INSTITUTE OF MATHEMATICS, ASTROPHYSICS AND PARTICLE PHYSICS

RADBOUD UNIVERSITY

PHD THESIS

The λ-R model:
Gravity with a preferred foliation and its consequences

Author:
Luís PIRES[a]

[a]email

Supervisor:
Prof. Renate LOLL

August 3, 2018

Cover: Photograph by Tess J. Post.

Printed by Gildeprint
ISBN: 978-94-92896-60-5

This work was sponsored by Fundação para a Ciência e Tecnologia through grant no. SFRH / BD / 76630 / 2011.

QUADRO DE REFERÊNCIA ESTRATÉGICO NACIONAL

UNIÃO EUROPEIA
Fundo Social Europeu

GOVERNO DA REPÚBLICA PORTUGUESA

QUALIFICAR É CRESCER

The λ-R model:

Gravity with a preferred foliation and its consequences

Proefschrift

ter verkrijging van de graad van doctor
aan de Radboud Universiteit Nijmegen
op gezag van de rector magnificus
prof. dr. J.H.J.M. van Krieken
volgens besluit van het college van decanen
in het openbaar te verdedigen op
dinsdag 11 september 2018
om 16.30 uur precies

door

Luís Pedro Moura Pena Pires

geboren op 8 mei 1988
te Oeiras, Portugal

Promotor: Prof. dr. R. Loll

Manuscriptcommissie:
Prof. dr. N. de Groot
Prof. dr. J. Ambjørn (Københavns Universitet, Denemarken)
Prof. dr. D. Giulini (Leibniz Universität Hannover / Universität Bremen, Duitsland)
Prof. dr. J. F. Barbero (Instituto de Estructura de la Materia, CSIC, Spanje)
Dr. F. Saueressig

The λ-R model:

Gravity with a preferred foliation and its consequences

Doctoral Thesis

to obtain the degree of doctor
from Radboud Unviersity Nijmegen
on the authority of the Rector Magnificus
prof. dr. J.H.J.M. van Krieken
according to the decision of the Council of Deans
to be defended publicly on
Tuesday, September 11, 2018
at 16.30 hours

by

Luís Pedro Moura Pena Pires

Born on May 8, 1988
in Oeiras, Portugal

Supervisor: Prof. dr. R. Loll

Doctoral Thesis Committee:
Prof. dr. N. de Groot
Prof. dr. J. Ambjørn (University of Copenhagen, Denmark)
Prof. dr. D. Giulini (University of Hannover / University of Bremen, Gemany)
Prof. dr. J. F. Barbero (Instituto de Estructura de la Materia, CSIC, Spain)
Dr. F. Saueressig

Contents

1

Introduction

General relativity [36,79] and the Standard Model of particle physics [20,96] are the pillars of our current understanding of the fundamental forces. While work for this thesis was under way, two spectacular discoveries have strengthened the evidence for these two theories even further. The Higgs boson was observed at the Large Hadron Collider (LHC) [4,28] and, more recently, gravitational waves were detected at the Laser Interferometer Gravitational-Wave Observatory (LIGO) [70]. These phenomena had been predicted by the Standard Model and general relativity respectively.

This thesis deals with an alternative to Einstein's classical theory of gravity. Despite the fact that general relativity has been verified experimentally with exceptional success [95], there are still open problems in its description of gravitational phenomena. As long as these remain unresolved, the investigation of well-motivated alternatives to general relativity remains mandatory.

One particular issue arises when we use general relativity to describe the dynamics of the universe at large. Imposing the usual conditions of spatial homogeneity and isotropy leads directly to standard cosmology. To account for observational cosmological data, it then seems necessary to introduce the concepts of dark matter and dark energy. However, neither has been observed directly or justified convincingly on theoretical grounds. One possible resolution is that the description of gravitational phenomena in terms of general relativity at the relevant scales is not adequate and the theory needs to be modified.

Another open issue is the fact that we do not currently have a fundamental theory of gravity at all scales. The classical, general relativistic description breaks down at extreme Planckian scales, which are involved when we want to describe what happens at the very beginning of our universe or at the centre of a black hole. These are examples of singularities, which have been shown to be an unavoidable feature of general relativity [44]. Explaining the dynamics of gravity at the smallest scales seems to require a theory of quantum gravity [23]. Because of the perturbative non-renormalisability of general relativity [48], perturbative methods of quantisation do not lead to a fundamental theory of quantum gravity valid at the Planck scale. While there are several candidate theories of non-perturbative quantum gravity (see, for instance [1,33,60]), these are currently incomplete and no consensus has emerged.

It is in principle possible that the correct quantum theory of gravity has a classical limit which differs from general relativity, but still agrees with all known observations within measuring accuracy. This constitutes a specific motivation to consider parametric deformations of general relativity.

Two crucial elements in the construction of a fundamental theory of gravity are the status of diffeomorphism invariance and the role of time [53,54,61]. The latter stems from the different role played by time in general relativity and standard quantum field theory. In classical general relativity, the geometry of spacetime is dynamical and there is no notion of time that is preferred *a priori*. Applying a spacetime diffeomorphism will in general lead to a different time, without changing the physics. By contrast, time in quantum field theory refers to a fixed metric background structure, usually that of Minkowski space, where it is unique up to global Lorentz transformations. Depending on whether and how diffeomorphisms and time appear in quantum gravity, it is possible that its classical limit reflects this in some way. The model we study is a concrete example of a classical theory of gravity which is not invariant under four-dimensional diffeomorphisms $\mathrm{Diff}(M)$ and has a preferred foliation $\mathscr{F}$ by leaves of constant time. It can be seen as a classical limit of Hořava-Lifshitz gravity [50], a candidate for a theory of quantum gravity. Although the study of this model cannot solve the more fundamental questions appearing in the quantum theory, it can nevertheless illustrate the possible classical implications of the specific choices made in Hořava-Lifshitz gravity regarding diffeomorphism invariance and the role of time.

There are many ways to modify general relativity by adding higher-order curvature terms. A well-known example are the so-called $f(R)$-theories[1] (see [88] for a review), which are alternatives to general relativity whose Lagrangian is an arbitrary function f of the four-dimensional Ricci scalar. The λ-R model studied in this thesis is a different type of modification. Its action does not contain invariants of order higher than two in derivatives, but has a different invariance group than general relativity, given by the subgroup of foliation-preserving spatial diffeomorphisms $\mathrm{Diff}_{\mathscr{F}}(M)$.

Our aim is to describe the physical solutions of the model and to understand how the mathematical structure of general relativity is affected by the presence of a preferred foliation and the associated breaking of diffeomorphism symmetry. Furthermore, we want to quantify deviations from general relativity to put bounds on λ, which is of interest in the context of Hořava-Lifshitz gravity. Finally, we want to compare the λ-R model to other $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant theories to understand which of its properties are a general feature of theories with that symmetry group and which are specific to the λ-R model.

[1] The "R" in $f(R)$ refers to the Ricci scalar of a four-dimensional spacetime. Note that in the remainder of the thesis this quantity will be denoted by ${}^{(4)}R$.

1.1 The λ-R model

The theory we will investigate is the so-called λ-R model. It is a classical field theory of the gravitational field which differs from general relativity by the introduction of an extra parameter, the dimensionless constant λ. The "R" of the λ-R model refers to the Ricci scalar of a three-dimensional hypersurface. We will keep referring to the "λ-R model", a name coined in [9], despite the fact that we will denote the three-dimensional Ricci scalar by $\mathscr{R}$.

The model is defined for four-dimensional spacetimes M which admit a foliation by leaves of constant time. This is expressed by decomposing M as

$$M = \mathbb{R} \times \Sigma_t, \tag{1.1}$$

where Σ_t denotes three-dimensional hypersurfaces of constant time (see chapter 2 and appendix B for details).

The action S_λ of the λ-R model can be written as a sum of the Einstein-Hilbert action S_{EH} of general relativity and an extra term,

$$S_\lambda = S_{EH} + \frac{1}{16\pi G_N} \int dt \int d^3x \sqrt{g} N (1-\lambda) K^2, \tag{1.2a}$$

$$= \frac{1}{16\pi G_N} \int dt \int d^3x \sqrt{g} N \left(K^{ij}K_{ij} - \lambda K^2 + \mathscr{R} - 2\Lambda \right), \tag{1.2b}$$

where G_N denotes Newton's constant and Λ the cosmological constant, g is the determinant of the metric g_{ij} of the three-dimensional spatial slices, N the lapse function, and K the trace of the extrinsic curvature K_{ij} of the spatial hypersurfaces Σ_t. Setting $\lambda = 1$, S_λ reduces to the Einstein-Hilbert action. Our main aim is to understand the properties of the classical theory defined by S_λ for general values of λ.

An important quantity appearing in the λ-R model is a λ-dependent generalisation of the Wheeler-De Witt metric,

$$G^{ijkl} = \frac{1}{2} \left(g^{ik}g^{jl} + g^{il}g^{jk} \right) - g^{ij}g^{kl}, \tag{1.3}$$

which was introduced by DeWitt in [32]. It is an ultra-local[2] metric on $\mathrm{Riem}\,\Sigma_t$, the space of three-dimensional spatial metrics on Σ_t, with metrics on it called supermetrics, to distinguish them from individual elements of $\mathrm{Riem}\,\Sigma_t$. In the ADM formulation of general relativity [3], the kinetic term of the Einstein-Hilbert action depends on the supermetric (1.3). Recall that general relativity is a constrained Hamiltonian system[3] whose physical solutions must satisfy certain constraints on an initial hypersurface Σ_{t_0}. Among them is the Hamiltonian constraint, $\mathscr{H} \approx 0$, which includes a kinetic term that depends on the inverse of the supermetric (1.3),

$$G_{ijkl} = \frac{1}{2} \left(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl} \right). \tag{1.4}$$

[2]ultra-local means that it does not depend on spatial derivatives of the metric g_{ij}

[3]See reference [39] for a recent and comprehensive review of gravity as a constrained system.

The functional form of this supermetric follows from the requirement of four-dimensional diffeomorphism invariance. If one only demands invariance of the action under spatial diffeomorphisms or foliation-preserving diffeomorphisms instead, the expression (1.4) is no longer distinguished and the theory can be written in terms of a generalised Wheeler-DeWitt metric,

$$G_{\lambda}^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda\, g^{ij}g^{kl}. \tag{1.5}$$

As pointed out in references [32] and [40], this is the most general ultra-local supermetric on $\mathrm{Riem}\,\Sigma_t$.

Replacing the Wheeler-DeWitt metric by this generalised version in the Einstein-Hilbert action without altering the definition of any of the fields of the ADM formulation yields the λ-R model. It constitutes a one-parameter family of gravity theories, including general relativity for the special choice $\lambda = 1$.

Another way of introducing the model is via Hořava-Lifshitz gravity, whose symmetry group is givey by foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathscr{F}}(M)$. There are several versions of the theory, but the kinetic term always depends on the generalised Wheeler-DeWitt metric (1.5). Hořava-Lifshitz gravity actions in general contain many other terms of higher order in spatial derivatives which are compatible with $\mathrm{Diff}_{\mathscr{F}}(M)$, but not with $\mathrm{Diff}(M)$-invariance. When analysing the theory beyond the classical limit, all coupling constants, including λ, will become scale-dependent. In four dimensions, the appropriate renormalization group analysis to determine this behaviour has only been done for the projectable version of the theory [6,31], which as we later explain is not compatible with the λ-R model as we define it. In Hořava-Lifshitz gravity, we therefore in general do not know which values λ can assume in the infrared. This leaves open the possibility that values other than $\lambda = 1$ can occur, providing an additional motivation to understand the implications of $\lambda \neq 1$ in the purely classical theory. Note that in the main body of the thesis, we will concentrate on the case of gravity without matter coupling, allowing at most for the presence of a cosmological constant. We will determine properties of the model in this context and compare it with general relativity.

How one should quantify the role of λ is in principle a straightforward question. One should obtain concrete observational predictions of the model and compare them to observational data. For the λ-R model, it is not sufficient to have a detailed comparison with general relativity to obtain observational constraints on λ. This happens because under certain conditions the λ-R model includes predictions of general relativity among its solutions for general values of λ, as we will see. We therefore must quantify the predictions which differ from those of general relativity and understand whether and how they can be used to obtain observational restrictions on λ.

1.2 Overview of the thesis

The remainder of the thesis is structured as follows. In chapter 2, we begin by defining the λ-R model. We write the action of the model using the ADM decomposition of the four-dimensional metric and specify the symmetries which form its invariance group. After an overview of past results on the model, we perform its Hamiltonian constraint analysis *à la* Dirac. We show that the λ-R model is a second-class constrained system, with the constant mean curvature condition appearing as a tertiary constraint when we impose that the Hamiltonian constraint should be preserved in time. Preserving the constant mean curvature condition in time yields a lapse-fixing equation as a quaternary constraint. We perform this analysis for closed and compact hypersurfaces and present a generalisation for a set of open hypersurfaces in Sec. 2.3. The chapter closes with a discussion of the time evolution equations of the model and a preliminary comparison with general relativity.

This comparison is expanded upon in chapter 3, where we adapt the conformal method for solving the initial value problem in general relativity to the λ-R model. This yields a generalisation of the so-called Lichnerowicz-York equation. We study the existence and uniqueness properties of its solutions, which enables us to compare explicitly the spacetime geometries solving the model with those of general relativity. This allows us to determine whether the λ-R model has a well-posed Cauchy problem, which we show to be true for most cases. We also show that the only way to obtain the same constraint-solving data is by relating the initial data in both theories, which implies that the constraint-solving data evolves differently in time, unless either $\lambda = 1$ or the maximal slicing condition is imposed. We finish the chapter by proving that the λ-R model and general relativity are not equivalent for general choices of initial data and values of the couplings.

We study a set of explicit solutions to the λ-R model in chapter 4, where we assume that the spatial leaves of the preferred foliation are spherically symmetric. Due to the lack of full diffeomorphism invariance, Birkhoff's theorem [15, 58] does not apply. To solve the λ-R model in this setting, we integrate out the angular dependence and study the Hamiltonian formulation of the reduced theory. Solving the constraints and the equations of motion, we find that the constant mean curvature condition implies generically non-static and non-asymptotically flat solutions. They comprise a one-function family of solutions parametrised by the spatially constant trace $K(t)$ of the extrinsic curvature. We show that the four-dimensional Ricci scalar of the solutions in general does not vanish and is proportional to $(1-\lambda)$. Geometric quantities that are gauge parameters in the standard Schwarzschild solution can become physical in this setting.

One of the motivations for studying the λ-R model is Hořava-Lifshitz gravity, a recent proposal for a theory of quantum gravity. In chapter 5 we review aspects of the theory that are particularly relevant to the work in this thesis. One version of Hořava-Lifshitz gravity has a classical limit which is very similar to the λ-R model, differing only in the spacetime dependence of the lapse. We show that it describes accurately results on the acceleration of the volume of a spatial hypersurface [40], obtained in an earlier, independent study of a one-parameter family of gravitational theories whose kinetic term also depends on the generalised Wheeler-DeWitt metric. We also compare our results on the λ-R model with other $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant classical theories related to Hořava-Lifshitz gravity.

In the final chapter 6, we present a discussion of the main results of the thesis and elaborate on their implications for the role of λ in the λ-R model. This is followed by a brief outlook, where we discuss possible follow-up work.

There are six appendices. In appendix A, we review the Hamiltonian formulation of constrained systems. Appendix B consists of two sections. In Sec. B.1, we present an overview of foliated spacetimes within a geometrical setting, and in Sec. B.2 we review the ADM formulation of general relativity. Appendix C is a recap of the conformal method for solving the initial value problem of general relativity. In appendix D, we discuss the constraint algebra of the λ-R model in the presence of matter. In appendix E, we treat two simple solutions of the λ-R model, perturbations around Minkowski spacetime and the FLRW solution.

2

The λ-R model

We define the λ-R model, a one-parameter family of classical gravitational theories with a preferred time foliation, and study it by analysing its constraint algebra[1]. We first consider the model with a cosmological constant Λ on closed and compact hypersurfaces Σ_t. Later, we extend our results to open hypersurfaces, provided they are either asymptotically flat or asymptotically null. The breaking of diffeomorphism symmetry implied by the presence of a preferred foliation is reflected in the set of constraints obtained for $\lambda \neq 1$, when the Hamiltonian constraint is no longer first class. Instead, it forms a pair of second-class constraints with the equation that defines either the maximal slicing or the constant mean curvature condition of general relativity. In Einstein's gravity, the maximal slicing and constant mean curvature conditions play the role of gauge-fixing conditions, while in the λ-R model they define the set of preferred foliations that all solutions of the model must belong to. The constraint analysis is also used to determine the number of local physical degrees of freedom, which turns out to be two. Since this matches the situation in general relativity, it raises the interesting question whether general relativity and the λ-R model describe the same physics. However, as will become clear in subsequent chapters, the two theories are in general inequivalent.

2.1 Introduction

We begin this chapter by outlining some of the geometric structures required to study the λ-R model, which we then define through its action. After an overview of past work on the model in subsection 2.1.1, the bulk of the chapter focuses on the determination of the constraint structure of the theory. We first define momentum variables, write down the

[1]This chapter is based in part on R. Loll and L. Pires: *Role of the extra coupling in the kinetic term in Hořava-Lifshitz gravity*, Phys.Rev. **D90** (2014) 124050.

Hamiltonian of the theory, and determine the constraint algebra associated with it[2]. We follow this with a discussion of the time evolution equations and a computation of the number of local physical degrees of freedom. These results are used to perform a first comparison between solutions of the model and general relativity.

The geometric quantities introduced now are discussed in depth in the appendices at the end of the thesis, namely, in appendix A, where we outline the Hamiltonian treatment of constrained systems based on [34, 45, 92], and in appendix B, where we elaborate on the geometry of foliations and explain the ADM decomposition of general relativity [3, 42].

We consider a $(3+1)$-dimensional pseudo-Riemannian manifold M with a metric ${}^{(4)}g_{\mu\nu}$ of signature $(-,+,+,+)$. Let M be globally hyperbolic, that is, assume M admits a foliation of co-dimension one by leaves of constant time. Then, it can be decomposed as

$$M = \mathbb{R} \times \Sigma_t, \tag{2.1}$$

where Σ_t is a three-dimensional Riemannian manifold. We will generally consider Σ_t to be a spatial manifold, although this will not be true for the entire hypersurface when dealing with open Σ_t in the asymptotically null case. More concretely, we shall consider separately the cases where Σ_t is closed and compact, open with asymptotically flat boundary conditions, and open with asymptotically null boundary conditions. In the latter case, the hypersurfaces cease to be spacelike at spatial infinity, because the metric g_{ij} on Σ_t becomes degenerate in that limit, and the hypersurface signature becomes $(0,+,+)$. For the remainder of the current chapter, we assume that Σ_t is closed and compact unless otherwise specified.

Concerning coordinates on M, we denote the time coordinate by t, with $t \in \mathbb{R}$. Coordinates on Σ_t are denoted by x^i, $i = 1, \cdots, 3$. Four-dimensional objects such as the $(3+1)$-dimensional metric on M are written with a dimensional prefix and Greek letters are used for the indices, e.g. ${}^{(4)}g_{\mu\nu}$. Objects defined intrinsically on Σ_t are written with no dimensional prefix and their indices are denoted by Latin letters. We write the spatial metric on Σ_t as g_{ij} (with inverse g^{ij}), the three-dimensional Ricci scalar on Σ_t as $\mathscr{R}$, and the extrinsic curvature of Σ_t in M as K_{ij}. The relationship between the ADM field variables defined in appendix B and the four-metric ${}^{(4)}g_{\mu\nu}$ can be expressed through the line element of M,

$$ds^2 = {}^{(4)}g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{2.2}$$

where N and $N^i = g^{ij}N_j$ denote the lapse function and the shift vector respectively, the last two ingredients required to fully specify the foliation of M by leaves Σ_t. In terms of these ADM variables, the extrinsic curvature is given by

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right), \tag{2.3}$$

where ∇_i denotes the covariant derivative with respect to the three-dimensional metric g_{ij}, and the dot denotes partial derivatives with respect to time. For further details regarding

[2]This constraint algebra is unchanged when we add matter according to the procedure outlined in [47], as we show in appendix D.

the decomposition of a $3+1$ spacetime in terms of an embedding of three-dimensional hypersurfaces, see appendix B and references therein.

The λ-R model is a classical field theory described by the action

$$S_\lambda = \frac{1}{16\pi G_N}\int dt \int d^3x \sqrt{g}N\left(K^{ij}K_{ij} - \lambda K^2 + \mathscr{R} - 2\Lambda\right), \tag{2.4}$$

where G_N denotes Newton's constant, g is the determinant of g_{ij}, λ is the dimensionless constant after which the model is named, and Λ is the cosmological constant. In the remainder of this chapter, we set $16\pi G_N = 1$. Recall the form of the Einstein-Hilbert action with a cosmological constant in the ADM variables [3],

$$S_{EH} = \int dt \int d^3x \sqrt{g}N\left(K^{ij}K_{ij} - K^2 + \mathscr{R} - 2\Lambda\right). \tag{2.5}$$

Comparing eqs. (2.4) and (2.5), we can explicitly write their difference as

$$S_\lambda - S_{EH} = \int dt \int d^3x \sqrt{g}N(1-\lambda)K^2. \tag{2.6}$$

Note that we have implicitly assumed that the cosmological constant in both actions is the same. One should keep in mind that there is no *a priori* reason to do so. In chapter 3, we briefly discuss the possibility of having two distinct values for Λ, one in S_λ and one in S_{EH}. In this case, equation (2.6) would of course depend on the difference between these cosmological constants.

Analogous to writing the kinetic term of the Einstein-Hilbert action S_{EH} in terms of the Wheeler-DeWitt metric G^{ijkl} [32] (see appendix B for details), we can also write the kinetic term of the λ-R action using a supermetric,

$$S_\lambda = \int dt \int d^3x \sqrt{g}N\left(G_\lambda^{ijkl}K_{ij}K_{kl} + \mathscr{R} - 2\Lambda\right), \tag{2.7}$$

where G_λ^{ijkl} is the so-called generalised Wheeler-DeWitt metric,

$$G_\lambda^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda\, g^{ij}g^{kl}. \tag{2.8}$$

As the notation suggests, the original Wheeler-DeWitt metric is obtained from eq. (2.8) by setting $\lambda = 1$. Its inverse G^λ_{ijkl} only exists for $\lambda \neq \frac{1}{3}$ and is given by

$$G^\lambda_{ijkl} = \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk}\right) - \frac{\lambda}{3\lambda - 1}g_{ij}g_{kl}. \tag{2.9}$$

The lack of invertibility at $\lambda = \frac{1}{3}$ is related to a change in signature at that point. For $\lambda > 1/3$, the generalised Wheeler-DeWitt metric is indefinite, for $\lambda = 1/3$ it is degenerate, and for $\lambda < 1/3$ it is positive definite. This behaviour extends to any number d of spatial dimensions [49], with $\frac{1}{3}$ replaced by $\frac{1}{d}$. In that case, the inverse generalised Wheeler-DeWitt metric is given by

$$G^\lambda_{ijkl} = \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk}\right) - \frac{\lambda}{d\lambda - 1}g_{ij}g_{kl}. \tag{2.10}$$

Returning to three spatial dimensions, we will restrict our discussion to the case $\lambda \neq \frac{1}{3}$ unless otherwise specified.

As alluded to earlier, the λ-R model is not invariant under four-dimensional diffeomorphisms $\mathrm{Diff}(M)$, but only under the subgroup of foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathscr{F}}(M)$. The infinitesimal generators of the latter are

$$\delta t = f(t), \qquad \delta x^i = \zeta^i(t, x^k), \tag{2.11}$$

and their action on the ADM fields is

$$\delta g_{ij} = \zeta^k \partial_k g_{ij} + f \dot{g}_{ij} + \left(\partial_i \zeta^k\right) g_{jk} + \left(\partial_j \zeta^k\right) g_{ik}, \tag{2.12a}$$

$$\delta N_i = \left(\partial_i \zeta^j\right) N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + \dot{f} N_i + f \dot{N}_i, \tag{2.12b}$$

$$\delta N = \zeta^j \partial_j N + \dot{f} N + f \dot{N}. \tag{2.12c}$$

The absence of space-dependent time reparametrisations of the type $\delta t = f(t, x^i)$ will play a crucial role in chapter 4 when we discuss the predictions of the λ-R model for a set of foliated spacetimes with spherical symmetry.

The λ-R model is commonly studied in the context of Hořava-Lifshitz gravity. There exists a so-called projectable version of Hořava-Lifshitz gravity, in which the lapse is defined as a function of time only. This means the lapse is effectively projected along the foliation and its value only depends on the leaf. To appreciate the motivation behind this choice, recall that in general relativity the lapse function and shift vector are often treated as Lagrange multipliers and not as fields. As Lagrange multipliers, their presence is associated with the Hamiltonian and momentum constraints respectively. Taking this point of view and applying it to a theory invariant under $\mathrm{Diff}_{\mathscr{F}}(M)$, we come to the projectable version of the λ-R model. More concretely, the momentum constraints arise due to the invariance of the theory under spatial diffeomorphisms. Since the latter are generated by the three arbitrary functions $\zeta^i(t, x^k)$ in eq. (2.11), the Lagrange multipliers associated with this symmetry, the components of the shift vector should have a matching spacetime dependence. Analogously, the time-dependent transformation generated by $f(t)$ in eq. (2.11) gives rise to a global constraint whose Lagrange multiplier, the lapse function, is therefore a function of time only. Because our main aim in this thesis is to study modified theories of gravity, we will assume that the lapse function has the same *a priori* spacetime dependence as in general relativity. For instance, the Schwarzschild spacetime in the standard Schwarzschild coordinates has a space-dependent lapse. The only exception are some comments on the λ-R model in the context of projectable Hořava-Lifshitz gravity in chapter 5.

To summarise, we treat the λ-R model as a one-parameter family of gravitational theories which for every value except $\lambda = 1$ break full diffeomorphism invariance. Despite working with the $(3+1)$-decomposition of the metric, we assume that the four-metric has the same *a priori* spacetime dependence as in general relativity. This is done to keep the deviations from Einstein's theory to a minimum and to better isolate the differences stemming from the presence of λ alone. It means that the three-metric, lapse, and shift are all functions of spacetime. Finally, we take the four-dimensional metric ${}^{(4)}g_{\mu\nu}$ to be the fundamental field of the theory and therefore treat all its components as such, implying that lapse and shift will not be treated as Lagrange multipliers.

2.1.1 Overview of past results

Giulini and Kiefer [40] were the first to study a model described by the action (2.4) in 1994. Their motivation was to understand certain structures associated with general relativity, namely, the space of three-metrics Riem Σ and the signature of the supermetric on it. Their work uses the generalised Wheeler-DeWitt metric given in eq. (2.8) and its inverse in eq. (2.9). Recall that the generalised supermetric exhausts all possible ultra-local metrics on Riem Σ. Taking the geometrodynamical [47] point of view that general relativity can be obtained by deformations of purely three-dimensional objects, the question - addressed in [40] - is what would happen if one abandoned full four-dimensional covariance and used the generalised Wheeler-DeWitt metric instead.

To justify using the generalised Wheeler-DeWitt metric, it is necessary to abandon invariance under four-dimensional diffeomorphisms, since the choice $\lambda = 1$ is the only one compatible with this symmetry. An action with G_{λ}^{ijkl} in the kinetic term is still compatible with invariance under the subgroup of spatial diffeomorphisms. In this setting, it is therefore natural to take the generalised supermetric to define a one-parameter family of gravity theories and study them. Note that we have only mentioned invariance under spatial diffeomorphisms, not under the larger group of foliation-preserving diffeomorphisms as we did when we described the λ-R model. While it is true that eq. (2.4) is invariant under both symmetry groups, the point of view taken in [40] was to merely demand invariance under spatial diffeomorphisms.

The work presented in [40] investigated the sign of the acceleration of the three-volume and its possible cosmological implications. In general relativity, the three-volume

$$V = \int d^3x \sqrt{g} \tag{2.13}$$

is in general a coordinate-dependent quantity, although it is possible to argue that the sign of its acceleration is physical (see for instance [40] and older work by DeWitt [32]). A key ingredient of the computation of $\ddot{V}$ in [40] was a specific gauge choice, the canonical or proper-time gauge defined by

$$N = 1, \qquad N^i = 0. \tag{2.14}$$

As will become clear below in Sec. 2.2, this is not a valid choice for our version of the λ-R model. However, it is perfectly valid in the context of projectable Hořava-Lifshitz gravity, where one can study a λ-R model with a lapse function that only depends on time. In any case, the result for the acceleration of the three-volume in general relativity is given by

$$\ddot{V} = -\int d^3x \sqrt{g} \left(\mathscr{R} - 3\Lambda\right), \tag{2.15}$$

whereas the λ-dependent result obtained in [40] is

$$\ddot{V} = -\frac{2}{3\lambda - 1} \int d^3x \, (\mathscr{R} - 3\Lambda) \,, \tag{2.16}$$

which displays a change of sign when $\lambda < 1/3$, precisely matching the change in signature of G_λ^{ijkl} at that value.

To understand the interpretation of eq. (2.16), note that in general relativity eq. (2.15) implies that a positive spatial curvature contributes negatively to the acceleration of the three-volume, a feature which is reversed when $\lambda < 1/3$ in eq. (2.16). In chapter 5, we will show that to derive eq. (2.16), the theory must possess a global and λ-dependent Hamiltonian constraint. This global constraint appears in the projectable version of Hořava-Lifshitz gravity, which we consider only in chapter 5.

After Hořava proposed his model for a power-counting perturbatively renormalisable theory of gravity with anisotropic scaling in the ultraviolet [50], the λ-R model was seen as one of its possible classical limits[3]. This motivated Bellorín and Restuccia to study it in [9], coining the term "λ-R model" along the way. They considered the model for the particular case of asymptotically flat spatial hypersurfaces Σ_t, and showed that it is equivalent to a gauge-fixed version of general relativity. By then it was already known that the time evolution of the modified Hamiltonian constraint $\mathscr{H}_\lambda$ is not trivial but yields a tertiary constraint [16, 46]. In reference [46], this constraint was interpreted as a lapse-fixing equation that could only be satisfied by setting the lapse to zero, which by construction is not allowed since it would result in a degenerate, unphysical 4-metric. However, it was argued in [9] and noted around the same time in [35] that instead of setting the lapse to zero, one can rewrite the tertiary constraint as a condition on the trace $\pi \equiv g_{ij}\pi^{ij}$ of the momentum π^{ij}, namely,

$$\nabla_i \pi = 0, \tag{2.17}$$

which in the asymptotically flat setting can only be solved by the so-called "maximal slicing" condition,

$$\pi = 0. \tag{2.18}$$

Preserving condition (2.18) in time yields a lapse-fixing equation whose preservation in time finally determines the Lagrange multiplier associated with the momentum of the lapse. Both the lapse-fixing equation and the equation that determines the Lagrange multiplier associated with the momentum of the lapse are also required in general relativity in order for the gauge $\pi = 0$ to be consistent. The difference is that in general relativity the gauge $\pi = 0$ is only one possible choice among others. By contrast, in the λ-R model with asymptotically flat boundary conditions, $\pi = 0$ is a necessary condition for the closure of the constraint algebra, which occurs because of the presence of λ in the Hamiltonian. However, λ drops out of the equations of motion because any λ-dependent term must come from a Poisson bracket with the Hamiltonian constraint and the latter is quadratic in π. The result of a Poisson bracket with a term that depends on π^2 is at least linear in π and therefore vanishes on

[3] As we will explain in more detail in chapter 5, there is not a unique "Hořava-Lifshitz gravity", but a number of variants. Not for all of these choices is the classical limit given by the λ-R model.

the constraint surface. The resulting equations for the lapse and the Lagrange multiplier associated with its momentum are then the same as those appearing in the maximal slicing version of general relativity, as are the equations of motion.

Because λ drops out of all equations of motion, reference [9] concluded that the λ-R model is equivalent to a gauge-fixed version of general relativity and λ is therefore a spurious parameter. However, reference [40] showed that the use of the generalised Wheeler-DeWitt metric leads to physical quantities that are explicitly λ-dependent. This apparent contradiction partly motivated our study of the model in [73].

2.2 The constraint algebra of the λ-R model

Following the approach outlined in appendix A, we begin by defining momentum variables for the spatial metric, lapse and shift, obtaining

$$\pi^{ij} := \frac{\delta S}{\delta \dot{g}_{ij}} = \sqrt{g}\, G_{\lambda}^{ijkl} K_{kl}, \tag{2.19a}$$

$$\phi := \frac{\delta S}{\delta \dot{N}} = 0, \qquad \phi_i := \frac{\delta S}{\delta \dot{N}^i} = 0. \tag{2.19b}$$

Just as in general relativity, the vanishing of both ϕ and ϕ_i described by eq. (2.19b) defines the primary constraint surface of the model. It is worth noting that the definition of π^{ij} is λ-dependent. This means that the correspondence between the momentum tensor π^{ij} and the extrinsic curvature is not the same as in general relativity. A simple consequence of this difference is an altered relationship between π and the trace of the extrinsic curvature $K = K^{ij} g_{ij}$. We can compute K by taking the trace of eq. (2.19a),

$$\pi = (1 - 3\lambda)\sqrt{g}\, K, \tag{2.20}$$

while in general relativity we have $\pi = -2\sqrt{g}K$. Assuming the same values for g_{ij} and π^{ij} in both the λ-R model and general relativity, we see that they describe hypersurfaces with different extrinsic curvatures.

Eq. (2.19a) can be inverted if $\lambda \neq 1/3$, as we are assuming. This allows us to write $\dot{g}_{ij}$ in terms of canonical variables,

$$\dot{g}_{ij} = \frac{2N}{\sqrt{g}} G^{\lambda}_{ijkl} \pi^{ij} + \nabla_i N_j + \nabla_j N_i. \tag{2.21}$$

This in turn enables us to write the Hamiltonian as

$$H = \int d^3x \left(N \mathscr{H}_{\lambda} + N^i \mathscr{H}_i \right) + \int_{\partial \Sigma_t} ds_i\, \pi^{ij} N_j, \tag{2.22}$$

where ds_i denotes the surface element on the boundary $\partial\Sigma_t$ of Σ_t, and $\mathscr{H}_{\lambda}$ and $\mathscr{H}_i$ are the following functionals of the metric g_{ij} and its momentum π^{ij},

$$\mathscr{H}_{\lambda} = \frac{1}{\sqrt{g}} G^{\lambda}_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{g}\left(\mathscr{R} - 2\Lambda\right), \tag{2.23a}$$

$$\mathscr{H}_i = -2 g_{ij} \nabla_k \pi^{jk}. \tag{2.23b}$$

In the remainder of this chapter we will discard the boundary term in eq. (2.22), stemming from the partial integration of covariant derivatives acting on the shift, and assume Σ_t to be closed and compact. When discussing the open case, this term is restored and a further boundary Hamiltonian $H_{\partial\Sigma_t}$ is introduced. We postpone this part of the analysis to Sec. 2.3, focusing now only on obtaining the local constraint algebra of the λ-R model.

