 0$ and $b_{2,0,0} = 1/2$ with sign $(-1)^{2-0}$. For $B_{2,1}\big(\mathbf{p}_2(x_1,x_2)\big) = -\frac{1}{2}p_2(x_1,x_2)$ one has $k = 1$ and $b_{2,1,1} = -\frac{1}{2}$ with sign $(-1)^{2-1}$.

Now we continue with the induction step and, again, we will use the identity (17). Fix $l \geq 0, 0 \leq l \leq \lfloor l/2 \rfloor$ and $1 \leq h \leq k+1$. Let $c_{h,j'} \in \mathbb{Q}$ denote the coefficient of $p_1^{h-2j'}p_2^{j'}$ in the expansion of p_h. Then by the case analysis as stated in Lemma 7 one obtains that

$$b_{l,k,k} = -1/l(b_{l-1,k,k}c_{1,0} + b_{l-2,k-1,k-1}c_{2,1}), \tag{22}$$

where the indices are computed by setting

- $h = 1$ (or $b_{2N-1,N,N} = 0$ for even $l = 2N$ and $k = N$) in the first summand;
- $h = 2$ (or $b_{l-2,k-1,k-1} = 0$ for $k = 0$) in the second summand.

Since $c_{1,0} = 1$ is the coefficient of $p_1^1p_2^0$ in the expansion of p_1 in the basis of $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=l}$ and $c_{2,1} = 1$ is the coefficient of $p_1^0p_2^1$ in the expansion of p_2, we get

$$b_{l,k,k} = -1/l(b_{l-1,k,k} + b_{l-2,k-1,k-1}). \tag{23}$$

Now, either by the induction assumption for $l-1$ one has $b_{l-1,k,k} \neq 0$ with its sign equal to $(-1)^{l-1-k}$ if $k < l/2$ (that is $k \leq \lfloor (l-1)/2 \rfloor$), or $b_{2N-1,N,N} = 0$ otherwise for $l = 2N, k = N$. The value $b_{l-2,k-1,k-1} \neq 0$ is either zero for $k = 0$ or it has sign $(-1)^{l-2-(k-1)} = (-1)^{l-1-k}$ by the induction assumption for $l-2$ and $k-1 \leq \lfloor (l-2)/2 \rfloor$, following from $k \leq l/2$. Therefore the sum of these two coefficients has either the *same* sign $(-1)^{l-1-k}$ or one of them is zero. Note that both of them cannot be zero simultaneously, because $k = 0 = N$ implies $l = 0$ and this is not the case for the induction step. In any case, both coefficients do not cancel each other and therefore $b_{l-1,k,k}$ has sign $(-1)(-1)^{l-1-k} = (-1)^{l-k}$.

□

Lemma 9. Any polynomial of the form $p_1^{l-2k}p_2^k$ where k ranges from 0 to $\lfloor l/2 \rfloor$, is a $\mathbb{Q}$-linear combination of the polynomials

$$B_{l,l}\big(\mathbf{p}_1(x_1,x_2)\big), B_{l,l-1}\big(\mathbf{p}_l(x_1,x_2)\big), \ldots, B_{l,l-k}\big(\mathbf{p}_l(x_1,x_2)\big).$$

Proof. Fix an arbitrary l and run the induction on k using Lemma 8. For the base case $k = 0$ from equality (21) one trivially obtains $p_1^l(x_1, x_2) = B_{l,l}\big(\mathbf{p}_l(x_1, x_2)\big)/b_{l,0,0}$, where $b_{l,0,0} \neq 0$ has sign $(-1)^l$.

For the induction step we use again equation (21) of Lemma 8:

$$B_{l,l-k}\big(\mathbf{p}_l(x_1, x_2)\big) = b_{l,k,k} p_1^{l-2k}(x_1, x_2) p_2^k(x_1, x_2) + \sum_{j=0}^{k-1} b_{l,k,j} p_1^{l-2j}(x_1, x_2) p_2^j(x_1, x_2)$$

From this equality and $b_{l,k,k} \neq 0$ it follows that the product $p_1^{l-2k} p_2^k$ is a linear combination of $B_{l,l-k}\big(\mathbf{p}_l(x_1, x_2)\big)/b_{l,k,k}$ and $p_1^{l-2j} p_2^j$ where $j < k$. By the induction assumption any such $p_1^{l-2j} p_2^j$ is a $\mathbb{Q}$-linear combination of polynomials

$$B_{l,l}\big(\mathbf{p}_l(x_1, x_2)\big), B_{l,l-1}\big(\mathbf{p}_l(x_1, x_2)\big), \ldots, B_{l,l-j}\big(\mathbf{p}_l(x_1, x_2)\big).$$

The conclusion is obvious. □

Lemma 10. The collection $B_{l,l-k}\big(\mathbf{p}_l(x_1, x_2)\big)$ where k ranges from 0 to $\lfloor l/2 \rfloor$ is a generator set of the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=l}$.

Proof. The statement follows from Lemma 9 which shows that any canonical generator $p_1^{l-2k} p_2^k$ for $D = 2$ is a $\mathbb{Q}$-linear combination of $\{B_{l,l-j}\big(\mathbf{p}_l(x_1, x_2)\big)\}_{j=0}^k$. □

Theorem 4. Let $D = 2$. Let $S_l(u_0, \mathbf{0}_l)$ be a non-zero polynomial and let the polynomials $S_0(u_0, \mathbf{0}_l)$,..., $S_{l-1}(u_0, \mathbf{0}_l)$ be all equal to the zero polynomial. Then $S_l(u_0, \mathbf{u}_l) = S_l(u_0, \mathbf{0}_l)$. Moreover, either $d \leq l$, or $d < \deg(G_0)$ or $S_l(d, \mathbf{0}_l) = 0$.

Proof. Let us assume that $d > l$ and $d \geq \deg(G_0)$, which altogether implies that $2d - l > d \geq \deg(G_0)$. It means that $S_l\big(d, \mathbf{p}_l(\mathbf{r}_d)\big) = 0$.

Every polynomial $A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{x}_D)\big)(u_0)$ belongs to the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_D} \rangle_{|\mathbf{j}_D|=l-|\mathbf{i}_l|}$, where $|\mathbf{i}_l| \geq 1$, by Lemma 5. We set $k := |\mathbf{i}_l|$. The fact that

$$S_{l-k}(u_0, \mathbf{0}_{l-k}) = \sum_{(t_1,t_2)\in T} \alpha_{t_1 t_2} A_{\mathbf{0}_{l-k}}\big(D, \mathbf{p}_l(t_1, t_2)\big)(u_0)$$

is the zero polynomial for $l' = l - k < l$ means that each of its coefficients of u_0^m vanishes, that is

$$\sum_{(t_1,t_2)\in T} \alpha_{t_1 t_2} B_{l-k,m}\big(\mathbf{p}_l(t_1, t_2)\big) = 0 \tag{24}$$

for all $m = 1, \ldots, l - k$. Since the collection $\{B_{l-k,m}\big(\mathbf{p}_l(x_1, x_2)\big)\}_{m=1}^{l-k}$ contains the generator set $\{B_{l-k,(l-k)-k'}\big(\mathbf{p}_l(x_1, x_2)\big)\}_{k'=0}^{\lfloor (l-k)/2 \rfloor}$ of the $\mathbb{K}[u_0]$-module $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_2|=l-k}$ then the polynomial $A_{\mathbf{i}_l}\big(2, \mathbf{p}_l(x_1, x_2)\big)(u_0)$ is a non-trivial $\mathbb{K}[u_0]$-linear combination of the polynomials $B_{l-k,m}\big(\mathbf{p}_{l-k}(x_1, x_2)\big)$ where $k = |\mathbf{i}_l|$, and together with (24) this implies that for $k \geq 1$ the coefficient of $u_1^{i_1}, \ldots, u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$, which is equal to

$$\sum_{(t_1,t_2)\in T} \alpha_{t_1 t_2} A_{\mathbf{i}_l}\big(2, \mathbf{p}_l(t_1, t_2)\big)(u_0),$$

vanishes. Indeed, since $A_{\mathbf{i}_l}\big(2,\mathbf{p}_l(x_1,x_2)\big)(u_0) = \sum_{m=1}^{l-k} c_{\mathbf{i}_l,m}B_{l,m}\big(\mathbf{p}_l(x_1,x_2)\big)$ for some $c_{\mathbf{i}_l,m} \in K[u_0]$ then

$$\begin{aligned}
&\textstyle\sum_{(t_1,t_2)\in T} \alpha_{t_1t_2} A_{\mathbf{i}_l}\big(2,\mathbf{p}_l(t_1,t_2)\big)(u_0) = \\
&\textstyle\sum_{(t_1,t_2)\in T} \alpha_{t_1t_2} \sum_{m=1}^{l-k} c_{\mathbf{i}_l,m} B_{l-k,m}\big(\mathbf{p}_l(t_1,t_2)\big) = \\
&\textstyle\sum_{m=1}^{l-k} c_{\mathbf{i}_l,m} \sum_{(t_1,t_2)\in T} \alpha_{t_1t_2} B_{l-k,m}\big(\mathbf{p}_l(t_1,t_2)\big) = \\
&\textstyle\sum_{m=1}^{l-k} c_{\mathbf{i}_l,m} 0 = 0.
\end{aligned}$$

Therefore, $S_l(u_0,\mathbf{0})$ is an indicial polynomial of the difference equation (2), where $D = 2$, because this polynomial must be equal to zero at d. $\square$

5. Completing the procedure of bounding the degree of a solution for quadratic real difference equations

Everywhere in this section it is assumed that $\mathbb{K} = \mathbb{R}$, that is the polynomials G and G_0 have real coefficients and τ_i are real numbers. Moreover, as in the section above, it is assumed that $D = 2$.

Without loss of generality one can assume that $\tau_1 < \cdots < \tau_s$ are positive. Otherwise one can consider a "shifted" difference equation:

$$G\big(P(x-\tau_1'),\ldots,P(x-\tau_s')\big) + G_0(x-\Delta) = 0, \tag{25}$$

where $\tau_i' = \tau_i + \Delta$ and Δ is some element of $\mathbb{R}$ such that all $\tau_i + \Delta > 0$. It easy to see that the original difference equation has a polynomial solution P if and only if the shifted one has the same solution, using the fact that equation (2) holds for all x and therefore for all $x - \Delta$.

The following auxiliary lemma will be used in the proof of Lemma 12 below where we will see that if $S_l(u_0,\mathbf{0}_l)$ are the zero polynomials for a sufficiently large number $l \geq 0$ then the quadratic part of G vanishes.

Lemma 11. Let $N > 0$ be an integer number and let $M = \{(t_1^{(m)}, t_2^{(m)})\}_{m=1}^N$ be a collection of positive real numbers such that all the ratios $t_2^{(m)}/t_1^{(m)} \geq 1$ are pairwise distinct. Then the $N \times N$ linear system

$$\sum_{m=1}^{N} x_m B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(m)}, t_2^{(m)})\big) = 0, \tag{26}$$

where $0 \leq k \leq N-1$ ranges over the rows of the corresponding matrix, has only the trivial (i.e. all zero's) solution.