Note that if we were to work with the special value $\lambda = 1/3$, $\pi = 0$ would appear as a primary constraint of the theory. Since it generates infinitesimal conformal transformations of the metric g_{ij} and the momentum π^{ij}, it is possible to write theories with higher-order derivatives in the potential, which for $\lambda = 1/3$ are Weyl invariant, as was done in [50].

As outlined in our discussion of constrained systems, we include primary constraints in the total Hamiltonian. We do this by associating Lagrange multipliers (α, α^i) to the four primary constraints $(\phi = 0, \phi_i = 0)$. Adding these pieces to the Hamiltonian in eq.(2.22), we have

$$H_{tot} = \int d^3x \left(N\mathscr{H}_\lambda + N^i \mathscr{H}_i + \alpha\phi + \alpha^i \phi_i \right). \tag{2.24}$$

The resulting total Hamiltonian H_{tot} is linear in both lapse N and shift N^i, implying that preserving the primary constraints in time yields four secondary constraints, namely, the three momentum constraints $\mathscr{H}_i \approx 0$ and a modified Hamiltonian constraint $\mathscr{H}_\lambda \approx 0$,

$$\dot{\phi} \approx 0 \Rightarrow \{\phi, H_{tot}\} = \mathscr{H}_\lambda \approx 0, \tag{2.25a}$$

$$\dot{\phi}_i \approx 0 \Rightarrow \{\phi_i, H_{tot}\} = \mathscr{H}_i \approx 0, \tag{2.25b}$$

where $\{A, B\}$ denotes the Poisson bracket of the phase space functionals A and B (see appendix A for a definition). We have also introduced the notation "$\approx$" to denote weak equality. Two quantities are said to be weakly equal if they are equal on the constraint surface (once again, see appendix A for details).

The λ-R model differs from general relativity; it is not a first-class theory, since not all of its constraints are first class. Recall that a constraint is first class if it Poisson-commutes with all other constraints (see subsection 2.2.1 for a more precise definition). The second-class property of the model becomes apparent in the next step of the Dirac algorithm when we impose preservation in time of the secondary constraints given in eq. (2.25), more specifically, when setting $\dot{\mathscr{H}}_\lambda \approx 0$ for $\lambda \neq 1$. Because spatial diffeomorphisms are still a symmetry of the model, the algebra associated with the $\mathscr{H}_i \approx 0$ constraints is unchanged, leading to

$$\dot{\mathscr{H}}_i = \mathscr{H}_\lambda \nabla_i N + \mathscr{H}_j \nabla_i N^j + N^j \nabla_j \mathscr{H}_i + \mathscr{H}_i \nabla_j N^j, \tag{2.26}$$

which vanishes on the constraint surface.

For the time derivative of $\mathscr{H}_\lambda$ we have

$$\dot{\mathscr{H}}_\lambda = \{\mathscr{H}_\lambda, H_{tot}\} = \left\{ \mathscr{H}_\lambda, \int d^3x N \mathscr{H}_\lambda \right\} + \left\{ \mathscr{H}_\lambda, \int d^3x \left(N^i \mathscr{H}_i + \alpha\phi + \alpha^i \phi_i \right) \right\}, \tag{2.27}$$

where we have split the Poisson bracket into two contributions to highlight that the tertiary constraint stems from the bracket $\{\mathscr{H}_\lambda, \mathscr{H}_\lambda\}$. Consider the smeared-out variation of $\mathscr{H}_\lambda$ with respect to the metric,

$$\begin{aligned}\int d^3x N\delta_{g_{ij}}\mathscr{H}_\lambda =& \frac{2N}{\sqrt{g}}\left(\pi^{ik}\pi^j{}_k - \frac{\lambda}{3\lambda-1}\pi\pi^{ij}\right) - \frac{N}{2\sqrt{g}}G^\lambda_{mnkl}\pi^{mn}\pi^{kl}g^{ij}\\ &-\sqrt{g}N\left(\frac{g^{ij}}{2}(\mathscr{R}-2\Lambda)-\mathscr{R}^{ij}\right)-\sqrt{g}G^{ijkl}\nabla_k\nabla_l N,\end{aligned} \tag{2.28}$$

where G^{ijkl} is the original Wheeler-DeWitt supermetric. Note that the smeared-out variation of the Ricci scalar $\mathscr{R}$ on Σ_t, a quantity invariant under three-dimensional spatial diffeomorphisms, yields terms that depend on derivatives of the lapse. The coefficients of these terms can be interpreted as a supermetric with $\lambda = 1$, as we did in the last term on the right-hand side of (2.28).

To see the tertiary constraint emerge, we compute the Poisson bracket of two smeared-out Hamiltonian constraints,

$$\begin{aligned}\left\{\int d^3y\,\eta\mathscr{H}_\lambda, \int d^3x N\mathscr{H}_\lambda\right\} = \int d^3z \Bigg\{&\int d^3y\,\eta\,\delta_{g_{ij}}\mathscr{H}_\lambda \int d^3x N\delta_{\pi^{ij}}\mathscr{H}_\lambda\\ &-\int d^3y\,\eta\,\delta_{\pi^{ij}}\mathscr{H}_\lambda \int d^3x N\delta_{g_{ij}}\mathscr{H}_\lambda\Bigg\}.\end{aligned} \tag{2.29}$$

Since there are no spatial derivatives of π^{ij} in $\mathscr{H}_\lambda$ that need a partial integration when computing the variation, the only possible non-vanishing contribution to eq. (2.29) comes from the last term in eq. (2.28), since the derivatives acting on η must be partially integrated in order to obtain $\mathscr{H}_\lambda$, making this term asymmetric under an exchange of η and N. The explicit computation reads

$$\left\{\int d^3y\,\eta\mathscr{H}_\lambda, \int d^3x N\mathscr{H}_\lambda\right\} = 2\int d^3z\, G^\lambda_{ijmn}G^{ijkl}\pi^{mn}\left(\eta\nabla_k\nabla_l N - N\nabla_k\nabla_l\eta\right) \tag{2.30a}$$

$$= 2\int d^3z\left(\pi^{kl} - \frac{\lambda-1}{3\lambda-1}g^{kl}\pi\right)\left(\eta\nabla_k\nabla_l N - N\nabla_k\nabla_l\eta\right) \tag{2.30b}$$

$$= -\int d^3z\left(g^{ij}\mathscr{H}_i + 2\frac{\lambda-1}{3\lambda-1}g^{kl}\nabla_k\pi\right)\left(N\nabla_l\eta - \eta\nabla_l N\right) \tag{2.30c}$$

$$= \int d^3z\,\eta\Bigg\{g^{ij}\left(2\mathscr{H}_i\nabla_j N + N\nabla_j\mathscr{H}_i\right) \tag{2.30d}$$

$$+2\frac{\lambda-1}{3\lambda-1}g^{kl}\left(2\nabla_k\pi\nabla_l N + N\nabla_l\nabla_k\pi\right)\Bigg\}, \tag{2.30e}$$

where we have merely expanded the supermetrics in eq. (2.30b), and performed consecutively the required partial integrations in the last two lines. The first term in eq. (2.30c) is the one we would have obtained for $\lambda = 1$, while the term proportional to $(\lambda - 1)$ is new. It comes from the fact that the generalised Wheeler-DeWitt metric G^λ_{ijkl} appearing in the variation of the kinetic term $\delta_{\pi^{ij}}\mathscr{H}_\lambda$ is not the inverse of its $\lambda = 1$ counterpart G^{ijkl}, which comes from the metric variation of the three-dimensional Ricci scalar. Adding the smeared-out version of the second term in eq. (2.27), we can read off the time evolution of $\mathscr{H}_\lambda$,

$$\left\{\int d^3x\,\eta\,\mathscr{H}_\lambda, H_{tot}\right\} = \int d^3z\,\eta\left\{g^{ij}\left(2\mathscr{H}_i\nabla_j N + N\nabla_j\mathscr{H}_i\right) - \mathscr{H}_\lambda N^i\nabla_i N\right. \tag{2.31a}$$

$$\left. +2\frac{\lambda-1}{3\lambda-1}g^{kl}\left(2\nabla_k\pi\nabla_l N + N\nabla_l\nabla_k\pi\right)\right\}, \tag{2.31b}$$

$$\Rightarrow \dot{\mathscr{H}_\lambda} = g^{ij}\left(2\mathscr{H}_i\nabla_j N + N\nabla_j\mathscr{H}_i\right) - \mathscr{H}_\lambda N^i\nabla_i N \tag{2.31c}$$

$$+2\frac{\lambda-1}{3\lambda-1}g^{kl}\left(2\nabla_k\pi\nabla_l N + N\nabla_l\nabla_k\pi\right). \tag{2.31d}$$

Since all terms in eq. (2.31c) vanish on the constraint surface, imposing $\dot{\mathscr{H}_\lambda} \approx 0$ yields a tertiary constraint given by

$$\dot{\mathscr{H}_\lambda} \approx 2\frac{\lambda-1}{3\lambda-1}g^{kl}\left(2\nabla_k\pi\nabla_l N + N\nabla_l\nabla_k\pi\right) \approx 0. \tag{2.32}$$

As we will show next, eq. (2.32) is an equation for the trace of the momentum π^{ij}. Since it is a constraint, it must be preserved in time; this will be done after the constraint is solved, which is allowed since both approaches coincide on the constraint surface. Discarding the numerical pre-factors, we multiply eq. (2.32) by N and use the Leibniz rule to obtain[4]

$$\dot{\mathscr{H}_\lambda} \approx 0 \Rightarrow g^{ij}\nabla_i\left(N^2\nabla_j\pi\right) \approx 0. \tag{2.33}$$

It is clear from the last equation that $\pi = 0$ is a solution of $\dot{\mathscr{H}_\lambda} \approx 0$. To obtain other solutions, assume $\pi \neq 0$, multiply eq. (2.33) by $\frac{\pi}{\sqrt{g}}$ (the $g^{-1/2}$ factor keeps the density character of the equation unchanged) and integrate over Σ_t to obtain

$$\int d^3x\frac{\pi}{\sqrt{g}}g^{ij}\nabla_i\left(N^2\nabla_j\pi\right) = -\int d^3x\frac{N^2}{\sqrt{g}}g^{ij}\nabla_i\pi\nabla_j\pi \approx 0. \tag{2.34}$$

Since $\frac{N^2}{\sqrt{g}}$ must not vanish anywhere in Σ_t, eq. (2.34) is solved by requiring that the inner product of $\nabla_i\pi$ with itself vanishes, that is,

$$\nabla_i\pi \approx 0 \;\Rightarrow\; \frac{\pi}{\sqrt{g}} - a(t) \approx 0, \tag{2.35}$$

where $a(t)$ can be any smooth function of time. Writing the solution to $\dot{\mathscr{H}_\lambda} \approx 0$ this way includes the case $\pi = 0$ in which $a(t)$ vanishes identically, avoiding the need for discussing these solutions separately. The conditions we have derived correspond to two well-known gauge-fixings of general relativity, the so-called maximal slicing condition $\pi = 0$ and the constant mean curvature condition $\pi = a(t)\sqrt{g}$ [99]. Combining both into a general function $a(t)$, we write the tertiary constraint as $\omega \approx 0$, where

$$\omega := \pi - a(t)\sqrt{g} \approx 0. \tag{2.36}$$

The result that the tertiary constraint is given by eq. (2.36) was obtained independently by Donnelly and Jacobson in [35] and by us in [73], although the consequences of eq. (2.36)

[4] This is allowed without any loss of generality because $N \neq 0$, $\forall x^i \in \Sigma_t$ by assumption.

derived in both papers were different. Since the aim of [35] was not to study the constraint algebra of the model, its authors did not pursue the consequences of eq. (2.36) in the same way as we did[5].

When eq. (2.36) is used to fix coordinates in general relativity, consistency requires it to be preserved in time, which leads to a lapse-fixing equation. Similarly, the tertiary constraint $\omega \approx 0$ of the λ-R model yields a λ-dependent lapse-fixing equation. It comes exclusively from the Poisson bracket of ω with the $\mathscr{H}_\lambda$-term in the total Hamiltonian H_{tot}, since the Poisson brackets between the new constraint and the shift-smeared momentum constraints all vanish on the constraint surface. The non-trivial part of $\dot{\omega} \approx 0$ reads

$$\dot{\omega} = \frac{3}{2} N \mathscr{H}_\lambda + 2\sqrt{g}\left(\mathscr{R} - 3\Lambda + \frac{a^2}{2(3\lambda - 1)} - \nabla^2\right) N - \dot{a}\sqrt{g} \approx 0, \tag{2.37}$$

where the $\dot{a}$-term must be included since $a(t)$ in general cannot be expressed in terms of phase space variables[6]. For a general choice of $a(t)$, the λ-dependent lapse-fixing equation (2.37) is a second-order non-homogeneous differential equation for the lapse function N.

For the closed and compact case we are considering, it is straightforward to write $a(t)$ in terms of phase space variables by integrating eq. (2.36) over Σ_t and writing

$$a(t) = \frac{1}{V}\int d^3x\, \pi, \tag{2.38}$$

where $V = \int d^3x \sqrt{g}$ is the volume of the hypersurface Σ_t. Making use of eq. (2.38), one can explicitly write the $\dot{a}$-term in eq. (2.37) purely in terms of phase space variables,

$$\dot{a} = \frac{1}{V}\int d^3x \left(\frac{3}{2} N \mathscr{H}_\lambda + 2\sqrt{g}\left(\mathscr{R} - 3\Lambda + \frac{a^2}{2(3\lambda - 1)} - \nabla^2\right) N\right). \tag{2.39}$$

We can use eq. (2.39) to rewrite the quaternary constraint imposing the lapse-fixing equation, eq. (2.37), which we denote by $\mathscr{M} \approx 0$. Explicitly, we have

$$\mathscr{M} := D_\lambda N - \frac{\sqrt{g}}{V}\int d^3x D_\lambda N \approx 0, \tag{2.40}$$

where we have introduced D_λ as a shorthand for the differential operator

$$D_\lambda := \sqrt{g}\left(\mathscr{R} - 3\Lambda + \frac{a^2}{2(3\lambda - 1)} - \nabla^2\right). \tag{2.41}$$

The final step in the Dirac algorithm consists in demanding that $\mathscr{M} \approx 0$ be preserved in time. Since $\mathscr{M} \approx 0$ is an equation for the lapse, its Poisson bracket with the total Hamiltonian has a non-trivial contribution from the $\alpha\phi$-term in H_{tot}. The resulting equation therefore determines the Lagrange multiplier α and reads

$$F + D_\lambda \alpha - \frac{\sqrt{g}}{V}\int d^3x \left(F + D_\lambda \alpha\right) \approx 0, \tag{2.42}$$

[5]The fact that in closed hypersurfaces the tertiary constraint is given by $\pi = a(t)\sqrt{g}$ led the authors of [35], an article about the Hamiltonian structure of the full non-projectable theory, to (wrongly) claim in passing that the λ-R model is equivalent to general relativity in the constant mean curvature gauge. This happened because they did not consider the preservation of the tertiary constraint in time and the time evolution equations [57].

[6]From this point onward, we depart from [73], which considered the simpler case $\dot{a} = 0$.

with F yet another shorthand for the scalar density

$$F := \left(2\pi^{kl} - \pi g^{kl}\frac{2\lambda-1}{3\lambda-1}\right)\left(N\nabla_k\nabla_l N + \nabla_k(N\nabla_l N) - N^2\mathscr{R}_{kl}\right) + \frac{N^2\pi\mathscr{R}}{3\lambda-1} - \frac{aN}{3\lambda-1}D_\lambda N. \quad (2.43)$$

Note that if we had treated the lapse as a Lagrange multiplier, the Dirac algorithm would have ended with eq. (2.40). As outlined in appendix A, by deriving an equation for the Lagrange multiplier α, without further conditions to check at this stage, we have successfully determined the complete constraint algebra of the model.

To obtain eq. (2.43) in this form, we implicitly redefined the Lagrange multiplier α appearing in the total Hamiltonian. The reason is that the momentum constraints $\mathscr{H}_i$ in eq. (2.23b) only generate infinitesimal spatial diffeomorphisms of the metric g_{ij} and its momentum π^{ij}. However, their Poisson brackets with the lapse vanish. As we show below in eq. (2.47), the fact that the $\mathscr{H}_i$ do not generate infinitesimal spatial diffeomorphisms of the lapse and its momentum ϕ is reflected by their brackets with $\mathscr{M}$. It turns out that including the generators of infinitesimal spatial diffeomorphisms of the lapse N and its momentum ϕ in the $\mathscr{H}_i$ makes them explicitly first class. Once this is done, the Poisson brackets between $\mathscr{H}_i$ and $\mathscr{M}$ do indeed vanish on the constraint surface as one would expect from a scalar invariant under spatial diffeomorphisms.

The three first-class constraints which coincide with $\mathscr{H}_i$ on the constraint surface are

$$\tilde{\mathscr{H}}_i := \mathscr{H}_i + \phi\nabla_i N. \quad (2.44)$$

We can write the total Hamiltonian in terms of the constraints $\tilde{\mathscr{H}}_i$ without changing it, by simply redefining the Lagrange multiplier α such that

$$N^i\mathscr{H}_i + \alpha\phi = N^i\tilde{\mathscr{H}}_i + \tilde{\alpha}\phi, \quad (2.45)$$

which implies

$$\tilde{\alpha} = \alpha - N^i\nabla_i N. \quad (2.46)$$

The Lagrange multiplier α in our earlier eq. (2.42) should be identified with the redefined parameter $\tilde{\alpha}$. The reason for this redefinition in the total Hamiltonian is to make eq. (2.42) independent of the shift. As we will see later, the shift dependence appears in a simple way in the equation for $\dot{N}$. Throughout this chapter, we will continue to denote the Lagrange multiplier by α, keeping in mind the redefinition just explained.

2.2.1 Classification of constraints

Recall that to classify n constraints $\Phi_i \approx 0$ as first and second class, we compute the $n \times n$ matrix $\mathbf{M}_{ij} = \{\Phi_i, \Phi_j\}$ whose entries are the pairwise Poisson brackets among the constraints. The rank of $\mathbf{M}$ corresponds to the number of second-class constraints, denoted by $\mathscr{C}_2$, while $n - \mathscr{C}_2$ is the number of first-class constraints, which we denote by $\mathscr{C}_1$.

Knowing what happens when the constant mean curvature condition is imposed in general relativity, one would expect ϕ, $\mathscr{H}_\lambda$, ω and $\mathscr{M}$ to all be second-class constraints while $\mathscr{H}_i$

and ϕ_i remain first class. However, as can be seen from the smeared-out Poisson bracket of $\mathscr{M}$ and $\mathscr{H}_i$, this is not immediately the case, since we have

$$\left\{\int d^3x\,\eta\mathscr{M},\int d^3y N^i\mathscr{H}_i\right\}=\int d^3z N^i\left(\nabla_i\eta\mathscr{M}-\nabla_i N\left(D_\lambda\eta-\frac{1}{V}\int d^3x\sqrt{g}D_\lambda\eta\right)\right). \quad (2.47)$$

To circumvent the fact that eq. (2.47) does not vanish on the constraint surface, we use the redefined constraint $\tilde{\mathscr{H}}_i$ of eq. (2.44) instead, such that eq. (2.47) becomes

$$\left\{\int d^3x\,\eta\mathscr{M},\int d^3y N^i\tilde{\mathscr{H}}_i\right\}=\int d^3z\,\mathscr{M}N^i\nabla_i\eta. \quad (2.48)$$

This is exactly the behaviour expected of a scalar under spatial diffeomorphisms. Moreover, the Poisson brackets of the new momentum constraints and all other constraints remain unchanged on the constraint surface, that is,

$$\left\{\tilde{\mathscr{H}}_i,\Phi_j\right\}\approx\left\{\mathscr{H}_i,\Phi_j\right\},\qquad \Phi_j\neq\mathscr{M}. \quad (2.49)$$

However, there is no need to redefine α to turn the momentum constraints $\mathscr{H}_i$ in the total Hamiltonian H_{tot} into the modified constraints $\tilde{\mathscr{H}}_i$. As mentioned previously, this would imply a differential equation for α with a shift dependence. To avoid this and obtain the relatively simpler eq. (2.42), we have implicitly imposed a partial determination of the Lagrange multiplier. This means that we have replaced the α that multiplied ϕ in the total Hamiltonian by the combination $\alpha+N^i\nabla_i N$. As a result, $\dot{N}$ is no longer given by α but by $\alpha+N^i\nabla_i N$, and the remaining contribution of the Lagrange multiplier in $\dot{N}$ satisfies eq. (2.42).

Note that the rank of M is not affected by this procedure and consequently rank $(\mathbf{M})=4$, implying

$$\mathscr{C}_2=4,\qquad \mathscr{C}_1=n-\mathscr{C}_2=6. \quad (2.50)$$

What we have accomplished by defining $\tilde{\mathscr{H}}_i$ via eq. (2.44) is to exhibit explicitly the momentum constraints as first class and make the discussion clearer. The model possesses six first-class constraints,

$$\phi_i\approx 0,\qquad \tilde{\mathscr{H}}_i\approx 0, \quad (2.51)$$

and four second-class constraints,

$$\phi\approx 0,\qquad \mathscr{H}_\lambda\approx 0,\qquad \omega\approx 0,\qquad \mathscr{M}\approx 0. \quad (2.52)$$

This information is used to determine the number of local physical degrees of freedom $\mathscr{N}$ in terms of the dimension of the phase space $\mathscr{P}$ and the number of each type of constraints. In our case, we have $\dim\mathscr{P}=20$, while $\mathscr{C}_1$ and $\mathscr{C}_2$ are given by eq. (2.50), which implies that

$$\mathscr{N}=\frac{1}{2}\left(\dim\mathscr{P}-2\mathscr{C}_1-\mathscr{C}_2\right)=2. \quad (2.53)$$

We conclude that the theory possesses the same number of local degrees of freedom as general relativity and closely resembles a gauge-fixed version of general relativity in the following sense. It is a classical field theory with the same field content and, as we have just shown, the same number of local physical degrees of freedom. Moreover, for any value of $a(t)$, the solution of its tertiary constraint $\omega \approx 0$ in eq. (2.36) corresponds to a gauge-fixing condition in general relativity. However, the lapse-fixing equation $\mathscr{M} \approx 0$ only yields the lapse function corresponding to the condition $\omega \approx 0$ of general relativity when $a(t) = 0$. This raises the important question whether and how the four-dimensional spacetimes solving the λ-R model for some $a(t) \neq 0$ differ from the general relativistic solutions satisfying the constant mean curvature condition $\omega \approx 0$. Since $a(t)$ is not observable in general relativity, the λ-R model with a given $a(t)$ could be equivalent to general relativity in constant mean curvature coordinates with a different function $\tilde{a}(t)$. In appendix E, we show that this is indeed what happens for the simpler case of linear perturbations around Minkowski spacetime and therefore λ does not imply any new physics in that case. However, general solutions of the λ-R model correspond to physically distinct spacetimes as we will demonstrate in subsequent chapters.

2.3 Note on open spatial hypersurfaces

In this section, we outline the generalisation of the results on the constraint analysis of the model to open hypersurfaces. To this end, let us go back to eq. (2.22) where we first introduced the Hamiltonian of the system. We saw that for open hypersurfaces there should be a boundary contribution coming from the partial integration of the $\nabla_i N_j$-terms in the extrinsic curvature, which gives rise to the momentum constraints in the Hamiltonian. Let us restore it in the Hamiltonian, such that

$$H = \int d^3x \left(N\mathscr{H}_\lambda + N^i \mathscr{H}_i \right) + \int_{\partial\Sigma_t} ds_i \, \pi^{ij} N_j. \tag{2.54}$$

As was argued by Regge and Teitelboim in [86], for the variational problem to lead to well-defined equations of motion, it is necessary that the variation δH can be written without any boundary contributions, that is,

$$\delta H = \int d^3x \left(A^{ij} \, \delta g_{ij} + B_{ij} \, \delta \pi^{ij} \right), \tag{2.55}$$

where A^{ij} and B_{ij} are defined as

$$A^{ij} := \frac{\delta H}{\delta g_{ij}}, \qquad B_{ij} := \frac{\delta H}{\delta \pi^{ij}}. \tag{2.56}$$

In this way one obtains the equations of motion in a unique manner, as

$$\dot{g}_{ij} = \frac{\delta H}{\delta \pi^{ij}} = B_{ij}, \qquad \dot{\pi}^{ij} = -\frac{\delta H}{\delta g_{ij}} = A^{ij}. \tag{2.57}$$

However, performing the variation of the Hamiltonian given in eq. (2.54) does not yield a result of the form of eq. (2.55) when dealing with open Σ_t[7], but instead

$$\delta H = \int d^3x \left(A^{ij}\,\delta g_{ij} + B_{ij}\,\delta\pi^{ij}\right) - \int_{\partial\Sigma_t} ds_i\, G^{ijkl}\left(N\,\delta\left(\partial_j g_{kl}\right) - \partial_j N\,\delta g_{kl}\right)$$
$$- \int_{\partial\Sigma_t} ds_i\,\left(N_j\,\delta\pi^{ij} + \left(2N^k\pi^{ij} - N^i\pi^{jk}\right)\delta g_{jk} + \pi^{ij}\,\delta N_j\right). \tag{2.58}$$

The variation of π^{ij} in the last term of eq. (2.54) appears with the wrong sign because $\delta N^i \mathscr{H}_i$ comes with a factor of 2. Moreover, we have omitted the local terms $\mathscr{H}_\lambda\,\delta N + \mathscr{H}_i \delta N^i$ because they are proportional to constraints. Finally, note that the supermetric G^{ijkl} comes from the variation of the three-dimensional Ricci scalar given in eq. (2.28) and is therefore λ-independent.

The goal is to define a boundary Hamiltonian $H_{\delta\Sigma_t}$ such that

$$\delta\left(H + H_{\delta\Sigma_t}\right) = \int d^3x\,\left(A^{ij}\delta g_{ij} + B_{ij}\delta\pi^{ij}\right). \tag{2.59}$$

The particular form of $H_{\delta\Sigma_t}$ depends on the fall-off conditions obeyed by the canonical variables. Let us for instance consider the asymptotically flat case, for which we have

$$g_{ij} \to \delta_{ij} + \mathscr{O}(r^{-1}), \qquad \pi^{ij} \to \mathscr{O}(r^{-2}), \tag{2.60a}$$
$$N \to 1 + \mathscr{O}(r^{-1}), \qquad N_i \to \mathscr{O}(r^{-1}), \tag{2.60b}$$

where $r = \infty$ is the two-dimensional boundary of Σ_t. Under these conditions, the only surviving variation at infinity of the corresponding terms in eq. (2.58) is

$$- \int_{\partial\Sigma_t} ds_i\, G^{ijkl}\,\delta\left(\partial_j g_{kl}\right) = -\delta \int_{\partial\Sigma_t} ds_i\, G^{ijkl}\partial_j g_{kl}\,, \tag{2.61}$$

where we could pull the variation out of the integral because the term $\partial_j g_{kl}\delta G^{ijkl}$ vanishes due to the boundary conditions (2.60). To cancel this variation, we define the boundary Hamiltonian as

$$H_{\delta\Sigma_t} := \int_{\partial\Sigma_t} ds_i\, G^{ijkl}\partial_j g_{kl}\,, \tag{2.62}$$

thus ensuring a well-defined variational principle.

The same logic applies in the asymptotically null case, although the treatment is technically more cumbersome. An explicit example of this case is given in subsection 4.3.4 of chapter 4, where we discuss the λ-R model for spherically symmetric hypersurfaces Σ_t. The important point is that while defining a boundary Hamiltonian might require additional conditions at infinity, both the local constraint algebra and the equations of motion presented in the current chapter hold for both asymptotically flat and asymptotically null hypersurfaces. The reason why we emphasise these two types of hypersurfaces is that they are compatible with solutions to the constant mean curvature condition - asymptotically flat for $\pi = 0$ and asymptotically null for $\pi = a(t)\sqrt{g}$ with a function $a(t)$ that does not vanish identically.

[7]The same holds for closed hypersurfaces with a boundary.

2.4 Time evolution and comparison with general relativity

We have shown that the constraint algebra of the λ-R model closes, with the momentum constraints of the model remaining first class. The second-class constraints are the Hamiltonian constraint, the vanishing momentum of the lapse, the constant mean curvature (or maximal slicing) condition, and the lapse-fixing equation. Together with the counting of physical degrees of freedom undertaken in the previous section, this implies a strong similarity of the λ-R model with the corresponding gauge-fixed versions of general relativity.

To extend the analysis beyond a single (initial) hypersurface Σ_t, consider the time evolution equations for the metric g_{ij} and its conjugate momentum density π^{ij},

$$\dot{g}_{ij} = \frac{2N}{\sqrt{g}}\left(\pi_{ij} - \frac{\lambda}{3\lambda-1}\pi\, g_{ij}\right) + g_{ik}\nabla_j N^k + g_{jk}\nabla_i N^k\,, \tag{2.63a}$$

$$\begin{aligned}\dot{\pi}^{ij} = &-\frac{2N}{\sqrt{g}}\left(g_{kl}\pi^{ik}\pi^{jl} - \frac{\lambda}{3\lambda-1}\pi\pi^{ij}\right) - N\sqrt{g}\left(\mathscr{R}^{ij} - g^{ij}\left(\Lambda - \frac{\lambda}{2\,(3\lambda-1)}\,a^2\right)\right)\\ &+\sqrt{g}\,g^{ik}g^{jl}\nabla_k\nabla_l N + \nabla_a\left(N^a\,\pi^{ij}\right) - \pi^{ai}\,\nabla_a N^j - \pi^{aj}\,\nabla_a N^i\,,\end{aligned} \tag{2.63b}$$

where the equation for $\dot{\pi}^{ij}$ has been simplified by using the Hamiltonian constraint. As for the foliation-defining fields N, N^i and their corresponding momentum variables, we have

$$\dot{N} = \alpha + N^i\nabla_i N, \qquad \dot{\phi} = \mathscr{H} + \nabla_i\left(N^i\phi\right) \approx 0\,, \tag{2.64a}$$

$$\dot{N}^i = \alpha^i, \qquad \dot{\phi}_i = \tilde{\mathscr{H}}_i \approx 0\,. \tag{2.64b}$$

Note that setting $\lambda = 1$ in eqs. (2.63) and (2.64) yields the equations of motion of general relativity in constant mean curvature coordinates. Finally, the N^i-dependent term in eq. (2.64) for $\dot{N}$ is a consequence of our redefinition of α in eq. (2.46). If we had opted to not redefine α, the equation would read $\dot{N} = \alpha$, but the equation for α (2.42) would be shift-dependent.

The equations of motion (2.63) and (2.64) constitute a λ-dependent version of the equations of motion for general relativity in the constant mean curvature gauge. This suggests that one may try to absorb λ in a re-definition of $a(t)$. In appendix E we show that for linear perturbations around Minkowski space this is indeed possible, which implies that in this approximation the λ-R model agrees with general relativity. However, as we will show in the next chapters, it is in general impossible to absorb λ simultaneously in the initial constraint surface and the equations of motion.

3

The initial-value formulation of the λ-R model

This chapter deals with the initial-value problem of the λ-R model[1]. After having shown in chapter 2 that the model is equivalent to vacuum general relativity when the trace of the extrinsic curvature vanishes, we now adapt the conformal method for the initial-value formulation of general relativity to show explicitly the inequivalence between the theories. We will obtain a generalised Lichnerowicz-York equation and study the existence and uniqueness properties of its solutions. While there are some exceptions for which the λ-R model can describe a gauge-fixed version of general relativity, we will show that matching the constraint-solving data at an initial hypersurface for general values of fixed parameters such as λ and Λ implies that this data evolves differently in time for both theories, yielding different spacetimes.

3.1 Introduction

In chapter 2, we discussed the λ-R model, a one-parameter family of gravitational theories, as a constrained Hamiltonian system. This model has a notion of preferred foliation, which becomes manifest in the solutions of the constraint algebra. The so-called constant mean curvature condition[2] appears as a tertiary constraint and a lapse-fixing equation as a quaternary constraint. Although the constant mean curvature condition is a well-known gauge choice in general relativity, we showed in chapter 2 that the λ-R model is equivalent to this gauge-fixed version of general relativity in the special case where the mean extrinsic curvature of Σ_t on M vanishes identically, satisfying the so-called maximal slicing condition.

Due to the closure of the constraint algebra, if all constraints are satisfied at some initial hypersurface Σ_0, the time evolution of the variables will remain within the constraint sur-

[1] This chapter is based on N. Ó Murchadha, and L. Pires: *Initial value formulation for the λ-R model*, in preparation [81].

[2] Since the maximal slicing condition can be seen as a special case of the constant mean curvature condition, defined by $a(t) = 0$ everywhere in Σ_t, we will no longer refer to both slicing conditions separately unless the distinction is important for the argument at hand.

face. In other words, satisfying all constraints at some initial time and evolving the fields according to their Poisson brackets with the total Hamiltonian guarantees that the constraints are satisfied at all times. Focusing on the arbitrary initial hypersurface, one can ask under what conditions these spacetime solutions exist and are unique. We also want to know which field components are constrained and thus fixed when solving the constraints, and which are freely specifiable. There is more than one scheme to address this issue in general relativity [42]. Among them, the so-called conformal method is particularly suited for our case because its first step consists in the choice of constant mean curvature coordinates, which we are also forced to adopt to satisfy the constraint algebra of the λ-R model. Although the use of maximal slicing coordinates may be seen as a particular case of a constant mean curvature slicing, it has very different properties in the context of the initial-value problem. This is already true in general relativity, as outlined in appendix C.

The application of the conformal method to the λ-R model and general relativity follows the same steps. One first solves the momentum constraints $\mathscr{H}_i \approx 0$ by a suitable choice of variables, which decouples them from the Hamiltonian constraint $\mathscr{H}_\lambda \approx 0$. One then performs a conformal transformation on the metric and the momentum, which turns the Hamiltonian constraint into a functional differential equation for the conformal factor of the transformation. This equation is called the Lichnerowicz-York equation in general relativity. For $\lambda \neq 1$, we will obtain a generalised version. A more detailed treatment of the original conformal method for general relativity can be found in appendix C and the references therein.