Proof. Let us assume the opposite, that is system (26) has a nontrivial solution, which we denote $(x_1^0,\ldots,x_N^0)$. Then the rows of the matrix are linearly dependent, that is for the vector-rows

$$\big(B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(1)}, t_2^{(1)})\big),\ldots,B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(N)}, t_2^{(N)})\big)\big), \tag{27}$$

there exists a nontrivial linear combination of them, equal to zero. This means that there exists a collection of $a_k \in \mathbb{R}$ such that

$$\sum_{k=0}^{N-1} a_k B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(t_1^{(m)}, t_2^{(m)})\big) = 0, \text{ for all } 1 \le m \le N, \tag{28}$$

where m ranges over the columns of the matrix. Consider the polynomial

$$F(x_1, x_2) := \sum_{k=0}^{N-1} a_k B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(x_1, x_2)\big).$$

It is homogeneous in (x_1, x_2) of degree $2N-2$ and symmetric in x_1, x_2 as a linear combination of homogeneous and symmetric polynomials. Moreover, it is given that it vanishes on the set $(t_1^{(m)}, t_2^{(m)})$ of N nodes, see (28), such that all the corresponding N lines connecting the points $(0,0)$ and $(t_1^{(m)}, t_2^{(m)})$ are pairwise distinct. We apply Lemma 3 with $l = 2N-2$ and $l/2+1 = N$ for the polynomial $F(x_1, x_2)$ to see that it vanishes everywhere. Therefore there is a nontrivial linear combination of polynomials $B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(x_1, x_2)\big)$, where $k = \overline{0, N-1}$, such that it is equal to the zero polynomial, which contradicts the fact that for $D =$ the collection $\{B_{2N-2,2N-2-k}\big(\mathbf{p}_{2N-2}(x_1, x_2)\big)\}_{k=0}^{N-1}$ is a generator set for $\langle \pi^{\mathbf{j}_2} \rangle_{|\mathbf{j}_D|=2N-2}$ where $D = 2$, see Lemma 10.

Therefore the assumption at the beginning of the proof is wrong, and the system has only the trivial solution.

□

Now we return to difference equation (2). Let R be the set of all the ratios t_2/t_1 where $\alpha_{(t_1,t_2)} \neq 0$. Take some $r \in R$. Let $M_r := \{(t_1, t_2) | \tau_1 \le t_1 \le t_2 \le \tau_s, \alpha_{(t_1,t_2)} \neq 0,\ t_2/t_1 = r\}$ be the collection of all the pairs (t_1, t_2) with the ratio $t_2/t_1 = r$. Select from each set M_r some representative pair (t_{r1}, t_{r2}). It is easy to check that for the k'-th pair $(t_1, t_2) \in M_r$ there is a number $\lambda_{rk'} \in \mathbb{R}$ such that $t_1 = \lambda_{rk'} t_{r1}$ and $t_2 = \lambda_{rk'} t_{r2}$, where k' runs over all pairs from the finite set M_r enumerated in some way.

Lemma 12. Let the sets M_r be singletons, except possibly M_1, containing all $\alpha_{(t_1,t_2)}$ for which $t_1 = t_2$. Let $N := |R|$ be the cardinality of the set R. Then there exists $0 \le l \le \max\{2N-2, s-1\}$ such that $S_l(u_0, \mathbf{0}_l)$ is a non-zero polynomial.

Proof. Fix some $r \in R$. For any $j_1 + 2j_2 = l$ one has

$$\sum_{(t_1,t_2)\in M_r} \alpha_{(t_1,t_2)} p_1^{j_1}(t_1, t_2) p_2^{j_2}(t_1, t_2) =$$
$$\sum_{k'=1}^{|M_r|} \alpha_{(\lambda_{rk'}t_{r1}, \lambda_{rk'}t_{r2})} (\lambda_{rk'}t_{r1} + \lambda_{rk'}t_{r2})^{j_1} (\lambda_{rk'}^2 t_{r1}^2 + \lambda_{rk'}^2 t_{r2}^2)^{j_2} =$$
$$\sum_{k'=1}^{|M_r|} \alpha_{(\lambda_{rk'}t_{r1}, \lambda_{rk'}t_{r2})} \lambda_{rk'}^{j_1+2j_2} (t_{r1} + t_{r2})^{j_1} (t_{r1}^2 + t_{r2}^2)^{j_2} =$$
$$\sum_{k'=1}^{|M_r|} \alpha_{(\lambda_{rk'}t_{r1}, \lambda_{rk'}t_{r2})} \lambda_{rk'}^{l} p_1^{j_1}(t_{r1}, t_{r2}) p_2^{j_2}(t_{r1}, t_{r2}).$$

Recall that $B_{l,l-k}\big(\mathbf{p}_l(x_1, x_2)\big)$ is a linear combination of the form

$$\sum_{j=0}^{k} b_{l,k,j} p_1^{l-2j}(x_1,x_2) p_2^{j}(x_1,x_2),$$

where $b_{l,k,j} \in \mathbb{Q}$ are defined in Lemma 8. Using the equations above we will compute the coefficient of u_0^{l-k} in

$$S_l(u_0,\mathbf{0}_l) = \sum_{(t_1,t_2)\in T} \alpha_{(t_1,t_2)} A_{\mathbf{0}_l}\big(\mathbf{p}_l(t_1,t_2)\big)(u_0) = \sum_{(t_1,t_2)\in T} \alpha_{(t_1,t_2)} \sum_{k=0}^{l} B_{l,l-k}\big(\mathbf{p}_l(t_1,t_2)\big) u^{l-k} = \sum_{k=0}^{l} u^{l-k} \sum_{(t_1,t_2)\in T} \alpha_{(t_1,t_2)} B_{l,l-k}\big(\mathbf{p}_l(t_1,t_2)\big).$$

This coefficient is equal to

$$\sum_{(t_1,t_2)\in T} \alpha_{(t_1,t_2)} B_{l,l-k}\big(\mathbf{p}_l(t_1,t_2)\big) = \sum_{r\in R} \alpha'_{(t_{r1},t_{r2})} B_{l,l-k}\big(\mathbf{p}_l(t_{r1},t_{r2})\big),$$

where $\alpha'_{(t_{r1},t_{r2})} := \sum_{k'=1}^{|M_r|} \alpha_{\lambda_{rk'}\ (t_{r1},t_{r2})} \lambda_{rk'}^{l}$. Indeed,

$$\begin{aligned}
&\sum_{(t_1,t_2)\in T} \alpha_{(t_1,t_2)} B_{l,l-k}\big(\mathbf{p}_l(t_1,t_2)\big) = \\
&\sum_{(t_1,t_2)\in T} \alpha_{(t_1,t_2)} \big(\sum_{j=0}^{k} b_{l,k,j} p_1^{l-2j}(t_1,t_2) p_2^{j}(t_1,t_2)\big) = \\
&\sum_{r\in R} \sum_{(t_1,t_2)\in M_r} \alpha_{(t_1,t_2)} \Big(\sum_{j=0}^{k} b_{l,k,j} p_1^{l-2j}(t_1,t_2) p_2^{j}(t_1,t_2)\Big) = \\
&\sum_{r\in R} \Big(\sum_{j=0}^{k} b_{l,k,j} \sum_{(t_1,t_2)\in M_r} \alpha_{(t_1,t_2)} p_1^{l-2j}(t_1,t_2) p_2^{j}(t_1,t_2)\Big) =^{\text{by equation 5}} \\
&\sum_{r\in R} \Big(\sum_{j=0}^{k} b_{l,k,j} \sum_{k'=1}^{|M_r|} \alpha_{\lambda_{rk'}(t_{r1},t_{r2})} \lambda_{rk'}^{l} p_1^{l-2j}(t_{r1},t_{r2}) p_2^{j}(t_{r1},t_{r2})\Big) = \\
&\sum_{r\in R} \Big(\sum_{k'=1}^{|M_r|} \alpha_{\lambda_{rk'}(t_{r1},t_{r2})} \lambda_{rk'}^{l} \sum_{j=0}^{k} b_{l,k,j} p_1^{l-2j}(t_{r1},t_{r2}) p_2^{j}(t_{r1},t_{r2})\Big) = \\
&\sum_{r\in R} \sum_{k'=1}^{|M_r|} \alpha_{\lambda_{rk'}(t_{r1},t_{r2})} \lambda_{rk'}^{l} B_{l,l-k}\big(\mathbf{p}_l(t_{r1},t_{r2})\big) = \\
&\sum_{r\in R} B_{l,l-k}\big(\mathbf{p}_l(t_{r1},t_{r2})\big) \alpha'_{(t_{r1},t_{r2})},
\end{aligned}$$

where r ranges over all possible ratios t_2/t_1 for $\alpha_{(t_1,t_2)} \neq 0$.

Now, we assume that $S_l(u_0,\mathbf{0}_l)$ is the zero polynomial for any $0 \leq l \leq \max\{2N-2, s-1\}$ and introduce the linear system $\sum_{r\in R} x_r B_{l,l-k}\big(\mathbf{p}_l(t_{r1},t_{r2})\big) = 0$ w.r.t. x_r, where

$0 \le k \le \lfloor l/2 \rfloor$ ranges of the rows of the matrix.

Fix $l = 2N-2$. By Lemma 11, setting $M = \{(t_{r1}, t_{r2}) | r \in R\}$ there, one immediately obtains that all $\alpha'_{(t_{r1},t_{r2})}$ are zero. Therefore $\sum_{k'=1}^{|M_r|} \alpha_{\lambda_{rk'}(t_{r1},t_{r2})} \lambda_{rk'}^{2N-2} = 0$, where k' ranges over all the pairs $(t_1, t_2) = \lambda_{rk'}(t_{r1}, t_{r2})$ from M_r. For any $r \ne 1$ the corresponding set M_r is a singleton of the form $M_r = \{\alpha_{(t_{r1},t_{r2})}\}$, and therefore

$$0 = \sum_{k'=1}^{|M_r|} \alpha_{\lambda_{rk'}(t_{r1},t_{r2})} \lambda_{k'}^{2N-2} = \alpha_{(t_{r1},t_{r2})}.$$

That is for any $t_1 \ne t_2$, $r \ne 1$, the corresponding coefficient $\alpha_{(t_1,t_2)}$ vanishes.

Therefore, we have obtained that $M = M_1 = \{\alpha_{(t,t)} \ne 0\}$. Since all the polynomials $S_l(u_0, \mathbf{0}_l)$, where $0 \le l \le s-1$, are equal to the zero polynomial w.r.t. u_0, one has that the coefficient $\sum_{t \in \{\tau_1,\ldots,\tau_s\}} \alpha_{(t,t)} B_{l,l}\big(\mathbf{p}_l(t,t)\big)$ of u_0^l in $S_l(u_0, \mathbf{0}_l)$ is zero. Since, $B_{l,l}\big(\mathbf{p}_l(x_1,x_2)\big) = b_{l,0,0} p_1^l(x_1,x_2)$ by Lemma 8 for $k=0$, this coefficient is equal to $b_{l,0,0} \sum_{t \in \{\tau_1,\ldots,\tau_s\}} \alpha_{(t,t)} p_1^l(t,t) = b_{l,0,0} \cdot 2^l \sum_t \alpha_{(t,t)} t^l$. The system

$$\sum_{t \in \{\tau_1,\ldots,\tau_s\}} x_t t^l = 0, \text{ where } l = 0, 1, \ldots, s-1$$

has the Vandermonde determinant which is non-zero because all the $\tau_1, \ldots, \tau_s$ are pairwise distinct. Therefore all $\alpha_{(t_1,t_2)}$, where $t_1 = t_2$, vanish as well.

Therefore the assumption about absence of some $l \le \max\{2N-2, s-1\}$ such that $S_l(u_0, \mathbf{0}_l)$ is a non-zero polynomial, leads to vanishing of the quadratic part of the difference equation, which contradicts to the assumption that we consider quadratic equations. Therefore there must be $l \le \max\{2N-2, s-1\}$ such that $S_l(u_0, \mathbf{0}_l)$ is a non-zero polynomial. □

To see that the condition of Lemma 12 does not influence the generality of the approach, one needs to consider a shifted equation of the form (25) with some properly chosen Δ. To provide the reader with an intuition we start with the running example of equation (3). In that equation the ratios $\frac{2}{1}$ and $\frac{4}{2}$ for two corresponding product $P(x-1)P(x-2)$ and $P(x-2)P(x-4)$ coincide. One can obtain an equation with a polynomial solution of the same degree as a solution for equation (3) by finding such Δ that makes ratios $\frac{2+\Delta}{1+\Delta}$ and $\frac{4+\Delta}{2+\Delta}$ distinct. For instance, with $\Delta = 1$ one has $\frac{2+1}{1+1} = \frac{3}{2}$ and $\frac{4+1}{2+1} = \frac{5}{3}$. In general, the following lemma holds.