In what follows, we will deal with two sets of phase space variables, which are related by a conformal transformation with conformal factor ϕ to be specified in Sec. 3.2 below. We will denote the first one by the usual symbols $\left(g_{ij}, \pi^{ij}\right)$ and the conformally related one by barred versions, $\left(\bar{g}_{ij}, \bar{\pi}^{ij}\right)$. We then write the Hamiltonian constraint in terms of the barred variables, $\mathscr{H}_\lambda\left[\bar{g}_{ij}, \bar{\pi}^{ij}\right] \approx 0$. Re-expressing $(\bar{g}_{ij}, \bar{\pi}^{ij})$ as functions of the conformal factor ϕ and the phase space variables (g_{ij}, π^{ij}) in the Hamiltonian constraint, we obtain the modified Lichnerowicz-York equation, $\mathscr{H}_\lambda\left[\phi, g_{ij}, \pi^{ij}\right] \approx 0$. When this equation has a solution ϕ, the barred variables solve the constraints. Therefore, we will refer to $(\bar{g}_{ij}, \bar{\pi}^{ij})$ as "constraint-solving data", despite the fact that their constraint-solving property only holds when the modified Lichnerowicz-York equation has a solution. The unbarred variables which are substituted into the equation to determine ϕ will be referred to as "initial data". Note that for given constraint-solving data $(\bar{g}_{ij}, \bar{\pi}^{ij})$, there is a whole family of configurations (g_{ij}, π^{ij}) related to it by a conformal transformation.

We can use this method to make several comparisons between general relativity and the λ-R model. The most straight-forward way is to solve the Lichnerowicz-York equation and its modified version for identical initial data and compare the resulting constraint-solving data. For a given set of initial data, we can also study the role of λ in the solutions to the Hamiltonian constraint by studying how the conformal factor for fixed initial data depends on λ. Another possibility is to compare both theories for identical constraint-solving data. As we will show, this requires relating the initial data of general relativity with those of the λ-R model. This can be done, but the constraint solving data will turn out to evolve differently in the two theories. The final comparison that can be made is by trying to match the time evolution equations through a relationship between the initial data of the λ-R model and of general relativity. As we will show, this is also possible but implies that the constraint-solving data in general relativity and in the λ-R model is manifestly different.

The remainder of this chapter deals with the application of the conformal method to the λ-R model. Following the steps outlined in appendix C, we obtain a generalised Lichnerowicz-York equation and study the existence and uniqueness of its solutions. We distinguish three cases, defined by the sign of the ϕ^5-term in the equation, which is the term containing both λ and Λ. We then use this knowledge to compare the λ-R model to general relativity, ending with a summary of the results obtained.

3.2 The conformal method in the λ-R model

We apply the conformal method to the λ-R model discussed in the previous chapter in the Hamiltonian formalism. This differs from the Lagrangian formulation of the original presentation of the method and its associated use of configuration space variables.

Recall that we showed in eq. (2.36) of chapter 2 that the λ-R model has the constant mean curvature condition as a tertiary constraint

$$\nabla_i \pi = 0\,, \tag{3.1}$$

whose solution we wrote as

$$\pi = a(t)\sqrt{g}\,, \tag{3.2}$$

where $a(t)$ is a constant for each Σ_t. These are the phase space versions of the constant-K condition mentioned in section C.2 of appendix C. Similar to what happens there, it is possible to decompose the momentum tensor density in terms of its trace π and transverse traceless components π^{ij}_{TT} according to

$$\pi^{ij} = \pi^{ij}_{TT} + \frac{1}{3} g^{ij}\pi\,, \tag{3.3}$$

where π satisfies eq. (3.2). This is not a fully general decomposition for such a tensor, which would be a densitised version of eq. (C.3) in appendix C. However, like in the discussion following eq. (C.3), it can be shown that the momentum constraints are solved by removing the vector parts in the decomposition, leading to the decomposition (3.3).

Recall the functional form of the momentum and Hamiltonian constraints given in eq. (2.23) of chapter 2,

$$\mathscr{H}_i = -2g_{ij}\nabla_k \pi^{jk} \approx 0\,, \tag{3.4a}$$

$$\mathscr{H}_\lambda = \frac{1}{\sqrt{g}} G^\lambda_{ijkl}\,\pi^{ij}\pi^{kl} - \sqrt{g}\,(\mathscr{R} - 2\Lambda) \approx 0\,. \tag{3.4b}$$

It is clear that the decomposition (3.3) with π satisfying eq. (3.2) solves the momentum constraints $\mathscr{H}_i \approx 0$. Substituting this decomposition into the Hamiltonian constraint, it reads

$$\mathscr{H}_\lambda = \frac{1}{\sqrt{g}}\left(\pi^{ij}_{TT}\,\pi^{TT}_{ij} - \frac{1}{3}\frac{\pi^2}{3\lambda-1}\right) - \sqrt{g}\,(\mathscr{R} - 2\Lambda) \approx 0\,. \tag{3.5}$$

Recall the conformal transformation

$$\bar{g}_{ij} = \phi^4\, g_{ij}\,, \tag{3.6}$$

of the metric given in eq. (C.5) to obtain the Lichnerowicz-York equation. As stated in appendix C, ϕ is a strictly positive function on Σ_0. The momentum-space version of the extrinsic curvature transformations (eq. (C.6)) and $\bar{K} = K$ can be deduced from eq. (3.6) and the Legendre transformation (2.19), yielding

$$\bar{\pi}^{ij}_{TT} = \phi^{-4}\pi^{ij}_{TT}, \tag{3.7a}$$

$$\bar{\pi} = \phi^6\pi\,. \tag{3.7b}$$

Note that the transformation of π reflects its density nature, that is, the fact that $\frac{\pi}{\sqrt{g}}$ transforms as a scalar, but not π itself. We now write the Hamiltonian constraint as a functional of the barred variables, that is,

$$\mathscr{H}_\lambda\left[\bar{g}_{ij}, \bar{\pi}^{ij}_{TT}, \bar{\pi}\right] \approx 0, \tag{3.8}$$

and substitute the barred variables by their expressions in terms of initial data and conformal factor given in eqs. (3.6) and (3.7),

$$\mathscr{H}_\lambda\left[\phi, g_{ij}, \pi^{ij}_{TT}, \pi\right] \approx 0. \tag{3.9}$$

After a few algebraic manipulations, eq. (3.9) becomes the modified Lichnerowicz-York equation,

$$8\nabla^2\phi = \phi\mathscr{R} - \phi^{-7}\,\frac{\pi^{ij}_{TT}\,\pi^{TT}_{ij}}{g} + \phi^5\left(\frac{1}{3\,(3\lambda-1)}\frac{\pi^2}{g} - 2\Lambda\right). \tag{3.10}$$

Next, we study the existence of solutions of eq. (3.10), treating separately the regimes in which the ϕ^5-term on the right-hand side has different signs.

3.2.1 Existence of solutions to the modified Lichnerowicz-York equation

We begin by discussing which parts of the classical general relativistic treatment still hold. First, integrating the left-hand side of eq. (3.10) over Σ_0 yields a vanishing result due to Stokes' theorem. The integral of the right-hand side must therefore also vanish. It is convenient to write the right-hand side as a polynomial $P(\phi)$ in ϕ, which we define as

$$P(\phi) := \phi\,\mathscr{R} - \phi^{-7}\mathscr{A} + \phi^5\,\mathscr{C}\,, \tag{3.11}$$

where we have introduced the shorthand $\mathscr{A}$ for the term including all transverse-traceless data,

$$\mathscr{A} := \frac{\pi^{ij}_{TT}\,\pi^{TT}_{ij}}{g}. \tag{3.12}$$

and the spatial constant $\mathscr{C}$,

$$\mathscr{C} := \frac{1}{3\,(3\lambda - 1)}\,\frac{\pi^2}{g} - 2\,\Lambda\,. \tag{3.13}$$

We will demonstrate that the sign of $\mathscr{C}$ determines the behaviour of eq. (3.10). Second, we take the scalar curvature $\mathscr{R}$ to be a spatial constant. How this can be achieved by a conformal transformation of the type (3.6) is described in appendix C. While not relevant for discussing the modified Lichnerowicz-York equation for $\mathscr{C} > 0$, the existence of solutions for $\mathscr{C} \leq 0$ will generally depend on the Yamabe class of the initial data metric g_{ij}. A metric g_{ij} can belong to the positive, negative, or vanishing Yamabe class, as we discuss in appendix C. Since all the λ-dependent information is encoded in $\mathscr{C}$ and we want to understand how the solutions to the constraints depend on λ, we proceed to discuss the properties of eq. (3.10) separately for $\mathscr{C} \geq 0$ and $\mathscr{C} < 0$.

Positive and vanishing $\mathscr{C}$

The reason for discussing these two cases together is that establishing existence and uniqueness of solutions follows from the general relativistic analysis in appendix C, without any additional considerations. If $\mathscr{C} = 0$, eq. (3.10) reduces to

$$8\,\nabla^2\,\phi = \phi\,\mathscr{R} - \phi^{-7}\mathscr{A}\,, \tag{3.14}$$

which is the Lichnerowicz equation [69] in Hamiltonian language. Hence, as long as g_{ij} belongs to the positive Yamabe class, there is a unique solution to eq. (3.14). Note that for $\Lambda \neq 0$, $\mathscr{C} = 0$ is only possible for a non-vanishing choice of $\frac{\pi}{\sqrt{g}}$. The constraint-solving data will therefore not resemble any set obtained from the original Lichnerowicz equation. We discuss this point further when we compare the solutions of the λ-R model to those of general relativity.

With respect to the existence of solutions, the case of $\mathscr{C} > 0$ is related to the Lichnerowicz-York equation in the same way as the $\mathscr{C} = 0$ is related to the Lichnerowicz equation. For $\mathscr{C} > 0$, there almost always exist unique solutions to the modified Lichnerowicz-York equation with no restrictions on the initial data[3]. Consider a set of initial data $\{g_{ij}, \pi^{ij}_{TT}, \pi\}$ and given values $\{\lambda^*, \Lambda^*\}$ for the constants λ and Λ, such that $\mathscr{C} > 0$. Denote the particular value of $\mathscr{C}$ for this configuration by $\mathscr{C}_0$,

$$\mathscr{C}_0 = \frac{1}{3\,(3\lambda^* - 1)}\,\frac{\pi^2}{g} - 2\Lambda^* > 0\,. \tag{3.15}$$

Then there exists a π_0 satisfying the constant mean curvature condition and such that

$$\frac{1}{6}\left(\frac{\pi_0}{\sqrt{g}}\right)^2 = \mathscr{C}_0\,. \tag{3.16}$$

This means that the same ϕ that uniquely solves the Lichnerowicz-York equation with initial data $\{g_{ij}, \pi^{ij}_{TT}, \pi_0\}$ also uniquely solves the modified Lichnerowicz-York equation with initial

[3] The set of restrictions associated with the "almost always " is discussed in appendix C, and also holds in the present case.

data $\{g_{ij}, \pi^{ij}_{TT}, \pi\}$ and couplings $\{\lambda^*, \Lambda^*\}$. The physical configurations obtained in both cases are generally different, as we will elaborate further after discussing the case of negative $\mathscr{C}$. For now, suffice it to say that for $\mathscr{C} > 0$ there always exists a unique solution to eq. (3.10) regardless of the given (values of) initial data, while for $\mathscr{C} = 0$ the initial data is restricted to the positive Yamabe class.

Negative $\mathscr{C}$

In terms of the existence of solutions, the most interesting case is $\mathscr{C} < 0$, since $P(\phi)$ behaves differently than in the cases studied previously. Recalling the form of $P(\phi)$ from eq. (3.11), we see that the only possibly non-negative contribution comes from the term linear in ϕ. Hence, solutions can only exist for initial metrics belonging to the positive Yamabe class. Let us assume $\mathscr{R} > 0$ for the remainder of this discussion. Although this is a necessary condition to ensure the existence of a solution, it is not sufficient. Recall that for solutions to exist, $P(\phi)$ must have at least one zero. As we can see from Figs. 3.1 and 3.2, it is possible to change the number of zeros of $P(\phi)$ from two to zero by changing the values of π^{ij}_{TT} and π. Thus, for a given initial choice of π and π^{ij}_{TT}, $\mathscr{R}$ must be large enough for a bounded interval to exist in which $P(\phi) > 0$.

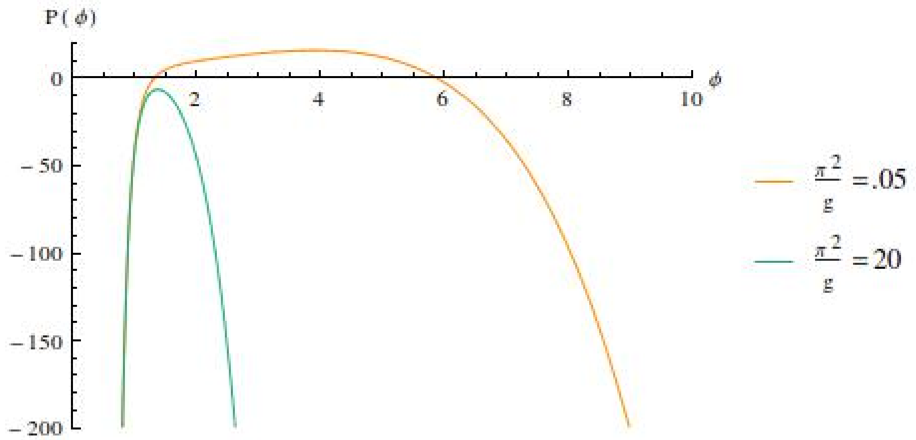

Figure 3.1: Comparison of $P(\phi)$ for two different values of $\frac{\pi^2}{g}$, $\frac{\pi^2}{g} = 0.5$ and $\frac{\pi^2}{g} = 20$. The other parameters are kept fixed and are given by $(\mathscr{R}, \mathscr{A}, \lambda, \Lambda) = (5, 50, -1, 0)$.

Let us now restrict our attention to the sets of initial data for which $P(\phi)$ does have two zeros. As discussed in section C.2 of appendix C, to apply the theorems guaranteeing the existence of a unique solution to the Lichnerowcizc-York equation, there must exist a bounded interval (ϕ_-, ϕ_+) with constants ϕ_-, ϕ_+, and such that $P(\phi_-) < 0$ and $P(\phi_+) > 0$ hold. This means that we can apply Theorem 3 only around the first zero of $P(\phi)$, which we denote by ϕ_1. Suppose we are given some initial values of π^{ij}_{TT} and π. For such an interval to exist, there must be a finite interval to which ϕ_+ belongs and for which

$$\mathscr{R} > \mathscr{A}\,\phi^{-8} - \mathscr{C}\phi^4 \tag{3.17}$$

holds for every $x^i \in \Sigma_0$.

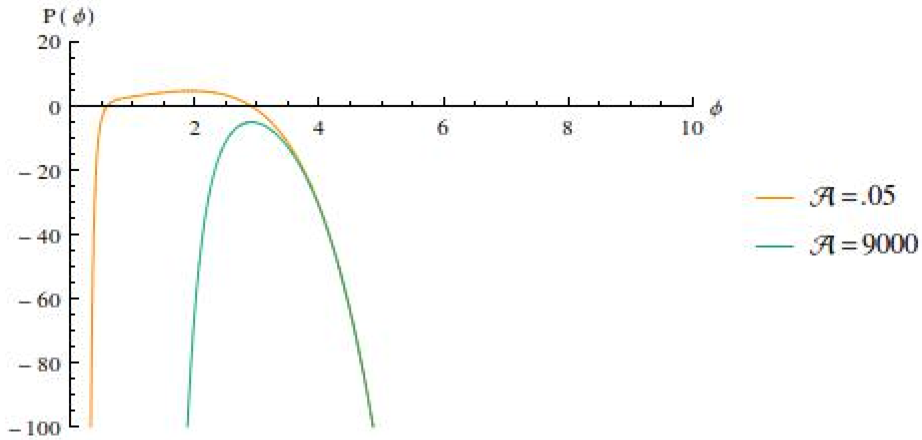

Figure 3.2: Comparison of $P(\phi)$ for two different values of $\mathscr{A}$, $\mathscr{A} = 0.5$ and $\mathscr{A} = 9000$. The other parameters are kept fixed and are given by $\left(\mathscr{R}, \frac{\pi^2}{g}, \lambda, \Lambda\right) = (3, 0.5, -1, 0)$.

Since π^{ij}_{TT} is not necessarily a constant, the minimal $\mathscr{R}$ for which eq. (3.17) is valid depends on the point $x^i \in \Sigma_0$. It is more convenient to write the inequality in terms of the maximum norm of the transverse-traceless initial data. Requiring that $\mathscr{A}$ has a maximum A on Σ_0,

$$A := \max_{x^i \in \Sigma_0} \mathscr{A}, \tag{3.18}$$

inequality (3.17) can be rephrased as

$$\mathscr{R} > A\phi^{-8} - \mathscr{C}\phi^4. \tag{3.19}$$

This guarantees not only that $\mathscr{R}$ is large enough to ensure the existence of both zeros on all Σ_0, but also that the position of the first zero is bounded from above. Note that if $\pi^{ij}_{TT} = 0$ for some $x^i \in \Sigma_0$, there cannot exist $P(\phi_-) < 0$ as required to ensure the existence of the solution. We will therefore assume that $\mathscr{A}$ is also bounded from below.

We have thus established three conditions that must be satisfied simultaneously for the equation to have at least one solution when $\mathscr{C} < 0$:

- g_{ij} belongs to the positive Yamabe class,
- $\mathscr{A}$ is bounded on Σ, and
- inequality (3.19) is satisfied.

Under these conditions we are guaranteed the existence of at least one solution. However, the polynomial has a second zero, ϕ_2. Because its derivative with respect to ϕ at ϕ_2 is negative, it is not possible to apply Theorem 3 around ϕ_2. If we could apply the theorem, we would be able to prove that a second solution exists around ϕ_2. However, the inapplicability of the theorem does not necessarily mean that no solution exists around ϕ_2. In fact, we can show that in some limiting cases, such a solution exists.

Suppose that instead of bounded transverse-traceless initial data, we start with $\pi^{ij}_{TT} = 0$. In this case, $P(\phi)$ reduces to

$$P(\phi) = \phi\mathscr{R} + \phi^5\mathscr{C}, \tag{3.20}$$

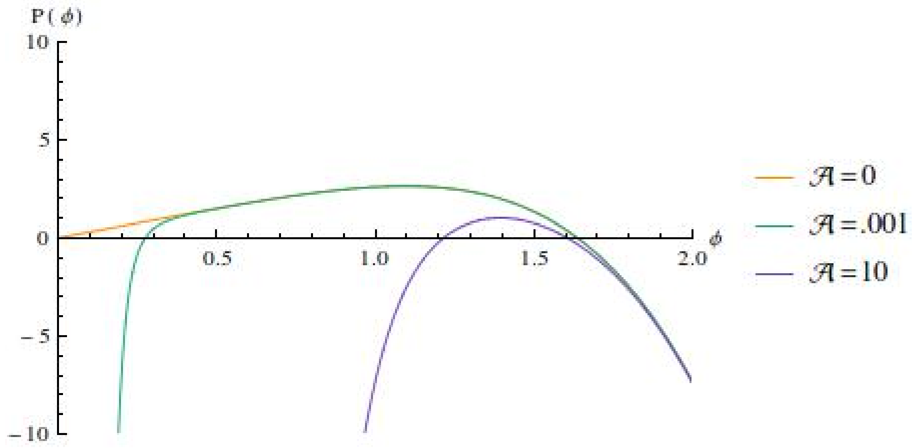

Figure 3.3: Comparison of $P(\phi)$ for three different values of $\mathscr{A}$, namely, $\mathscr{A}=0$, $\mathscr{A}=0.001$, and $\mathscr{A}=10$. The other parameters are kept fixed and are given by $\left(\mathscr{R},\frac{\pi^2}{g},\lambda,\Lambda\right)=(3,5,-1,0)$.

and since both $\mathscr{R}$ and $\mathscr{C}$ are constants, there is a constant solution ϕ_c given by

$$\phi_c=\left(\frac{\mathscr{R}}{-\mathscr{C}}\right)^{1/4}, \tag{3.21}$$

which exists as long as $\mathscr{R}>0$. One can then ask what happens to the solution once a small perturbation around $\pi^{ij}_{TT}=0$ is introduced. Let δA denote an infinitesimal ϕ^{-7}-contribution to $P(\phi)$. We make the replacements

$$\mathscr{A}=\delta A, \tag{3.22a}$$

$$\phi=\phi_c+\delta\phi, \tag{3.22b}$$

in eq. (3.10), reducing it to

$$8\nabla^2\delta\phi=-4\mathscr{R}\,\delta\phi-\delta A\left(\frac{-\mathscr{C}}{\mathscr{R}}\right)^{7/4}. \tag{3.23}$$

This equation always has solutions as long as $\delta\phi<0$. The fact that $\delta\phi$ is negative comes about because a non-vanishing value of π^{ij}_{TT} will decrease the value of ϕ_2, as illustrated by Fig. 3.3.

For the general situation in which the polynomial $P(\phi)$ has two zeros, ϕ_1 and ϕ_2, we have not been able to prove that two solutions always exist. Using Theorem 3, we proved that for an initial g_{ij} belonging to the positive Yamabe class, if $\mathscr{A}$ is bounded on Σ and inequality (3.19) is satisfied, there exists a solution to the modified Lichnerowicz-York equation around ϕ_1. When $\mathscr{A}$ vanishes, there is a unique constant solution at ϕ_c given in eq. (3.21). When we introduce a small, non-vanishing $\mathscr{A}$, the first solution re-appears, co-existing in that case with a perturbed version of ϕ_c, which is located around ϕ_2. Unfortunately, the behaviour of the polynomial around ϕ_2 prevents us from applying Theorem 3 to establish the existence of a second solution beyond the perturbative regime. We nevertheless believe that our perturbative results are suggestive that this is the case.

3.3 Comparison with general relativity

We have shown that all choices of initial data and couplings compatible with $\mathscr{C} > 0$ yield a unique solution to the modified Lichnerowicz-York equation (3.10). Other values of $\mathscr{C}$ require a choice of the base metric in the positive Yamabe class. For $\mathscr{C} = 0$, no further restrictions exist, while for negative $\mathscr{C}$ the value of the spatial curvature must be large enough for $P(\phi)$ to have two zeros; in particular, it must obey eq. (3.17). In this case, where $\mathscr{C} < 0$ and $\mathscr{R}$ is large enough for $P(\phi)$ to have two zeros, if the transverse traceless data is bounded and non-vanishing, there is at least one solution. A constant solution also exists when the transverse traceless data vanishes, $\mathscr{A} = 0$. We have also shown that this second solution persists up to first order in perturbations when we allow a very small but non-vanishing transverse-traceless momentum tensor. In the latter case, the solution is not unique, because it coexists with the one discussed previously. Having established the existence and in some cases uniqueness of these solutions, we will now compare them with those obtained in general relativity.

We must discuss first which ranges of initial data are associated with these particular values of $\mathscr{C}$. To make matters simpler, we begin by considering a vanishing cosmological constant $\Lambda = 0$, for which $\mathscr{C} > 0$ translates to

$$\lambda > \frac{1}{3} \tag{3.24}$$

for all values of $\frac{\pi}{\sqrt{g}}$.

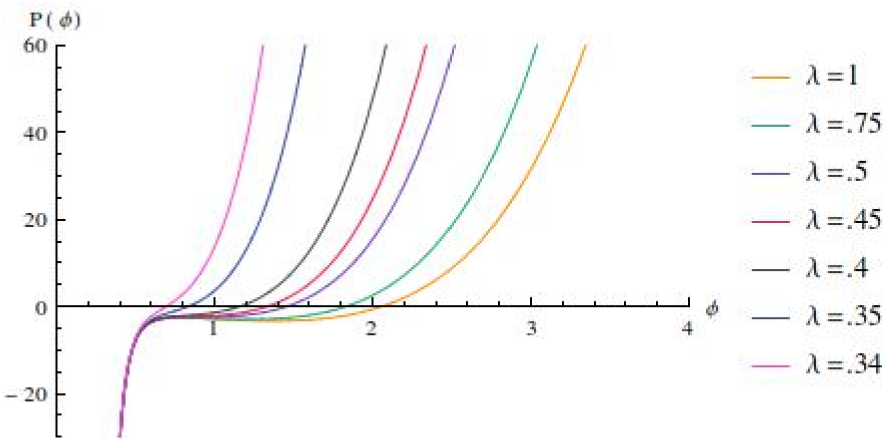

Figure 3.4: Comparison of $P(\phi)$ as λ decreases from 1 to $\frac{1}{3}$. All other parameters are kept fixed and are given by $\left(\mathscr{R}, \mathscr{A}, \frac{\pi^2}{g}\right) = (-3, 0.05, 1)$.

This implies that we are in the same regime as general relativity in the constant mean curvature gauge and can address the behaviour of ϕ as λ changes away from its general relativistic value of 1. One way of understanding this is to start by setting the scalar curvature of the initial hypersurface Σ_0 to zero, $\mathscr{R} = 0$, and imposing $P(\phi) = 0$, yielding

$$-\phi^{-7}\mathscr{A} + \phi^5\mathscr{C} = 0 \Rightarrow \mathscr{A} = \mathscr{C}\phi^{12}. \tag{3.25}$$

Taking $\mathscr{A}$ to be finite and non-vanishing means that the same must hold for the right-hand side. However, $\mathscr{C}$ goes to infinity as λ approaches $\frac{1}{3}$ from above, and to keep the product $\mathscr{C}\phi^{12}$ finite, ϕ must scale as

$$\phi \propto \mathscr{C}^{-1/12}. \tag{3.26}$$

In other words, as $\mathscr{C}$ approaches infinity, the value ϕ_s for which $P(\phi_s) = 0$ decreases according to eq. (3.26), as illustrated by Fig. 3.4. This behaviour is explained by the fact that the solution must be located in a finite neighbourhood of ϕ_s. Another limit one may be interested in is $\lambda \to \infty$. This implies $\mathscr{C} \to 0$, which translates to an increase of the position of the zero of $P(\phi)$ as λ increases. For given choices of $\mathscr{A}$ and g_{ij}, the maximum value of ϕ solving the modified Lichnerowicz-York equation is obtained for $\mathscr{C} = 0$, regardless of the value of π. This is an important point, since in this limit one obtains a conformal factor which in general relativity would be associated with maximal slicing coordinates, while having a non-vanishing π in the constraint-solving data. The consequences of this behaviour are best illustrated by recalling the time evolution equations for $\bar{g}_{ij}$ and $\bar{\pi}^{ij}_{TT}$. They correspond in this case to a maximal slicing initial metric and transverse-traceless momentum at Σ_0,

$$\dot{\bar{g}}_{ij} = \frac{2N}{\sqrt{\bar{g}}}\left(\bar{\pi}^{TT}_{ij} - \frac{\bar{g}_{ij}\,\bar{\pi}}{3\,(3\lambda-1)}\right), \tag{3.27a}$$

$$\begin{aligned}\dot{\bar{\pi}}^{ij}_{TT} = {}& \frac{N}{\sqrt{\bar{g}}}\left(\frac{2}{3\,(3\lambda-1)}\,\bar{\pi}^{ij}_{TT}\bar{\pi} - 2\,\bar{g}_{kl}\,\bar{\pi}^{ik}_{TT}\,\bar{\pi}^{jl}_{TT}\right) - \sqrt{\bar{g}}N\left(\bar{\mathscr{R}}^{ij} - \frac{1}{3}\bar{g}^{ij}\bar{\mathscr{R}}\right)\\ & - \sqrt{\bar{g}}\left(\bar{g}^{ik}\bar{g}^{jl} - \frac{1}{3}\bar{g}^{ij}\bar{g}^{kl}\right)\bar{\nabla}_k\bar{\nabla}_l N,\end{aligned} \tag{3.27b}$$

where $\bar{\nabla}_i$ denotes the covariant derivative with respect to $\bar{g}_{ij}$.

The behaviour of ϕ discussed previously in the limit of $\mathscr{C} \to 0$ when $\pi \neq 0$ can occur in two ways. First, as λ goes to infinity, ϕ gets asymptotically close to its maximal slicing value. The second possibility requires a non-vanishing cosmological constant and can therefore occur for any value of λ. Starting from a given, non-vanishing scalar $\frac{\pi}{\sqrt{g}}$, there is one λ for each value of Λ such that $\mathscr{C} = 0$ in the initial hypersurface Σ_0. In this case, the conformal factor ϕ obtained from the modified Lichnerowicz-York equation for a given choice of $(g_{ij}, \mathscr{A})$ in the positive Yamabe class is identical to the one we would have obtained in general relativity in the maximal slicing gauge. However, $\pi \neq 0$ implies $\bar{\pi} \neq 0$, and eq. (3.27b) does not describe the time evolution of a transverse-traceless momentum density in the maximal slicing gauge. Instead, it describes the evolution of a transverse-traceless momentum density in the constant mean curvature gauge of general relativity, with an effective trace term $\bar{\pi}_{eff}$ given by

$$\bar{\pi}_{eff} = \frac{2}{3\lambda - 1}\,\bar{\pi} = \frac{2\,\phi^6}{3\lambda - 1}\,\pi\,. \tag{3.28}$$

The same is not true in the limit $\lambda \to \infty$, because then $\bar{\pi}_{eff} \to 0$ as one can readily see from eq. (3.28)

As we have highlighted, substituting the same set of initial data into the Lichnerowicz-York and the modified Lichnerowicz-York equations yields different conformal factors. However, there is no *a priori* reason why we should compare solutions to the two equations with the same initial data, since the initial data has no direct physical meaning, in the sense explained at the beginning of this chapter. Instead, we could ask whether there is a way of obtaining the same constraint-solving data with the two different equations, at least for $\lambda > 1/3$. This is indeed possible and can be achieved by relating the mean curvature $\pi_{\lambda R}$ in the modified Lichnerowicz-York equation to the mean curvature π_{GR} in the original equation via

$$\pi_{\lambda R} = \sqrt{\frac{3\lambda - 1}{2}}\,\pi_{GR}. \tag{3.29}$$

This transformation effectively removes λ from $\mathscr{C}$. As a result, when eq. (3.29) is satisfied, the modified Lichnerowicz-York equation with initial data $\{g_{ij}, \mathscr{A}, \pi_{\lambda R}\}$ has the same solution as the original Lichnerowicz-York equation with initial data $\{g_{ij}, \mathscr{A}, \pi_{GR}\}$. However, we see that the $\bar{\pi}$-term in eq. (3.27b) is still λ-dependent. This happens because $\frac{\pi^2}{3\lambda-1}$ is the combination determining the constraint surface, while time evolution is determined by the combination $\frac{\bar{\pi}}{3\lambda-1}$.

Before comparing general relativity and the λ-R model for $\mathscr{C} < 0$, let us discuss the effect of a non-vanishing Λ in $\mathscr{C} > 0$. Considering $\frac{\pi}{\sqrt{g}}$ and $\lambda > 1/3$ as given, there is a bound on Λ for which $\mathscr{C} > 0$, namely,

$$\Lambda < \frac{1}{6\,(3\lambda - 1)}\frac{\pi^2}{g}. \tag{3.30}$$

Assuming eq. (3.30) is satisfied, the only effect of a non-vanishing cosmological constant is to shift $\mathscr{C}$, and therefore ϕ, away from its $\Lambda = 0$ value. However, it happens for both $\lambda = 1$ and $\lambda \neq 1$ and therefore the arguments used to compare the solutions for $\Lambda = 0$ are still valid. The only difference occurs if one wants to generalise the $\lambda \to \infty$ behaviour to open hypersurfaces. In this case our previous conclusions are no longer valid, since in this limit $\mathscr{C} \to -2\Lambda$ (note that due to eq. (3.30), Λ is necessarily negative in this limit). This means that for $\Lambda \neq 0$, $\mathscr{C}$ does not even vanish in the limit, although ϕ still reaches its maximum as $\lambda \to \infty$ because this limit corresponds to the minimum value of $\mathscr{C} > 0$ for that particular choice of initial data. Finally, note that for $\Lambda < 0$ it is possible to have $\mathscr{C} > 0$ even when $\lambda < 1/3$. In this case $\bar{\pi}$ takes a value that can be matched to a constant mean curvature configuration in general relativity, but appears with a different sign in the equations of motion due to $\lambda < 1/3$. It implies that $\dot{\pi}^{ij}_{TT}$ behaves differently in time than it would in Einstein's gravity.

Regarding the regime of negative $\mathscr{C}$, the comparison with general relativity is rather tenuous, because this case is absent from the original Lichnerowicz-York equation. However, if a positive cosmological constant is added to general relativity, there is, for each choice of $\frac{\pi}{\sqrt{g}}$, a minimal Λ such that $\mathscr{C} < 0$, and therefore our discussion for negative $\mathscr{C} < 0$ applies. Matching the constraint-solving data in both models is easily seen to be impossible, since eq. (3.29) is only valid when $\lambda > 1/3$.

3.3.1 No equivalence for general λ and Λ

As we have seen, whenever we are able to match the constraint-solving data of the λ-R model with that of general relativity, the evolution equations are manifestly different and therefore the theories with those choices of initial data are not equivalent. However, we can attempt to match the evolution data, which means matching the constraint-solving data $\bar{g}_{ij}$ and $\bar{\pi}^{ij}_{TT}$ in both theories, while relating the trace terms via

$$\bar{\pi}_{\lambda R} = \frac{3\lambda - 1}{2}\,\bar{\pi}_{GR}\,. \tag{3.31}$$

Since we are matching constraint-solving data, we can take the barred λ-R Hamiltonian constraint and write it as a function of ϕ_{GR} and the remaining general relativistic initial data, using eq. (3.31) for the trace term. We thus obtain the following version of the modified Lichnerowicz-York equation,

$$8\nabla^2\phi_{GR} = \phi_{GR}\mathscr{R} - \phi_{GR}^{-7}\mathscr{A} + \phi_{GR}^5\left(\frac{3\lambda-1}{12}\frac{\pi^2}{g} - 2\Lambda\right). \tag{3.32}$$

Using the fact that ϕ_{GR} solves the usual Lichnerowicz-York equation, we obtain

$$\phi_{GR}^5\,\frac{\pi^2}{g}\frac{\lambda-1}{4} = 0\,, \tag{3.33}$$

which is only true if either $\lambda = 1$ or $\pi = 0$, the two cases already known to yield equivalence between the theories. Similarly, we can also allow for a constant additive shift between the two cosmological constants, because there is no reason to assume that both models should be written with the same value of the cosmological constant. Setting $\Lambda_{\lambda R} = \Lambda_{GR} + \Lambda'$ effectively turns equation (3.32) into

$$\phi_{GR}^5\left(\frac{\pi^2}{g}\frac{\lambda-1}{4} - 2\Lambda'\right) = 0\,. \tag{3.34}$$

Since $\frac{\pi}{\sqrt{g}}$ is a spatial constant, there always exists a Λ' such that eq. (3.34) is valid on Σ_0. Imposing eq. (3.34) does not spoil the matching of time evolution, because the cosmological constant drops out from the equations. Moreover, although $\frac{\pi}{\sqrt{g}}$ is in general a function of time, eq. (3.34) only refers to the initial data and therefore to its value at that particular point in time. In terms of comparing the initial value formulations of both models, this would imply including either λ or Λ' in the initial data. For general values of these couplings, there is no way to match both theories unless one fine-tunes the values of these parameters as we have just illustrated.