Lemma 13. Given a finite set of pairs $U \subset \mathbb{R}$ such that it does not contain pairs of the form $(0, t)$ and pairs of the form (t, t), one can effectively define $\Delta \in \mathbb{R}^+$ such that for any two elements $(t_1, t_2) \ne (t'_1, t'_2) \in U$ one has $\frac{t_2+\Delta}{t_1+\Delta} \ne \frac{t'_2+\Delta}{t'_1+\Delta}$.

Proof. First, for any two distinct elements $(t_1, t_2) \ne (t'_1, t'_2) \in U$ we will find $\Delta_{t_1,t_2,t'_1,t'_2}$ such that $\frac{t_2+\Delta_{t_1,t_2,t'_1,t'_2}}{t_1+\Delta_{t_1,t_2,t'_1,t'_2}} = \frac{t'_2+\Delta_{t_1,t_2,t'_1,t'_2}}{t'_1+\Delta_{t_1,t_2,t'_1,t'_2}}$ holds. Since this equation is equivalent for a polynomial equation w.r.t. $\Delta > 0$, there will be a finite number of the corresponding solutions $\Delta_{t_1,t_2,t'_1,t'_2}$ for all pairs of pairs (t_1, t_2) and (t'_1, t'_2). Second, due to the finiteness of the set of all such solutions $\Delta_{t_1,t_2,t'_1,t'_2}$ where $(t_1,t_2) \neq (t'_1,t'_2) \in U$, we will be able to pick up an arbitrary Δ which is distinct from all these $\Delta_{t_1,t_2,t'_1,t'_2}$, and this Δ will satisfy the condition of the lemma.

We start with solving $\frac{t_2+\Delta}{t_1+\Delta} = \frac{t'_2+\Delta}{t'_1+\Delta}$ w.r.t. $\Delta > 0$. This equation is equivalent to $(t_2+\Delta)(t'_1+\Delta) = (t'_2+\Delta)(t_1+\Delta)$ which is reduced to a linear one

$$(t_2 + t'_1 - t'_2 - t_1)\Delta = t'_2 t_1 - t_2 t'_1.$$

We consider all possible cases for this equation setting $K := (t_2 + t'_1 - t'_2 - t_1)$ and $L := t'_2 t_1 - t_2 t'_1$ for the sake of convenience:

- $K \neq 0$; then the only solution is $\Delta_{t_1,t_2,t'_1,t'_2} = L/K$;
- $K = 0, L \neq 0$, this case is impossible since $0 \cdot \Delta = L$ implies $L = 0$;
- $K = L = 0$; this case leads to contradictions with the conditions of the lemma. Namely, $K = L = 0$ implies that the following system of equalities holds:

$$\begin{aligned} t_2 + t'_1 - t'_2 - t_1 &= 0 \\ t'_2 t_1 - t_2 t'_1 &= 0 \end{aligned}$$

  Since U does not contain pairs of the form $(0,t)$, we use $t'_1 \neq 0$ and apply the substitution $t'_2 = t_2 t'_1/t_1$ (derived from the second equation) into the first equation. We obtain

$$t_2 - t_1 = t'_2 - t'_1 = t_2 t'_1/t_1 - t'_1 = t'_1(t_2/t_1 - 1) = t'_1(t_2 - t_1)/t_1$$

  which implies that $t_2 = t_1$ or $t'_1 = t_1$. The second option implies $t'_2 = t_2$ via the first equation. Therefore, both options contradict the condition of the lemma.

Now, take any Δ which is distinct from $\Delta_{t_1,t_2,t'_1,t'_2}$ for any $(t_1,t_2) \neq (t'_1,t'_2) \in U$. This Δ makes $\frac{t_2+\Delta}{t_1+\Delta} \neq \frac{t'_2+\Delta}{t'_1+\Delta}$ for any $(t_1,t_2) \neq (t'_1,t'_2) \in U$. $\square$

The shifted equation for a running example, where $\Delta = 1$ has the form

$$\begin{aligned} &P(x-2)P(x-2) - 3P(x-2)P(x-3) + \\ &\tfrac{5}{2}P(x-3)P(x-3) - \tfrac{1}{2}P(x-3)P(x-5) + \\ &\big(-P(x-1)\big) + 2P(x-2) - \tfrac{1}{8}P(x-3) = 0 \end{aligned} \tag{29}$$

which does satisfy the condition of the Lemma 12. It is a routine to show that the values $S_0(u_0)$, $S_1(u_0,0)$ and $S_2(u_0,0,0)$ for the shifted equation (as well for the shifted equation for an arbitrary Δ) are exactly the same as for the original one, that it 0, 0 and $\frac{1}{2}u_0(3-u_0)$.

Now we are ready to prove the main result of this article.

Theorem 5. If $\mathbb{K} = \mathbb{R}$ then for any difference equation of the form (2) with $D = 2$ there exist a number $l \leq 2N-2, s-1$ and a non-zero polynomial $f(u_0)$ such that

- either $d \leq l$,
- or $d < \deg(G_0)$,
- or d is a root of $f(u_0)$.

Proof. A bound of the degree d of a polynomial solution of difference equation (2) is the same as the bound for a shifted equation of the form (25). Using Lemma 13 one can construct Δ such that the corresponding shifted difference equation satisfies the conditions of Lemma 12. We apply this lemma to obtain a polynomial $f_l(u_0) = S_l(u_0, \mathbf{0}_l)$ where l is the minimal index l' such that $S_{l'}(u_0, \mathbf{0}_l)$ is not the zero polynomial. Now one applies Theorem 4 to see that the conclusion of the current theorem holds.

In more detail, we take $U = T$ in Lemma 13 and obtain Δ such that the shifted difference equation satisfies the condition of Lemma 12, since all $\frac{t_2+\Delta}{t_1+\Delta}$ are pairwise distinct or $\frac{t_2+\Delta}{t_1+\Delta} = 1$. Therefore, that lemma can be applied to obtain the statement of this theorem. □

Note that $\alpha^{\text{shifted}}_{(t_1+\Delta, t_2+\Delta)} = \alpha_{(t_1,t_2)}$ since $G\big(p(x-\tau'_1), \dots, p(x-\tau'_s)\big)$ is obtained from $G\big(P(x-\tau_1), \dots, P(x-\tau_s)\big)$ via replacing every product $P(x-t_1)P(x-t_2)$ with the corresponding product $P\big(x-(t_1+\Delta)\big)P\big(x-(t_2+\Delta)\big)$ in the quadratic part of G and applying the corresponding substitutions in the linear part of G.

6. Algebraic difference equations of degree D with variable coefficients

Contrary to quadratic difference equations with constant coefficients, there are quadratic difference equations with variate (polynomial) coefficients that have a solution of any degree, and therefore the degree of polynomial solutions for such difference equations cannot be bounded. Consider, for instance, the equation

$$P_n(x)P_n(x-1) - xP_n^2(x-1) + (x-1)P_n(x)P_n(x-2) = 0 \tag{30}$$

It is a routine to check that for an arbitrary positive integer number n the falling factorial $P_n(x) := x(x-1)\dots\big(x-(n-1)\big)$, which is a polynomial of degree n, solves this equation. Indeed, one has:

$$\begin{aligned}
&P_n(x) - P_n(x-1) = \\
&x(x-1)\cdots\big(x-(n-1)\big) - (x-1)\cdots(x-n) = \\
&(x-1)\cdots\big(x-(n-1)\big)\big(x-(x-n)\big) = \\
&(x-1)\cdots\big(x-(n-1)\big)n = \\
&\tfrac{nP_n(x)}{x}.
\end{aligned}$$

This implies that $n = \frac{x\left(P_n(x)-P_n(x-1)\right)}{P_n(x)}$ for all x, and therefore

$$\begin{aligned}
&0 = n - n = \\
&\tfrac{x\left(P_n(x)-P_n(x-1)\right)}{P_n(x)} - \tfrac{(x-1)\left(P_n(x-1)-P_n(x-2)\right)}{P_n(x-1)} = \\
&\tfrac{xP_n(x-1)\left(P_n(x)-P_n(x-1)\right)-(x-1)P_n(x)\left(P_n(x-1)-P_n(x-2)\right)}{P_n(x)P_n(x-1)}
\end{aligned}$$

which is equivalent to equation (30). Indeed

$0 =$

$xP_n(x-1)\big(P_n(x)-P_n(x-1)\big)-(x-1)P_n(x)\big(P_n(x-1)-P_n(x-2)\big) =$

$P_n(x)P_n(x-1)-xP_n^2(x-1)+(x-1)P_n(x)P_n(x-2)$

This proves that $P_n(x)$ solves that equation.

However the earlier results for polynomial difference equations with constant coefficients of an arbitrary degree $D \geq 2$ (Shkaravska and van Eekelen, 2014), to some extend still can be generalised for equations with polynomial coefficients.

The set of shifts $\{\tau_1, \ldots, \tau_s\} \subseteq \mathbb{K}$ is finite and therefore can be totally ordered. Let $\preccurlyeq$ denote a total order on this set. If $\mathbb{K} \subseteq \mathbb{R}$ then we assume that $\preccurlyeq$ is the usual order $\leq$ on real numbers.

Let T_m denote the set of all non-decreasing m-tuples of the elements from the set $\{\tau_1, \ldots, \tau_s\}$. Formally, $T_m = \{(t_1, \ldots, t_m) | \tau_1 \preccurlyeq t_1 \preccurlyeq \cdots \preccurlyeq t_m \preccurlyeq \tau_s\}$. Let $\mathbf{t}$ range over the tuples from all the sets $T_1, \ldots, T_D$. Then equation (1) with polynomial coefficients has the following presentation:

$$\sum_{m=1}^{D} \sum_{\mathbf{t} \in T_m} \alpha_{\mathbf{t}}(x) \cdot P(x-t_1) \cdots P(x-t_m) + G_0(x) = 0 \tag{31}$$

where $\mathbf{t} = (t_1, \ldots, t_m)$ and the coefficient of the m-fold product $P(x-t_1) \ldots P(x-t_m)$ is the polynomial $\alpha_{\mathbf{t}}(x) = w_{0,\mathbf{t}} x^{n_{\mathbf{t}}} + w_{1,\mathbf{t}} x^{n_{\mathbf{t}}-1} + \cdots + w_{n_{\mathbf{t}},\mathbf{t}}$ with the number $n_{\mathbf{t}}$ being the degree of the polynomial $\alpha_{\mathbf{t}}(x)$ and $w_{k,\mathbf{t}} \in \mathbb{K}$ being the coefficient of $x^{n_{\mathbf{t}}-k}$ in $\alpha_{\mathbf{t}}(x)$. For instance, for equation (30) one has $n_{(0,1)} = 0$, $n_{(1,1)} = 1$ and $n_{(0,2)} = 1$ with $\alpha_{(0,1)}(x) = 1, \alpha_{(1,1)}(x) = -x$ and $\alpha_{(0,2)}(x) = x-1$ respectively.

As in Notation 1, $\mathbf{t}_D$ abbreviates an ordered D-tuple of the shifts. Let also $\mathbf{w}_l$ denote the $(l+1)$-tuple of variables $(w_0, \ldots, w_l)$, and $\mathbf{w}_{l,\mathbf{t}_D}$ denote the $(l+1)$-tuple of values $(w_{0,\mathbf{t}_D}, \ldots, w_{l,\mathbf{t}_D}) \in \mathbb{K}^{l+1}$.

Let N and M denote the maximal degrees of the polynomial coefficients of the D-fold products $P(x-t_1) \cdots P(x-t_D)$ and the $(D-1)$-fold products $P(x-t_1) \cdots P(x-t_{D-1})$ respectively. For instance, for equation (30) one has $N = 1$ and $M = 0$.

We consider now a product of the form $\alpha_{\mathbf{t}_D}(x) \cdot P(x-t_1) \cdots P(x-t_D)$. If the degree of $\alpha_{\mathbf{t}_D}$ is some $n < N$, we assign $w_{k,\mathbf{t}_D} = 0$, where $k < N-n$. To compute the product's coefficients of x^{Dd+N-l} in this product, where $0 \leq l \leq Dd+N$, one will need the following definition, based on the rule of multiplication of two polynomials applied to the polynomial $\alpha_{\mathbf{t}_D}(x)$ and the symbolic polynomial $P(x-t_1) \cdots P(x-t_D)$:

Definition 8. $E_l^*(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) := \sum_{k=0}^{l} E_k\big(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l\big) w_{l-k}$.

Using the rule of the multiplication of two polynomials, it easy to prove the following lemma.

Lemma 14. Let $0 \leq l \leq Dd+N$. The coefficient of x^{Dd+N-l} in the product $\alpha_{\mathbf{t}_D}(x) \cdot P(x-t_1) \cdots P(x-t_D)$ is equal to $E_l^*\big(\mathbf{w}_{l,\mathbf{t}_D}, D, \mathbf{p}_l(\mathbf{t}_D), d, \mathbf{p}_l(\mathbf{r}_d)\big)$.

Proof. Fix some integer numbers k_1 and k_2 such that $0 \leq k_1 \leq N$ and $0 \leq k_2 \leq Dd$. Since $w_{N-k_1,\mathbf{t}_D}$ is the coefficient of $x^{N-(N-k_1)} = x^{k_1}$ in $\alpha_{\mathbf{t}_D}(x)$ and the value

$E_{Dd-k_2}\big(D, \mathbf{p}_{Dd-k_2}(\mathbf{t}_D), d, \mathbf{p}_{Dd-k_2}(\mathbf{r}_d)\big)$ is the coefficient of $x^{Dd-(Dd-k_2)} = x^{k_2}$ in the symbolic product $P(x-t_1)\cdots P(x-t_D)$, by the polynomial-multiplication rule one has that the coefficient of x^{Dd+N-l} in $\alpha_{\mathbf{t}_D}(x)\cdot P(x-t_1)\cdots P(x-t_D)$ is equal to:

$$\sum_{k_1+k_2=Dd+N-l} w_{N-k_1,\mathbf{t}_D} E_{Dd-k_2}\big(D, \mathbf{p}_{Dd-k_2}(\mathbf{t}_D), d, \mathbf{p}_{Dd-k_2}(\mathbf{r}_d)\big) =^{l_1:=N-k_1, l_2:=Dd-k_2}$$
$$\sum_{Dd+N-l_1-l_2=Dd+N-l} w_{l_1,\mathbf{t}_D} E_{l_2}\big(D, \mathbf{p}_{l_2}(\mathbf{t}_D), d, \mathbf{p}_{l_2}(\mathbf{r}_d)\big) =$$
$$\sum_{l_1+l_2=l} w_{l_1,\mathbf{t}_D} E_{l_2}\big(D, \mathbf{p}_{l_2}(\mathbf{t}_D), d, \mathbf{p}_{l_2}(\mathbf{r}_d)\big),$$

where $0 \le l_1 \le N$ and $0 \le l_2 \le Dd$. The conclusion of the lemma follows directly when one sets $l := l_1 + l_2$ and $k := l_2$ in the equality above. □

For the sake of convenience we assume that $w_{-m,\mathbf{t}} = 0$ for all integer numbers $m > 0$. The coefficient of x^{Dd+N-l} on the left-hand side of equation (1) is computed using the following function:

Definition 9. $S^*_l(u_0, \mathbf{u}_l) := \sum_{\mathbf{t}_D \in T} E^*_l(\mathbf{w}_{l,\mathbf{t}_D}, D, \mathbf{p}_l(\mathbf{t}_D), u_0, \mathbf{u}_l)$.

It is easy to see that $S^*_l\big(d, \mathbf{p}_l(\mathbf{r}_d)\big)$ is the coefficient of x^{Dd+N-l} on the left-hand side of equation (1) if $Dd + N - l > (D-1)d + M$ and $(D-1)d + M \ge \deg(G_0)$. Therefore, it must vanish when these inequations hold. Using a computer algebra system it is easy to prove the following statement.

Theorem 6. Let an ADE of the form (1) be given. If there exists $0 \le l \le 2$ such that $S^*_l(u_0, \mathbf{0}_l)$ is a non-zero polynomial and for any $0 \le l' \le l-1$ the corresponding $S^*_{l'}(u_0, \mathbf{0}_{l'})$ is the zero polynomial, then one can tell the following about the degree d of a polynomial solution of equation (1):
- either $d \le l - N + M$;
- or $d < \big(\deg(G_0) - M\big)/(D-1)$,
- or d is a root of $S^*_l(u_0, \mathbf{0}_l)$.

Proof. Let the first two alternative conclusions of the lemma do not hold. Therefore the degree of x^{Dd+N-l} on the left-hand side of equation (1) must vanish.

The schema of the proof is the same as the schema of the proof of Theorem 1 for the polynomials with constant coefficients. We consider $E^*_l(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ as a polynomial in $\mathbb{K}[v_0][\mathbf{w}_l, \mathbf{v}_l][u_0][\mathbf{u}_l]$ and define the polynomial $A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$ as its coefficient of $u_1^{i_1}\cdots u_l^{i_l}$. Consequently, the polynomial $B^*_{l,m}(\mathbf{w}_l, \mathbf{v}_l)$ is the coefficient of u_0^m in the polynomial $A^*_{\mathbf{0}_l}(\mathbf{w}_l, \mathbf{v}_l)(u_0)$.

Using symbolic computations it is easy to check that for $l = 0, 1, 2$ and $\mathbf{i}_l \ne \mathbf{0}_l$ the polynomial $A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$ is a $\mathbb{K}[u_0, v_0]$-linear combination of the polynomials $B^*_{l,m}(\mathbf{w}_l, \mathbf{v}_l)$, where $0 \le m \le l-1$. In Subsection A.5 in the Appendix one can find the tables which contain the expressions for the polynomials mentioned above. Moreover the symbolic coefficients for the corresponding $\mathbb{K}[u_0, v_0]$-linear combinations are given as well.

For $l = 3$ the expression for $A^*_{100}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$ is not a $\mathbb{K}[u_0, v_0]$-linear combination of the polynomials $B^*_{l,m}(\mathbf{w}_l, \mathbf{v}_l)$, where $0 \le l \le 2$ and $0 \le m \le l-1$. It can be shown by solving the linear system w.r.t. unknowing coefficients of the hypothetical $\mathbb{K}[u_0, v_0]$-linear combination for A^*_{100}. The system is derived by equating the coefficients of the monomials $w_0^{k_0} w_1^{k_1} w_2^{k_2} v_1^{j_1} v_2^{j_2}$ in A^*_{100} and in the linear combination. The system is inconsistent and therefore the linear combination doe snot exist.

It follows that if $S^*_0(u_0) = \sum_{\mathbf{t}_D} \alpha_{\mathbf{t}_D} w_{0,\mathbf{t}_D} = 0$ then the dependency on u_1 vanishes in $S^*_1(u_0, u_1) = \sum_{\mathbf{t}_D} \alpha_{\mathbf{t}_D} E^*_1(\mathbf{w}_{l,\mathbf{t}_D}, D, \mathbf{p}_l(\mathbf{r}_d), u_0, u_1)$. If $S^*_1(u_0, 0)$ is a non-zero polynomial then it is an indicial polynomial for the difference equation under consideration. Otherwise the dependencies on u_1 and u_2 vanish in $S^*_2(u_0, u_1, u_2)$. If $S^*_2(u_0, 0, 0)$ is a non-zero polynomial then it is an indicial polynomial. Otherwise the method does not give an answer. In this case the coefficient of u_1 in $S^*_3(u_0, \mathbf{u}_3)$ is equal to $-1/2 \sum_{\mathbf{t} \in T} p_2(\mathbf{t}_D) w_{0,\mathbf{t}_D}$ and it is not reducible to zero in general. Therefore $S^*_3(u_0, \mathbf{u}_3)$ is not reducible to a 1-variate polynomial of u_0 which can be taken as an indicial polynomial. □

It is worth to note that in general, the coefficients $w_k(y_1, \ldots, y_D)$ considered as functions given by their values $w_k(\mathbf{t}_D) = w_{k,\mathbf{t}_D}$ are not necessarily homogeneous symmetric polynomial functions and moreover they are not necessary polynomials or any other analytic functions at all. Therefore refinements via homogeneous symmetric polynomials are not applicable in the general case.

7. Constructing polynomial solutions given an upper bound of their degrees

As it follows from equations (15), for the running example of equation (3) the degree of its possible polynomial solution satisfies $d \le 3$. Using a computer algebra system one can find all possible values of parameters a_3, a_2, a_1, a_0 that send to zero the coefficients of $x^6, \ldots, x^1, x^0$ on the l.h.s. of equation (3), when it is instantiated with $P_{a_3,a_2,a_1,a_0}(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. The coefficients of $x^6 = x^{2d-0}$, $x^5 = x^{2d-1}$ and $x^4 = x^{2d-2}$, where $d = 3$, vanish since they are equal to $S_0(3) = 0$, $S_1(3, 0)$ and $S_2(3, \mathbf{0}_2) = 0$ respectively. The symbolic coefficients of $x^3, x^2, x, 1$ are equal to

$$\begin{aligned}
C_3(a_0, a_1, a_2, a_3) &= (168a_3^2 + 7a_3)/8 \\
C_2(a_0, a_1, a_2, a_3) &= -(888a_3^2 + (-168a_2 + 24a_1 + 42)a_3 - 8a_2^2 - 7a_2)/8 \\
C_1(a_0, a_1, a_2, a_3) &= (1584a_3^2 + (-592a_2 + 168a_1 - 72a_0 + 36)a_3)/8+ \\
&\quad ((8a_1 - 28)a_2 + 7a_1)/8 \\
C_0(a_0, a_1, a_2, a_3) &= -(952a_3^2 + (-528a_2 + 224a_1 - 168a_0 + 8)a_3)/8+ \\
&\quad (24a_2^2 + (24a_0 - 12)a_2 - 8a_1^2 + 14a_1 - 7a_0)/8
\end{aligned} \tag{32}$$

respectively. Solving the system $C_3(a_0, a_1, a_2, a_3) = 0, \ldots, C_0(a_0, a_1, a_2, a_3) = 0$ w.r.t. $a_3, \ldots, a_0$, yields an infinite number of solutions amongst of which there are complex ones and the trivial one $a_3 = \cdots = a_0 = 0$. Real and rational tuples solving this system exist as well. For instance, there is a subfamily of solutions defined by $a_3 = -1/24, a_1 = -(192a_2^2 + 5)/24, a_0 = (256a_2^3 + 20a_2 - 5)/12$, where a_2 is free.

The procedure of the search for polynomial solutions, given an upper band on their degrees, works in the same way for ADE with finite number of solutions.

In general, the following statement holds.

Lemma 15. If the first-order theory of $\mathbb{K}$ is decidable then for any ADE of the form (1) there exists a finite deterministic algorithm that for an arbitrary nonnegative integer d answers if this ADE has a polynomial solution or not.