3.4 Summary

Let us summarise briefly the results of this chapter. We studied the initial value formulation of the λ-R model by applying the conformal method developed by Lichnerowicz, York and Ó Murchadha. It is particularly suited to our case since its underlying condition $\nabla_i\pi = 0$ is a constraint of the model. Analogous to what happens in general relativity, the Hamiltonian constraint becomes an equation for the conformal factor of the metric, which we referred to as the modified Lichnerowicz-York equation (3.10). This equation differs from its $\lambda = 1$ counterpart only in the ϕ^5-term, which we denoted by $\mathscr{C}$. In the absence of a cosmological constant, the range of $\mathscr{C}$ therefore differs from its general relativistic counterpart. More importantly, for given values of $\frac{\pi}{\sqrt{g}}$ and Λ, the sign of $\mathscr{C}$ is λ-dependent.

For vanishing $\mathscr{C}$, the solutions to the modified equation are the same as those of the traditional one for initial data obeying the maximal slicing condition and base metric in the positive Yamabe class. We further argued that unless $\pi = 0$ and $\Lambda = 0$, the time evolution of the model does not match that of general relativity in the constant mean curvature gauge, since the equations of motion for g_{ij} and π^{ij}_{TT} depend on π and λ in a manifestly different way.

For positive $\mathscr{C}$, the existence and uniqueness of solutions follows straightforwardly from the general relativistic case. We argued that in the limit $\lambda \to 1/3$ (and therefore $\mathscr{C} \to \infty$), the conformal factor scales as $\phi \propto \mathscr{C}^{-1/12}$. When $\lambda \to \infty$ and $\mathscr{C} \to 0$, general relativity is recovered since λ drops out of the equation. We have also explained how it is possible to scale the initial data in order to have the same constraint-solving data both in the λ-R model and in general relativity. This makes explicit that, unless $\pi = 0$, the constraint surfaces match only at the initial hypersurface, because the time evolution of both theories is manifestly different. In addition, we have shown that the only way to obtain matching constraint-solving data whose time evolution is the same is for either $\lambda = 1$ or $\pi = 0$.

Finally, we studied the case of negative $\mathscr{C}$. This regime can occur when $\lambda = 1$, if Λ is large enough compared to the choice of $\frac{\pi}{\sqrt{g}}$. Similar to the case of vanishing $\mathscr{C}$, only metrics belonging to the positive Yamabe class can yield solutions. Even then, the allowed choices of base metric depend on the initial value of the momentum tensor, since the spatial curvature $\mathscr{R}$ must be large enough for solutions to exist. We have shown that for a bounded choice of transverse-traceless initial data there always exists a solution. When $\pi^{ij}_{TT} = 0$ everywhere on Σ, there is a constant solution to the equation regardless of the value of $\mathscr{R}$ (as long as it is admissible). Moreover, for a very small but non-vanishing π^{ij}_{TT}, perturbative arguments show that a small (negative) perturbation around the constant solution remains a solution, coexisting with the one mentioned previously.

Comparing general relativity and the λ-R model for $\mathscr{C} < 0$ is more subtle than for $\mathscr{C} > 0$, although the conclusions are similar. The only way to have $\mathscr{C} < 0$ in general relativity is when Λ is sufficiently large. In this case the conditions that $\mathscr{R}$ must be sufficiently large and that π^{ij}_{TT} must be bounded still apply, and solutions can be found. Naturally, one can fine-tune π and Λ to find the same value of $\mathscr{C}$, regardless of the value of λ. However, for the same reason that no equivalence was obtained when $\mathscr{C} > 0$ unless $\pi = 0$, no equivalence is found here. In this case, the evolution equations change even more drastically if $\lambda < 1/3$, because the $\bar{\pi}$ in the equations of motion not only picks up a different pre-factor, but changes sign altogether. Moreover, the same $\mathscr{C} < 0$ that in general relativity requires a non-vanishing Λ, can be obtained in the λ-R model for $\Lambda = 0$.

4

Spherical symmetry in the λ-R model

In this chapter, we study spherically symmetric solutions of the λ-R model[1]. We focus on the case where the Killing vectors satisfying the $SO(3)$-algebra lie entirely in the tangent space of the hypersurfaces associated with the preferred foliation. Since the model is not invariant under four-dimensional diffeomorphisms, there is no reason to assume these are its most general spherically symmetric solutions. Nevertheless, they already exhibit very interesting properties such as the non-vanishing of the four-dimensional curvature and a change in physical status of the degrees of freedom characterising the transverse-traceless components of the extrinsic curvature. Starting from the action of the model, we perform a reduction to variables suited to spherically symmetric hypersurfaces and study the reduced model in its Hamiltonian formulation. The resulting constraint algebra satisfies reduced versions of the constant mean curvature condition and the associated lapse-fixing equation. In this setting, we can explicitly solve all constraints and time evolution equations, obtaining two sets of solutions. The solutions in the first set satisfy the maximal slicing condition and coincide with the general relativistic solution written in those coordinates, while those in the second set have a non-vanishing constant mean curvature and exhibit λ-dependent corrections to the constant mean curvature version of the Schwarzschild spacetime. The solutions with non-vanishing constant mean curvature also have a non-vanishing four-dimensional curvature, which is proportional to $(\lambda - 1)$. Moreover, we show that gauge parameters in the general relativistic description of the Schwarzschild solution become potentially physical, since the four-dimensional curvature depends on them in a nontrivial way.

4.1 Introduction

When considering spherically symmetric solutions of general relativity without a cosmological constant, Birkhoff's theorem tells us that there is a unique one-parameter family of

[1]This chapter is based on R. Loll and L. Pires: *Spherically symmetric solutions of the λ-R model*, Phys. Rev. D**96** (2017) 044030, arXiv:1702.08362v1 [gr-qc], [74].

solutions, in which the parameter is the mass M of the central body. In the λ-R model, the reduced symmetry group prevents the applicability of Birkhoff's theorem. It is nevertheless instructive to look at some of the initial steps of its derivation, to the extent they are applicable in this generalised context.

In order to proceed with the discussion in a well-defined setting, let us define what we mean by spherical symmetry. A spacetime is said to be spherically symmetric if it possesses three linearly independent spacelike Killing vectors obeying an $SO(3)$-algebra. Due to the diffeomorphism invariance of general relativity, it is always possible to define coordinates such that the three Killing vectors lie in the tangent space of the spatial hypersurfaces chosen to foliate the spacetime. This is in general not possible in the λ-R model because time reparametrisations are not sufficient to align the hypersurfaces with the orbits of the $SO(3)$-algebra. For now, we will restrict ourselves to the case where the Killing vectors are aligned with the preferred foliation. In other words, we consider a four-dimensional spacetime whose preferred foliation is given in terms of spherically symmetric hypersurfaces Σ_t. A definition of a spherically symmetric three-dimensional Riemannian manifold can be found in chapter IV of Choquet-Bruhat's book [25], say. For our purposes, it is sufficient to know that such a manifold can be represented by a chart whose image[2] is $\mathbb{R}^3$ and that its metric in spherical coordinates reads

$$dS^2 = \mu^2(r)dr^2 + R^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.1}$$

where $\mu(r)$ and $R(r)$ are functions of r and the spherical coordinates (r,θ,ϕ) are defined in the usual manner with respect to the canonical coordinates (x,y,z) on $\mathbb{R}^3$.

Consider a spacetime manifold foliated by spherically symmetric hypersurfaces Σ_t. It can be shown [25] that the associated lapse and shift must depend only on (t,r), implying that the four-dimensional metric can be written as

$$ds^2 = -a^2(t,r)\,dt^2 + 2\,b(t,r)\,dt\,dr + \mu^2(t,r)\,dr^2 + R^2(r,t)\,d\Omega^2. \tag{4.2}$$

In general relativity, it is possible to eliminate the $b(t,r)$-term from (4.2) by defining a new time coordinate τ via

$$e^{-2\nu(t,r)}\Big(a(t,r)\,dt - b(t,r)\,dr\Big) = d\tau\,, \tag{4.3}$$

where the function $\nu(t,r)$ is defined such that its product with $(a(t,r)\,dt - b(t,r)\,dr)$ is the differential of a function τ. However, this is a space-dependent time reparametrization, which is not a symmetry of the λ-R model. To study spherically symmetric solutions of the λ-R model, even in the reduced setting we are considering, we are therefore forced to work with a non-vanishing radial shift.

Birkhoff's theorem is also not applicable in Hořava-Lifshitz gravity (see chapter 5 for a review) for the reasons we just outlined. In this theory, spherically symmetric solutions have been obtained from more restrictive ansätze than the one we consider in eq. (4.4) below, namely, assuming staticity, asymptotic flatness and a vanishing shift [5,18,59,62,63,75].

[2]The definition also works if the image is the exterior of a ball in $\mathbb{R}^3$.

4.2 Reduced λ-R model

For the remainder of this chapter, we assume that all leaves Σ_t of the foliation are spherically symmetric. This means that their line element is given by eq. (4.1) while the four-dimensional line element ds^2 is given by eq. (4.2). Since we will analyse the reduced model in a Hamiltonian setting, it is preferable to work with the lapse function $N(t,r)$ and radial shift $N^r := \xi(t,r)$ instead of the functions $a(t,r)$ and $b(t,r)$. The four-dimensional line element ds^2 then becomes

$$ds^2 = -\left(N^2 - \mu^2\xi^2\right)dt^2 + 2\mu^2\xi\, dr\, dt + \mu^2 dr^2 + R^2\, d\Omega^2, \tag{4.4}$$

where N, ξ, μ, and R are all functions of t and r. Moreover, we take μ, N and R to be strictly positive.

We will use dotted and primed quantities to denote partial derivatives with respect to t and r. Under radial transformations, that is, (r,t)-dependent redefinitions of the coordinate r, R behaves like a scalar while μ is a scalar density of positive unit weight. This implies that both the radial shift ξ and $R'(t,r)$ are scalar densities, the former of weight -1 and the latter of weight $+1$. We want to use these quantities to write the action of the λ-R model without a cosmological constant, given by

$$S_\lambda = \frac{1}{16\pi G_N}\int dt \int d^3x \sqrt{g} N \left(K^{ij}K_{ij} - \lambda K^2 + \mathscr{R}\right), \tag{4.5}$$

in a simplified form, valid for the spherically symmetric metrics given in eq. (4.4). As before, G_N denotes Newton's constant, and $\mathscr{R}$ the scalar curvature of Σ_t. In order to rewrite eq. (4.5) in terms of the metrics given in eq. (4.4), we need the determinant of the three-metric, the extrinsic curvature tensor K_{ij}, and the scalar curvature. The latter reads

$$\mathscr{R} = \frac{2}{R^2}\left(1 - \frac{(R')^2}{\mu^2} - 2\frac{R}{\mu}\left(\frac{R}{\mu}\right)'\right), \tag{4.6}$$

while for the extrinsic curvature we obtain

$$K_{rr} = \frac{1}{N}\left(\mu\dot{\mu} - \mu^2\xi' - \mu\mu'\xi\right), \tag{4.7a}$$

$$K_{\theta\theta} = \frac{1}{N}\left(R\dot{R} - RR'\xi\right) = \frac{K_{\phi\phi}}{\sin^2\theta}, \tag{4.7b}$$

and $\sqrt{g}$ is given by

$$\sqrt{g} = \mu R^2 \sin^2\theta\,. \tag{4.8}$$

We can now substitute eqs. (4.6), (4.7), and (4.8) into the action S_λ and integrate out the angular dependence,

$$S = \frac{1}{16\pi G_N}\int dt \int_{-\infty}^{+\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, \sqrt{g} N \left(K_{ij}K^{ij} - \lambda K^2 + \mathscr{R}\right) \tag{4.9a}$$

$$= \frac{1}{4G_N}\int dt \int_{-\infty}^{+\infty} dr\, \mu R^2 N \left(K_{ij}K^{ij} - \lambda K^2 + \mathscr{R}\right), \tag{4.9b}$$

where we have left $K_{ij}K^{ij} - \lambda K^2$ and $\mathscr{R}$ untouched since they have no angular dependence. We have chosen the range $r \in (-\infty, +\infty)$, implying that Σ_t runs from the left to the right wedge of the Kruskal diagram, matching the constant mean curvature treatment of the Schwarzschild spacetime given in [78] to which we will compare our results later on.

In the next section, we discuss the Hamiltonian formulation of the reduced λ-R model, which we have just defined. We will set the prefactor $\frac{1}{4G_N}$ in eq. (4.9b) to 1.

4.3 Phase space analysis

We begin by defining conjugate momentum variables from the action given in eq. (4.9b). As in the general case, the momenta associated with the lapse and shift vanish, thereby defining the primary constraints of the theory,

$$\phi_N := \frac{\delta S}{\delta \dot{N}} = 0\,, \qquad \phi_\xi := \frac{\delta S}{\delta \dot{\xi}} = 0\,. \tag{4.10}$$

The momenta associated with μ and R are non-vanishing and invertible for $\lambda \neq 1/3$,

$$\pi_\mu := \frac{\delta S}{\delta \dot{\mu}} = \frac{2R}{N}\left[(1-\lambda)\frac{R}{\mu}\left(\dot{\mu} - \mu\xi' - \mu'\xi\right) - 2\lambda\left(\dot{R} - R'\xi\right)\right], \tag{4.11a}$$

$$\pi_R := \frac{\delta S}{\delta \dot{R}} = \frac{4\mu}{N}\left[(1-2\lambda)\left(\dot{R} - R'\xi\right) - \lambda\frac{R}{\mu}\left(\dot{\mu} - \mu\xi' - \mu'\xi\right)\right]. \tag{4.11b}$$

Inverting eqs. (4.11), the Hamiltonian without any primary constraints becomes

$$H = \int dr \Big(\xi \mathscr{H}_r + N \mathscr{H}_\lambda\Big) + H_{\partial\Sigma}\,, \tag{4.12}$$

where $\mathscr{H}_r$ and $\mathscr{H}_\lambda$ denote the phase space functions

$$\mathscr{H}_r = \pi_R R' - \mu {\pi_\mu}'\,, \tag{4.13a}$$

$$\mathscr{H}_\lambda = \frac{2\lambda - 1}{4(3\lambda - 1)}\frac{\mu\pi_\mu^2}{R^2} + \frac{\lambda - 1}{8(3\lambda - 1)}\frac{\pi_R^2}{\mu} - \frac{\lambda}{2(3\lambda - 1)}\frac{\pi_\mu\pi_R}{R} - 2\left(\mu - \frac{(R')^2}{\mu} - 2R\left(\frac{R'}{\mu}\right)'\right). \tag{4.13b}$$

The final term in eq. (4.12), which we denoted by $H_{\partial\Sigma}$, is the boundary Hamiltonian previously discussed in Sec. 2.3. It will be chosen later in a way to ensure that the Hamiltonian is sufficiently differentiable in the sense of Regge and Teitelboim [86]. We will discuss its precise form in subsection 4.3.4 below when we address the boundary and fall-off conditions of the fields. Adding the primary constraints, the total Hamiltonian reads

$$H_{tot} = \int dr \left(N\mathscr{H}_\lambda + \xi\mathscr{H}_r + \phi_N\alpha + \phi_\xi\beta\right) + H_{\partial\Sigma}\,. \tag{4.14}$$

We now determine the constraint algebra associated with this reduced Hamiltonian, beginning as usual by demanding that the primary constraints defined by eq. (4.10) should be preserved in time.

4.3.1 Constraint algebra

As is the case in general relativity, the total Hamiltonian is linear in both lapse and radial shift. Since the primary constraints of the theory are given by the vanishing of their respective momenta, the Poisson brackets between the total Hamiltonian and the primary constraints yield the radial momentum and Hamiltonian constraints,

$$\dot{\phi}_\xi = \{\phi_\xi, H_{tot}\} = -\mathscr{H}_r \approx 0\,, \tag{4.15a}$$

$$\dot{\phi}_N = \{\phi_N, H_{tot}\} = -\mathscr{H}_\lambda \approx 0\,. \tag{4.15b}$$

The model thus possesses two secondary constraints, $\mathscr{H}_r \approx 0$ and $\mathscr{H}_\lambda \approx 0$, both of which must be preserved in time. Due to the remaining invariance under spatial diffeomorphisms, ensuring that $\dot{\mathscr{H}_r}$ vanishes at all times yields the same expression as in general relativity, namely,

$$\dot{\mathscr{H}_r} = \{\mathscr{H}_r, H_{tot}\} = 2\mathscr{H}_r\xi' + \xi\mathscr{H}_r{}' + \mathscr{H}_\lambda N' \approx 0, \tag{4.16}$$

which vanishes straightforwardly on the constraint surface. Computing the time derivative of the Hamiltonian constraint yields a reduced version of eq. (2.32), which we previously obtained in chapter 2,

$$\dot{\mathscr{H}_\lambda} = (\xi\mathscr{H}_\lambda)' + \frac{2N' + N\partial_r}{3\lambda - 1}\left[2\lambda\frac{\mathscr{H}_r}{\mu^2} + (\lambda - 1)\left(-2\frac{\pi_\mu}{\mu R}R' + \frac{R}{\mu}\left(\frac{\pi_R}{\mu}\right)'\right)\right]. \tag{4.17}$$

Like its counterpart in the full model, eq. (4.17) only vanishes on the constraint surface for the general relativistic value $\lambda = 1$. For other values of λ, a tertiary constraint emerges when demanding the weak vanishing of its right-hand side. After some algebraic manipulations, this can be shown to imply

$$\frac{R^2}{\mu}\left(N^2\left(\frac{\pi_\mu}{R^2} + \frac{\pi_R}{R\mu}\right)'\right)' \approx 0\,, \tag{4.18}$$

which is solved by the reduced version of the constant mean curvature condition, namely,

$$\omega := \mu\pi_\mu + R\pi_R - A(t)\mu R^2 \approx 0\,, \tag{4.19}$$

where $A(t)$ is a function of time only. When we change our description to extrinsic curvature variables in subsection 4.3.3, we will show that $A(t)$ is proportional to K, thereby confirming that eq. (4.19) is indeed the constant mean curvature condition. We proceed by imposing $\dot{\omega} \approx 0$, including an explicit time derivative of $A(t)$, because we do not have a description of $A(t)$ in terms of phase-space variables,

$$\begin{aligned}\dot{\omega} &= \frac{\partial}{\partial t}\omega + \{\omega, H_{tot}\} = -\dot{A}\mu R^2 + \{\omega, H_{tot}\} \\ &\approx 4\mu R^2\left\{\left(\mathscr{R} + \frac{A^2}{8(3\lambda - 1)} - \frac{1}{\mu R^2}\partial_r\left(\frac{R^2}{\mu}\partial_r\right)\right)N - \frac{\dot{A}}{4}\right\} \approx 0.\end{aligned} \tag{4.20}$$

Unsurprisingly, this is just the lapse-fixing equation (2.37) of chapter 2 in reduced variables. As we showed then, the Dirac algorithm ends when imposing the preservation of the lapse-fixing equation in time. This yields an equation for the Lagrange multiplier α, which when solved determines the time evolution $\dot{N}$ of the lapse function. Instead of proceeding by Poisson-commuting eq. (4.20) with the total Hamiltonian, we will do the same we did with the tertiary constraint, which we first solved in eq. (4.19) before Poisson-commuting. Before that, we will also solve both the radial momentum and the Hamiltonian constraints.

Solving the constraints

We begin by addressing the radial momentum constraint $\mathscr{H}_r \approx 0$. Using the constant mean curvature condition (4.19), we can eliminate π_R from eq. (4.13a), obtaining

$$R'\left(AR\mu - \frac{\mu\pi_\mu}{R}\right) - \mu\pi'_\mu \approx 0\,, \tag{4.21}$$

which is solved by

$$\pi_\mu = \frac{C}{R} + \frac{A}{3}R^2\,, \tag{4.22}$$

where $C = C(t)$ is a new integration constant, which describes the transverse-traceless degrees of freedom of the extrinsic curvature tensor K_{ij}, as we will show in subsection 4.3.3. Once again using eq. (4.19), we can also write π_R in terms of C and A,

$$\pi_R = \mu\left(\frac{2}{3}AR - \frac{C}{R^2}\right), \tag{4.23}$$

implying that we have successfully written both momentum variables in terms of the metric and two integration constants.

Substituting both eqs. (4.22) and (4.23) into the Hamiltonian constraint $\mathscr{H}_\lambda \approx 0$ (4.13b) and performing some algebraic manipulations reduces it to a total derivative,

$$\left(\left(R\left(\frac{R'}{\mu}\right)^2\right) - R - \frac{C^2}{16R^3} - \frac{A^2}{72\,(3\lambda-1)}R^3\right)' = 0 \tag{4.24a}$$

$$\Rightarrow \left(R\left(\frac{R'}{\mu}\right)^2\right) - R - \frac{C^2}{16R^3} - \frac{A^2}{72\,(3\lambda-1)}R^3 = -8\,m\,, \tag{4.24b}$$

where we have again introduced a possibly time-dependent integration constant denoted by m. Inverting eq. (4.24b), we can write μ in terms of R, its spatial derivative and integration constants as

$$\frac{\mu^2}{(R')^2} = \frac{1}{B(R)}\,, \tag{4.25}$$

where we have introduced $B(R)$ as a shorthand for the function

$$B\,(R;m,A,C) := 1 - \frac{8m}{R} + \frac{C^2}{16R^4} + \frac{A^2R^2}{72\,(3\lambda-1)}\,. \tag{4.26}$$

If we set the constants A and C to zero and impose $R = r$, we recover the metric component $g_{rr} = \mu^2$ of the standard Schwarzschild solution with mass $M_s = 16m$. However, for $A \neq 0$ we obtain a restriction for the allowed values of λ, because eq. (4.25) implies $B > 0$, which for large values of R is only possible for $\lambda > 1/3$. This is an important restriction and will be discussed further at the end of this chapter, as well as in chapter 6.

Having solved the radial momentum, Hamiltonian, and tertiary constraints, we now turn to the lapse-fixing equation obtained as a quaternary constraint in eq. (4.20), which is a non-homogeneous second-order differential equation. Our strategy will be to first obtain the most general solution of the associated homogeneous equation and add to it a particular solution of the full, inhomogeneous equation. The first step consists in finding two linearly independent solutions of the homogeneous differential equation. To obtain the first one, we consider the ansatz

$$N = B(R)^n. \tag{4.27}$$

Substituting this ansatz into eq. (4.20) with $\dot{A} = 0$, we find that eq. (4.27) is a solution of the homogeneous equation provided $n = 1/2$. To obtain the second solution, we use a different ansatz, namely,

$$N = \sqrt{B(R)}\, f\,. \tag{4.28}$$

Knowing that $\sqrt{B}$ is a solution, we only need to consider the terms in eq. (4.20) containing spatial derivatives of f. Solving the resulting equation we obtain

$$f = \int_{r_0}^{r} d\tilde{r}\, \frac{R'}{B^{3/2}} \frac{1}{R^2}\,. \tag{4.29}$$

Introducing possibly time-dependent integration constants n_1 and n_2 for the first and second solutions, the general solution of the homogeneous equation associated with eq. (4.20) is

$$N = \sqrt{B}\left(n_1 + \int_{r_0}^{r} d\tilde{r}\, \frac{R'}{B^{3/2}} \frac{n_2}{R^2} \right). \tag{4.30}$$

To obtain a particular solution to the inhomogeneous system, we employ the same strategy used to obtain the term proportional to n_2 in eq. (4.30), that is, we once again resort to the ansatz of eq. (4.28). This yields the general solution N_{sol} to eq. (4.20),

$$N_{sol} = \sqrt{B}\left(n_1 + \int_{r_0}^{r} d\tilde{r}\, \frac{R'}{B^{3/2}} \left(\frac{n_2}{R^2} - \frac{\dot{A}R}{12} \right) \right). \tag{4.31}$$

Later, we will fix $r_0 = \infty$ and show that for this choice n_1 determines the behaviour of the lapse at spatial infinity. Moreover, the time evolution equations will show that n_2 is directly proportional to the time derivatives of the transverse-traceless components of the extrinsic curvature K_{ij}.

We can now perform the remainder of the Dirac algorithm. Solving $\dot{\omega} \approx 0$ with the lapse given by eq. (4.31), we rewrite the quaternary constraint as

$$\mathscr{M} := N - N_{sol} \approx 0\,, \tag{4.32}$$

and demand that it is preserved in time, i.e. $\dot{\mathscr{M}} \approx 0$. To ensure the consistency of this step, we use the original total Hamiltonian H_{tot}, without substituting any of the solutions of the constraints we have obtained. The same holds for eq. (4.32), which means that we must restore the original phase space dependence of N_{sol} also. This leads to

$$N_{sol} \approx \pm \frac{R'}{\mu} \left(n_1 + \int_{r_0}^{r} d\tilde{r}\, \frac{\mu^3}{(R')^2}\, b(R) \right), \tag{4.33}$$

where we have introduced yet another shorthand $b(R)$ for the quantity

$$b(R) := \frac{n_2}{R^2} - \frac{\dot{A}R}{12}\,, \tag{4.34}$$

which will appear often in the remainder of this chapter. Recall that we imposed $\mu > 0$ when we defined the phase space variables. This means that for $\sqrt{B}$ to be well defined by eq. (4.25), we must use the plus sign in equation (4.33) when $R' > 0$ and the minus sign when $R' < 0$. Despite our goal of writing eq. (4.33) in terms of the original phase space variables, N_{sol} still depends on n_1, n_2, and $\dot{A}$. We therefore add explicit time derivatives of these quantities when computing $\dot{\mathscr{M}}$, obtaining

$$\dot{\mathscr{M}} = \frac{\partial}{\partial t}\mathscr{M} + \{\mathscr{M}, H_{tot}\} = \alpha - \frac{\partial}{\partial t}N_{sol} - \{N_{sol}, H_{tot}\} \approx 0\,. \tag{4.35}$$

After a long but unenlightening computation, the remaining Poisson bracket $\{N_{sol}, H_{tot}\}$ in eq. (4.35) is found to be

$$\begin{aligned}&\{N_{sol}, H_{tot}\}\\ =&\xi N' - \frac{N}{R'}\left(\frac{AR}{6\,(3\lambda-1)} + \frac{C}{4R^2}\right)\left(N' + \frac{bR'}{B}\right) + \sqrt{B}\int_{r_0}^{r} d\tilde{r}\,\frac{3R'b^2}{B^{5/2}}\left(\frac{AR}{6\,(3\lambda-1)} + \frac{C}{4R^2}\right).\end{aligned} \tag{4.36}$$

In the process of obtaining eq. (4.36), we have discarded all boundary terms evaluated at r_0. As mentioned above, we will later set $r_0 = \pm\infty$, which are the limits in which these terms vanish for the adopted boundary conditions (see subsection 4.3.4 for details). Using this result, we can write eq. (4.35) as

$$\alpha = \sqrt{B}\int_{r_0}^{r} d\tilde{r}\left(\frac{1}{B^{3/2}}\left(R'\left(\frac{\dot{n}_2}{R^2} - \frac{\ddot{A}R}{12}\right)\right) + \frac{3R'b^2}{B^{5/2}}\left(\frac{AR}{6\,(3\lambda-1)} + \frac{C}{4R^2}\right)\right) \tag{4.37a}$$

$$+\sqrt{B}\,\dot{n}_1 + \xi N' - \frac{N}{R'}\left(\frac{AR}{6\,(3\lambda-1)} + \frac{C}{4R^2}\right)\left(N' + \frac{bR'}{B}\right), \tag{4.37b}$$

for the Lagrange multiplier α. As was the case in general relativity and in the general λ-R model discussed in chapter 2, determining α through the constraint algebra implies that the time evolution of the lapse is no longer arbitrary[3]. For this particular case, we will see that imposing $\alpha = \dot{N}$, with α given by eq. (4.37), yields no non-trivial conditions, provided the equations of motion for the metric g_{ij} and momenta π^{ij} are satisfied.

Let us briefly discuss the first- and second-class nature of the constraints before moving on to the discussion of the time evolution.

Classification of the constraints

Due to the integration of angular coordinates we performed before defining the phase space, the λ-R model with spherical symmetry is parametrised by eight phase space variables

$$(\mu, R, N, \xi, \pi_\mu, \pi_R, \phi_N, \phi_\xi) \tag{4.38}$$

[3] In general relativity this is true when a gauge fixing of the Hamiltonian constraint is imposed, for example, in terms of the constant mean curvature condition.

and six constraints

$$\phi_\xi = 0\,, \qquad \phi_N = 0\,, \qquad \mathscr{H}_r \approx 0\,, \qquad \mathscr{H}_\lambda \approx 0\,, \qquad \omega \approx 0\,, \qquad \mathscr{M} \approx 0\,. \tag{4.39}$$

From our earlier discussion, only one of the constraints in (4.39) is trivially first class, namely, $\phi_\xi = 0$, since no other constraint depends on ξ. The same is not immediately true for $\mathscr{H}_r$ for the same reason it was also not immediately true in the full model. From our earlier computations of $\dot{\mathscr{H}}_r$ in (4.16), $\dot{\mathscr{H}}_\lambda$ in (4.17), and $\dot{\omega}$ in (4.20), we deduce that the radial momentum constraint $\mathscr{H}_r$ has a weakly vanishing Poisson bracket with $\mathscr{H}_\lambda$ and ω, as well as with ϕ_ξ and ϕ_N. However, this does not hold for the constraint $\mathscr{M} \approx 0$. By virtue of eq. (4.36), we have

$$\{N - N_{sol}, \mathscr{H}_r\} = -N'\,, \tag{4.40}$$

which does not vanish on the constraint surface. This is the same situation we faced in chapter 2 and can be summarised by saying that in its current form, the constraint $\mathscr{H}_r$ only generates infinitesimal spatial diffeomorphisms of μ, R and their conjugate momenta. Again, we solve the issue by adding to the momentum constraint a term linear in the other constraints, which is always allowed. The modified momentum constraint we will use from now on is

$$\tilde{\mathscr{H}}_r := \mathscr{H}_r + \phi_N N' \approx 0\,. \tag{4.41}$$

It generates infinitesimal diffeomorphisms of the lapse and its momentum, and Poisson-commutes with $\mathscr{M}$ on the constraint surface since

$$\left\{N - N_{sol}, \tilde{\mathscr{H}}_r\right\} \approx N' - N' = 0\,. \tag{4.42}$$

This implies that $\tilde{\mathscr{H}}_r \approx 0$ is first class because the additional term $\phi_N N'$ does not have a non-vanishing Poisson bracket with any of the other constraints.

In summary, we have two first-class constraints, $\tilde{\mathscr{H}}_r \approx 0$ and $\phi_\xi = 0$. The remaining four constraints are second class, implying that the model has no local degrees of freedom.

4.3.2 Time evolution equations

Before stopping to comment on the first- and second-class nature of the constraints, we had determined and completely solved the constraint algebra of the system. We now want to determine what conditions are imposed by evolving the constraint-solving data in time. We start with the metric variables μ and R, finding

$$\dot{\mu} = \{\mu, H_{tot}\} = \frac{N}{2\,(3\lambda - 1)}\left((2\lambda - 1)\,\frac{\mu\pi_\mu}{R^2} - \lambda\,\frac{\pi_R}{R}\right) + \xi'\mu + \xi\mu'\,, \tag{4.43a}$$

$$\dot{R} = \{R, H_{tot}\} = \frac{N}{4\,(3\lambda - 1)}\left((\lambda - 1)\,\frac{\pi_R}{\mu} - 2\lambda\,\frac{\pi_\mu}{R}\right) + R'\xi\,. \tag{4.43b}$$

Substituting the expressions for the canonical momenta π_μ and π_R from eqs. (4.22) and (4.23) into eq. (4.43b), we obtain an expression for the radial component of the shift,

$$\xi = \frac{\dot{R}}{R'} + \frac{N}{R'}\left(\frac{C}{4R^2} + \frac{AR}{6\,(3\lambda - 1)}\right). \tag{4.44}$$

Using in addition the solutions for μ^2, N, and ξ, obtained in eqs. (4.25), (4.31), and (4.44) respectively, and substituting everything into expression (4.43a) for $\dot{\mu}$ yields

$$\frac{R'}{BR}\left\{\left(\frac{n_2 A}{3(3\lambda-1)}-\frac{C\dot{A}}{24}-8\dot{m}\right)+\frac{C}{2R^3}\left(\frac{\dot{C}}{4}+n_2\right)\right\}=0\,, \tag{4.45}$$

whose only acceptable solution is

$$8\dot{m}=\frac{n_2 A}{3(3\lambda-1)}-\frac{C\dot{A}}{24} \quad \wedge \quad \left(\dot{C}=-4n_2 \quad \vee \quad C=0\right), \tag{4.46}$$

since we have $R>0$ by definition and, due to condition (4.25), B and R' cannot vanish either. We will show in the next section that C describes the transverse-traceless components of the extrinsic curvature. Taken together with $n_2=-\frac{\dot{C}}{4}$, this implies the interpretation of n_2 alluded to before, namely, as a measure of the time derivative of the transverse-traceless components of K_{ij}. The condition on $\dot{m}$ will later be used to define a mass M for which $\dot{M}=0$ and which is proportional to the Schwarzschild mass M_s when $\lambda=1$ or $A=0$. Note that these properties are already satisfied when $A=0$, with $\dot{m}=0$ also holding for the special case $\dot{A}=\dot{C}=0$.