Proof. Given an ADE and an arbitrary integer $d \geq 0$, the decision procedure for $\mathbb{K}$ takes as an input the finite system of equations w.r.t. $a_d, \ldots, a_1, a_0$ induced by equating to zero the corresponding coefficients on the l.h.s. of the ADE, which is instantiated with the parametric polynomial $P_{a_d,\ldots,a_1,a_0}(x)$. The procedure decides if the system is solvable or not. Moreover, if the procedure instantiates existential quantifiers constructively, e.g. in the form of CAD for $a_d, \ldots, a_1, a_0$, then the corresponding polynomials $P_{a_d,\ldots,a_1,a_0}(x)$ are solutions of the ADE, by their construction. □

If the first-order theory of $\mathbb{K}$ is not decidable then in general it is not decidable if a given ADE in $\mathbb{K}$ has a polynomial solution in $\mathbb{K}[x]$ of degree at most d. It can be shown by establishing connection between Diophantine equations and algebraic difference equations. How it is done in general is shown in subsection A.6 Appendix. Here we consider a simple example which gives an idea behind the connection between ADE and Diophantine equations related to Great Fermat Theorem. We will construct an ADE which has a polynomial solution of degree $d = 1$ in $\mathbb{Q}[x]$ if and only if the corresponding Fermat equation $a_0^D + a_1^D = 1$ has rational solutions $(a_0, a_1) \in \mathbb{Q}^2$. It is known that this equation does not have solutions w.r.t. (a_0, a_1), except $(0, 1)$ and $(1, 0)$. For the equation $a_0^D + a_1^D = 1$ the corresponding ADE is derived in the following way. First, one considers the parametric system w.r.t. a_0 and a_1:

$$\begin{aligned} a_1 x + a_0 &= P(x) \\ a_1(x-1) + a_0 &= P(x-1) \end{aligned} \tag{33}$$

The corresponding determinants from Cramer's rule are $\Delta(x) = x - (x - 1) = 1$, $\Delta_1(x, P) = P(x) - P(x - 1)$, and $\Delta_0(x, P) = xP(x - 1) - (x - 1)P(x)$. The ADE that corresponds the Fermat equation is obtained via the substitutions $a_0 := \Delta_1(x, P)/\Delta(x)$ and $a_1 := \Delta_1(x, P)/\Delta(x)$:

$$\big(xP(x-1) - (x-1)P(x)\big)^D + \big(P(x) - P(x-1)\big)^D = 1. \tag{34}$$

This example illustrates the complexity of the problem of solving ADE in $\mathbb{Q}[x]$ even when an upper bound of the degree of a possible polynomial solution is given.

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A. Appendix

A.1. Notations and definitions

This section contains a table of notations and a table the definitions used in this article. The main notations used in this article are as follows:

Notation	Meaning	Page
$\mathbb{K}$	a field of characteristic zero	1
$\mathbb{R}$	the field of real numbers	2
$\mathbf{t}_D, \mathbf{r}_d$	$(t_1, \dots, t_D), (r_1, \dots, r_d)$	5
$\mathbf{u}_l, \mathbf{v}_l$	$(u_1, \dots, u_l), (v_1, \dots, v_l)$	7
$\mathbf{p}_l(\mathbf{t}_D), p_l(\mathbf{r}_d))$	the tuples of the values of the power-sum polynomials $\big(p_1(\mathbf{t}_D), \dots, p_l(\mathbf{t}_D)\big)$, $\big(p_1(\mathbf{r}_d), \dots, p_l(\mathbf{r}_d)\big)$ respectively	7
$\mathbf{i}_l, \mathbf{j}_l, \mathbf{0}_l$	$(i_1, \dots, i_l), (j_1, \dots, j_l), (0, \dots, 0)$	8
$\lvert\mathbf{i}_l\rvert$	$i_1 + 2i_2 + \cdots + li_l$, the weight of $\mathbf{i}_l$	7
$\mathbb{Q}$	the field of rational numbers	9
$\mathbf{x}_n$	the tuple of the variables $(x_1, \dots, x_n)$	10
$\pi^{\mathbf{j}_D}$	the product of the power-sum polynomials $p_1^{j_1} \cdots p_D^{j_D}$	8
$\mathbf{w}_l$	the tuple of the variables $(w_0, \dots, w_l)$	24
$\mathbf{w}_{l,\mathbf{t}_D}$	the tuple of the values $(w_{0,\mathbf{t}_D}, \dots, w_{l,\mathbf{t}_D})$	24

(A.1)

The main definitions introduced in this article are as follows.

Definition	Brief description	Page
φ	is a map from the set of s-tuples such that $\varphi:\ (i_1, \dots, i_s) \mapsto \big(\underbrace{\tau_1, \dots, \tau_1}_{i_1}, \underbrace{\tau_2, \dots, \tau_2}_{i_2}, \dots, \underbrace{\tau_s, \dots, \tau_s}_{i_s}\big)$	5
T	the image $\varphi(\{\mathbf{i} = (i_1, \dots, i_s) \mid \sum_j i_j = D\})$	5
$E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$	$-(1/l)\big(\sum_{\kappa=1}^{l} E_{l-\kappa}(v_0, \mathbf{v}_{l-\kappa}, u_0, \mathbf{u}_{l-\kappa})$ $(\sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda})\big)$	7
$S_l(u_0, \mathbf{u}_l)$	$\sum_{\mathbf{t} \in T} \alpha_{\mathbf{t}_D \in T} E_l\big(D, \mathbf{p}_l(\mathbf{t}_D), u_0, \mathbf{u}_l\big)$	7
$A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$	is the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$.	8

	For $\mathbf{i}_l = \mathbf{0}_l$ this polynomial does not depend on v_0, therefore one can write $A_{\mathbf{0}_l}(u_0)(\mathbf{v}_l)$	
$B_{l,m}(\mathbf{v}_l)$	is the coefficient of u_0^m in $A_{\mathbf{0}_l}(u_0)(\mathbf{v}_l)$	8
$E_l^*(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$	$\sum_{k=0}^{l} E_k(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) w_{l-k}$	24
$A_{\mathbf{i}_l}^*(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$	is the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $E_l^*(\mathbf{w}_l, v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$	25, the proof of Theorem 6
$B_{l,m}^*(\mathbf{w}_l, \mathbf{v}_l)$	is the coefficient of u_0^m in $A_{\mathbf{0}_l}^*(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$	ibid.

A.2. Bridge from the old result to the present ones

If the condition of Theorem 1 does not hold, that is, for all $0 \le l \le 5$ the polynomials $S_l(u_0, \mathbf{0}_l)$ are equal to the zero polynomial, then, in general, $S_6(u_0, \mathbf{u}_6)$ and $S_6(u_0, \mathbf{0}_6)$ do not have to be equal as polynomials and therefore $S_6(u_0, \mathbf{0}_6)$ cannot be taken as an indicial polynomial. More precisely, the following statement holds (Shkaravska and van Eekelen, 2014).

Lemma 16. If for any $0 \le l \le 5$ the polynomial $S_l(u_0, \mathbf{0}_l)$ is the zero polynomial, then

$$S_6(u_0, \mathbf{u}_6) = S_6(u_0, \mathbf{0}_6) + (1/8)(u_1^2 - u_2 u_0) \sum_{\mathbf{t}_D \in T} p_2^2(\mathbf{t}_D)\alpha_{\mathbf{t}_D}. \tag{A.2}$$

As one can see, $S_6(u_0, \mathbf{u}_6)$ depends not only on the variable u_0 which represents the degree d but on the term $u_1 u_0 - u_2^2$ with variables u_1 and u_2 representing the power-sums $p_1(\mathbf{r}_d)$ and $p_2(\mathbf{r}_d)$ of the unknown roots of a solution. However it was proven in Corollary 2 of (Shkaravska and van Eekelen, 2014) that for $D = 2$ the coefficient $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2)$ of $u_1 u_0 - u_2^2$ vanishes if $S_0(u_0), \ldots, S_5(u_0, \mathbf{0}_5)$ are all equal to the zero polynomial. We reconsider the proof from that article to provide the reader with an intuition behind the arguments used in the work under consideration. Moreover we will see why the same result does not hold for $D > 2$.

We observe that the condition $S_4(u_0, \mathbf{0}_4) \equiv 0$ induces the system of equations

$$\begin{cases} 1/24 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_1^4(t_1, t_2) = 0, \text{ the coefficient of } u_0^4 \\ -1/4 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2(t_1, t_2) p_1^2(t_1, t_2) = 0, \text{ the coefficient of } u_0^3 \\ 1/3 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_3(t_1, t_2) p_1(t_1, t_2) + 1/8 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1, t_2) = 0, \\ \qquad\qquad\qquad\qquad \text{the coefficient of } u_0^2 \\ -1/4 \sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_4(t_1, t_2) = 0, \text{ the coefficient of } u_0^1. \end{cases} \tag{A.3}$$

Note that for $D = 2$ the product $p_3 p_1$ is equal to $-1/2 p_1^4 + 3/2 p_2 p_1^2$. This can be shown by direct calculations, using the definition of $p_l(t_1, t_2) = t_1^l + t_2^l$. Indeed $p_1^3 = p_3 + 3t_1^2 t_2 + 3t_1 t_2^2 = p_3 + 3t_1 t_2(t_1 + t_2)$. Now, use $t_1 t_2 = 1/2(p_1^2 - p_2)$ to obtain $p_3 = p_1^3 - 3/2 p_1(p_1^2 - p_2) = -1/2 p_1^3 + 3/2 p_1 p_2$. This implies that $B_{4,2}(p_1, p_2) = 1/3 p_3 p_1 + 1/8 p_2^2 = -1/6 p_1^4 + 1/2 p_1^2 p_2 + 1/8 p_2^2$.

From this equality and the equations for the coefficients of u_0^4, u_0^3 and u_0^2 in the system above it follows that $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1,t_2) = 0$. Therefore the term with u_1 and u_2 in S_6 vanishes and $S_6(u_0, \mathbf{0}_6)$ is an indicial polynomial, unless it is the zero polynomial.

For $D = 3$ equating coefficients of $u_0^4, \dots, u_0^1$ in $S_4(u_0, \mathbf{0}_4)$ to zero does not imply that $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1,t_2) = 0$. To see this, again consider the system (A.3). Note that $p_4 = 1/6p_1^4 + 1/2p_2^2 + 4/3p_1p_3 - p_1^2p_2$. This can be checked by direct calculations, e.g. using a computer algebra one can check that $1/6(x+y+z)^4 + 1/2(x^2+y^2+z^2)^2 + 4/3(x+y+z)(x^3+y^3+z^3) - (x+y+z)^2(x^2+y^2+z^2) = x^4+y^4+z^4$. Therefore, the last equation can be discarded because it is an linear combination of the first three ones, where the first and the third equations are multiplied by -1, and the second equation is multiplied by 4. Therefore one obtains a system of 3 equations with 4 variables:

$$\begin{cases} 1/24X = 0 \\ -1/4Y = 0 \\ 1/3U + 1/8V = 0, \end{cases}$$

where X stands for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_1^4(t_1,t_2)$, Y stands for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2(t_1,t_2) p_1^2(t_1,t_2)$, U stands for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_3(t_1,t_2) p_1(t_1,t_2)$ and V is for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1,t_2)$. It is obvious that $X = Y = 0$, but then either U or V is a free variable and $V = 0$ cannot be established.

Now, consider $D \geq 4$. Again, equating coefficients of $u_0^4, \dots, u_0^1$ in $S_4(u_0, \mathbf{0}_4)$ to zero, in general does not imply that $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_2^2(t_1,t_2) = 0$. Indeed, in this case one obtains the system of 4 equations with 5 variables, where X, Y, U, V are defined as above and Z stays for $\sum_{(t_1,t_2)} \alpha_{(t_1,t_2)} p_4(t_1,t_2)$, with no auxiliary equations between these variables.

A.3. Auxiliary lemma

The following lemma below is used in the proof of Lemma 8.

Lemma 17. Let $l \geq 1$ and $0 < k < l$. Then the identity

$$B_{l,l-k}(\mathbf{v}_l) = -1/l \sum_{h=1}^{k+1} B_{l-h,l-k-1}(\mathbf{v}_{l-h}) v_h$$

holds and $B_{l,0}(\mathbf{v}_l) = 0$.