Turning our attention to the time evolution equations for the momentum variables π_μ and π_R, we obtain

$$\dot{\pi}_\mu = N\left(2+2\frac{(R')^2}{\mu^2}+\frac{1}{4(3\lambda-1)}\left(\frac{\lambda-1}{2}\frac{\pi_R^2}{\mu^2}-(2\lambda-1)\frac{\pi_\mu^2}{R^2}\right)\right)-4\frac{R'}{\mu^2}\left(N'R+R'N\right)+\xi\pi_\mu'\,, \tag{4.47a}$$

$$\dot{\pi}_R = N\left(\frac{1}{2(3\lambda-1)}\left((2\lambda-1)\frac{\mu\pi_\mu^2}{R^3}-\lambda\frac{\pi_\mu\pi_R}{R^2}\right)-4\frac{R''}{\mu}+4\frac{R'\mu'}{\mu^2}\right)-4\left(\frac{R}{\mu}N'\right)'+(\xi\pi_R)'\,. \tag{4.47b}$$

Substituting the results for π_μ, π_R, μ, N, and ξ in terms of R into eq. (4.47a) results in the familiar condition

$$\frac{\dot{C}}{R}+\frac{\dot{A}}{3}R^2=-4\frac{n_2}{R}+\frac{\dot{A}}{3}R^2\,, \tag{4.48}$$

in other words, $\dot{C}=-4n_2$. A lengthy algebraic computation shows that eq. (4.47b) is satisfied if

$$\frac{(R')^2}{B^2}\left(P_0+P_{-2}R^{-2}+P_{-3}R^{-3}\right)=0\,, \tag{4.49}$$

where the P_k are polynomials of degree k in the metric function R and otherwise functions of A, $\dot{A}$, C, $\dot{C}$, m, $\dot{m}$, n_2 and λ. Explicitly, they are given by

$$P_0=\frac{n_2 A^2}{6(3\lambda-1)}-\frac{C\dot{A}A}{72}-\frac{8}{3}A\dot{m}+\frac{\dot{C}A^2}{72(3\lambda-1)}\,, \tag{4.50a}$$

$$P_{-2}=4n_2+\dot{C}\,, \tag{4.50b}$$

$$P_{-3}=\frac{3\lambda-2}{6(3\lambda-1)}CAn_2+\frac{C^2\dot{A}}{16}-32mn_2-8m\dot{C}+4C\dot{m}-\frac{C^2\dot{A}}{24}+\frac{AC\dot{C}}{24}\,. \tag{4.50c}$$

Since $\frac{(R')^2}{B^2}$ cannot vanish everywhere[4], the individual $P_k(R)$ must vanish identically on Σ_t. Setting $P_{-2}=0$ yields $\dot{C}=-4n_2$ once again. Substituting this relation into the expressions for P_0 and P_{-3} reduces both equations to the condition for $\dot{m}$ obtained in eq. (4.46) when solving the equation of motion for μ, up to an overall factor. In short, eqs. (4.49) are solved by

$$P_0 = 0 \Rightarrow A = 0 \quad \vee \quad 8\dot{m} = \frac{n_2 A}{3(3\lambda - 1)} - \frac{C\dot{A}}{24}, \tag{4.51a}$$

$$P_{-2} = 0 \Rightarrow \dot{C} = -4n_2, \tag{4.51b}$$

$$P_{-3} = 0 \Rightarrow C = 0 \quad \vee \quad 8\dot{m} = \frac{n_2 A}{3(3\lambda - 1)} - \frac{C\dot{A}}{24}. \tag{4.51c}$$

We conclude that the equations of motion for all phase space variables are solved by the two conditions

$$8\dot{m} = \frac{n_2 A}{3(3\lambda - 1)} - \frac{C\dot{A}}{24} \quad \wedge \quad \dot{C} = -4n_2. \tag{4.52}$$

Finally, we should solve $\dot{N}=\alpha$, with α given by eq. (4.37). Expanding $\dot{N}$, we obtain

$$\dot{N} = \frac{\partial N}{\partial R}\dot{R} + \frac{\partial N}{\partial n_1}\dot{n}_1 + \frac{\partial N}{\partial n_2}\dot{n}_2 + \frac{\partial N}{\partial m}\dot{m} + \frac{\partial N}{\partial A}\dot{A} + \frac{\partial N}{\partial C}\dot{C}. \tag{4.53}$$

To solve it, we begin by substituting the radial shift ξ given in eq. (4.44) into eq. (4.37) for α. The term $\frac{N}{R'}\left(\frac{C}{4R^2} + \frac{AR}{6(3\lambda-1)}\right)$ in the shift then cancels the last N'-term in eq. (4.37b), and the α-equation reduces to

$$\alpha = \sqrt{B}\int_{r_0}^{r} d\tilde{r}\left(\frac{1}{B^{3/2}}\left(R'\left(\frac{\dot{n}_2}{R^2} - \frac{\ddot{A}R}{12}\right)\right) + \frac{3R'b^2}{B^{5/2}}\left(\frac{AR}{6(3\lambda-1)} + \frac{C}{4R^2}\right)\right) \tag{4.54a}$$

$$+\sqrt{B}\,\dot{n}_1 + \frac{N'\dot{R}}{R'} - \frac{bN}{B}\left(\frac{AR}{6(3\lambda-1)} + \frac{C}{4R^2}\right). \tag{4.54b}$$

Imposing $\alpha = \dot{N}$ with $\dot{N}$ given by eq. (4.53), we see by applying the chain rule that the $\dot{R}$-dependence cancels. Similarly, it is straightforward to establish the cancellation between the terms proportional to $\dot{n}_1$, $\dot{n}_2$, and $\ddot{A}$ in the α-equation with the terms $\frac{\partial N}{\partial n_1}$, $\frac{\partial N}{\partial n_2}$, and $\frac{\partial N}{\partial \dot{A}}$ in $\dot{N}$ respectively. The remainder of the $\dot{N} = \alpha$ equation reads

$$\alpha = \dot{N} \Leftrightarrow b\left(\frac{C}{4R^2} + \frac{AR}{6(3\lambda-1)}\right) = -\frac{1}{2}\left(\frac{\partial B}{\partial m}\dot{m} + \frac{\partial B}{\partial A}\dot{A} + \frac{\partial B}{\partial C}\dot{C}\right), \tag{4.55}$$

which is immediately satisfied once eq. (4.52) is substituted into the expanded right-hand side.

Although we have solved all constraints and equations of motion, two quantities remain undetermined. These are the canonical coordinate R as a function of (t,r), and the Lagrange multiplier $\beta(t,r)$ associated with the radial momentum constraint $\mathscr{H}_r \approx 0$. Recall that β is a Lagrange multiplier associated with spatial diffeomorphism symmetry in the radial direction. Since the spherically symmetric ansatz we have been using does not fix this symmetry,

[4]As we will see later, for $A \neq 0$ this combination vanishes in the $r \to \pm\infty$ limit, as a result of which the hypersurface Σ becomes asymptotically null.

β has remained arbitrary up to this point. This coordinate freedom can be used to fix R as a function of (t,r), which in turn fixes β. In order to make this idea concrete, we start with a gauge-fixing condition on the shift, written as $\xi - \xi_{gf} \approx 0$. Demanding that this choice is preserved in time leads us to an equation for β, namely,

$$\frac{d}{dt}\left(\xi - \xi_{gf}\right) = \beta - \dot{\xi}_{gf} = \beta - \frac{\partial \xi_{gf}}{\partial t} - \left\{\xi_{gf}, H_{tot}\right\} \approx 0\,, \tag{4.56}$$

regardless of the functional form of ξ_{gf}. Given that we have determined the radial shift when dealing with the equation of motion for R, it is necessary to ensure that any expression for ξ_{gf} is compatible with eq. (4.44) under the substitution $\xi \to \xi_{gf}$. In fact, our gauge choice for ξ is directly inspired by eq. (4.44) and reads

$$\xi_{gf} := pN_{sol}\left(\frac{C}{4R^2} + \frac{AR}{6\,(3\lambda-1)}\right), \tag{4.57}$$

where p is a real number that will be chosen separately for $r > 0$ and $r < 0$. As will become clear below, p is an unphysical parameter introduced for mere convenience. On the constraint surface, ξ_{gf} can be written equivalently as

$$\xi_{gf} = \frac{pN_{sol}}{2\,(3\lambda-1)}\left(\lambda\,\frac{\pi_\mu}{R} + \frac{1-\lambda}{2}\,\frac{\pi_R}{\mu}\right). \tag{4.58}$$

Substituting expression (4.58) into eq. (4.56), the latter becomes

$$\beta \approx \frac{\alpha\,\xi_{gf}}{N_{sol}} + \frac{pN_{sol}}{2\,(3\lambda-1)}\left\{\left(\lambda\,\frac{\pi_\mu}{R} + \frac{1-\lambda}{2}\,\frac{\pi_R}{\mu}\right), H_{tot}\right\}. \tag{4.59}$$

Note that the only contribution to $\frac{\partial \xi_{gf}}{\partial t}$ comes from $\frac{\partial N_{sol}}{\partial t}$. Combined with $\{N_{sol}, H_{tot}\}$, this yields the α-dependent term on the right-hand side of eq. (4.59). Computing the Poisson bracket term in eq. (4.59), this equation becomes

$$\beta \approx \frac{\alpha\,\xi_{gf}}{N_{sol}} + \frac{N_{sol}^2}{2\,(3\lambda-1)}\left(\frac{A^2R}{18\,(3\lambda-1)} - \frac{3\lambda-1}{4}\frac{C^2}{R^5} - \frac{AC}{12R^2}\right)\left(pR'-1\right) \tag{4.60a}$$

$$+\,pN_{sol}\left(\frac{\dot{A}R}{6\,(3\lambda-1)} - \frac{n_2}{R^2}\right). \tag{4.60b}$$

The remaining step consists in computing $\dot{\xi}_{gf}$ and substituting it into eq. (4.56) together with the expression just obtained for β, obtaining

$$\beta - \dot{\xi}_{gf} \approx \frac{N_{sol}^2}{2\,(3\lambda-1)}\left(\frac{A^2R}{18\,(3\lambda-1)} - \frac{3\lambda-1}{4}\frac{C^2}{R^5} - \frac{AC}{12R^2}\right)\left(pR'-1\right) \tag{4.61a}$$

$$-\,pN_{sol}\dot{R}\left(\frac{A}{6\,(3\lambda-1)} - \frac{C}{2R^3}\right) \approx 0\,, \tag{4.61b}$$

which is solved by

$$\dot{R} = 0\,, \qquad R' = \frac{1}{p}\,. \tag{4.62}$$

These are precisely the conditions obtained from demanding consistency between eqs. (4.57) and (4.44), that is, between our gauge choice for the radial shift and the condition obtained when solving the equation of motion for R.

In the remainder of this chapter, we will set $\dot{R}=0$ but not fix R as function of the coordinate r. Due to eq. (4.62), we could pick a value for p, determine $R(r)$ and use it throughout. However, we wish to emphasise the validity of our results for general $R(r)$. We will make an exception to this when discussing the boundary conditions of the model in subsection 4.3.4, where we will set $R=|r|$, that is, choose $p=1$ for $r>0$ and $p=-1$ for $r<0$. This is motivated by our wish to have the same spacetime for both positive and negative r. As can be seen from the definition of B in terms of R in eq. (4.26), this requires R to be even with respect to the inversion $r\to -r$. Moreover, this choice is needed to force the vanishing of the integrand in the solution of the lapse-fixing equation given in eq. (4.31) for $r\to-\infty$. By choosing R the way we did and $r_0=\infty$, we see that n_1 determines the behaviour of the lapse at both spatial infinities as mentioned previously.

Note that setting $\dot{R}=0$ does not remove all time dependence from the metric and thus does not imply a static solution. This would only be true if all of $\dot{A}$, $\dot{C}$, $\dot{n}_1$, and $\dot{m}$ vanished as well, and would imply a considerable restriction on the space of solutions. However, we can still use the conditions expressed in eq. (4.52) to define a quantity M that is conserved, $\dot{M}=0$, and in such a way that B contains a term of the form $1-\frac{2M}{R}$. For the general relativistic case in which $\lambda=1$, this is achieved in a straightforward manner by noting that eq. (4.52) simplifies to

$$8\dot{m}=-\frac{1}{24}\left(\dot{C}A+C\dot{A}\right), \tag{4.63}$$

implying that we can define a quantity M that satisfies $\dot{M}=0$ by simply writing

$$2M:=8m+\frac{CA}{24}. \tag{4.64}$$

For the general case $\lambda\neq 1$, we define the conserved quantity M by

$$2M:=8m+\frac{CA}{12\,(3\lambda-1)}+\frac{\lambda-1}{8\,(3\lambda-1)}\int_{-\infty}^{t}d\tilde{t}\,C\dot{A}, \tag{4.65}$$

where we have set the lower integration limit to $-\infty$ in order to have $\dot{M}(t)=0$ for all times t. For the integral in eq. (4.65) to exist and be finite, we must demand in addition that the functions $A(t)$ and $C(t)$ are such that $C\dot{A}$ goes to zero faster than $1/t$ in the limit $t\to-\infty$. In the following, we assume this to be the case. We can now write the function $B(R)$ as

$$B=1-\frac{2M}{R}+\frac{1}{3\lambda-1}\left(\frac{CA}{12}+\frac{\lambda-1}{8}\int_{-\infty}^{t}d\tilde{t}\,C\dot{A}\right)\frac{1}{R}+\frac{C^2}{16R^4}+\frac{A^2R^2}{72\,(3\lambda-1)}. \tag{4.66}$$

Before discussing the properties of the solutions obtained, let us finally substitute $n_2=-\dot{C}/4$ into the lapse function (4.31) and set $r_0=+\infty$, obtaining

$$N_{sol}=\sqrt{B}\left(n_1+\frac{1}{4}\int_{r}^{\infty}d\tilde{r}\,\frac{R'}{B^{3/2}}\left(\frac{\dot{C}}{R^2}+\frac{\dot{A}R}{3}\right)\right). \tag{4.67}$$

Inspecting (4.67), we reconfirm that the function $n_1(t)$ determines the behaviour of the lapse function at radial infinity, as stated earlier below eq. (4.31).

In the next section, we address the geometric properties of the solutions we have obtained. Since we want to disentangle the genuine λ-dependence from the one induced by the Legendre transformation, we begin by computing the extrinsic curvature of the constant-time slices. This will lead us to a geometric interpretation of the functions $A(t)$ and $C(t)$ introduced earlier. Following this, we discuss the boundary and fall-off conditions that must be imposed on the fields, and determine the boundary Hamiltonian introduced earlier. Implementing these steps enables us to write the four-dimensional metrics corresponding to the λ-R solutions in a form where they can be compared explicitly to their general relativistic counterparts. Finally, we compute the four-dimensional scalar curvature ${}^{(4)}R$ of the λ-R model and find it to be nonvanishing and proportional to $(\lambda - 1)$, provided that the trace of the extrinsic curvature does not vanish, a situation which is very different from the one found in standard gravity.

4.3.3 Curvature variables

We begin by re-expressing the extrinsic curvatures of eqs. (4.7) in terms of the parameters of the reduced phase space,

$$K_{rr} = \mu^2 \left(\frac{C}{2R^3} - \frac{A}{6(3\lambda - 1)} \right), \tag{4.68a}$$

$$K_{\theta\theta} = \frac{K_{\phi\phi}}{\sin^2\theta} = -\frac{AR^2}{6(3\lambda - 1)} - \frac{C}{4R}. \tag{4.68b}$$

Using this result, we can compute the so-called mean curvature, that is, the trace $K = g^{ij}K_{ij}$ of the extrinsic curvature tensor, in a straightforward manner. Up to a λ-dependent prefactor, it turns out to be equal to the integration constant $A(t)$ first introduced in eq. (4.19) above,

$$K = -\frac{A}{2(3\lambda - 1)} \;\Rightarrow\; A = -2(3\lambda - 1)K. \tag{4.69}$$

This verifies our earlier claim that the spatial constant $A(t)$ is proportional to the mean curvature of the slices of the preferred foliation. Using eq. (4.68), we can now also justify our previous assertion that C measures the transverse-traceless components of the extrinsic curvature K_{ij}. Defining the traceless extrinsic curvature tensor K^T_{ij} by

$$K^T_{ij} := K_{ij} - \frac{1}{3} g_{ij} K, \tag{4.70}$$

the principal curvatures ${K^T}_i{}^i$ – the coordinate-independent eigenvalues of the Weingarten map – are found to be

$${K^T}_r{}^r = \frac{C}{2R^3}, \qquad {K^T}_\theta{}^\theta = {K^T}_\phi{}^\phi = -\frac{C}{4R^3}. \tag{4.71}$$

Since they only depend on C and R, it follows that C carries all the transverse-traceless information of the extrinsic curvature, as we stated earlier.

Since we want to compare our results with the general constant mean curvature foliations of the Schwarzschild geometry, we now introduce the same variables as in [78], replacing A by K everywhere. This leads to the following expressions for B^{-1}, N, and ξ,

$$\frac{\mu^2}{(R')^2} = \frac{1}{B} = \left(1 - \frac{2M}{R} + \left(\frac{KR}{3} - \frac{C}{4R^2}\right)^2 + (\lambda - 1)\left(\frac{K^2R^2}{6} - \frac{1}{4R}\int_{-\infty}^{t} dt'\, C\dot{K}\right)\right)^{-1}, \tag{4.72a}$$

$$N = \sqrt{B}\left(n_1 + \frac{1}{4}\int_r^\infty d\tilde{r}\, \frac{R(\tilde{r})'}{B^{3/2}}\left(\frac{\dot{C}}{R(\tilde{r})^2} - \frac{4}{3}\dot{K}R(\tilde{r}) - 2(\lambda-1)\dot{K}R(\tilde{r})\right)\right), \tag{4.72b}$$

$$\xi = \frac{N}{R'}\left(\frac{C}{4R^2} - \frac{KR}{3}\right). \tag{4.72c}$$

It can be shown that the equations of motion of the *full* λ-R model as derived in chapter 2 are also satisfied by our solutions. To perform this check, we choose the three-metric g_{ij} as given in eq. (4.1), with μ and R given by eqs. (4.72a) and (4.62) (with $p = 1$) respectively, while lapse N and radial shift ξ are given by eqs. (4.72b) and (4.72c). Then, we compute the momentum tensor π^{ij} from the extrinsic curvatures obtained in eqs. (4.68) and substitute in these quantities in all constraints and time evolution equations to show that they are all satisfied.

4.3.4 Fall-off conditions and boundary Hamiltonian

As previously mentioned in Sec. 4.3 and outlined in Sec. 2.3, the total Hamiltonian H_{tot} must include a boundary term term $H_{\partial\Sigma}$ to make the variational principle well defined, in the sense that its variation δH can be written without any boundary contributions,

$$\delta H = \int d^3x\, \left(A^{ij}\,\delta g_{ij} + B_{ij}\,\delta\pi^{ij}\right). \tag{4.73}$$

This means that the equations of motion

$$\dot{g}_{ij} = \frac{\delta H}{\delta\pi^{ij}} := B_{ij}\,, \qquad \dot{\pi}^{ij} = -\frac{\delta H}{\delta g_{ij}} := A^{ij}\,, \tag{4.74}$$

follow from it in a unique manner [86] (see also the related discussion in [66]). In our reduced setting, eq. (4.73) becomes

$$\delta H = \int_{-\infty}^{+\infty} dr\, \left(A_\mu\,\delta\mu + A_R\,\delta R + B_\mu\,\delta\pi_\mu + B_R\,\delta\pi_R\right). \tag{4.75}$$

A straightforward variation of eq. (4.12) does not yield an equation of this form, because the Hamiltonian contains spatial derivatives of some of the phase space variables. They require a partial integration when computing their variation and thereby introducing boundary contributions. To address this issue, we shall collect all boundary contributions generated in this way, impose the coordinate condition $R = |r|$ motivated in subsection 4.3.2 above, substitute the solutions of the local equations into the same boundary contributions, and finally determine the boundary Hamiltonian whose variation cancels these unwanted terms.

It is important to note that to obtain the Hamiltonian of eq. (4.12), we have performed a partial integration to make it linear in the radial shift ξ, namely,

$$\int_{-\infty}^{+\infty} dr\, \pi_\mu \,(\mu\xi)' = -\int_{-\infty}^{+\infty} dr\, \mu\, \xi\, \pi_\mu' + \mu\, \xi\, \pi_\mu \big|_{-\infty}^{+\infty}, \tag{4.76}$$

In other words, a boundary Hamiltonian had to be present from the outset to cancel the variations of the second term on the right-hand side of eq. (4.76). In addition, we find a boundary contribution

$$\xi\,(\pi_R\, \delta R - \mu\, \delta\pi_\mu)\big|_{-\infty}^{+\infty} \tag{4.77}$$

from the variation of the shift-dependent term in (4.12) and a contribution

$$4\left(\frac{NR}{\mu}\,\delta\,(R') - \frac{N'R}{\mu}\,\delta R - \frac{NRR'}{\mu^2}\,\delta\mu\right)\Bigg|_{-\infty}^{+\infty} \tag{4.78}$$

from varying the lapse-dependent term. Adding equations (4.76) and (4.77), we see that the shift-dependent boundary variation is given by

$$\left(\,\xi\pi_R\,\delta R + \pi_\mu\,\delta(\xi\mu)\right)\big|_{-\infty}^{+\infty}. \tag{4.79}$$

Let us now implement the previously motivated gauge fixing $R = |r|$. It implies $R' = -1$ for $r < 0$ and $R' = 1$ for $r > 0$. In line with our comments below eq. (4.33), this means that we choose the sign of the square root for μ differently in both regimes. For $r > 0$ we have $\mu = R'/\sqrt{B}$, while for $r < 0$ we must use $\mu = -R'/\sqrt{B}$. For all $r \in \mathbb{R}$, this leads to $\mu = B^{-1/2}$, ensuring μ is well defined as a function of R, provided that for $K \neq 0$ we restrict ourselves to $\lambda > 1/3$.

With this choice both δR and $\delta R'$ vanish. We can now substitute our solutions for μ, R, N, ξ, and π_μ into the equations, obtaining

$$\begin{aligned} 4\,NRR'\,\delta\frac{1}{\mu}\Bigg|_{-\infty}^{+\infty} &= 2\,n_1\,|r|\,|r|'\,\delta\left(1 - \frac{8m}{|r|} + \frac{C^2}{16r^4} + \frac{3\lambda-1}{18}K^2r^2\right)\Bigg|_{-\infty}^{+\infty} \\ &= 2\lim_{r\to\infty} n_1\left(\frac{2}{9}(3\lambda-1)\,|r|^3\,K\,\delta K - 16\,\delta m\right) \end{aligned} \tag{4.80}$$

for eq. (4.78), while expression (4.79) yields

$$\begin{aligned} \pi_\mu\,\delta(\xi\mu)\big|_{-\infty}^{+\infty} &= \left(\frac{C}{|r|} - \frac{2}{3}(3\lambda-1)\,Kr^2\right)\frac{1}{|r|\,'}\,\delta\left[n_1\left(\frac{C}{4r^2} - \frac{K\,|r|}{3}\right)\right]\Bigg|_{-\infty}^{+\infty} \\ &= 2\lim_{r\to\infty}\Bigg\{\delta n_1\left(\frac{2}{9}(3\lambda-1)\,K^2\,|r|^3 - \frac{3\lambda+1}{6}CK\right) \\ &\quad + n_1\left(\frac{2}{9}(3\lambda-1)\,|r|^3\,K\,\delta K - \frac{\delta\,(KC)}{3} + \frac{1-\lambda}{2}K\delta C\right)\Bigg\}. \end{aligned} \tag{4.81}$$

Let us focus first on the variation of m in the last term of equation (4.80),

$$-\,32\lim_{r\to\infty} n_1\,\delta m = -32\,n_1\,\delta m, \tag{4.82}$$

which depends only on time because both m and n_1 are spatially constant. To remove this boundary variation from the variational principle, one could in principle demand that $n_1(t) = 0$. This is not an acceptable choice, since it would imply that the lapse vanishes at spatial infinity and that no time evolution takes place there. As discussed in appendix B, this is not physically acceptable and is inconsistent from a geometrical point of view. Alternatively, we can include a term $32 n_1 m$ in the boundary Hamiltonian $H_{\partial\Sigma}$. Taking the variation of the boundary Hamiltonian then leads to a cancellation of the δm-term in equation (4.80). However, this introduces a term proportional to δn_1 which should vanish,

$$32 m\, \delta n_1 = 0, \tag{4.83}$$

since it is not cancelled by any term coming from the variation of the Hamiltonian. There are two ways of removing this variation. The first consists in setting $m=0$. In the asymptotically flat case, this would imply $M=0$. This condition appears too restrictive, since it would not even allow for the standard Schwarzschild solution to be recovered. We are left with one last possible way to satisfy eq. (4.83). Following the line of reasoning presented in [66], it consists in assuming that n_1 is a prescribed function at radial infinity (and thus everywhere), which we therefore do not vary.

Adopting this prescription and setting $\delta n_1 = 0$, we can add the remaining nonvanishing variations from expressions (4.80) and (4.81), leading to

$$2 \lim_{r\to\infty} n_1 \left(\frac{2}{9}(3\lambda-1)\,|r|^3\,\delta\left(K^2\right) - \frac{\delta\,(KC)}{3} + \frac{1-\lambda}{2} K\delta C \right). \tag{4.84}$$

Allowing for arbitrary variations of m, K, and C, it is not possible to write down a boundary Hamiltonian whose variation cancels all terms in eq. (4.84). The main obstruction comes from the term proportional to $(1-\lambda)$, which cannot be written as a total variation. A second issue is that the boundary term necessary to cancel the term proportional to $\delta(K^2)$ in eq. (4.84) is manifestly divergent. Both of these issues are resolved by setting δK to zero at the radial infinities,

$$\delta K\,|_{|r|\to\infty} = 0\,, \tag{4.85}$$

and therefore everywhere. Together with the condition $\delta n_1|_{|r|\to\infty}=0$ this implies

$$\delta N|_{|r|\to\infty} = 0, \tag{4.86}$$

as can be seen by inspecting eqs. (4.66) and (4.67).

Taking all of these considerations into account, we arrive at a finite expression for the boundary Hamiltonian,

$$H_{\partial\Sigma} = n_1 \left(32 m - \frac{3\lambda-1}{3} KC \right), \tag{4.87}$$

accompanied by the conditions

$$\delta K = 0\,, \qquad \delta n_1 = 0\,. \tag{4.88}$$

Note that expression (4.87) coincides with the expressions given in reference [66] for the general relativistic case upon setting $\lambda=1$ and $K=C=0$.

We have so far not mentioned a subtlety of this treatment, which comes from the fact that the coordinate system used is ill-defined for $r = 0$. Because of this special point, we have been working implicitly with two distinct coordinate patches for every spatial hypersurface, defined by $r > 0$ and $r < 0$. However, there is no reason why the integration constants chosen for both patches should be the same. To achieve full generality, the set of constants should be twice as large. For instance, m should be replaced by m_+ for $r > 0$, and m_- for $r < 0$. Doubling all constants in this manner leads to a boundary Hamiltonian of the form

$$H_{\delta\Sigma} = \lim_{r\to+\infty} n_{1+}\left[16m_+ - \frac{3\lambda-1}{6}K_+C_+\right] + \lim_{r\to-\infty} n_{1-}\left[16m_- - \frac{3\lambda-1}{6}K_-C_-\right], \tag{4.89}$$

with conditions (4.88) replaced by

$$\delta K_\pm = 0\,, \qquad \delta n_{1\pm} = 0\,. \tag{4.90}$$

In the remainder of the text, we will refrain from distinguishing between integration constants for the different charts, although it should be understood that there is in principle one set of distinct constants for each.

Finally, note that while g_{rr} vanishes as $r \to \pm\infty$, the vector ∂_t does not become null in this limit, as can be seen by computing $g_{00} \equiv N^iN_i - N^2$,

$$\lim_{r\to\pm\infty}\left(N^iN_i - N^2\right) = \lim_{r\to\pm\infty}\left(\mu^2\xi^2 - N^2\right) = \frac{n_1^2A^2r^2}{24\,(3\lambda-1)^2}\,(1-\lambda)\,. \tag{4.91}$$

This implies that ∂_t is timelike for $\lambda > 1$ and spacelike for $\lambda < 1$.[5] As pointed out in appendix B, the fact that the vector ∂_t associated with the time coordinate t can become spacelike when the shift is large is related to the choice of foliation. This feature is familiar from general relativity, as illustrated by the Painlevé-Gullstrand representation of the Schwarzschild metric inside the event horizon. It nevertheless illustrates how different values of the parameter λ can affect aspects of the foliation structure. While the time vector ∂_t can cease to be timelike, the normal evolution vector $\vec{m} = \vec{n}N$, with $\vec{n}$ the unit normal to the hypersurface, will of course remain timelike whenever the hypersurface is spacelike (or null, when the hypersurface is null).

4.4 Four-dimensional spacetime reconstruction

In this section, we focus on a four-dimensional point of view, which is particularly suited for comparisons with general relativity. First, we reconstruct the four-dimensional metric and compare it to its general relativistic counterpart, further discussing the role played by the parameters introduced while solving the constraint algebra and time evolution equations. Secondly, we use the Ricci and Gauss equations discussed in appendix B to obtain the four-dimensional Ricci scalar curvature, which we will show to be non-vanishing and λ-dependent.

[5]The case $\lambda = 1$ must be considered separately; the leading term on the right-hand side of eq. (4.91) in this case is of order r^0 and negative, implying a timelike vector ∂_t.

4.4.1 Four-dimensional metric

Using the expressions (4.72) for μ^2, N, and ξ, we can write the ${}^{(4)}g_{0\mu}$-components of the four-dimensional metric of the solutions of the λ-R model as

$$\begin{aligned} {}^{(4)}g_{00} &= -N^2 + \mu^2\xi^2 = -\frac{N^2}{B}\left(B - \left(\frac{C}{4R^2} - \frac{KR}{3}\right)^2\right) \\ &= -\frac{N^2}{B}\left(1 - \frac{2M}{R} + (\lambda - 1)\left(\frac{K^2R^2}{6} - \frac{1}{4R}\int_{-\infty}^{t} dt' C\dot{K}\right)\right), \qquad (4.92a) \\ {}^{(4)}g_{0r} &= \mu^2\xi = \frac{R'N}{B}\left(\frac{C}{4R^2} - \frac{KR}{3}\right) \\ &= \frac{R'}{\sqrt{B}}\left(\frac{C}{4R^2} - \frac{KR}{3}\right)\left(n_1 + \frac{1}{4}\int_r^\infty d\tilde{r}\,\frac{R(\tilde{r})'}{B^{3/2}}\left(\frac{\dot{C}}{R(\tilde{r})^2} - \frac{4}{3}\dot{K}R(\tilde{r}) - 2(\lambda - 1)\dot{K}R(\tilde{r})\right)\right), \qquad (4.92b) \end{aligned}$$

where the quotient $\frac{N^2}{B}$ in eq. (4.92a) is given by

$$\frac{N^2}{B} = n_1 + \frac{1}{4}\int_r^\infty d\tilde{r}\,\frac{R(\tilde{r})'}{B^{3/2}}\left(\frac{\dot{C}}{R(\tilde{r})^2} - \frac{4}{3}\dot{K}R(\tilde{r}) - 2(\lambda - 1)\dot{K}R(\tilde{r})\right). \qquad (4.93)$$

The ${}^{(4)}g_{rr}$-entry of the metric is given by $\mu^2 = (R')^2/B$, which was given in eq. (4.72a). From there, one can straightforwardly read off that ${}^{(4)}g_{rr}$ goes to zero as $|r| \to \infty$, implying that the hypersurfaces of constant time become asymptotically null in this limit.

For the inverse metric, we find

$$\begin{aligned} {}^{(4)}g^{00} &= -\frac{1}{N^2}, \qquad g^{0r} = \frac{\xi}{N^2} = \frac{1}{R'N}\left(\frac{C}{4R^2} - \frac{KR}{3}\right), \qquad (4.94a) \\ {}^{(4)}g^{rr} &= \frac{1}{\mu^2} - \frac{\xi^2}{N^2} = \frac{1}{(R')^2}\left(1 - \frac{2M}{R} + (\lambda - 1)\left(\frac{K^2R^2}{6} - \frac{1}{4R}\int_{-\infty}^{t} dt' C\dot{K}\right)\right). \qquad (4.94b) \end{aligned}$$

Summarising, the four-dimensional metric ${}^{(4)}g_{\mu\nu}$ and its inverse ${}^{(4)}g^{\mu\nu}$ are given by

$${}^{(4)}g_{\mu\nu} = \begin{pmatrix} -\frac{N^2}{B}\left(B - \left(\frac{C}{4R^2} - \frac{KR}{3}\right)^2\right) & \frac{R'N}{B}\left(\frac{C}{4R^2} - \frac{KR}{3}\right) & 0 & 0 \\ \frac{R'N}{B}\left(\frac{C}{4R^2} - \frac{KR}{3}\right) & \frac{(R')^2}{B} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{pmatrix}, \qquad (4.95a)$$

$${}^{(4)}g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{1}{R'N}\left(\frac{C}{4R^2} - \frac{KR}{3}\right) & 0 & 0 \\ \frac{1}{N}\left(\frac{C}{4R^2} - \frac{KR}{3}\right) & \frac{1}{(R')^2}\left(B - \left(\frac{C}{4R^2} - \frac{KR}{3}\right)^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}. \qquad (4.95b)$$

Comparison with constant mean curvature description of the Schwarzschild solution

Now that we have written the four-dimensional metric, we can compare our λ-dependent spacetimes with the standard $\lambda = 1$ constant mean curvature description of the Schwarzschild spacetime given in reference [78][6]. The first constant mean curvature formulation of the Schwarzschild spacetime dates back to 1980 [19], but it is still a topic of ongoing research as illustrated by references [67, 68, 93].

By using K and isolating the λ-dependence into terms proportional to $(\lambda - 1)$ in expressions (4.92a), (4.92b), (4.93), and (4.94b), we have made explicit how the spacetime metric $g_{\mu\nu}$ differs from its general relativistic counterpart. As we will show below in subsection 4.4.2, these extra contributions lead to a nonvanishing four-dimensional curvature for $K \neq 0$ and $\lambda \neq 1$.

The four-dimensional metric we have derived depends on five parameters, two constants (λ, M) and three functions of time (C, K, n_1). Let us discuss their role and interpretation in turn. The coupling constant λ only occurs in the prefactors $(\lambda - 1)$ of terms that do not appear in the Schwarzschild solution.

The constant M was defined in eq. (4.65) from the integration constant $m(t)$, which was in turn introduced earlier when solving the Hamiltonian constraint. This was done to have a genuinely conserved quantity that reduces to a constant multiple of the Schwarzschild mass M_s for $\lambda = 1$. It can be checked that in this latter case we have $M = M_s/4$ with our choice of units.

When $\lambda = 1$, neither C nor K play a direct physical role. However, they determine the range of R for which the function $B(R)$ is positive, which in turn determines the spacetime covered by the slices of the foliation. More concretely, as was shown in [76, 77], for $\dot{K} = 0$ and later in [78] for $\dot{K} \neq 0$, the number and location of the roots of B depends on the value of both parameters. Keeping $K > 0$ fixed, there are three possibilities. If $C = 0$, B has only one root, the minimal radius, in keeping with the conventions of references [76–78]. In this case, the foliation extends from null infinity to this minimal radius and re-emerges on the other side of the “throat", continuing from there all the way to the other null infinity. For small $C > 0$, there are two roots and two regions for which B is positive. One is in the interior black hole region of the Kruskal diagram, extending from the singularity $R = 0$ to some maximal radius and then returning to the singularity, another retains the $C = 0$ behaviour. Lastly, if C is large enough, there is a critical point for which the two roots coincide. Beyond it, the leaves of the foliation start at either of the null infinities and end again in the singularity. We expect a qualitatively similar behaviour in our solutions, certainly for small deviations from the general relativistic case, although the roots of B will of course become λ-dependent.

Regarding the role played by C and K in the λ-R model, recall that the former is obtained when solving the radial momentum constraint and is therefore associated with radial diffeomorphisms, while the latter is associated with the second-class tertiary constraint $\omega \approx 0$ and parametrises the implementation of the constant mean curvature condition. As we have seen, physical differences between the λ-R model and general relativity come from differences in the Hamiltonian constraint part of the constraint algebra. Because the λ-R model

[6]There is a minor discrepancy between our result in eq. (4.94b) for g^{rr} (regardless of the value of λ) and that of [78], presumably because the authors used g^{ij} instead of the correct four-dimensional inverse ${}^{(4)}g^{ij} = g^{ij} - \frac{N^i N^j}{N^2}$.

is still invariant under spatial diffeomorphisms, the symmetry associated with the momentum constraints, one would suppose that C is not a physical quantity while K is. However, the argument turns out to be more involved. Unlike what happens in general relativity, the lapse function N is not determined by making a gauge choice but by solving the quaternary constraint $\mathscr{M} \approx 0$, and necessarily depends on both C and n_1. We will show in the next subsection that K is a physical quantity, in the sense that the four-dimensional scalar curvature – a scalar under local diffeomorphisms – depends on it. However, a similar logic applies to C and n_1, by virtue of their appearance in the lapse function: changing either C or n_1 while keeping all other parameters fixed will alter the lapse and consequently yield a different four-dimensional Ricci scalar. On the other hand, K, C, and n_1 have the same geometric interpretation they had in general relativity, namely, as the trace of the extrinsic curvature, the transverse-traceless part of the extrinsic curvature, and the leading-order behaviour of the lapse at infinity when $K \neq 0$ and $\lambda \neq 1$.

4.4.2 Spacetime curvature

One way to obtain the four-dimensional Ricci scalar ${}^{(4)}R$ is to start from the explicit expression (4.95a) of the four-metric and perform a full, four-dimensional calculation. Instead, we will use an expression for ${}^{(4)}R$ in terms of three-dimensional quantities. Following [42], this is derived in appendix B by combining the contracted Ricci and Gauss equations[7]. It reads

$$ {}^{(4)}R = \mathscr{R} + K^2 + K_{ij}K^{ij} + \frac{2}{N}\mathscr{L}_{N\vec{n}}K - \frac{2}{N}g^{ij}\nabla_i\nabla_j N \,, \tag{4.96} $$

where $\mathscr{L}_{N\vec{n}}$ is the Lie derivative along the normal evolution vector $N\vec{n}$, and $\vec{n}$ the unit normal to the hypersurface Σ,

$$ \vec{n} = N^{-1}\left(1\,, -N^i\right). \tag{4.97} $$

We first substitute the solutions obtained for the phase space variables into the expression (4.6) for the scalar three-curvature $\mathscr{R}$, resulting in

$$ \mathscr{R} = \frac{2}{R^2}\left(1 - B - R\,\frac{\partial B}{\partial R}\right). \tag{4.98} $$

The term with the Lie derivative is given by

$$ \frac{2}{N}\mathscr{L}_{N\vec{n}}K = \frac{2\dot{K}}{N}\,, \tag{4.99} $$

while the $K_{ij}K^{ij}$-term can be obtained in a straightforward way from eqs. (4.68),

$$ K_{ij}K^{ij} = \frac{3}{8}\frac{C^2}{R^6} + \frac{K^2}{3}\,. \tag{4.100} $$

To determine the last term in eq. (4.96), we recall the form of the lapse N given in eq. (4.72b) as a function of $\dot{K}$ and R, and compute its Laplacian as

$$ -\frac{2}{N}\nabla_i\nabla^i N = -\frac{3\lambda - 1}{N}\dot{K} - \left(\frac{\partial^2 B}{\partial R^2} + \frac{2}{R}\frac{\partial B}{\partial R}\right). \tag{4.101} $$

[7]Note that the sign of the term linear in K on the right-hand side of (4.96) is opposite to that given by Gourgoulhon [42]. This happens because his definition of the extrinsic curvature has the opposite sign to ours.

Combining all contributions finally yields the four-dimensional scalar curvature

$$\begin{aligned} {}^{(4)}R &= -3(\lambda-1)\frac{\dot{K}}{N}+\frac{3}{8}\frac{C^2}{R^6}+\frac{4K^2}{3}+\frac{2}{R^2}\left(1-B-2R\frac{\partial B}{\partial R}-\frac{R^2}{2}\frac{\partial^2 B}{\partial R^2}\right) \\ &= (1-\lambda)\left(2K^2+\frac{3\dot{K}}{N}\right). \end{aligned} \quad (4.102)$$

This expression vanishes in the general relativistic case $\lambda = 1$, as it should, and also for vanishing mean curvature, $K = 0$. The latter is consistent with the fact that in the asymptotically flat case the λ-R model is equivalent to the maximal slicing description of general relativity. If neither $\lambda = 1$ nor $K = 0$, the scalar curvature is necessarily nonzero because the non-trivial radial dependence of the lapse ($N' \neq 0$) prevents a tuning of the initial data K and $\dot{K}$ such that ${}^{(4)}R$ vanishes.

Comparing with general relativity, the fact that vacuum solutions without a cosmological constant can have a non-vanishing scalar curvature is certainly a surprise.

4.5 Spherical symmetry in other Diff$_{\mathscr{F}}(M)$-invariant models

The results we obtained in this chapter were not the first to be derived for gravitational theories with a preferred foliation. To provide a context for the suggestions we will make in Sec. 6.2 of chapter 6, we briefly list some of these results. We refer to chapter 5 for terminology relating to Hořava-Lifshitz gravity.

The first set of spherically symmetric results in Hořava-Lifshitz gravity was obtained in [59] and [75] for a version of the theory that satisfies a softly broken detailed-balance condition by adding to the potential (5.20) a term proportional to $\mathscr{R}$. The difference between the two references is that the solutions of [59] were obtained perturbatively from quadratic fluctuations around the Minkowski vacuum. Common to both analyses was the ansatz

$$ds^2 = -N(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (4.103)$$

where $f(r)$ is some function of r, which plays the same role in the rr-component of the four-dimensional metric as $B(r)$ does in our solutions. This ansatz differs from ours in chapter 4 in assuming staticity, a vanishing shift and $R(r,t) = r$. Moreover, unlike ours, their analysis assumed asymptotic flatness. Other static solutions for the version with the softly broken detailed-balance condition can be found in [21], but without the assumption of a vanishing radial shift. The same ansatz was used in [63] for the non-projectable version of the theory without detailed balance and without derivatives of the lapse in the potential, and in [62], this time with derivatives of the lapse in the potential. Additional asymptotically flat and spherically symmetric solutions to non-projectable Hořava-Lifshitz gravity were obtained in [5,18], and studied in the context of the relationship between Hořava-Lifshitz gravity and Einstein-aether theory [90]. In these articles, the concept of universal horizon appeared for the first time (see [71] for a recent analysis). These are horizons from which no modes can escape, even if their dispersion relation allows them to travel at arbitrarily high speeds with respect to the aether. It would be interesting to see what happens to universal horizons for non-static solutions.

In [22], the motion of test particles in the spherically symmetric solutions of [59] was analysed and conditions for circular orbits obtained. These constitute a generalisation of earlier studies of particle orbits in the solutions of [59] that had already been performed in [43, 64, 85].

4.6 Summary

In this chapter, we have described the general solutions to the λ-R model when the spatial hypersurfaces associated with the preferred foliation possess spherical symmetry. As one could have predicted, there is no analogue of Birkhoff's theorem, since the solutions are in general non-flat, non-static, incompatible with asymptotic flatness, and parametrised not only by their conserved mass M, but also by the mean extrinsic curvature $K(t)$ of the leaves of the foliation, as well as by prescribed functions $C(t)$ and $n_1(t)$.

Our solutions have a generally nonvanishing radial shift ξ which we cannot remove by an allowed diffeomorphism. This happens because the model is only invariant under foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathscr{F}}(M)$. As a consequence and in agreement with our findings for the full λ-R model, only constant mean curvature and maximal slicings are permitted by the dynamics. Solving the (second-class) constraint algebra, imposing appropriate fall-off conditions and requiring that the time evolution equations are satisfied, we have derived the explicit functional form of the four-metric ${}^{(4)}g_{\mu\nu}$ of the spherically symmetric solution of λ-R gravity of the class considered, given in eqs. (4.95).

The λ-dependent constant mean curvature solutions ($K \neq 0$) are *not* physically equivalent to the ones with maximal slicing ($K = 0$). Moreover, only the latter correspond to vacuum solutions of general relativity, as follows from the nonvanishing of the four-dimensional Ricci scalar ${}^{(4)}R$ given in eq. (4.102) for the constant mean curvature case. The Ricci scalar is of course a local invariant, as it is in general relativity. The fact that the model predicts a nonvanishing value for $\lambda \neq 1$, even in the absence of matter, has to do with the fact that the λ-R model possesses a local invariant *not* present in general relativity, namely, the trace K of the extrinsic curvature of the distinguished foliation.

In general relativity, the constant mean curvature foliations of the Schwarzschild geometry can be obtained from the usual asymptotically flat metric description (with $K = 0$ and $C = 0$) by means of space-dependent time reparametrisations, as described in [76]. They are therefore equivalent to their asymptotically flat counterparts. While these diffeomorphisms generate nonvanishing values for K and C, they do not change the geometry of the spacetime, but only the way in which the $3+1$ split is implemented. In that case, one concludes that K and C can be thought of as unphysical gauge parameters. By contrast, these space-dependent time reparametrisations are precisely the diffeomorphisms that are absent from the symmetry group of the λ-R model. Therefore, spacetimes related by them will in general correspond to physically inequivalent solutions. For each $\lambda > 1/3$, $\lambda \neq 1$, the function $K(t)$ becomes effectively physical and parametrises physically distinct spacetimes, as is clear from the functional form of the scalar curvature (4.102) in terms of $K(t)$. In other words, the gauge orbit of general relativity parametrised by different values of K (different space-dependent time reparametrisations) becomes a one-function family of inequivalent spacetimes in the λ-R model.

Although the standard, general relativistic solution is included among those of the λ-R model (for initial data $K=0$), it is not unique, not even for $\lambda \neq 1$. Moreover, as a consequence of the preferred foliation, it can only be attained in a restricted set of coordinate charts. There are two regimes where the general relativistic solution is the only solution of the reduced spherically symmetric λ-R model we studied, namely, asymptotic flatness and $\lambda < 1/3$. The reason why $K = 0$ is the only acceptable solution in the $\lambda < 1/3$ case is that we defined the phase-space variable μ to be strictly positive. From eq. (4.25) and $R = |r|$, we see that $\mu = B^{-1/2}(|r|)$. In the limit $|r| \to \infty$, B becomes negative when $K \neq 0$ and $\lambda < 1/3$, which is not physically acceptable.

In physical theories, it is not always true that one can obtain the most general solutions satisfying certain isometries through the reduced action associated with those symmetries [84]. That is, a reduction procedure based on a set of isometries, such as the one we performed in this chapter, does not always lead one to the correct solutions of the full model that satisfy those isometries. While there are cases where these type of procedures have been shown to work, this does not cover the spherically symmetric sectors of general relativity and the λ-R model, because of their Lorentzian instead of Riemannian signature. It is therefore important that we have checked that our symmetric solutions obtained in the reduced case are indeed symmetric solutions of the full model, as mentioned earlier.

As we have mentioned earlier in this chapter, spherically symmetric solutions of Hořava-Lifshitz gravity have so far been obtained only in a static and asymptotically flat setting. One of the first papers with spherically symmetric solutions to Hořava-Lifshitz gravity [59], states that it is "not at all obvious" that for general values of λ, all spherically symmetric vacuum solutions are static. The solutions we obtained in this chapter are non-static for $\lambda \neq 1$ and therefore show explicitly that more general solutions exist.

Lastly, to avoid misunderstandings, some care should be taken when referring to solutions of general relativity in the context of $\text{Diff}_{\mathcal{F}}(M)$-invariant theories. One should take into account that a metric ${}^{(4)}g_{\mu\nu}$ which solves general relativity is understood as a representative of an equivalence class of four-dimensional metrics under $\text{Diff}(M)$, while for $\text{Diff}_{\mathcal{F}}(M)$-invariant theories the same metric can only be a representative of an equivalence class under $\text{Diff}_{\mathcal{F}}(M)$. For example, rather than talking generically about the status of "the Schwarzschild spacetime" in the λ-R model, a better and more concrete statement is to say that the Schwarzschild spacetime in $K = 0$ coordinates is a solution of the model, while the same spacetime in $K(t) \neq 0$ coordinates is not.

5

Some remarks on Hořava-Lifshitz gravity

In this chapter, we review some aspects of Hořava-Lifshitz gravity and highlight which assumptions and choices lead to the different versions of the theory. The λ-R model is the classical limit of the original non-projectable version of the theory, if we discard the assumption of detailed balance. However, the non-projectable version of the theory without detailed balance has since been extended to include in its potential also terms with spatial derivatives of the lapse. The λ-R model does not constitute the classical limit of this more general theory. Although the λ-R model that we have studied is incompatible with projectable Hořava-Lifshitz gravity, we will also discuss the classical limit of this version of the theory. This creates the context for the subsequent analysis of the results on the acceleration of the three-volume obtained by Kiefer and Giulini, which can be interpreted in terms of the projectable λ-R model.

This chapter is organised as follows. In Sec. 5.1, we review the general construction of Hořava-Lifshitz gravity, culminating in the actions defining both projectable and non-projectable versions of the theory in $3+1$ dimensions. Their classical limits are presented in Sec. 5.2. Readers already familiar with Hořava-Lifshitz gravity can skip directly to subsection 5.2.1, where we show that the classical limit of the projectable theory describes the results of Giulini and Kiefer in [40]. Finally, in Sec. 5.3, we compare the λ-R model with other classical models of Hořava-Lifshitz gravity.

5.1 A short review of Hořava-Lifshitz gravity

In a 2009 article [50], Hořava proposed a new theory of quantum gravity, which has since become known as Hořava-Lifshitz gravity. The goal behind the proposal was to establish a stand-alone, perturbatively renormalisable theory of quantum gravity in four dimensions, using both the field content of general relativity and standard quantum field theory methods. Hořava states a two-fold motivation for his original construction. Firstly, he argues that it may well be meaningful to study quantum gravity in a self-contained way, without invoking the larger context of string theory. This is of course the attitude taken all along in non-stringy approaches to quantum gravity. Secondly, tools and concepts developed in condensed matter physics are used to build the theory, a motivation which is further explained in his earlier article [49].

Since general relativity is not perturbatively renormalisable [48], to achieve the goal of Hořava-Lifshitz gravity one must abandon at least one of the standard assumptions of either perturbative quantum field theory or general relativity. The main hypothesis of Hořava-Lifshitz gravity is the existence of a Gaussian ultraviolet fixed point with an anisotropic scaling between space and time, implying the absence of local Lorentz symmetry in this regime. Specifically, unitarity and locality are kept intact while local Lorentz symmetry is broken at high energies.

As usual, we denote by x^i the coordinates on the spatial hypersurfaces Σ_t of constant time t. The anisotropic scaling relation at the fixed point is described by its action on coordinates,

$$t \to b^z t, \qquad x^i \to b x^i, \tag{5.1}$$

where b is a scaling parameter, and z the so-called dynamical critical exponent. A choice of z defines a specific model. Note that eq. (5.1) with $z \neq 1$ assumes the existence of a preferred foliation and is therefore not compatible with invariance under four-dimensional diffeomorphisms. Hořava-Lifshitz gravity is defined to be invariant under foliation-preserving diffeomorphisms with respect to a foliation $\mathscr{F}$ by leaves of constant time t, a group we have denoted by $\mathrm{Diff}_{\mathscr{F}}(M)$ throughout the text. Recall that its infinitesimal generators, previously introduced in eq. (2.11) of chapter 2, are given by

$$\delta t = f(t), \qquad \delta x^i = \zeta^i(t, x^k), \tag{5.2}$$

and their action on the ADM field variables, introduced in eq. (2.12), is

$$\delta g_{ij} = \zeta^k \partial_k g_{ij} + f \dot{g}_{ij} + \left(\partial_i \zeta^k\right) g_{jk} + \left(\partial_j \zeta^k\right) g_{ij}, \tag{5.3a}$$

$$\delta N_i = \left(\partial_i \zeta^j\right) N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + \dot{f} N_i + f \dot{N}_i, \tag{5.3b}$$

$$\delta N = \zeta^j \partial_j N + \dot{f} N + f \dot{N}, \tag{5.3c}$$

where the lapse N and shift N^i are those defined in appendix B.

In the original article by Hořava [50], the theory is designed as a field theory for the spatial metric g_{ij}. This contrasts with the four-dimensional point of view we have taken in this thesis, although in general relativity these two viewpoints are of course equivalent. By four-dimensional point of view, we mean that instead of considering the three-dimensional geometries as the fundamental quantities and studying their evolution in time, we assume that four-dimensional geometries are the fundamental objects. We use a $3+1$ decomposition merely because it simplifies the formalism for theories invariant under $\mathrm{Diff}_{\mathscr{F}}(M)$. Taking the four-dimensional geometry as fundamental, its expression in terms of the three-dimensional metric g_{ij}, lapse function N, and shift vector N^i follows from the geometric considerations outlined in appendix B. When we consider full four-dimensional diffeomorphism invariance, both stances are equivalent and lead to general relativity, as was shown in [47]. In this work, one starts with the spatial metric g_{ij}, performs an infinitesimal deformation of a hypersurface, and then shows that the lapse function and shift vector must be introduced to satisfy four-dimensional diffeomorphism invariance.

Following a similar logic, in an earlier article [49] Hořava considered a gravitational action constructed only from the spatial metric and its derivatives. He showed that such an action cannot be invariant under foliation-preserving diffeomorphisms, thus requiring the presence of analogues of lapse and shift. By "analogues", we mean that since these quantities are viewed as the gauge fields associated with the infinitesimal transformations generated by δt and δx^i of eq. (5.2), they are required to have the same space-time dependence as these generators. This does not have any consequences for the shift vector. However, a "lapse" that only depends on time is a restricted version of the usual lapse, which is the one defined in appendix B from purely geometrical considerations.

As a result, following the logic of [47] does not lead to the introduction of the remaining components of the four-dimensional metric, but only a restricted version of them. This implies that in Hořava-Lifshitz gravity the two viewpoints which we alluded to above are not equivalent and one must choose between them. If we want to match the field content of general relativity and keep the geometric origin of the lapse, we must set $N := N(t,x^i)$. This choice leads to the theory called non-projectable Hořava-Lifshitz gravity. On the other hand, if the lapse is chosen to be a projectable function[1] for the reasons outlined above, we obtain the so-called projectable Hořava-Lifshitz gravity (see [91] for a review).

Before we discuss further the different versions of the theory, it is convenient to present the scaling arguments that lead to a choice of the dynamical critical exponent z characterising the anisotropic scaling relation (5.1). To write down an action invariant under $\mathrm{Diff}_{\mathscr{F}}(M)$, we must establish the scaling dimensions of the fields. We determine first the scaling dimensions of space and time at the ultraviolet fixed point. Following reference [89], we use the symbol κ as a placeholder for an object with the scaling dimension of momentum. From the anisotropic scaling relation (5.1) at the ultraviolet fixed point, space and time at these energies should scale as

$$\left[dx^i\right] = [\kappa]^{-1}, \qquad [dt] = [\kappa]^{-z}, \tag{5.4}$$

which implies that c, the speed of light in vacuum, is not dimensionless unless $z = 1$. This is not in contradiction with our choice of $c = 1$ elsewhere in the thesis, where we deal with a classical field theory far away from the high-energy regime. It should be noted that z is not a variable of the theory, but a fixed parameter, and that each choice of z defines a different theory.

To appreciate the implications of eq. (5.4) for the spatial metric, lapse function, and shift vector, recall that before setting $c = 1$ the line element had the form

$$ds^2 = -N^2c^2dt^2 + g_{ij}\left(dx^i - N^i dt\right)\left(dx^j - N^j dt\right). \tag{5.5}$$

For eq. (5.5) to be consistent with the scaling (5.4), we must have $\left[N^i\right] = [\kappa]^{z-1}$. Further

[1]A quantity which takes constant values on each leaf of the foliation is called projectable.

imposing $[ds] = [dx^i]$, we are left with the following scaling dimensions for the metric, lapse, and shift,

$$[g_{ij}] = [1], \qquad [N] = [1], \qquad [N^i] = [\kappa]^{z-1}. \tag{5.6}$$

This implies that the spacetime volume element $dV = N\sqrt{g}\,dt\,d^3x$ scales as

$$[dV] = [\kappa]^{-(z+3)}, \tag{5.7}$$

while the extrinsic curvature tensor K_{ij} scales as

$$[K_{ij}] = [\kappa]^z. \tag{5.8}$$

We split the action S into a kinetic term S_K and a potential term $S_{\mathscr{V}}$,

$$S = S_K - S_{\mathscr{V}}. \tag{5.9}$$

The kinetic term S_K is defined as the most general scalar invariant under $\mathrm{Diff}_{\mathscr{F}}(M)$ with at most two time derivatives of the metric g_{ij}, no constant terms, and no spatial derivatives of g_{ij}. From an effective field theory perspective, the potential should consist of all $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant scalars constructed from spatial derivatives of the metric or containing no derivatives at all. In addition, it should include all terms whose scaling dimension is equal to or lower than that of the kinetic term.

The kinetic term is the same we have been using throughout, namely,

$$S_K = g_\kappa \int dt \int d^3x\,\sqrt{g}\,N\,K_{ij}G_\lambda^{ijkl}K_{kl}, \tag{5.10}$$

where g_κ is the overall coupling of the action, which should reduce to $g_\kappa = \frac{1}{16\pi G_N}$ in the infrared. In this context, the presence of the dimensionless coupling λ comes from the fact that both the $K_{ij}K^{ij}$- and the K^2-terms are separately invariant under $\mathrm{Diff}_{\mathscr{F}}(M)$. Recall that in general relativity only the combination $G^{ijkl}K_{ij}K_{kl}$ is invariant under the full diffeomorphism group $\mathrm{Diff}(M)$, implying $\lambda = 1$.

To obtain a perturbatively renormalisable theory of quantum gravity, at least from power-counting arguments, we want to have $[g_\kappa] = [1]$. From the previous scaling discussion, we can compute $[g_\kappa]$, obtaining

$$[g_\kappa] = [\kappa]^{3-z}. \tag{5.11}$$

Hořava-Lifshitz gravity is defined by the choice $z = 3$, which fixes the anisotropic relative scaling of space and time at the ultraviolet fixed point[2]. Following reference [89], the same argument can be repeated for any number d of spatial dimensions, from which one concludes that g_κ becomes dimensionless for $z = d$.

The general form of the potential term of the action is

$$S_{\mathscr{V}} = g_\kappa \int dt \int d^3x\,\sqrt{g}\,N\,\mathscr{V}, \tag{5.12}$$

[2]It is also possible to set $z > 3$, which would imply even higher-order spatial derivatives. Reference [50] contains a discussion of a $z = 4$ model in $3+1$ dimensions, but we will not consider this possibility any further.

where $\mathscr{V}$ is a functional of the fields of the theory and their spatial derivatives. We will specify its precise form below when we discuss the projectable and non-projectable versions of the theory. Applying eqs. (5.4), (5.7) and (5.11), and setting $z = 3$, we see that the scaling dimension of $\mathscr{V}$ is

$$[\mathscr{V}] = [\kappa]^6 . \tag{5.13}$$

Therefore, $\mathscr{V}$ should include terms of dimension $[\kappa]^n$, with $n = 0,2,4,6$. We exclude odd values of n because they violate parity. We therefore have the following independent sixth-order terms[3],

$$\mathscr{R}^3 , \qquad \mathscr{R}\mathscr{R}^i{}_j\mathscr{R}^j{}_i , \qquad \mathscr{R}^i{}_j\mathscr{R}^j{}_k\mathscr{R}^k{}_i , \qquad \mathscr{R}\nabla^2\mathscr{R} , \qquad \nabla_i\mathscr{R}_{jk}\nabla^i\mathscr{R}^{jk} , \tag{5.14}$$

where we have used the symbol $\mathscr{R}_{ij}$ to denote the three-dimensional Ricci tensor. In addition, there are four lower-order contributions,

$$\mathscr{R}^2 , \qquad \mathscr{R}^{ij}\mathscr{R}_{ij} , \qquad \mathscr{R} , \qquad 1 , \tag{5.15}$$

where the first two are of order four, and the other two of order two and zero respectively. These are precisely the terms we have been considering throughout this thesis.

Collecting all terms in eqs. (5.14) and (5.15), the most general potential term for the projectable version of the theory is

$$S_{proj} = g_\kappa \int dt \int d^3x \sqrt{g} N \left(K_{ij} G_\lambda^{ijkl} K_{kl} - \mathscr{V}_{proj}\left[g_{ij}\right] \right) , \tag{5.16}$$

with $\mathscr{V}_{proj}$ given by

$$\begin{aligned} \mathscr{V}_{proj} := {} & g_0 + g_1\mathscr{R} + g_2\mathscr{R}^2 + g_3\mathscr{R}_{ij}\mathscr{R}^{ij} + g_4\mathscr{R}^3 + g_5\mathscr{R}\mathscr{R}^i{}_j\mathscr{R}^j{}_i \\ & + g_6\mathscr{R}^i{}_j\mathscr{R}^j{}_k\mathscr{R}^k{}_i + g_7\mathscr{R}\nabla^2\mathscr{R} + g_8\nabla_i\mathscr{R}_{jk}\nabla^i\mathscr{R}^{jk}, \end{aligned} \tag{5.17}$$

where the g_i, $i = 0, \cdots, 8$, denote coupling constants. However, note that the potential term of eq. (5.17) does not coincide with that of the projectable version of the theory introduced in reference [50]. The discrepancy is due to the additional assumption of "detailed balance" in [50]. This extra condition was imposed primarily to reduce the large number of independent couplings in the potential of the theory, rather than on physical grounds. As a consequence, the potential must have the form

$$\mathscr{V}_{DB} := E^{ij} G^\lambda_{ijkl} E^{kl} , \tag{5.18}$$

where E^{ij} follows from a variational principle,

$$E^{ij} := \frac{1}{\sqrt{g}} \frac{\delta W\left[g_{kl}\right]}{\delta g_{ij}} , \tag{5.19}$$

for some action W. It was argued in [49, 50] that systems obeying the detailed-balance condition described by eqs. (5.18) and (5.19), with W an action for a d-dimensional system, are

[3] We omit any term that differs from these by a total derivative.

often simpler to study in the quantum regime than a generic $(d+1)$-dimensional theory. In the same work, W was determined indirectly by considering the properties E^{ij} should obey, resulting in the identification of E^{ij} with the Cotton tensor C^{ij} in three dimensions. As was shown subsequently in reference [90], the most general potential term obeying the detailed-balance condition has the form

$$\mathscr{V}_{DB} = \alpha^2 C_{ij}C^{ij} - \frac{2\alpha^2\beta\varepsilon^{ijk}}{\sqrt{g}}\mathscr{R}_{il}\nabla_j\mathscr{R}^l{}_k + \beta^2\mathscr{R}^{ij}\mathscr{R}_{ij} - \frac{\beta^2}{4}\frac{1-4\lambda}{1-3\lambda}\mathscr{R}^2 - \frac{\beta^2\zeta}{1-3\lambda}\mathscr{R} - \frac{3\beta^2\zeta^2}{1-3\lambda}, \quad (5.20)$$

where α, β, and ζ are coupling constants, and ε^{ijk} denotes the Levi-Civita symbol in three dimensions. Unlike in the potential (5.17) of the projectable theory, there are fewer couplings than terms in the potential (5.20). Moreover, the parameter λ now appears explicitly in the potential term, which indicates that it may have a different role than in the λ-R model.

We will now turn our attention to the non-projectable version of the theory, whose lapse depends on both space and time. As we will see shortly, even if we used the same potential (5.17) in both the projectable and the non-projectable versions of the theory, their classical behaviour would be different. In addition, if we consider the lapse to be part of the field content of the theory and not just a Lagrange multiplier, the potential of the non-projectable version will have more terms than $\mathscr{V}_{proj}$. As first pointed out in reference [17], there is then a further quantity transforming as a vector under $\mathrm{Diff}_{\mathscr{F}}(M)$ (but not under $\mathrm{Diff}(M)$), namely,

$$a_i := \nabla_i \log N. \quad (5.21)$$

Of course, any terms involving this vector can only be relevant for the non-projectable version of the theory, since a_i depends on the spatial derivative of the lapse, which in the projectable theory vanishes identically.

Following the logic of effective field theory, the potential of the full non-projectable Hořava-Lifshitz gravity should then also include all scalars constructed from a_i and its spatial derivatives with scaling dimension equal to or smaller than that of the kinetic term. This includes terms with spatial derivatives of both the lapse and the spatial metric, as long as they comply with the symmetry and scaling requirements. The most general action for the non-projectable version of Hořava-Lifshitz gravity without detailed balance reads

$$S_{nproj} = g_\kappa \int dt \int d^3x \sqrt{g} N \left(K_{ij} G_\lambda^{ijkl} K_{kl} - \mathscr{V}_{nproj}\left[g_{ij}, N\right] \right). \quad (5.22)$$

The potential $\mathscr{V}_{nproj}$ is a functional of the metric and the lapse defined by

$$\mathscr{V}_{nproj} := \mathscr{V}_{proj} + \tilde{b}\, a_i a^i + \mathscr{V}_4\left[g_{ij}, N\right] + \mathscr{V}_6\left[g_{ij}, N\right], \quad (5.23)$$

where $\tilde{b}$ is a coupling constant, and $\mathscr{V}_n\left[g_{ij}, N\right]$ denotes a sum of all invariants absent from $\mathscr{V}_{proj}$ whose scaling dimension is $[\kappa]^n$, including terms such as $\left(a_i a^i\right)^2$, $a_i a_j \mathscr{R}^{ij}$, and $a_i \nabla^2 a^i$.

In addition to the projectable and non-projectable versions, there are two other versions that should be mentioned, the so-called generally covariant extension of Hořava-Lifshitz gravity and the mixed-derivatives version. The former was introduced in [51, 52] and was motivated by the wish to remove the extra scalar mode that appears in the original theory due to the restriction from $\mathrm{Diff}(M)$ to $\mathrm{Diff}_{\mathscr{F}}(M)$. To remove this scalar mode, an additional $U(1)$-symmetry is imposed on the theory, leading to the introduction of a new gauge field and a Goldstone boson. The mixed-derivatives version introduced in [29] and further developed in [27,30] addresses the quadratic divergences that appear in the theory when coupled to matter. This behaviour is attenuated when considering invariants containing both time and spatial derivatives. For a recent review of these variants of Hořava-Lifshitz gravity, see [94] and references therein. Discussing them further is beyond the scope of this thesis.

Finally, we note that although Hořava-Lifshitz gravity was introduced as a perturbatively renormalisable theory of quantum gravity, this property is only established at the level of power-counting. A full proof of renormalisability has so far only been given for the projectable version of the theory [6].

5.2 Classical limits of Hořava-Lifshitz gravity

The classical limit of the action of Hořava-Lifshitz gravity is described by the terms of lowest order in spatial derivatives. Recall that by construction the action is only of second order in time derivatives. For the projectable version of the theory, this classical limit is a projectable variant of the λ-R model,

$$S_{proj,class} = \frac{1}{16\pi G_N} \int dt \int d^3x \sqrt{g} N(t) \left(K_{ij} G_\lambda^{ijkl} K_{kl} + \mathscr{R} - 2\Lambda \right), \tag{5.24}$$

where we have absorbed the coupling associated with the Ricci scalar by a coordinate rescaling and have written the cosmological constant term in the same way as in the Einstein-Hilbert action. Note that this is not the λ-R model we defined in chapter 2, because the lapse function depends on time only. In subsection 5.2.1 below, we will show that the results of reference [40] mentioned earlier describe the acceleration of the volume in this scenario.

Following the same reasoning, the classical limit of the non-projectable version of the theory is given by

$$S_{nproj,class} = \frac{1}{16\pi G_N} \int dt \int d^3x \sqrt{g} N \left(K_{ij} G_\lambda^{ijkl} K_{kl} + \mathscr{R} - 2\Lambda + \tilde{b}\, a_i a^i \right), \tag{5.25}$$

which is also not identical to the λ-R model due to the inclusion of the $a_i a^i$-term. The λ-R model is contained in eq. (5.25) as the special case $\tilde{b} = 0$. However, for the λ-R model to be the classical limit of non-projectable Hořava-Lifshitz gravity, the renormalisation group flow of $\tilde{b}$ must vanish sufficiently fast as one flows away from the ultraviolet fixed point.

The classical limit of versions of the theory satisfying the detailed-balance condition is less straightforward. As we can see from inspecting eq. (5.20), setting the couplings associated with all higher-order operators to zero would also remove the Ricci scalar and cosmological constant terms, and the same is true if we set the coupling of the term $\mathscr{R}_{ij}\mathscr{R}^{ij}$ to zero. An immediate consequence of this interdependence of couplings, which has been pointed out in many references (see, for instance, [90]), is that matching the cosmological constant to its observed value requires fine-tuning. Moreover, models satisfying detailed balance have been shown to have strong-coupling problems, where the additional scalar mode survives the fine-tuned classical limit of the theory [24].

5.2.1 The projectable λ-R model: a contradiction resolved

We now address the classical limit of the projectable theory[4]. Starting from the action (5.24) and performing the Legendre transformation in the usual manner yields the total Hamiltonian

$$H_{tot}^{proj} = \alpha\phi + N\int d^3x\,\mathscr{H}_\lambda + \int d^3x\left(N^i\mathscr{H}_i + \alpha^i\phi_i\right), \tag{5.26}$$

where we have taken the lapse N out of the integral to highlight that it takes constant values on each hypersurface Σ_t. The fact that the lapse is a projectable function implies that its momentum ϕ is also a function of time only. Since ϕ is also projectable, requiring it to be preserved in time involves a variation of the lapse outside the integral. This does not lead to a delta function in space, but instead results in an integrated version of the Hamiltonian constraint, namely,

$$\dot{\phi} = \left\{\phi(t), H_{tot}^{proj}\right\} \approx 0 \;\Rightarrow\; \int d^3x\,\mathscr{H}_\lambda \approx 0\,. \tag{5.27}$$

By contrast, the local momentum constraints arise in the usual fashion. Preserving the global Hamiltonian constraint in time does not lead to a tertiary constraint, unlike in the non-projectable case. This can be seen from eq. (2.28), whose last term is absent when N is a projectable function. Recall that this last term was precisely the one responsible for the appearance of the tertiary constraint in chapter 2. Without a tertiary constraint, the constraint algebra is fully determined and closes after obtaining the secondary constraints. As a consequence, one no longer needs to impose a lapse-fixing equation and can use the canonical gauge imposed in reference [40], which in the non-projectable case is in general not allowed.