Proof. Recall the definition of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ for $l \geq 1$:

$$E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) = -1/l \sum_{h=1}^{l} E_{l-h}(v_0, \mathbf{v}_{l-h}, u_0, \mathbf{u}_{l-h}) \sum_{\lambda=0}^{h} \binom{h}{\lambda} u_\lambda v_{h-\lambda}. \tag{A.4}$$

It implies that for $l \geq 1$ the coefficient $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ of the $\mathbf{u}_l$-free sub-expression of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ is equal to $-1/l \sum_{h=1}^{l} A_{\mathbf{0}_{l-h}}(\mathbf{v}_{l-h})(u_0) u_0 v_h$. First, form this it follows there are no u_0-free terms in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$, which means that $B_{l,0}(\mathbf{v}_l) = 0$.

Second, the equation above implies that for $m > 0$ the coefficient $B_{l,m}(\mathbf{v}_l)$ of u_0^m in $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ is defined for $m \geq 1$ by the recurrent formula

$$
\begin{aligned}
B_{l,m}(\mathbf{v}_l) &= -1/l \sum_{h=1}^{l} B_{l-h,m-1}(\mathbf{v}_{l-h})v_h \\
&= -1/l \sum_{l-h\geq m-1,h=1}^{l} B_{l-h,m-1}(\mathbf{v}_{l-h})v_h \qquad \text{(A.5)} \\
&= -1/l \sum_{h=1}^{l-m+1} B_{l-h,m-1}(\mathbf{v}_{l-h})v_h.
\end{aligned}
$$

We introduce the index k by assigning $k := l - m$. Then the identity above implies the statement the lemma:

$$
B_{l,l-k}(\mathbf{v}_l) = -1/l \sum_{h=1}^{k+1} B_{l-h,l-k-1}(\mathbf{v}_{l-h})v_h. \qquad \text{(A.6)}
$$

The lemma is proven. □

A.4. Other properties of E- and A-polynomials

In this section we consider a number of properties of the polynomials $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ and $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$. This properties are used when one considers the influence of the term $G_0(x)$ on the existence of an upper bound of the degree of the solutions of a given ADE. They may be used in the future research as well.

Lemma 18. Let $S_0(u_0)$ be the zero polynomial and $|\mathbf{i}_l| = l$. Then for all $l \geq 1$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l$. Recall that for any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l|+|\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 0$ which means that for any term that occurs in $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = \cdots = j_l = 0$. From this it follows that $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0) = A_{\mathbf{i}_l}(v_0)(u_0)$ since it does not contain terms with occurrences of v_k, where $k \geq 1$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}(D)(u_0) = A_{\mathbf{i}_l}(D)(u_0) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} = 0$ due to $S_0(u_0) = \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} = 0$. □

Lemma 19. Let $S_1(u_0, 0)$ be equal to the zero polynomial and $|\mathbf{i}_l| = l - 1$. Then for all $l \geq 2$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l-1$. Recall that for any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l| + |\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 1$ which means that for all the terms that occur in the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = 1$ and $j_2 = \cdots = j_l = 0$. This implies that $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is of the form $K(u_0, v_0)v_1$ for some $K \in \mathbb{K}[u_0, v_0]$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{t}_D)\big)(u_0) = K(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1(\mathbf{t}_D) = 0$ due to the fact that $S_1(u_0, 0) = u_0 \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1(\mathbf{t}_D)$ is the zero polynomial in u_0. □

Lemma 20. Let $S_2(u_0, \mathbf{0}_2)$ be equal to the zero polynomial and $|\mathbf{i}_l| = l - 2$. Then for all $l \geq 3$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l - 2$. Recall that any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l| + |\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 2$. From this it follows that for all the terms that occur in the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = 2$ and $j_2 = \cdots = j_l = 0$, or $j_2 = 1$ and $j_1 = j_3 = \cdots = j_l = 0$. This implies that $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ is of the form $K_1(u_0, v_0)v_1^2 + K_2(u_0, v_0)v_2$ for some $K_1, K_2 \in \mathbb{K}[u_0, v_0]$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{t}_D)\big)(u_0) = K_1(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1^2(\mathbf{t}_D) + K_2(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_2(\mathbf{t}_D) = 0$ due to the fact that $S_2(u_0, 0, 0)$ is the zero polynomial in u_0, because its coefficients of u_0 and u_0^2 are proportional to the sums $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_2(\mathbf{t}_D)$ and $\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1^2(\mathbf{t}_D)$ respectively.
□

Lemma 21. Let $S_3(u_0, \mathbf{0}_3)$ be equal to the zero polynomial and $|\mathbf{i}_l| = l - 3$. Then for all $l \geq 4$ the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ vanish.

Proof. Fix some $\mathbf{i}_l$, such that $|\mathbf{i}_l| = l-3$. Recall that for any monomial $u_1^{i_1} \cdots u_l^{i_l} v_1^{j_1} \cdots v_l^{j_l}$ that occurs in the polynomial $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ the equality $|\mathbf{i}_l| + |\mathbf{j}_l| = l$ holds (Shkaravska and van Eekelen, 2014). Then the condition of the lemma implies that $|\mathbf{j}_l| = 3$. From this it follows that for all the terms that occur in the polynomial $A_{\mathbf{i}_l}(v_0, \mathbf{v}_l)(u_0)$ one has $j_1 = 3$ and $j_2 = \cdots = j_l = 0$, or $j_1 = j_2 = 1$ and $j_3 = \cdots = j_l = 0$, or $j_3 = 1$ and $j_1 = j_2 = j_4 = \cdots = j_l = 0$. This implies that $A_{\mathbf{i}_l}(u_0)(v_0, \mathbf{v}_l)$ is of the form $K_1(u_0, v_0)v_3^2 + K_2(u_0, v_0)v_1 v_2 + K_3(u_0, v_0)v_3$ for some $K_1, K_2, K_3 \in \mathbb{K}[u_0, v_0]$. Therefore the coefficient of $u_1^{i_1} \cdots u_l^{i_l}$ in $S_l(u_0, \mathbf{u}_l)$ is equal to

$$
\begin{aligned}
\sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} A_{\mathbf{i}_l}\big(D, \mathbf{p}_l(\mathbf{t}_D)\big)(u_0) = \; & K_1(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1^3(\mathbf{t}_D) + \\
& K_2(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_1(\mathbf{t}_D) p_2(\mathbf{t}_D) + \\
& K_3(u_0, D) \sum_{\mathbf{t}_D \in T} \alpha_{\mathbf{t}_D} p_3(\mathbf{t}_D) \\
& = 0
\end{aligned}
$$

due to the fact that $S_3(u_0, \mathbf{0}_3)$ is the zero polynomial in u_0. □

Lemma 22. Let $S_0(u_0), \ldots, S_3(u_0, \mathbf{0}_3)$ be all equal to the zero polynomial and let $l \geq 5$. Then if some term of $S_l(u_0, \mathbf{u}_l)$ contains u_{l-4} then u_{l-4} occurs in this term only linearly. In other words, there are no terms in $S_l(u_0, \mathbf{u}_l)$ which contain the products of u_{l-4} and any other $u_\lambda^{i_\lambda}$ with $\lambda \geq 1$ and $i_\lambda > 0$, and, in particular, there are no terms with powers of u_{l-4} which are higher than 1.

Proof. The statement follows from Lemmata 18, 19, 20, 21. Indeed, if such a term had occurred in $S_l(u_0, \mathbf{u}_l)$ then due to $l - 4 \geq 1$ this term would have satisfied the inequation $i_1 + 2i_2 + \cdots + (l-4)i_{l-4} + \cdots + l i_l \geq (l-4) + 1 = l - 3$. Then either $|\mathbf{i}_l| = l$, or

$|\mathbf{i}_l| = l-1$, or $|\mathbf{i}_l| = l-2$, or $|\mathbf{i}_l| = l-3$, and one can apply one of the lemmata listed above. Therefore the term vanishes in $S_l(u_0, \mathbf{u}_l)$. □

Lemma 23. The $\mathbf{u}_l$-free subterm $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ is the recursive function of l and therefore the symbolic value $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ can be computed for any fixed $l \geq 1$ via its recursive presentation.

Proof. We recall the inductive definition (3) of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ to obtain the recursive function computing $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$:

$$E_0\big(v_0, (), u_0, ()\big) := 1,$$
$$E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l) := -(1/l) \sum_{\kappa=1}^{l} E_{l-\kappa}(v_0, \mathbf{v}_{l-\kappa}, u_0, \mathbf{u}_{l-\kappa}) \left(\sum_{\lambda=0}^{\kappa} \binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda} \right).$$

To obtain $\mathbf{u}_l$-free terms in $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ one needs to set the indices λ in the products of the form $\binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda}$ only to 0. In this way we obtain the following inductive representation of $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$:

$$A_{()}()(u_0) := 1,$$
$$A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0) = -(1/l) \sum_{\kappa=1}^{l} A_{\mathbf{0}_{l-\kappa}}(\mathbf{v}_{l-\kappa})(u_0) \cdot u_0 v_\kappa.$$

This expression is represented as a program in the computer algebra system so that the symbolic value $A_{\mathbf{0}_l}(\mathbf{v}_l)(u_0)$ can be obtained for any fixed $l \geq 0$. □

Lemma 24. Let $l \geq 5$. Then the coefficient $A_{(\mathbf{0}_{l-5},1,\mathbf{0}_4)}(v_0, \mathbf{v}_l)(u_0)$ of the term of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ with the linear occurrence of u_{l-4} is a function of l, u_0 and $v_0, \ldots, v_4$ of the form

$$\begin{aligned}
A_{(\mathbf{0}_{l-5},1,\mathbf{0}_4)}(v_0, \mathbf{v}_l)(u_0) = {} & -(v_4 l^4 + (-10v_4 - 4u_0v_1v_3)l^3 + \\
& (35v_4 + 36u_0v_1v_3 - 6u_0v_2^2 + 6u_0^2v_1^2v_2)l^2 + \\
& (-50v_4 - 112u_0v_1v_3 + 42u_0v_2^2 - 30u_0^2v_1^2v_2 - 4u_0^3v_1^4)l + \\
& (24 - 6u_0v_0)v_4 + (8u_0^2v_0 + 128u_0)v_1v_3 + (3u_0^2v_0 - 72u_0)v_2^2 + \\
& (24u_0^2 - 6u_0^3v_0)v_1^2v_2 + (u_0^4v_0 + 16u_0^3)v_1^4)/(24l - 96)
\end{aligned}$$

Proof. We use the inductive definition of $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$ to obtain a recursive formula for the coefficient $A_{(\mathbf{0}_{l-m-1},1,\mathbf{0}^m)}(v_0, \mathbf{v}_l)(u_0)$ for the all the monomials where $m \geq 0$ and u_{l-m} occurs linearly in $E_l(v_0, \mathbf{v}_l, u_0, \mathbf{u}_l)$. The recursion in this formula runs over m.