To obtain the acceleration of the volume $V = \int d^3x\sqrt{g}$ of a compact hypersurface Σ, we take two time derivatives by computing the Poisson brackets with the total Hamiltonian. In the first step, we find

$$\dot{V} = \left\{\int d^3x\sqrt{g}, H_{tot}^{proj}\right\} = -\frac{1}{3\lambda-1}\int d^3x\,\pi \tag{5.28}$$

for the "velocity" of the volume. Taking a second Poisson bracket yields

$$\ddot{V} = \frac{1}{3\lambda-1}\int d^3x\left(\frac{3}{2}\left(2\sqrt{g}\Lambda - \frac{G^\lambda_{ijkl}}{\sqrt{g}}\pi^{ij}\pi^{kl}\right) - \sqrt{g}\frac{\mathscr{R}}{2}\right). \tag{5.29}$$

Using the integrated Hamiltonian constraint, this can be simplified to

$$\ddot{V} = -\frac{2}{3\lambda-1}\int d^3x\sqrt{g}\left(\mathscr{R} - 3\Lambda\right), \tag{5.30}$$

[4] In the remainder of this chapter, we will set $16\pi G_N = 1$.

which is the result found in [40], and stated earlier in eq. (2.15). Furthermore, as we showed in [73], removing time reparametrisations from the symmetries of the theory means that there is no Hamiltonian constraint, global or local. Without this constraint, one can only derive eq. (5.29) for the acceleration of the volume. To obtain the result (5.30) requires the presence of a global Hamiltonian constraint.

This analysis clarifies that the results of [40] and [9] on the physical nature of λ do not contradict each other after all, but were merely derived in different contexts. Using the more recent language of Hořava-Lifshitz gravity, the results of [40] can be interpreted consistently within the projectable version of the theory. In this context, the conclusion that λ affects the acceleration of the three-volume and is thereby related to the attractivity of gravity remains valid.

5.3 Comparison with other $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant models

In this section, we argue that some features of the λ-R model are common to other classical $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant theories whose kinetic term depends on G_λ^{ijkl}, while others are specific to the λ-R model.

An immediate comparison can be made with the projectable λ-R model (5.24) discussed above. Although λ appears in its equations of motion, as demonstrated in [40], λ plays no role in the constraint algebra, which remains unchanged when we set $\lambda = 1$. As we showed in subsection 5.2.1, the fact that the lapse N is a projectable function implies the existence of a global Hamiltonian constraint. As a result, the number of local physical degrees of freedom is three, which differs from both general relativity and the λ-R model, with six first-class constraints and eighteen local phase-space variables.

Another model to compare to is what we may call the "λ-aR model". It was already mentioned in Sec. 5.2 and is described by the action (5.25)

$$S_{\lambda aR} = \frac{1}{16\pi G_N} \int dt \int d^3x \sqrt{g} N \left(K_{ij} G_\lambda^{ijkl} K_{kl} + \mathscr{R} - 2\Lambda + \tilde{b}\, a_i a^i \right). \tag{5.31}$$

The "a" in its name refers to the extra vector a_i which appears in the additional term in the potential. It has been shown to be equivalent to a version of the Einstein-aether theory where the unit timelike vector defining the dynamical preferred foliation [55] is hypersurface orthogonal [56]. The model was further studied in [10] for asymptotically flat hypersurfaces. Its Hamiltonian is not linear in the lapse and not a sum of constraints. As a consequence, demanding the preservation of the lapse-dependent Hamiltonian constraint in time yields an equation for the Lagrange multiplier α associated with the primary constraint $\phi = 0$. The Hamiltonian constraint and the momentum of the lapse ϕ form a pair of second-class constraints, leading to an additional physical degree of freedom. This is consistent with the idea of a dynamical foliation, which is absent from the λ-R model because its constraint algebra fixes the form of the foliation.

A third model, which we will refer to as the "λ-R^2 model", is described by the action [8]

$$S_{\lambda R^2} = \frac{1}{16\pi G_N}\int dt \int d^3x \sqrt{g} N \left(K_{ij}G_\lambda^{ijkl}K_{kl} + \mathscr{R} - 2\Lambda + \xi\mathscr{R}^2\right), \tag{5.32}$$

where ξ is a coupling constant. Unlike the models mentioned so far, the λ-R^2 model (5.32) is not the classical limit of any version of Hořava-Lifshitz gravity and therefore the $\xi\mathscr{R}^2$-term is expected to be highly suppressed at very low energies [90]. Nevertheless, given that the λ-R model has only two physical degrees of freedom, it is interesting to determine whether this continues to be the case for higher-order models of non-projectable Hořava-Lifshitz gravity. Like in the λ-R model, the total Hamiltonian of the λ-R^2 model is a sum of constraints and preserving the Hamiltonian constraint in time yields a tertiary constraint. As shown in reference [8], the only way to not obtain a tertiary constraint is to set $\lambda = 1$ and $\xi = 0$ simultaneously. Setting only $\xi = 0$ results in the constant mean curvature condition, while setting $\lambda = 1$ still yields a tertiary constraint. On the other hand, according to the same reference, the lapse appears as a source term in the differential equation defining the tertiary constraint. Its preservation in time thus yields an equation for α instead of a quaternary constraint. For the constraint algebra to be consistent, two out of the $\phi = 0$, the Hamiltonian constraint and the tertiary one must form a pair of second-class constraints while the other is first class[5], unlike what was argued in [8]. One concludes that the model has two local physical degrees of freedom, like general relativity and the λ-R model.

Generally speaking, applying the Dirac algorithm to Diff$_{\mathscr{F}}(M)$-invariant theories with the field content of general relativity, with at least second-order terms in the potential[6], with the kinetic term of the λ-R model, and without additional symmetries requires additional steps, compared to general relativity. In the λ-R model, we obtained three new conditions, a tertiary constraint, a quaternary one, and an equation for α. In the λ-R^2 model, the algebra closes after two new steps, with one new constraint [8] and an equation for α. In the λ-aR model, the Dirac algorithm closes after one additional step with no additional constraints [10]. In all three cases, the Lagrange multiplier α is determined in the last step of the Dirac algorithm and therefore $\phi = 0$ is no longer a first-class constraint (c.f. subsection A.2.2 of Appendix A). We expect this to be a general feature of theories of this type, because the local $\phi = 0$ constraint is not associated with a local symmetry.

The distinguishing features of the λ-R model are that it has two local degrees of freedom and that its tertiary constraint has the same functional form as a gauge-fixing condition of general relativity. In chapter 6, we will use these comparisons to suggest further research into Diff$_{\mathscr{F}}(M)$-invariant theories.

[5] We expect that one must re-define the constraints on the constraint surface to make this explicit, just as we did with the momentum constraints $\mathscr{H}_i$ in chapter 2.

[6] The so-called theory of ultra-local gravity contains only the cosmological constant in the potential and can be understood as a Hořava-Lifshitz gravity theory with $z = 0$ [50].

6

Discussion and outlook

In this final chapter, we summarise the results of this thesis in Sec. 6.1, by answering a question first raised in chapter 1, namely, about the role of λ in the λ-R model. We also provide an outlook for future research in Sec. 6.2 and end with our conclusions in Sec. 6.3.

6.1 The role of λ in the λ-R model

It is clear from our discussion that for $\lambda \neq 1$ the λ-R model is inequivalent to general relativity. We will now collect and juxtapose results obtained in various parts of the thesis to clarify the role of λ in a systematic manner. We will distinguish between three aspects. First, $\lambda \neq 1$ determines the particular class of foliations for which the model admits solutions. Second, given the same constraint-solving data for general relativity and the λ-R model on an initial hypersurface Σ_0, the presence of a general λ modifies the solutions of the lapse-fixing equation, as well as the time evolution of the metric, momenta, and lapse variables. This implies that its solutions yield different spacetimes from those obtained in general relativity for the same set of foliations. Finally, $\lambda \neq 1$ induces a proliferation of non-equivalent solutions by turning the constant mean curvature $K(t)$ of the foliation, which in general relativity is unphysical, into a physical quantity. In what follows, we will elaborate on these three aspects in turn.

6.1.1 λ determines the foliation

When we discussed the constraint algebra of the model in chapter 2, we saw how the presence of λ is related directly to the notion of a preferred foliation. We have shown in eqs. (2.29) to (2.36) that the λ-dependent Hamiltonian constraint $\mathscr{H}_\lambda \approx 0$ is only preserved in time if the constant mean curvature condition,

$$\nabla_i K = 0\,, \tag{6.1}$$

here written in terms of the trace of the extrinsic curvature K_{ij}, is imposed as a tertiary constraint. Eq. (6.1) not only determines the way in which the spatial hypersurfaces are embedded in spacetime, but demanding that it is preserved in time yields a λ-dependent lapse-fixing equation $\mathscr{M} \approx 0$, as shown in eq. (2.37).

Including λ in the action of general relativity to obtain the λ-R model implies a preferred foliation by leaves which satisfy eqs. (6.1) and the lapse-fixing equation (2.37). This is a property specific to the λ-R model and not of general $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant theories. There are two other classical limits of Hořava-Lifshitz gravity that do not require these equations to be satisfied, namely, the projectable λ-R model and the classical limit of non-projectable Hořava-Lifshitz gravity, which we called λ-aR model in chapter 5. As we pointed out in Sec. 5.3, neither has a preferred foliation by leaves of constant mean curvature. Moreover, the λ-R model is the only one of these classical limits with the same number of local physical degrees of freedom as general relativity.

6.1.2 λ modifies the spacetime geometry

As we saw in eq. (2.63) in chapter 2, the equations of motion for the metric g_{ij} and the momenta π^{ij} are λ-dependent. Because the constraints do not fix the coordinates completely, the equations of motion were not enough to prove that their λ-dependence cannot be eliminated by a gauge choice. Our treatment of the initial value problem in chapter 3 and the particular example of spherical symmetry in chapter 4 illustrated that the solutions of the theory are indeed λ-dependent. In the analysis of chapter 3, we decomposed the momentum tensor in terms of its trace π and its traceless-transverse components π^{ij}_{TT}, eq. (3.3). Combining this with the definition of π^{ij} from the Legendre transformation (2.19a) and its trace (2.20), we obtain

$$\pi^{ij}_{TT} = \sqrt{g}\left(K^{ij} - \frac{1}{3}g^{ij}K\right) := \sqrt{g}\,K^{ij}_{TT}, \tag{6.2}$$

which is λ-independent and shows that the quantity $\mathscr{A} = g^{-1}\pi^{ij}_{TT}\pi^{TT}_{ij}$ of eq. (3.12) appearing in the ϕ^{-7}-term of the modified Lichnerowicz-York equation is exactly the same as that in the original Lichnerowicz-York equation,

$$\mathscr{A} = \frac{\pi^{ij}_{TT}\pi^{TT}_{ij}}{g} = K^{ij}_{TT}K^{TT}_{ij}. \tag{6.3}$$

This implies that the analysis of chapter 3 separates the λ-dependence of the Legendre transformation, which then only appears in the relation $\pi = \sqrt{g}\,(1-3\lambda)\,K$, from the λ-dependence of the constraints and equations of motion.

We showed in chapter 3 that for an arbitrary set of general relativity initial data and arbitrary constants $\lambda > 1/3$ and Λ it is in general possible to choose a set of initial data in the λ-R model such that the constraint-solving data on some initial hypersurface Σ_0 coincides. Assuming that both theories have the same cosmological constant Λ, the choices of π are related by a factor of $\sqrt{3\lambda - 1}$, eq. (3.29). However, to match the time evolution equations the traces of π solving the constraints in both theories must be related by a factor of $(3\lambda - 1)$.

Attaining both simultaneously is only possible if either $\lambda = 1$ or $\pi = 0$, conditions that we know lead to solutions of Einstein's gravity. Similar matching conditions can be obtained in terms of the extrinsic curvature variables. The constraint-solving data is matched when the two choices of K are related by a factor of $(3\lambda - 1)^{-1/2}$, while the time evolution equations require that the two choices of K coincide.

The discrepancies between the spacetimes solving the model and solutions of general relativity were made even more explicit in the discussion of spherically symmetric solutions in chapter 4. In subsections 4.4.1 and 4.4.2, we computed the four-dimensional metric ${}^{(4)}g_{\mu\nu}$ and the Ricci scalar ${}^{(4)}R$ respectively, and found that both contain modifications to the constant mean curvature versions of the Schwarzschild solution that are proportional to $(\lambda - 1)$. The fact that the Ricci scalar ${}^{(4)}R$ is in general non-vanishing and proportional to $(\lambda - 1)$ is of particular significance, since it is an invariant under both Diff(M) and Diff$_{\mathscr{F}}(M)$, which means that it constitutes an observable both in general relativity and in the λ-R model.

In addition, λ also appeared in the definition of the mass M of the central object of the spacetime. To define M such that $\dot{M} = 0$ when $\lambda \neq 1$ (and $K \neq 0$), it was necessary to add a modification proportional to $(\lambda - 1)$ to the general relativistic expression (4.65).

Finally, note that the explicit λ-dependence of the geometry is less general than the preferred foliation, since for $K = 0$ it is still true that these solutions obey the constant mean curvature condition and lapse-fixing equation, but the spacetime geometry becomes λ-independent.

6.1.3 K becomes physical

In general relativity, the value of K is not physical. The same physical solution of the theory can be represented in different coordinate systems, with different values of K. A concrete example was given in [76–78], whose authors considered the standard Schwarzschild metric, which has $K = 0$, and then performed a space-dependent time reparametrisation to obtain the same spacetime in $K \neq 0$ constant mean curvature coordinates. However, this is not a symmetry of the λ-R model, where instead K becomes a physical quantity, as we will argue below. In other words, if two metrics are related by a space-dependent time reparametrisation, they are not in the same gauge orbit in a Diff$_{\mathscr{F}}(M)$-invariant theory and will therefore be physically distinct.

These features were illustrated by the spherically symmetric solutions of the model found in chapter 4. For the $K = 0$ solution of the tertiary constraint, we obtained the Schwarzschild spacetime in the standard Schwarzschild coordinates, regardless of the value of λ. However, they are physically inequivalent to the solutions with $K(t) \neq 0$, because their four-dimensional curvature, eq. (4.102), is given by

$$ {}^{(4)}R\left[K,\dot{K},N\right] = (1-\lambda)\left(2K^2 + \frac{3\dot{K}}{N}\right), \qquad \text{where} \qquad N = N\big(r;C,n_1\big], \tag{6.4} $$

where we used the mixed bracket notation to distinguish between arguments like r that are real numbers and those, such as $C(t)$ and $n_1(t)$ that are real functions of real numbers.

This equation shows that for $\lambda \neq 1$ the four-dimensional spacetime curvature becomes a functional of $K(t)$, its time derivative, and the lapse function. Since the latter also depends on K, as can be seen from eq. (4.72), two spacetimes with the same λ but different values of $K(t)$ are physically different.

Whenever $\lambda \neq 1$ and $K \neq 0$ simultaneously, also other parameters become potentially physical. One of them is the parameter $C(t)$ introduced in chapter 4, which describes the transverse and traceless components of the extrinsic curvature tensor K_{ij}. In general relativity with spherical symmetry, C is a gauge parameter and therefore its variation does not affect diffeomorphism-invariant quantities like the scalar curvature ${}^{(4)}R$. By contrast, a brief computation shows that varying C while keeping all other parameters constant alters the value of ${}^{(4)}R$ in eq. (6.4), provided that $\dot{K}$ is not identically zero. The same holds for the parameter n_1 describing the leading-order behaviour of the lapse at infinity.

The consequences of $\lambda \neq 1$ we have just described were derived in chapter 4 in the presence of $SO(3)$-isometry. However, it is straightforward to generalise them to the case without isometry by considering the equations of motion derived directly from the λ-R action,

$$\frac{\delta S_\lambda}{\delta^{(4)}g^{\mu\nu}} = 0 \Leftrightarrow \frac{\delta S_{EH}}{\delta^{(4)}g^{\mu\nu}} + (1-\lambda)\frac{\delta}{\delta^{(4)}g^{\mu\nu}}\int dt \int d^3x \sqrt{-{}^{(4)}g}\, K^2 \tag{6.5a}$$

$$= \sqrt{-{}^{(4)}g}\left({}^{(4)}R_{\mu\nu} - \frac{1}{2}\,{}^{(4)}g_{\mu\nu}\left({}^{(4)}R - 2\Lambda\right) + (1-\lambda)\left(A_{\mu\nu} - \frac{{}^{(4)}g_{\mu\nu}}{2}K^2\right)\right) = 0\,, \tag{6.5b}$$

where $A_{\mu\nu}$ is a shorthand for the variation of the K^2-term,

$$A_{\mu\nu} := \frac{2}{\sqrt{-{}^{(4)}g}}\int dt \int d^3x \sqrt{-{}^{(4)}g}\, K \frac{\delta K}{\delta^{(4)}g^{\mu\nu}}\,. \tag{6.6}$$

Taking the trace of $\frac{\delta S_\lambda}{\delta^{(4)}g^{\mu\nu}} = 0$, we obtain

$${}^{(4)}R = 4\Lambda + (1-\lambda)\left(A - 2K^2\right), \tag{6.7}$$

where A is the trace of the tensor $A_{\mu\nu}$.

Eq. (6.7) demonstrates that in the λ-R model, different values of $K(t)$ will in general correspond to physically inequivalent spacetimes. It also shows that setting $\Lambda = 0$ and discarding any matter fields is in general not sufficient to guarantee the existence of a flat, four-dimensional spacetime solution. When $K \neq 0$, the condition $A - 2K^2 = 0$ must be satisfied to obtain a vanishing four-dimensional Ricci scalar. However, as is clear from the example of the spherically symmetric solutions, this is not true in general.

6.2 Outlook

In this section, we discuss some possible directions of future research. We divide the discussion into four parts, addressing first some potential phenomenological implications of the work presented so far. We follow this with a suggestion motivated by the role of the Wheeler-DeWitt metric in the λ-R model. The last two subsections focus on more direct generalisations of the work presented so far, namely, obtaining a more general set of spherically symmetric solutions to the λ-R model, and investigating other $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant theories.

6.2.1 Phenomenological implications

It is clear that the λ-R model and general relativity are inequivalent. A possible follow-up to the work presented here is to determine whether and to what extent physical observables are sensitive to the presence of the parameter λ. Natural candidates to quantify this are the classic solar system tests of general relativity, such as light deflection or perehelion precession. This idea has been explored in [72] for solutions to a different model of Hořava-Lifshitz gravity, namely, the spherically symmetric solutions obtained in [59]. These solutions assume a vanishing shift, a static spacetime and asymptotic flatness, assumptions under which λ does not appear in solutions of the λ-R model[1]. The spherically symmetric solutions we have obtained allow us to make λ-dependent predictions for solar system tests and establish observational bounds on deviations of λ from its canonical value of 1. However, these predictions will not be unique for each value of λ, because for $K(t) \neq 0$, not only λ but also the choices of $K(t)$, $C(t)$, and $n_1(t)$ influence the geometry. One possibility to fix the value of $K(t)$ is to assume that the preferred foliation is aligned with the cosmological frame and to use observational cosmological data. As we showed in Sec. E.2 of appendix E, the λ-R model yields the same equations of motion as general relativity for the FLRW metric, for which $K(t) = 3\frac{\dot{a}_F}{a_F} = 3H_F$, where a_F is the scale factor of the FLRW solution and H_F the associated Hubble parameter. Although the Hubble parameter is usually associated with the FLRW solution, it could also be used in the present context to fix the function $a(t)$ appearing in the tertiary constraint of the theory, and therefore as a choice for $K(t)$. This choice does not yield unique spherically symmetric predictions[2] for $\lambda > 1/3$, since the lapse function depends on $C(t)$ and $n_1(t)$, but would constitute a specific proposal for identifying the preferred foliation and reduce the space of functions that needs to be fixed to obtain a prediction.

6.2.2 The Wheeler-DeWitt metric

One of the objectives of the work of Giulini and Kiefer in [40] was to understand better the role of the Wheeler-DeWitt metric by studying a gravitational theory that uses its generalised counterpart. In [38], the generalised Wheeler-DeWitt metric in the kinetic term was again considered, but in a different context, this time associated with superspace, the space

[1]These conditions do not eliminate λ completely from the solutions obtained in [59], because λ appears in the potential of Hořava-Lifshitz models that satisfy the detailed-balance condition.

[2]As we showed in chapter 4, for $\lambda < 1/3$ the theory does not admit spherically symmetric solutions with a non-vanishing $K(t)$.

of three-dimensional geometries obtained by taking the quotient of RiemΣ_t by spatial diffeomorphisms. In this work, it is shown that for $\lambda \neq 1$ it is impossible to satisfy the first-class constraint algebra of Diff(M) unless $K = 0$.

Two further instances of the Wheeler-DeWitt metric we have encountered in the course of our work suggest that studying Diff$_{\mathscr{F}}(M)$-invariant models in the context of superspace may be a fruitful direction of research. Firstly, the variation of the Hamiltonian constraint $\mathscr{H}_\lambda$ contains not only a term linear in the generalised Wheeler-DeWitt metric appearing in the kinetic term, but also a term linear in the original Wheeler-DeWitt metric with $\lambda = 1$. More concretely, the variation of the three-dimensional Ricci scalar $\mathscr{R}$ of the potential term with respect to the metric g_{ij} yields (2.28)

$$\delta_{g_{ij}} \int d^3x \sqrt{g} N \mathscr{R} = -\sqrt{g} N \left(\frac{g^{ij}}{2} \mathscr{R} - \mathscr{R}^{ij} \right) - \sqrt{g}\, G^{ijkl} \nabla_k \nabla_l N. \tag{6.8}$$

The contraction of this generalised supermetric with its $\lambda = 1$ counterpart[3] in eq. (6.8) then yields the tertiary constraint of the λ-R model.

Secondly, the variation of the Hamiltonian constraint of the λ-aR model of eq. (5.31) with respect to the metric[4] is given by

$$\delta_{g_{ij}} \int d^3x \sqrt{g} N a_k a^k = \sqrt{g} N G^{ijkl}_{1/2} a_k a_l, \tag{6.9}$$

and therefore depends on yet another version of the generalised Wheeler-DeWitt metric, this time with $\lambda = 1/2$.

We currently have no interpretation for the appearance of these specific supermetrics in eqs. (6.8) and (6.9). Unlike in the kinetic term, their value of λ is not dictated by the invariance group of the model. We also do not know whether the variation of the Hamiltonian constraint of other models of Hořava-Lifshitz gravity depends on other supermetrics. Because these are metrics on RiemΣ_t and on superspace, it might be worthwhile to investigate the properties of Diff$_{\mathscr{F}}(M)$-invariant theories with a non-projectable lapse in the setting of superspace. This analysis could provide not only an explanation for the appearance of these supermetrics with specific values of λ, but also illuminate the general constraint structure of Diff$_{\mathscr{F}}(M)$-invariant theories.

6.2.3 General spherically symmetric solutions of the λ-R model

The spherically symmetric solutions to the λ-R model that we obtained in chapter 4 are not the most general ones because we restricted ourselves to the case where the spatial hypersurfaces are spherically symmetric by construction. This motivates the search for a convenient framework to tackle the more general problem, where the $SO(3)$-orbits are not aligned

[3] Unlike what we do here, the metric prefactors of the derivatives of the lapse in eq. (6.8) are not usually written as a supermetric, as can be seen in [65].

[4] The Hamiltonian constraint of this model depends on a particular combination of an $a_i a^i$- and a $\nabla_i a^i$-term. The variation of the extra piece of the Hamiltonian constraint also depends on $G^{ijkl}_{1/2}$ in a way that does not cancel the one coming from the $a_i a^i$-term.

with the preferred foliation. One option may be the so-called covariant $1+1+2$ formalism developed in [26], which in addition to a preferred time direction uses a preferred spatial direction. For our purposes, the choice of preferred spatial direction would be the radial direction perpendicular to the shells of spherical symmetry. In the context of general relativity, this approach was already adapted to the study of systems with approximate spherical symmetry in references [37, 41] to obtain an approximate Birkhoff theorem for spacetimes with approximate spherical symmetry.

We want to consider the different case in which the whole spacetime possesses exact spherical symmetry but cannot be foliated by spherically symmetric hypersurfaces. The way we envision applying the $1+1+2$ formalism to this case is to parametrise the lack of alignment between the orbits of isometry and the preferred foliation, given by the inner product between the Killing vectors and the timelike unit normal to Σ_t, in terms of the small violations of spherical symmetry of the hypersurface, as is done in [41] for the whole spacetime. Obtaining these perturbative solutions will give us a first indication of what happens when $SO(3)$-orbits are not aligned with the preferred foliation.

6.2.4 Other models of Hořava-Lifshitz gravity

The constraint algebra of the full non-projectable Hořava-Lifshitz gravity has not yet been determined, although a few cases of models other than the λ-R model have been discussed in the literature. Early results include the finding that the Hamiltonian constraint is not trivially preserved in time for general potentials [35], and the analysis of the classical limit of the λ-aR model[5] for asymptotically flat hypersurfaces [10].

It would be interesting to derive the equations defining the foliation and the number of physical degrees of freedom for other classical models of Hořava-Lifshitz gravity. The analyses performed so far suggest that when a_i-dependent terms are included in the potential, the resulting theory has three physical degrees of freedom. When the potential is only a functional of the spatial metric and its derivatives, the situation is less clear. Both the λ-R and λ-R^2 model have only two physical degrees of freedom (despite suggestions to the contrary for the latter [8]). Although determining the full constraint algebra for arbitrary higher-order potentials of Hořava-Lifshitz gravity may not be technically feasible, understanding the properties of models that go beyond the λ-R model could provide a description of the foliation-defining equations for the more general theory and an understanding of the number of physical degrees of freedom of these classical models. Moreover, obtaining observational bounds for the couplings appearing in other models of Hořava-Lifshitz gravity could provide important constraints on the renormalization group flow of non-projectable Hořava-Lifshitz gravity.

A specific extension of our results would be to apply the spherically symmetric ansatz from chapter 4 to other models of non-projectable Hořava-Lifshitz gravity. This would en-

[5]The complete canonical analysis of this version of the theory for asymptotically flat hypersurfaces has been derived in [11] for the special value $\lambda = 1/3$, which we did not consider in this thesis.

able us to generalise several sets of solutions already obtained in the literature, which we summarised in Sec. 4.5 of chapter 4. It would also be interesting to apply the analysis of the motion of test particles in [22] to the solutions of chapter 4, to determine the influence of both λ and $K(t)$ on the existence and properties of circular orbits.

6.3 Conclusion

In this thesis we have performed a careful investigation of the λ-R model, a one-parameter family of theories of gravity invariant under foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathscr{F}}(M)$. In the context of $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant theories, the model can be seen as a minimal departure from general relativity. We have analysed how the presence of the parameter λ affects the physics of the model. This sets the stage for studying the phenomenology of the λ-R model, and we have made some suggestions in Sec. 6.2 of how observational bounds on λ may be obtained. We have also discussed which features of the λ-R model are exclusive to it and which are shared by other models with the same $\mathrm{Diff}_{\mathscr{F}}(M)$ symmetry group, field content, and kinetic term. Properties specific to the λ-R model are its foliation by leaves of constant mean curvature and the associated λ-dependent lapse-fixing equation. Features it has in common with other models include the λ-dependence of spacetime geometry and the second-class nature of the constraint $\phi = 0$. Accordingly, we have made suggestions for further research into classical $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant theories, which may lead to a better understanding of their physical properties. Underlying both our work and outlook is the idea that to understand the physics of $\mathrm{Diff}_{\mathscr{F}}(M)$-invariant theories of classical gravity, it is necessary to question whether each of the usual general relativistic assumptions holds true or must be modified according to the context.

Appendices

A

Constrained Hamiltonian systems

A.1 What is a constrained system?

In this appendix, we will provide a brief summary of the Hamiltonian formulation of constrained systems. We will deal mainly with the finite-dimensional case and only at the end generalise to the field-theoretical case, of which general relativity and the λ-R model are examples. We base our discussion on references [45] and [92], which in turn build on the work of Dirac [34] and Bergmann [2, 12–14]. The description of a constrained system contains unphysical redundancies. The formalism we will review allows one to deal with them in a systematic way.

To simplify the presentation, we consider a finite-dimensional system of dimension n, with position variables $q^i(t)$, $i = 1, \cdots, n$. Suppose the dynamics of such a system is encoded in a classical action of the form

$$S = \int_{t_1}^{t_2} dt\, L(q, \dot{q})\,, \tag{A.1}$$

where $L(q, \dot{q})$ denotes the Lagrangian and $\dot{q}$ the velocity. In eq. (A.1), we omitted the time dependence of positions and velocities, as we will do in most of the remainder of this discussion. The stationary points of the action (A.1) under variations δq^i of the position q^i correspond to the classical trajectories of the system, provided the δq^i vanish at the end points, i.e. $\delta q^i(t_1) = \delta q^i(t_2) = 0$. This variational principle yields the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0\,, \tag{A.2}$$

whose solutions are the classical trajectories. Performing the total time derivative on the left-hand side and using the chain rule leads to the equation

$$\ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial L}{\partial q^i} - \dot{q}^j \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\,. \tag{A.3}$$

In order to solve eq. (A.3) uniquely for the acceleration $\ddot{q}^i$, the Hessian

$$W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\,. \tag{A.4}$$

must be invertible. Otherwise, for $\det(W_{ij}) = 0$, arbitrary functions of time can appear in the solutions of the equations of motion (A.3). This motivates the following definition of a singular, or constrained, system.

Definition 1. *A system described by a given Lagrangian is called singular if its Hessian, eq.* (A.4)*, has vanishing determinant.*
After a Legendre transformation, such a system is called a constrained Hamiltonian system.

Note that being singular is a property of the Lagrangian or Hamiltonian description of the system, not of the underlying physical system itself. Since we are interested in the singular case, we will assume from now on that the determinant of W_{ij} vanishes unless otherwise specified. We further assume that the rank of W_{ij} is a constant in configuration space and given by $(n-m)$, with $0 < m < n$. As we will see, the non-invertibility of the Hessian leads to the presence of so-called constraints in the Hamiltonian picture.

A.1.1 Legendre transformation, primary constraints and the constraint surface

In what follows, we will denote by $\mathscr{C}_s$ the $2n$-dimensional space of configurations and velocities of the system[1]. A point in $\mathscr{C}_s$ represents a pair $\left(q^i(t), \dot{q}^i(t)\right)$. The momentum variables of the Hamiltonian formulation are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}\,. \tag{A.5}$$

This equation defines a map from $\mathscr{C}_s$ to the phase space $\mathscr{P}$ of the system, the space of positions $q^i(t)$ and momenta $p_i(t)$.

To obtain the Hamiltonian, it is necessary to invert eq. (A.5) to express the velocities $\dot{q}^i$ as functions $\dot{q}^i(q^j, p_j)$ on phase space. Because of the inverse function theorem, this is only possible when $\det\left(\frac{\partial p_i}{\partial \dot{q}^j}\right)$ does not vanish,

$$\det\left(\frac{\partial p_i}{\partial \dot{q}^j}\right) = \det\left(\frac{\partial}{\partial \dot{q}^j}\frac{\partial L}{\partial \dot{q}^i}\right) = \det W_{ij} \neq 0\,, \tag{A.6}$$

which is the condition on the Hessian that appeared already in the definition of a singular system. This shows that whenever we cannot solve uniquely for the accelerations in terms of the positions and velocities, it is impossible to obtain all the velocities of the system as functions of the phase space variables. In this case, the map from $\mathscr{C}_s$ to $\mathscr{P}$ defined by eq. (A.5) is not one-to-one and therefore not all the momenta p_i are independent. This is expressed

[1] The subscript "s" is used to distinguish $\mathscr{C}_s$ from the symbol $\mathscr{C}$ used in chapter 3.

through the existence of m' relations on the phase space variables, which are the so-called primary constraints. We write them as

$$\phi_\alpha(q,p) = 0\,, \qquad \alpha = 1,\ldots,m'. \tag{A.7}$$

Relations (A.7) reduce to identities when eqs. (A.5) are substituted into them. When all constraints are independent, we have $m = m'$ and the description of the system is said to be irreducible. Otherwise, it is said to be reducible and $m' > m$. We will assume from now on that the set of primary constraints is irreducible. Because both $\mathscr{C}_s$ and $\mathscr{P}$ are $2n$-dimensional, the primary constraints $\phi_\alpha = 0$ define a subspace of dimension $2n - m$.

Definition 2. *The set of points satisfying the constraints* (A.7) *defines a* $(2n-m)$*-dimensional subspace of* $\mathscr{P}$ *called the* **constraint surface** *and denoted by* $\mathscr{P}_c$.

The inverse image of a point $(q^i, p_i) \in \mathscr{P}_c$ under (A.5) is multivalued because the definition of momenta is given by a map from a $2n$-dimensional space $\mathscr{C}_s$ to $\mathscr{P}_c$, a space of dimension $(2n-m)$.

As an example, consider a simple system described by the Lagrangian

$$L = \frac{1}{2}\left(\dot{q}^1 - q^2\right)^2 - V\left(q^1\right), \tag{A.8}$$

where $V(q^1)$ is a potential that depends only on q^1. The Hessian (A.4) of (A.8) is given by

$$W_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \det W_{ij} = 0\,, \tag{A.9}$$

which implies that eq. (A.8) defines a singular Lagrangian system. Its momenta are

$$p_1 = \dot{q}^1 - q^2\,, \qquad p_2 = 0\,, \tag{A.10}$$

where $p_2 = 0$ is the single primary constraint of the system.

Note that there can be equivalent ways to specify the same constraint surface with different phase space functions. For instance, the constraint surface $p_2 = 0$ in the example above can also be obtained from the condition $p_2^2 = 0$. However, the infinitesimal symplectic transformation generated by p_2^2, unlike that of p_2, has a trivial action on the constraint surface, since all Poisson brackets involving p_2^2 are linear in p_2 and therefore vanish identically on the constraint surface (see Sec. A.2 below for the definition of Poisson brackets). This motivates the definition of regularity conditions.

Assume that the m constraints $\phi_\alpha = 0$ are linearly independent. The regularity condition that we impose from now on is that the rank of their Jacobian on the constraint surface $\mathscr{P}_c$ is m,

$$\mathrm{Rank}\left(\frac{\partial \phi_\alpha}{\partial\left(q^i, p_i\right)}\right)_{\phi_\alpha = 0} = m. \tag{A.11}$$

There are other, equivalent ways to state this condition (see [45] for instance), but this one suffices for our purposes.

The following is a useful theorem regarding functions on the phase space $\mathscr{P}$, whose proof can be found in reference [45] and requires the regularity condition.

Theorem 1. *Let $F(q,p)$ be a function on $\mathscr{P}$ which vanishes on the constraint surface $\mathscr{P}_c$,*

$$F(q,p)|_{\phi_\alpha=0} = 0. \tag{A.12}$$

Then, it can always be written as a linear combination of constraints,

$$F = f^\alpha \phi_\alpha, \tag{A.13}$$

for some functions $f^\alpha(q,p)$, not necessarily constants.