Let $m = 0$. We use the fact that u_l does not occur in $E_{l-\kappa}(v_0, \mathbf{v}_{l-\kappa}, u_0, \mathbf{u}_{l-\kappa})$ for any $1 \leq \kappa \leq l$. Then using the definition of E_l one obtains the formula for $A_{\mathbf{0}^{l-1}1}$, by setting λ to l in the products of the form $\binom{\kappa}{\lambda} u_\lambda v_{\kappa-\lambda}$ of that definition:

$$
\begin{aligned}
A_{\mathbf{0}_{l-1}1}(v_0, \mathbf{v}_l)(u_0) = & -(1/l)\sum_{\kappa=1}^{l} A_{\mathbf{0}_{l-\kappa}}(\mathbf{v}_{l-\kappa})(u_0)\binom{\kappa}{l} \cdot v_{\kappa-l} \\
& \overset{\kappa\geq l\ ,\kappa\leq l \Rightarrow \kappa=l}{=} (-1/l)A_{()}()(u_0)\cdot v_0 \\
& = -v_0/l.
\end{aligned}
$$

Let $m > 0$. One sets $\lambda := l - m$ for the products of $u_{l-m}v_{\kappa-(l-m)}$ and the terms of $E_{l-\kappa}$ with no occurrences of u_{l-m}, and $\lambda := 0$ for the products of $u_0 v_\kappa$ and the terms of $E_{l-\kappa}$ where $u_{l-m} = u_{(l-\kappa)-(m-\kappa)}$ occurs linearly. We obtain the following equalities:

$$
\begin{aligned}
A_{\mathbf{0}_{l-m-1}1\mathbf{0}_m}(v_0, \mathbf{v}_l)(u_0) = & -(1/l)\sum_{\kappa=1}^{l} A_{\mathbf{0}_{l-\kappa}}(\mathbf{v}_{l-\kappa})(u_0)\binom{\kappa}{l-m} v_{\kappa-(l-m)} \\
& -(1/l)\sum_{\kappa=1}^{l} A_{\mathbf{0}_{l-m-1}1\mathbf{0}_{l-\kappa-(l-m)}}(v_0, \mathbf{v}_{l-\kappa})(u_0)u_0 v_\kappa \\
& = -(1/l)\sum_{\kappa=l-m}^{l} A_{\mathbf{0}_{l-\kappa}}(\mathbf{v}_{l-\kappa})(u_0)\binom{\kappa}{l-m} v_{\kappa-(l-m)} \\
& -(1/l)\sum_{\kappa=1}^{m} A_{\mathbf{0}_{l-m-1}1\mathbf{0}_{l-\kappa-(l-m)}}(v_0, \mathbf{v}_{l-\kappa})(u_0)u_0 v_\kappa \\
& \overset{k:=l-\kappa}{=} -(1/l)\sum_{k=0}^{m} A_{\mathbf{0}_k}(\mathbf{v}_k)(u_0)\binom{l-k}{l-m} v_{m-k} \\
& -(1/l)\sum_{\kappa=1}^{m} A_{\mathbf{0}_{l-m-1}1\mathbf{0}_{l-\kappa-(l-m)}}(v_0, \mathbf{v}_{l-\kappa})(u_0)u_0 v_\kappa
\end{aligned}
$$

We have encoded this recursive over m definition for $A_{\mathbf{0}_{l-m-1}1\mathbf{0}_m}(v_0, \mathbf{v}_l)(u_0)$ as a program in the computer algebra system. To obtain the statement of the lemma one runs this program for $m = 4$. □

A.5. Tables of the coefficients for analysis of ADE with variable polynomial coefficients.

This subsection provides detailed technical information for Section 6. The expressions for E^*_l, $A^*_{\mathbf{i}_l}$ and $B^*_{l,m}$ for $0 \leq l \leq 3$ given here are used in the proof of Theorem 6. They are obtained by programming the corresponding recursive definitions in the CAS Maxima. One can download the corresponding script `VariateCoefficients` at `http://resourceanalysis.cs.ru.nl/#Algebraic Difference Equations`.

The expressions for E_l^*:

E_l^*	expression
$E_0^*(w_0, v_0, u_0)$	w_0
$E_1^*(w_0, w_1, v_0, v_1, u_0, u_1)$	$w_1 - u_0 w_0 v_1 - v_0 w_0 u_1$
$E_2^*(\mathbf{w}_2, v_0, \mathbf{v}_2, u_0, \mathbf{u}_2)$	$w_2 - (u_0 w_0 v_2)/2 - (v_0 w_0 u_2)/2 - u_0 v_1 w_1 - v_0 u_1 w_1 +$
	$(u_0^2 w_0 v_1^2)/2 + u_0 v_0 w_0 u_1 v_1 - w_0 u_1 v_1 + (v_0^2 w_0 u_1^2)/2$
$E_3^*(\mathbf{w}_3, v_0, \mathbf{v}_3, u_0, \mathbf{u}_3)$	$w_3 - (u_0 w_0 v_3)/3 - (v_0 w_0 u_3)/3 - u_0 v_1 w_2$
	$v_0 u_1 w_2 - (u_0 w_1 v_2)/2 + (u_0^2 w_0 v_1 v_2)/2 + (u_0 v_0 w_0 u_1 u_2)/2 -$
	$w_0 u_1 v_2 - (v_0 w_1 u_2)/2 +$
	$(u_0 v_0 w_0 v_1 u_2)/2 - w_0 v_1 u_2 + (v_0^2 w_0 u_1 u_2)/2 +$
	$(u_0^2 v_1^2 w_1)/2 + u_0 v_0 u_1 v_1 w_1 - u_1 v_1 w_1 + (v_0^2 u_1^2 w_1)/2 -$
	$(u_0^3 w_0 v_1^3)/6 - (u_0^2 v_0 w_0 u_1 v_1^2)/2 + u_0 w_0 u_1 v_1^2 -$
	$(u_0 v_0^2 w_0 u_1^2 v_1)/2 + v_0 w_0 u_1^2 v_1 - (v_0^3 w_0 u_1^3)/6$

(A.7)

Expressions for $A_{\mathbf{0}_l}^*$:

$A_{\mathbf{0}_l}^*(\mathbf{w}_l, \mathbf{v}_l)(u_0)$	expression
$A_{()}^*(w_0)(u_0)$	w_0
$A_0^*(w_0, w_1, v_1)(u_0)$	$w_1 - u_0 w_0 v_1$
$A_{00}^*(\mathbf{w}_2, \mathbf{v}_2)(u_0)$	$w_2 - (u_0 w_0 v_2)/2 - u_0 v_1 w_1 + (u_0^2 w_0 v_1^2)/2$

(A.8)

Expressions for $B_{l,m}^*$:

$B_{l,m}^*(\mathbf{w}_l, \mathbf{v}_l)$	expression
$B_{()}^*(w_0)$	w_0
$B_{1,0}^*(w_0, w_1, v_1)$	w_1
$B_{1,1}^*(w_0, w_1, v_1)$	$-w_0 v_1$
$B_{2,0}^*(\mathbf{w}_2, \mathbf{v}_2)$	w_2
$B_{2,1}^*(\mathbf{w}_2, \mathbf{v}_2)$	$-(w_0 v_2)/2 - v_1 w_1$
$B_{2,2}^*(\mathbf{w}_2, \mathbf{v}_2)$	$(w_0 v_1^2)/2$

(A.9)

Expressions for $A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$, where $\mathbf{i}_l \neq \mathbf{0}_l$:

$A^*_{\mathbf{i}_l}(\mathbf{w}_l, v_0, \mathbf{v}_l)(u_0)$	expression	presentation via $B^*_{l,m}$
$A^*_1(w_0, w_1, v_0, v_1)(u_0)$	$-v_0 w_0$	$-v_0 B_{0,0}(w_0)$
$A^*_{10}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$	$-v_0 w_1 + u_0 v_0 w_0 v_1 - w_0 v_1$	$(1 - u_0 v_0) B_{1,1}(w_0, w_1, v_1) -$ $v_0 B_{1,0}(w_0, w_1, v_1)$
$A^*_{20}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$	$(v_0^2 w_0)/2$	$(v_0^2)/2 B_{0,0}(w_0)$
$A^*_{01}(\mathbf{w}_2, v_0, \mathbf{v}_2)(u_0)$	$-(v_0 w_0)/2$	$-(v_0/2) B_{0,0}(w_0)$
$A^*_{100}(\mathbf{w}_3, v_0, \mathbf{v}_3, \mathbf{u}_3)(u_0)$	$-v_0 w_2 + (u_0 v_0 w_0 v_2)/2 - w_0 v_2 -$ $u_0 v_0 v_1 w_1 - v_1 w_1 -$ $(u_0^2 v_0 w_0 v_1^2)/2 + u_0 w_0 v_1^2$	$(-v_0) B_{2,0}(\mathbf{w}_2, \mathbf{v}_2) +$ $(2u_0 - u_0^2 v_0) B_{2,2}(\mathbf{w}_2, \mathbf{v}_2) +$ $(1 - u_0 v_0) B_{2,1}(\mathbf{w}_2, \mathbf{v}_2) -$ $(w_0 v_2)/2$

(A.10)

A.6. Connection between Diophantine equations and ADE

We consider a solution tuple $a_0, \ldots, a_m \in \mathbb{A}$ of a Diophantine equation

$$F(x_0, \ldots, x_m) = 0 \tag{A.11}$$

as the coefficients of the polynomial $P(x) = a_m x^m + \cdots + a_1 x + a_0$. Also, we consider an $(m+1) \times (m+1)$ linear system of symbolic equations w.r.t. $a_m, \ldots, a_0$:

$$\begin{aligned} &a_m x^m + \cdots + a_1 x + a_0 &&= P(x) \\ &a_m (x-1)^m + \cdots + a_1 (x-1) + a_0 &&= P(x-1) \\ &\ldots \\ &a_m (x-m)^m + \cdots + a_1 (x-m) + a_0 &&= P(x-m). \end{aligned} \tag{A.12}$$

Using Kramer's rule one obtains the rational symbolic algebraic expressions of the form $a_i = \Delta_i(x, P)/\Delta(x)$, where :

$$\Delta(x) = \begin{vmatrix} x^m & x^{m-1} & \ldots & x & 1 \\ (x-1)^m & (x-1)^{m-1} & \ldots & x-1 & 1 \\ \ldots \\ (x-m)^m & (x-m)^{m-1} & \ldots & x-m & 1 \end{vmatrix} \tag{A.13}$$

is the determinant of the system, and

$$\Delta_i(x,P) = \begin{vmatrix} x^m & \dots\ x^{m-(i-1)} & P(x) & x^{m-(i+1)} & \dots\ 1 \\ (x-1)^m & \dots\ (x-1)^{m-(i-1)} & P(x-1) & (x-1)^{m-(i+1)} & \dots\ 1 \\ \dots \\ (x-m)^m & \dots\ (x-m)^{m-(i-1)} & P(x-m) & (x-m)^{m-(i+1)} & \dots\ 1 \end{vmatrix} \quad \text{(A.14)}$$

is the minor corresponding to the variable a_i. Now we substitute these expressions into the original Diophantine equation and obtain the following rational algebraic equation

$$F\big(\tfrac{\Delta_0(x,P)}{\Delta(x)}, \dots, \tfrac{\Delta_m(x,P)}{\Delta(x)}\big) = 0 \quad \text{(A.15)}$$

which after the multiplication of both parts by the symbolic denominator $\Delta^D(x)$, where D is the total degree of the polynomial F, yields the ADE of the form

$$G_F(x)\big(P(x), P(x-1), \dots, P(x-m)\big) = 0. \quad \text{(A.16)}$$

Lemma 25. Diophantine equation (A.11) has an integer solution tuple if and only if the derived ADE (A.16) has a polynomial solution in $\mathbb{Z}[x]$.

Proof. Let the Diophantine equation have an integer solution tuple $a_0, \dots, a_m$. Introduce the polynomial $P(x) := a_m x^m + \dots + a_1 x + a_0$. It is a routine to show that by the construction of ADE (A.16) this polynomial is its solution. Indeed, from the symbolic equalities $a_i = \Delta_i(x,P)/\Delta(x)$, where P is considered as a symbol, it follows that equation (A.15) holds for $P(x)$, but this equation is equivalent to ADE (A.16), since $\Delta(x) \neq 0$ because it is the Vandermonde determinant with the entries $1, x, x-1, \dots x-m$. Now, let ADE (A.16) has a polynomial solution $P(x) := a_m x^m + \dots + a_1 x + a_0$. Then, again, it is easy to check that $(a_0, \dots, a_m)$ solves $F(y_0, \dots, y_m) = 0$. Indeed, ADE (A.16) is equivalent to rational equation (A.15) and since $\Delta_i(x,P)/\Delta(x) = a_i$ then $F(a_0, \dots, a_m) = 0$ as well. □

Theorem 7. There is no algorithm that for any m and any ADE with integer coefficients decides if it has and integer polynomial solutions of degree at most m or not.

Proof. The theorem follows from Lemma 25 and the fact that the problem of the existence of the roots of Diophantine equations is undecidable (Davis, 1973). In particular if an undecidable Diophantine equation $F(a_0, \dots, a_m)$ is given then there is no decision procedure that for the corresponding equation $G_F\big(P(x), P(x-1), \dots, P(x-m), x\big) = 0$ decides if it has an integer polynomial solution of degree at most m or not. □

Recall that the decidability of the Diophantine problem in the field $\mathbb{Q}$ of rational numbers is still open.

A.7. Undecidability of the positive existential theory for polynomial rings with the difference operator

Let $\mathbb{K}$ be a number field. In Section 7 we have shown that knowing the degree of a possible polynomial solution of an ADE does not guarantee that one can find this solution or prove it absence, if the first-order theory of $\mathbb{K}$ is undecidable. In this section we will show that whichever the first-order theory of $\mathbb{K}$ is, finding polynomial solutions of systems of equations involving ADE, is in general an undecidable problem.

Recall that a *positive existential theory* in a language L is a the set of all first-order existential sentences, containing only equations, in the language L which are true in $\mathbb{K}[x]$ (Pheidas and Zahidi, 2000). Let Δ denote the difference operator $\Delta(P)(x) := P(x) - P(x-1)$.

The following construction is used to show the representation of integer numbers via linear ADEs with variate coefficients. We consider an n-parametric family of the rising-factorial polynomials of the form

$$P_n(x) := (x+1)\cdots(x+n). \tag{A.17}$$

When the number n is fixed, for the polynomial $P_n(x)$ the following equation holds:

$$\big(P_n(x) - P_n(x-1)\big)(x+n) = nP_n(x) \tag{A.18}$$

which is easy to check by routine calculations:

$$\begin{aligned} P_n(x) - P_n(x-1) = \; & (x+1)\cdots(x+n) - x\cdots(x+n-1) = \\ & (x+1)\cdots(x+n-1)\big((x+n) - x\big) = \\ & \frac{P_n(x)}{x+n}n. \end{aligned} \tag{A.19}$$

Equation (A.18) gives an idea for the following definition of the integer number.

Lemma 26. A number $b \in \mathbb{K}$ is integer if and only if there is a non-zero polynomial solution $P \in \mathbb{K}[x]$ for the difference equation

$$\big(P(x) - P(x-1)\big)(x+b) = bP(x). \tag{A.20}$$

Moreover, b is the degree of this polynomial solution.

Proof. Let equation (A.20) does have a polynomial solution $P \in \mathbb{K}[x]$ and let $P(x) := a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$. The coefficients of x^d and of x^{d-1} in $P(x-1)$ are a_d and $(-a_d d + a_{d-1})$ respectively. This implies that the coefficient of x^d in $P(x) - P(x-1)$ is $a_d - a_d = 0$ and the coefficient of x^{d-1} is $a_{d-1} - (-a_d d + a_{d-1}) = a_d d$. Therefore the coefficient of x^d on the left-hand side of equation (A.20) is $a_d d$. Trivially the coefficient of x^d on the right-hand side is ba_d. Therefore one has $a_d d = ba_d$ and therefore $d = b$. So, b is an integer number, and moreover it is equal degree of the polynomial solution.

Now, let $b = n$ be an integer number. Then by the definition (A.17) and equation (A.18) one has that the rising-factorial polynomial of degree $b = n$ is a polynomial solution of equation (A.20). □

Now, we want to obtain the equivalent definition of b being an integer number in the positive existential theory for $\mathbb{K}[x]$ with the difference operator Δ, that is we want to exclude the universal quantifier in the formula $\exists\ P.\forall x.\big((x+b)\Delta[P](x) = bP(x)\big)$. Let $\mathbb{Id}$ denote the identity polynomial, which sends x to itself. Then formula for defining b as integer number above can be written as $\exists\ P.\forall x.\big(\mathbb{Id}(x) + b\big)\Delta[P](x) = bP(x)$ and the following statement holds:

$$b \in Z^+ \text{ if and only if } \exists\ P\ H.(\mathbb{Id} + b)\Delta[P] = bP \wedge PH = 1, \tag{A.21}$$

where $\exists\, H.PH = 1$ is equivalent to "P is a non-zero polynomial." Now it is a routine to prove the following theorem.

Theorem 8. The positive existential theory of $\mathbb{K}[x]$ in the language L_Δ is undecidable.

Proof. The proof mimics the proof of undecidability of the positive existential theory of complex rational functions in the language augmented with the derivative operator, (Pheidas and Zahidi, 1999).

Due to Lemma 26 and equation (A.21), a Diophantine equation $F(y_1, \ldots, y_m) = 0$ has an integer solution if and only if the formula $\exists\, b_1, \ldots, b_m, P_1, \ldots, P_m.\ F(b_1, \ldots, b_m) = 0 \wedge (\mathbb{Id}+b_1)\Delta[P] = b_1 P \wedge \cdots \wedge (\mathbb{Id}+b_m)\Delta[P] = b_m P \wedge P_1 H_1 = 1 \wedge P_m H_m = 1$ is true in $\mathbb{K}[x]$. Since the solvability of the Diophantine problem for integers reduces to the decidability of the positive theory of $\mathbb{K}[x]$ in the language L_Δ, the latter theory is undecidable. $\Box$

A.8. The influence of G_0 on the existence of an upper bound of the degree of a polynomial solution

Since the polynomial $G_0(x)$ is not involved in the main work behind the presented approach, it may be tempting to remove it from the formulation of the main results and to use assumptions like "without loss of generality assume that G_0 is the zero polynomial".

However we decided to keep $G_0(x)$ in the formulations because it does influence the existence of an upper bound of the degree of a polynomial solution of an ADE. An example is given by the following pair of the ADEs that differ only by $G_0(x)$, with $G_0(x) \equiv 0$ for the first one, and $G_0(x) \equiv -1$ for the second one:

$$\begin{aligned}&P(x)P(x-2)P(x-3) - 2P(x-1)P(x-1)P(x-3)+\\&P(x-1)P(x-2)P(x-2) + P(x)P(x-1)P(x-3)-\\&2P(x)P(x-2)P(x-2) + P(x-1)P(x-1)P(x-2) = 0,\end{aligned} \tag{A.22}$$

$$\begin{aligned}&P(x)P(x-2)P(x-3) - 2P(x-1)P(x-1)P(x-3)+\\&P(x-1)P(x-2)P(x-2) + P(x)P(x-1)P(x-3)-\\&2P(x)P(x-2)P(x-2) + P(x-1)P(x-1)P(x-2) = 1.\end{aligned} \tag{A.23}$$

There is no an upper bound for the degree of a polynomial solution for the first ADE because for any real a the corresponding falling factorial of the form $P_n(x) = (x-a)(x-a-1)\cdots\big(x-a-(n-1)\big)$ is its solution, see (Shkaravska and van Eekelen, 2014). The second equation has polynomial solutions of degree at most 2. Below we discover why this is the case.

Let $\mathbb{K}$ be a number field.

Lemma 27. For $l \geq 5$ the coefficient of the linear occurrence of u_{l-4} in $S_l(u_0, \mathbf{u}_l)$ for equations (A.22) and (A.23) is $K(u_0, l) = -u_0 l^2 + 9u_0 l - 20u_0$. (Note that this polynomial is equal to zero in $l = 4, 5$.)

Proof. Recall that this coefficient is equal to the sum $\sum_{\mathbf{t}_3 \in T} \alpha_{\mathbf{t}_3} A_{\mathbf{0}_{l-5}\mathbf{1}\mathbf{0}_4}(u_0)(3, \mathbf{p}_l(\mathbf{t}_3))$. The expression for $K(u_0, l)$ is calculated straightforwardly in the computer algebra system as this sum, using Lemma 24. $\Box$

Lemma 28. If $d \geq 3$ then, up to the leading coefficient, the only solutions of equation (A.22) are rising factorials of the form $(x + a)(x + a + 1) \cdots \bigl(x + a + (d - 1)\bigr)$.

Proof. We fix arbitrary $d \geq 3$ and some solution $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ of degree d. We are going to show all the values $p_2(\mathbf{r}_d), \ldots, p_d(\mathbf{r}_d)$ of the power-sum symmetric polynomials at the roots of the solution are defined uniquely as functions of $p_1(\mathbf{r}_d)$.

Indeed, for equation (A.22) the polynomials $S_0(u_0), S_1(u_0, 0), S_2(u_0, 0, 0), S_3(u_0, \mathbf{0}_3)$ are all equal to the zero polynomial, and therefore one can apply Lemmata 18, 19, 20,21 and Lemma 27. Therefore for each $l \geq 6$ there is the nonzero coefficients $K(d, l) = d(-l^2 + 9l - 20)$ and an algebraic expression $M(l, d, \mathbf{u}_{l-5})$ such that $S_l(d, \mathbf{u}_l) = K(d, l)u_{l-4} + M(l, d, \mathbf{u}_{l-5})$.

Since $d \geq 3$ one has that $3d - (d + 4) \geq 1$, and therefore for all $6 \leq l \leq d + 4$ the values $S_l\bigl(d, \mathbf{p}_l(\mathbf{r}_l)\bigr) = 0$. Together with the presentation above this implies that

$$p_{l-4}(\mathbf{r}_d) = -\frac{M\bigl(l, d, \mathbf{p}_{l-5}(\mathbf{r}_d)\bigr)}{K(d, l)} \tag{A.24}$$

By induction from this follows that $p_{l-4}(\mathbf{r}_d)$ is defined uniquely via its predecessors $p_1(\mathbf{r}_d), \ldots, p_{l-5}(\mathbf{r}_d)$ for $l = 6, \ldots, d + 4$. Therefore all the coefficients $a_l = (-1)^l e_l(\mathbf{r}_d)$ of a polynomial solution are defined uniquely via $p_1(\mathbf{r}_d)$ using the Newton-Girard identities and the identity (A.24) above. For the given a_{d-1} one can find such a that

$$a_{d-1} = -p_1(\mathbf{r}_d) = -\bigl(a + (a + 1) + \cdots + (a + d - 1)\bigr). \tag{A.25}$$

The rising factorial corresponding to this $a \in \mathbb{K}$ solves equation (A.22) and therefore the power-sums at its roots satisfy the identity (A.24) as well. Therefore $P(x)$ coincides with the rising factorial for this a. □

Now we can prove the main statement of this section.

Lemma 29. Equation (A.23) does not have solutions of degree $d \geq 3$.

Proof. The proof is similar to the proof of Lemma 28. Let us assume the opposite, that is for equation (A.23) there exists a polynomial solution $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ of degree $d \geq 3$. We note that for both equations (A.22) and (A.23) the functions $S_l(u_0, \mathbf{u}_l)$ are the same and, moreover, the coefficients of x^{3d-l} must vanish for all $6 \leq l \leq d + 4$ if we speak about their solutions of degree $d \geq 3$. This implies that for the roots of solutions of both equations identities (A.24) hold. For the given a_{d-1} one can find such a that equality (A.25) holds. Therefore by induction on $l = 1, \ldots, d$ we obtain that the corresponding coefficients of the polynomial $P(x)$ and of the rising factorial for this a are equal. Therefore both polynomials are equal and $P(x)$ cannot solve equation (A.23), since the rising factorial solves equation (A.22). □

Lemma 30. Equation (A.23) does have solutions of degree $d \leq 2$.

Proof. The solutions of degree $d \leq 2$ are obtained via the method of unknown coefficients using the computer algebra system. For instance, it can be shown that for $d = 2$ the normalised solutions are of the form $P(x) = x^2 + a_1 x + a_0$ where $a_0 = (2a_1^2 - 1)/8$. □