We will also need the notion of weak equality.

Definition 3. *Let F and G be two phase space functions on $\mathscr{P}$. If they coincide on $\mathscr{P}_c$, we say they are weakly equal and write*

$$F \approx G \;\Leftrightarrow\; F(q,p)|_{\phi_\alpha=0} = G(q,p)|_{\phi_\alpha=0}\,. \tag{A.14}$$

It follows from Theorem 1 that if $F \approx G$, we can write the difference between the two functions as a linear combination of constraints

$$F - G = c^\alpha \phi_\alpha. \tag{A.15}$$

The weak equality sign thus means "equality up to a linear combination of constraints". Since physics takes place on the constraint surface $\mathscr{P}_c$ and not on the whole phase space, the notion of weak equality is important for the methods we are discussing.

A.2 Hamiltonian, equations of motion and Poisson brackets

Before discussing the Hamiltonian, its equations of motion, and the Poisson bracket structure in detail, let us quickly recap the Hamiltonian formulation of non-singular systems. Consider $\det W \neq 0$ and define the Hamiltonian H in the usual manner as

$$H = \dot{q}^i p_i - L(q,\dot{q})\,. \tag{A.16}$$

To see that H is a function on $\mathscr{P}$, we take the variation of L, use the chain rule, and use the definition of momenta (A.5),

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i = \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q^i}\delta q^i + \delta\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) - \dot{q}^i\delta\left(\frac{\partial L}{\partial \dot{q}^i}\right)\\
\Leftrightarrow \delta\left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right) &= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q^i}\delta q^i - \dot{q}^i\delta\left(\frac{\partial L}{\partial \dot{q}^i}\right)\\
\Leftrightarrow \delta H &= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q^i}\delta q^i - \dot{q}^i\delta p_i\,.
\end{aligned} \tag{A.17}$$

This shows that the variation of H does not depend directly on the variations of $\dot{q}^i$.

To obtain the Hamiltonian equations of motion, we compute the variation of H directly and compare it with eq. (A.17),

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \tag{A.18a}$$

$$\frac{\partial H}{\partial p_i} = \dot{q}^i, \tag{A.18b}$$

$$\frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i} \underset{E-L\,eq}{\Rightarrow} \frac{\partial H}{\partial q^i} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = -\dot{p}_i\,, \tag{A.18c}$$

where we have imposed the Euler-Lagrange equations to obtain the last equalities in eq. (A.18c). Relations (A.18) can be used to write the total time derivative of any phase space function F as

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} := \frac{\partial F}{\partial t} + \{F,H\}\,, \tag{A.19}$$

where we have introduced the usual notation for Poisson brackets, which we define next.

Definition 4. *Let A and B be phase space functions. The Poisson bracket of A and B, denoted $\{A,B\}$, is a map from two functions A, B on $\mathscr{P}$ to another function $\{A,B\}$ on $\mathscr{P}$ given by the expression*[2]

$$\{A,B\}(q,p) := \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}\,. \tag{A.20}$$

It obeys the following properties:

- *antisymmetry,* $\{A,B\} = -\{B,A\}$,
- *linearity,* $\{A, c_1 B + c_2 C\} = c_1\{A,B\} + c_2\{A,C\}$, *for constants c_1 and c_2,*
- *Jacobi identity,* $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$,
- *Leibniz rule,* $\{A,BC\} = B\{A,C\} + \{A,B\}C$.

It is straightforward to compute the fundamental Poisson brackets among the canonical variables,

$$\{q^i,q^j\} = 0\,, \qquad \{p_i,p_j\} = 0\,, \qquad \{q^i,p_j\} = \delta^i_j\,. \tag{A.21}$$

From now on, we will again assume that $\det W = 0$. Because of the vanishing of the determinant of the Hessian, it is not possible to write the velocities $\dot{q}^i$ in terms of positions and momenta. Hence, we cannot obtain eqs. (A.17), (A.18), and (A.19). The following theorem shows how one can nevertheless obtain similar results (see [45] for a proof).

Theorem 2. *Let δq^i, δp_i denote arbitrary variations, tangent to the constraint surface. Then, if for some functions a_i, b^i the following expression holds*

$$a^i \delta q_i + b_i \delta p^i = 0, \tag{A.22}$$

[2] As with other phase space functions, we will mostly omit the explicit (q,p)-dependence of the Poisson bracket.

then a^i and b_i can be written as

$$a_i \approx u^\alpha \frac{\partial \phi_\alpha}{\partial q^i}\,, \tag{A.23a}$$

$$b^i \approx u^\alpha \frac{\partial \phi_\alpha}{\partial p_i}\,. \tag{A.23b}$$

for some functions u^α.

We now compute the variation of H as in eq. (A.17), obtaining

$$\delta H = p_i \delta \dot{q}^i + \dot{q}^i \delta p_i - \delta q^i \frac{\partial L}{\partial q^i} - \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = \dot{q}^i \delta p_i - \delta q^i \frac{\partial L}{\partial q^i}\,. \tag{A.24}$$

While this variation is independent of factors of $\delta \dot{q}^i$, not all of the variations δp_i are independent because they are required be tangent to the constraint surface. Therefore the δp_i are linear combinations of δq^i and $\delta \dot{q}^i$. By applying the chain rule to δH, we can rewrite eq. (A.24) as

$$\left(\frac{\partial H}{\partial q^i} + \frac{\partial L}{\partial q^i}\right) \delta q^i + \left(\frac{\partial H}{\partial p_i} - \dot{q}^i\right) \delta p_i = 0\,, \tag{A.25}$$

to which we can now apply Theorem 2 and write

$$-\frac{\partial L}{\partial q^i} \approx \frac{\partial H}{\partial q^i} + u^\alpha \frac{\partial \phi_\alpha}{\partial q^i}\,, \tag{A.26a}$$

$$\dot{q}^i \approx \frac{\partial H}{\partial p_i} + u^\alpha \frac{\partial \phi_\alpha}{\partial p_i}\,. \tag{A.26b}$$

Note that eq. (A.26a) allows us to recover the velocities in terms of canonical data belonging to the constraint surface and from the knowledge of the extra functions u^α. Due to the regularity conditions, we know that the vectors $\frac{\partial \phi_\alpha}{\partial p_j}$ are independent. It follows that two different functions u^α yield different velocities $\dot{q}^i$. These functions u^α are in principle obtainable in terms of positions and velocities, by solving the equation

$$\dot{q}^i = \frac{\partial H}{\partial p_i}\left(q^j, p_j\left(q^k, \dot{q}^k\right)\right) + u^m \frac{\partial \phi_\alpha}{\partial p_i}\left(q^j, p_j\left(q^k, \dot{q}^k\right)\right). \tag{A.27}$$

As can be seen in [45], it is possible to define an invertible Legendre transformation from the $2n$-dimensional space of positions and velocities to the $2n$-dimensional subspace of the $2n+m$-dimensional $\left(q^i, p_i, u^\alpha\right)$-space defined by $\phi_m\left(q^i, p_i\right) = 0$. In other words, in order to have an invertible Legendre transformation when there are m constraints, it is necessary to introduce m additional variables.

We can now insert the Euler-Lagrange equations in eq. (A.26b), which allows us to write the equations of motion of the system as

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + u^\alpha \frac{\partial \phi_\alpha}{\partial p_i}\,, \tag{A.28a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - u^\alpha \frac{\partial \phi_\alpha}{\partial q^i}\,, \tag{A.28b}$$

$$\phi_\alpha = 0\,, \tag{A.28c}$$

where we have added the conditions $\phi_\alpha = 0$ explicitly because they must hold independently of the equations of motion. Note that we cannot derive these equations via a variational principle of the form $\delta \int_{t_1}^{t_2} \left(\dot{q}^i p_i - H\right) = 0$ with the Hamiltonian H (A.16). However, we can obtain eqs. (A.28) from the variational principle

$$\delta \int_{t_1}^{t_2} dt \left(\dot{q}^i p_i - H - u^\alpha \phi_\alpha\right) = 0, \tag{A.29}$$

for arbitrary variations δq_i, δp_i, and δu^α, subject to $\delta q_i(t_1) = \delta q_i(t_2) = 0$, where the functions u^α play the role of Lagrange multipliers, which enforce the constraints $\phi_\alpha = 0$. Equivalently, we can obtain the equations of motion from a variational principle with a modified Hamiltonian H_{tot} of the form

$$H_{tot} = H + u^\alpha \phi_\alpha\,, \tag{A.30}$$

the so-called total Hamiltonian. Given Theorem 1, the Hamiltonians H and H_{tot} coincide on the constraint surface. However, the latter is suited for deriving equations of motion through a standard variational principle. Due to the fact that Poisson brackets obey the Jacobi identity, the Poisson brackets involving the Lagrange multipliers u^α vanish weakly, since for any function $F(q,p)$,

$$\{F, u^\alpha \phi_\alpha\} = u^\alpha \{F, \phi_\alpha\} + \phi_\alpha \{F, u^\alpha\} \approx u^\alpha \{F, \phi_\alpha\}\,. \tag{A.31}$$

Ignoring any explicit time dependence[3], we can compute $\dot{F}$ for some function $F(q,p)$ in the following manner

$$\dot{F} = \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} + u^\alpha \frac{\partial F}{\partial q^i}\frac{\partial \phi_\alpha}{\partial p_i} - u^\alpha \frac{\partial F}{\partial p_i}\frac{\partial \phi_\alpha}{\partial q^i} = \{F, H\} + u^\alpha \{F, \phi_\alpha\} \approx \{F, H_{tot}\}\,. \tag{A.32}$$

It is possible to specify which of the functions u^α remain arbitrary on the constraint surface and which ones are determined by consistency conditions, as we will now discuss.

A.2.1 Secondary constraints and the Dirac algorithm

So far, we have seen that singular Lagrangian systems give rise to constrained Hamiltonian systems. The fact that one cannot express all velocities $\dot{q}^i$ as functions of canonical data implies that there are at least m primary constraints, where $(n-m)$ is the rank of the Hessian W, which we assume is constant over the whole configuration space. Note that the argument leading to the appearance of the primary constraints does not rely in any way on the equations of motion.

The result of the Legendre transformation should not depend on time. Consistency of the procedure outlined above therefore requires that the constraints are valid at all times. We must thus impose that the time evolution of the primary constraints, computed through eq. (A.32), vanishes weakly,

$$\dot{\phi}_\alpha = \{\phi_\alpha, H_{tot}\} \approx 0\,. \tag{A.33}$$

[3] This will always be the case unless otherwise specified.

However, the weak equality in eq. (A.33) is not always satisfied automatically. When $\dot{\phi}_\alpha$ does not vanish weakly, we must impose $\dot{\phi}_\alpha \approx 0$ as an extra condition on either a Lagrange multiplier or a phase space function. The Poisson bracket between a primary constraint and the total Hamiltonian is given by

$$\dot{\phi}_\alpha = \{\phi_\alpha, H\} + u^\beta \left\{\phi_\alpha, \phi_\beta\right\} . \tag{A.34}$$

There are three possibilities, namely,

- both $\{\phi_\alpha, H\}$ and $\left\{\phi_\alpha, \phi_\beta\right\}$ vanish weakly, in which case eq. (A.33) holds without further restrictions,
- for some β, $\left\{\phi_\alpha, \phi_\beta\right\}$ does not vanish weakly, in which case eq. (A.33) yields an equation for the Lagrange multipliers associated with the non-commuting constraints (this holds regardless of the status of $\{\phi_\alpha, H\}$),
- $\left\{\phi_\alpha, \phi_\beta\right\}$ vanishes weakly for all β but $\{\phi_\alpha, H\}$ does not vanish weakly, in which case satisfying eq. (A.33) imposes an equation on phase space variables, that is, it imposes a new constraint.

Let a label the set of constraints $\phi_a = 0$ whose Poisson brackets with all other constraints are weakly vanishing but whose Poisson bracket with the Hamiltonian $\{\phi_a, H\} \approx 0$ yields non-trivial relations on phase space. These relations can be written as

$$g_a\left(q^i, p_i\right) \approx 0 , \tag{A.35}$$

and do not vanish on the constraint surface. Assume there are m' of these relations (A.35). They define the so-called secondary constraints of the theory. Unlike their primary counterparts, they only come about when imposing the Hamiltonian equations of motion.

Recall that the primary constraints were included in the Hamiltonian to have a well-defined variational principle, a fact which has already been established when the secondary constraints are obtained. Hence, the secondary constraints do not have to be added to the total Hamiltonian. By contrast, the argument for preserving the constraints in time and imposing (A.33) also applies to the secondary constraints, and we must demand that

$$\dot{g}_a\left(q^i, p_i\right) = \{g_a, H\} + u^\alpha \{g_a, \phi_\alpha\} \approx 0 , \tag{A.36}$$

where we have implicitly updated the meaning of constraint surface to include the new constraints. Whenever new constraints are derived, we call constraint surface the sub-space of the phase space defined by the vanishing of all constraints, including the new ones. The notion of weak equality is updated accordingly. Again, there are three possibilities to satisfy (A.36). It can be satisfied trivially, determine some Lagrange multiplier or impose further constraints. It is then a matter of choice whether to call these further constraints tertiary in order to reflect the stage at which they were obtained, or to collectively call secondary all constraints obtained by the computation of a time derivative. In this text, we will use the first convention. At each stage, we must impose that the time derivative of the constraints vanishes. This process finishes when no new constraints are generated at some given stage. This procedure is called the "Dirac algorithm" and only after its completion the equations of motion for position and momentum can be imposed and solved consistently, which was our original goal.

Once we have obtained the complete set of m' constraints, which we collectively denote by Φ_μ, $\mu = 1, \cdots, m'$, where

$$\Phi_\mu = \phi_\alpha, \qquad \text{for} \qquad \mu = 1, \cdots, m, \tag{A.37a}$$

$$\Phi_\mu = g_a, \qquad \text{for} \qquad \mu = m+1, \cdots, m', \tag{A.37b}$$

we have established that

$$\left\{\Phi_\mu, H\right\} + u^\alpha \left\{\Phi_\mu, \phi_\alpha\right\} \approx 0, \tag{A.38}$$

is satisfied for all μ. Eq. (A.38) defines a set of m' nonhomogeneous linear equations for the $m \leq m'$ unknown functions u^α, with coefficients that depend on positions q^i and momenta p_i. We can then write the Lagrange multipliers as $u^\alpha = U^\alpha + V^\alpha$, where the functions U^α are particular solutions to the inhomogeneous equations and the functions V^α are the general solutions to the associated homogeneous system. The latter is therefore given by a linear combination of linearly independent solutions V_b^α, with $b = 1, \cdots, B$. The number B of independent solutions is the same for all points on the constraint surface because we assume that the rank of the matrix $\left\{\Phi_\mu, \phi_\alpha\right\}$ is constant there. Therefore, u^α is given by

$$u^\alpha = U^\alpha + v^b V_b^\alpha, \tag{A.39}$$

with arbitrary coefficients v^b. As we will see in subsection A.2.2 below, the number B of arbitrary coefficients v^b is equal to the number of first-class constraints.

A.2.2 From first- and second-class constraints to physical degrees of freedom

Suppose now that we have performed the Dirac algorithm, resulting in m primary constraints and a total of k secondary and higher-order constraints. There is a further classification of this set of constraints into first and second class, which we give below[4].

Definition 5. *Let $\Phi_\mu \approx 0$ denote the complete set of constraints of a given system ($\mu = 1, \cdots, m'$). A constraint ϕ is said to be first class if*

$$\left\{\phi, \Phi_\mu\right\} \approx 0, \tag{A.40}$$

holds for all μ. Otherwise, ϕ is said to be second class.

[4] In [45], the definition of first and second class is given for phase space functions, of which constraints are a particular case. In this text we only apply this classification to constraints, obviating the need to present the more general definition.

Because one can redefine a constraint by adding a linear combination of other constraints, it is preferable to use the rank of the square matrix $\mathbf{M}_{\mu\nu}$ defined in terms of all pairwise Poisson brackets by,

$$\mathbf{M}_{\mu\nu} := \left\{\Phi_\mu, \Phi_\nu\right\}, \tag{A.41}$$

to establish the number of first- and second-class constraints. The rank of $\mathbf{M}_{\mu\nu}$ is then equal to the number $\mathscr{C}_2$ of second-class constraints. Due to the antisymmetry of $\mathbf{M}_{\mu\nu}$, $\mathscr{C}_2$ has to be an even number. It follows that the number $\mathscr{C}_1$ of first-class constraints is given by $\mathscr{C}_1 = m + k - \mathscr{C}_2$.

Recall now the decomposition (A.39) of the Lagrange multipliers u^α in terms of U^α and V^α. By definition, we have that $V^\alpha \left\{\Phi_\mu, \phi_\alpha\right\} \approx 0$. It follows that the constraints

$$\phi_b := V_b^\alpha \phi_\alpha, \tag{A.42}$$

form a complete set of first-class constraints. In other words, it can be shown that any first-class constraint is a linear combination of the ϕ_b. This proves that the number B of arbitrary coefficients v^b remaining in the Lagrange multipliers after finishing the Dirac algorithm is equal to the number of first-class constraints.

Let $\dim\mathscr{P} = 2n$ denote the dimension of the phase space $\mathscr{P}$. Since physical configurations lie within the constraint surface, the physics of a constrained system takes place on a $2n - (m+k)$-dimensional subspace of the phase space. However, even when we impose all the constraints on the equations of motion, there are still up to m arbitrary functions appearing in these equations. The number of arbitrary functions is equal to the number of undetermined Lagrange multipliers at the end of the Dirac algorithm. Only the Lagrange multipliers associated with primary second-class constraints are determined, since the first-class constraints generate transformations which leave the physical state of the system invariant. Hence, the dimension of the space where physical motion takes place is further reduced by the number of independent first-class constraints. Recall that for an unconstrained system, we associate a degree of freedom with half the dimensionality of its phase space, that is, a system with n local physical degrees of freedom has a phase space of dimension $2n$. Therefore, the number of physical degrees of freedom $\mathscr{N}$ for a constrained system is given by

$$\mathscr{N} = \frac{1}{2}\left(\dim\mathscr{P} - 2\mathscr{C}_1 - \mathscr{C}_2\right). \tag{A.43}$$

A.2.3 Field-theoretical generalisations

We have so far discussed finite-dimensional constrained systems with a $2n$-dimensional phase-space $\mathscr{P}$. Since we want to apply these results to general relativity, we now discuss the case of classical field theories.

Instead of n position variables $q^i(t)$, we have as field configurations the metric tensor $g_{ij}(x,t)$, the shift vector $N^i(x,t)$, and the lapse function $N(x,t)$, where x refers to coordinates on a three-dimensional spatial hypersurface Σ_t, and Latin indices correspond to tangent directions of Σ_t. In what follows, we will address the generalisation of constrained systems to a theory of the metric g_{ij}, because the inclusion of vectors and scalars is straightforward.

Let $F[g]$ be a functional of the metric g_{ij}. A variation of F with respect to $g_{kl}(z,t)$ is given by a generalisation of the chain rule

$$\delta_{g_{kl}(z,t)} F[g(x,t)] = \frac{\partial F[g(x,t)]}{\partial g_{ij}(x,t)} \frac{\delta g_{ij}(x,t)}{\delta g_{kl}(z,t)}, \tag{A.44}$$

where $\frac{\delta g_{ij}(x,t)}{\delta g_{kl}(z,t)}$ is given by

$$\frac{\delta g_{ij}(x,t)}{\delta g_{kl}(z,t)} = \delta^{kl}_{ij}\delta^3(x,z) := \frac{1}{2}\left(\delta^k_i\delta^l_j + \delta^k_j\delta^l_i\right)\delta^3(x,z), \tag{A.45}$$

where δ^a_b stands for the Kronecker delta and $\delta^3(x,z)$ is the three-dimensional Dirac delta function between x and z, behaving as a scalar in the argument x and a density in z. Denote the momentum density conjugate to g_{ij} by π^{ij}. For simplicity, we introduce the following short-hand

$$A[x] := A\left[g_{ij}(x,t), \pi^{ij}(x,t)\right], \tag{A.46}$$

for the functional dependence of some functional A of the metric g_{ij} and its momentum π^{ij}. The field-theoretical Poisson bracket between functionals A and B is given by

$$\{A[x], B[y]\} = \int d^3z \left(\delta_{g_{ij}(z)}A[x]\,\delta_{\pi^{ij}(z)}B[y] - \delta_{\pi^{ij}(z)}A[x]\,\delta_{g_{ij}(z)}B[y]\right). \tag{A.47}$$

Because the functional derivative (A.44) is given in terms of distributions, eq. (A.47) is not the most suitable expression for practical computations. It is preferable to work with finite, well-defined expressions rather than distributions, by introducing arbitrary test functions $\eta(x,t)$ and $\chi(x,t)$ on Σ. The Poisson bracket between A and B is replaced by its "smeared-out" version

$$\{A,B\} \to \left\{\int d^3x\, \eta(x,t) A[x], \int d^3y\, \chi(y,t) B[y]\right\}, \tag{A.48}$$

The original Poisson bracket between the functionals A and B can then be read off from the integrand on the right-hand side of

$$\left\{\int d^3x\, \eta(x,t) A[x], \int d^3y\, \chi(y,t) B[y]\right\} = \int d^3z\, \eta(z,t)\chi(z,t)\{A[x], B[y]\}. \tag{A.49}$$

Note that it may be necessary to perform partial integrations in order to cast the left-hand side of eq. (A.49) in the form of the right-hand side. This process is particularly useful when considering the time evolution of a system. Unlike in the finite-dimensional case, the total Hamiltonian is now the spatial integral over Σ of a total Hamiltonian density. Hence, computing a time evolution corresponds to a Poisson bracket of a local field functional $A\left[g_{ij}(x,t), \pi^{ij}(x,t)\right]$ with a global functional H_{tot}. Because of this global property of H_{tot}, the time derivative $\dot{A}$ of A can also be recovered from inside an integral, namely,

$$\left\{\int d^3x\, \eta(x,t) A[x], H_{tot}\right\} = \int d^3z\, \eta(z,t)\dot{A}. \tag{A.50}$$

For the field-theoretical case, the counting of degrees of freedom refers to the local physical degrees of freedom at a point, which we denote by $\mathscr{N}$. In that local sense, the phase space $\mathscr{P}$ has a local dimension of $2n$, where we assume n independent field configurations per point - the conjugate momentum density has the same number of independent components. In the same way, we denote by $\mathscr{C}_i$ the number of local constraints of i^{th}-class. Then, we can apply eq. (A.43) to obtain the number of local degrees of freedom for field theories. In the case of the spatial metric discussed above, for 3 spatial dimensions, there are $\frac{3(3+1)}{2}$ independent components at each point and therefore $\dim \mathscr{P} = 12$.

Finally, note that there are subtleties regarding the validity of the theorems provided for the finite-dimensional case in the field-theoretical case discussed here, which we did not address.

B

Foliations and the ADM decomposition

In this appendix, we discuss foliated spacetimes and their application to general relativity. In Sec. B.1, we discuss only geometrical aspects, independent of any gravitational theory. In particular, we discuss the notion of three-dimensional spatial hypersurfaces Σ embedded in a four-dimensional spacetime M and later extend the discussion to foliations of spacetime. The last part of this first section is the derivation of an expression for the Ricci scalar of a four-dimensional manifold in terms of the Ricci scalar of a spatial hypersurface and information about its embedding in M, such as the extrinsic curvature of Σ and the lapse function N.

In Sec. B.2, we apply this construction to general relativity. We finish this section with a brief recap of the Hamiltonian formulation of general relativity. Because general relativity is a constrained Hamiltonian system, we will make use of the results of appendix A here. Finally, note that in appendix C we use the results of this appendix to discuss the initial value formulation of general relativity.

We distinguish between quantities defined on four- and three-dimensional spaces by using Greek indices for the former and Latin ones for the latter. Scalar quantities such as the Ricci scalar $\mathscr{R}$ are distinguished by using a dimensional prefix for the four-dimensional versions, e.g. ${}^{(4)}R$.

B.1 Geometry of foliated spacetimes

This section is based on chapters 2, 3, and 4 of Gourgoulhon's notes [42]. It provides merely an overview of the general ideas presented there, not a comprehensive review. The goal is to clarify the conventions used throughout the thesis and provide a self-contained explanation of the expressions used.

As in the rest of the thesis, we denote a four-dimensional smooth manifold by M and a Lorentzian metric of signature $(-,+,+,+)$ on M by ${}^{(4)}g_{\mu\nu}$. A spacetime is a pair $\left(M, {}^{(4)}g_{\mu\nu}\right)$, although we will often refer only to M as a spacetime when it is clear that there is a specific four-dimensional metric associated with it. We begin by defining a three-dimensional hypersurface Σ of a manifold M.

Definition 6. *A hypersurface Σ is the image of a three-dimensional manifold $\bar{\Sigma}$ by an embedding map Φ,*

$$\Phi : \bar{\Sigma} \longrightarrow M. \tag{B.1}$$

We can also define Σ locally as the set of points in M for which some scalar field t is constant,

$$\Sigma := \{p \in M \,|\, t(p) = 0\}. \tag{B.2}$$

The three-metric g_{ij} on Σ is formally defined as the pullback of ${}^{(4)}g_{\mu\nu}$ under the embedding Φ. A hypersurface Σ is said to be spacelike if and only if its induced metric g_{ij} is positive definite, and null if its induced metric is degenerate, that is, its signature is $(0,+,+)$ everywhere. From now on, we will assume that Σ is spacelike unless otherwise specified.

We denote by D_μ the covariant derivative associated with ${}^{(4)}g_{\mu\nu}$ and by ∇_i the covariant derivative associated[1] with the metric g_{ij} on Σ.

A foliation only exists if the spacetime is globally hyperbolic, which is equivalent to saying that it admits a Cauchy surface Σ.

Definition 7. *A Cauchy surface is a spacelike hypersurface Σ in M which is intersected exactly once by each timelike or null curve without endpoint.*

This allows us to define a foliation of M.

Definition 8. *Let M be globally hyperbolic. Then there exists a smooth scalar field $\hat{t}$ on M whose gradient never vanishes and for each $t \in \mathbb{R}$, there is a hypersurface Σ_t defined by*

$$\Sigma_t := \{x \in M,\ \hat{t}(x) = t\}. \tag{B.3}$$

Because of the gradient of $\hat{t}$ does not vanish, different hypersurfaces do not intersect,

$$\Sigma_t \cap \Sigma_{t'} = \emptyset, \qquad \text{for } t \neq t'. \tag{B.4}$$

The family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ constitutes a foliation of M and each Σ_t is called a leaf of the foliation.

In what follows, we will not distinguish between the scalar field $\hat{t}$ and its values $t \in \mathbb{R}$. We will also assume that the foliation covers the entire spacetime manifold,

$$\bigcup_{t\in\mathbb{R}} \Sigma_t = M. \tag{B.5}$$

Given the scalar field t on M, its gradient one-form dt is normal to Σ_t, in the sense that its action on vectors tangent to Σ_t vanishes. Its dual $D^\mu t$ is a normal vector with respect to Σ_t and we use it to define the unit normal n^μ to Σ_t as

$$n^\mu = \left(-{}^{(4)}g_{\alpha\beta}\, D^\alpha t\, D^\beta t\right)^{-1/2} D^\mu t\,. \tag{B.6}$$

[1]This convention is opposite to the one used in [42].

It follows that ${}^{(4)}g_{\mu\nu}n^{\mu}n^{\nu} = -1$. The multiplying factor between n^{μ} and $D^{\mu}t$ is called the lapse function N,

$$N := \left(-{}^{(4)}g_{\alpha\beta}\, D^{\alpha}t\, D^{\beta}t\right)^{-1/2} . \tag{B.7}$$

Note that its definition in eq. (B.7) is such that

$$N > 0 \tag{B.8}$$

for any spacelike hypersurface.

To introduce coordinates adapted to the foliation, consider coordinates $x^i = \left(x^1, x^2, x^3\right)$ on each hypersurface Σ_t, defined such that they vary smoothly between hypersurfaces. Then, $x^{\mu} = \left(t, x^i\right)$ is a coordinate system on M. We denote the natural basis of T_pM associated with x^{μ} by $\partial_{\mu} = (\partial_t, \partial_i)$. Note that $\partial_i \in T_p\Sigma_t$, while the time vector ∂_t is tangent to the lines of constant x^i.

The extrinsic curvature K_{ij} of Σ_t on M is defined from the covariant derivative of the normal vector with respect to the four-metric,

$$K_{ij} = \left(D_{\nu}n_{\mu}\right)(\partial_i)^{\mu}\left(\partial_j\right)^{\nu} , \tag{B.9}$$

where $(\partial_i)^{\mu}$ denotes the components of the vector ∂_i with respect to the basis ∂_{μ} defined above. Note that eq. (B.9) differs from the definition of the extrinsic curvature in reference [42] by an overall minus sign and the same applies to all terms linear in the extrinsic curvature appearing in this chapter.

The space of all four-dimensional vectors at each point $p \in M$ can be decomposed as

$$T_pM = T_p\Sigma_t \oplus \mathrm{Vect}(\vec{n}) , \tag{B.10}$$

where $\mathrm{Vect}(\vec{n})$ denotes the one-dimensional subspace of T_pM spanned by the unit normal to Σ_t. To project a vector onto Σ_t, it is necessary to use an orthogonal projector $\vec{\gamma}$. In coordinate-free notation, this projector is defined as

$$\begin{aligned} \gamma^{\mu} : T_pM &\to T_p\Sigma_t , \\ v^{\mu} &\mapsto v^{\mu} + v^{\nu}n_{\nu}n^{\mu} . \end{aligned} \tag{B.11}$$

In coordinates, it reads

$$\gamma^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + n^{\mu}n_{\nu} . \tag{B.12}$$

The projector can be used, through its pullback, to extend the induced metric g_{ij} on Σ_t to act on vectors in the tangent space T_pM of the manifold. Consequently, the orthogonal projector (B.12) is just the extended metric $g_{\mu\nu}$ with its first index raised by ${}^{(4)}g_{\mu\nu}$, that is,

$$g_{\mu\nu} := \gamma_{\mu\nu} = {}^{(4)}g_{\mu\nu} + n_{\mu}n_{\nu} , \tag{B.13}$$

where the $g_{\mu\nu}$ on the left-hand side of eq. (B.13) is not a metric on M, but the induced metric g_{ij} pulled back through $\vec{\gamma}$ to act on vectors in T_pM. We will refer to the extended version of

g_{ij} as $\gamma_{\mu\nu}$ to avoid confusion. We can also extend the extrinsic curvature by a pullback to act on vectors in the tangent space of M, obtaining

$$K_{\mu\nu} = D_\nu n_\mu + a_\mu n_\nu, \tag{B.14}$$

where a_μ stands for

$$a_\mu = \nabla_\mu \ln N, \tag{B.15}$$

and ∇_μ denotes the extension of the covariant derivative with respect to the spatial metric via the orthogonal projection. Since n^μ is a unit timelike vector, it can be seen as the four-velocity of some observer. Such an observer is called an Eulerian observer and it follows that its acceleration is given by a^μ.

We now define the concept of Lie dragging.

Definition 9. *A vector v^μ is said to Lie drag a hypersurface Σ_t if for any point p in Σ_t, the point $(p')^\mu = p^\mu + \delta t v^\mu$ belongs to $\Sigma_{t+\delta t}$,*

$$t\left(p^\mu + \delta t v^\mu\right) = t(p) + \delta t. \tag{B.16}$$

A vector which has precisely this property is the so-called normal evolution vector m^μ,

$$m^\mu := N n^\mu. \tag{B.17}$$

Recall the expression of the Lie derivative of $\vec{v}$ along $\vec{u}$, $\mathscr{L}_{\vec{u}}\vec{v}$,

$$\mathscr{L}_{\vec{u}} v^\mu = u^\nu \frac{\partial v^\mu}{\partial x^\nu} - v^\nu \frac{\partial u^\mu}{\partial x^\nu} = [\vec{u}, \vec{v}]^\mu . \tag{B.18}$$

The Lie dragging of Σ_t by $\vec{m}$ implies that the Lie derivative along $\vec{m}$ of every $\vec{v} \in T_p\Sigma_t$, denoted by $\mathscr{L}_{\vec{m}} v^\mu$, is still tangent to Σ_t. Taking the Lie derivative of the extended metric along $\vec{m}$, we obtain

$$\mathscr{L}_{\vec{m}}\, \gamma_{\alpha\beta} = 2N K_{\alpha\beta}\,, \tag{B.19}$$

a result which we will need when we discuss the decomposition of the Riemann tensor.

Another vector which Lie drags the leaves of the foliation is the time vector ∂_t introduced earlier in this section. If the coordinates $\left(x^i\right)$ on Σ_t are such that the lines of constant x^i are orthogonal to Σ_t for all t, then $\vec{m}$ and ∂_t coincide. If not, their difference is called the shift vector $\vec{N}$,

$$N^\mu = (\partial_t)^\mu - m^\mu. \tag{B.20}$$

The fact that the shift belongs to $T_p\Sigma_t$, together with eq. (B.20) imply that the timelike unit normal n^μ can be decomposed in terms of N and N^i as

$$n^\mu = \left(\frac{1}{N}, \frac{-N^i}{N}\right). \tag{B.21}$$

Moreover, the time vector can be written as

$$(\partial_t)^\mu = N n^\mu + N^\mu. \tag{B.22}$$

Finally, note that the time vector can either be timelike, null, or spacelike depending on the relative size of shift and lapse. This can be seen by computing its scalar square

$$ {}^{(4)}g_{\mu\nu}\,(\partial_t)^\mu\,(\partial_t)^\nu = -N^2 + N^i N_i\,, \tag{B.23} $$

where we have written the shift in spatial coordinates because it is tangent to Σ_t and therefore $g_{ij}N^iN^j = g_{\mu\nu}N^\mu N^\nu$.

In the remainder of this section, we review the decomposition of ${}^{(4)}R^\rho{}_{\sigma\mu\nu}$ in terms of three-dimensional quantities. The derivation can be found in full detail in [42].

B.1.1 The $3+1$ decomposition of the Riemann tensor

Due to the symmetry properties of the Riemann tensor, there are three $3+1$-projections of ${}^{(4)}R^\rho{}_{\sigma\mu\nu}$ that can yield non-vanishing results. One can fully project it onto Σ_t, three times onto Σ_t and once along the normal n^μ, and twice onto Σ_t and twice along the normal n^μ. The first option yields the so-called Gauss relation,

$$ \gamma^\mu{}_\alpha\gamma^\nu{}_\beta\gamma^\gamma{}_\rho\gamma^\sigma{}_\delta\ {}^{(4)}R^\rho{}_{\sigma\mu\nu} = \mathscr{R}^\gamma{}_{\delta\alpha\beta} + K^\gamma{}_\alpha K_{\delta\beta} - K^\gamma{}_\beta K_{\alpha\delta}\,. \tag{B.24} $$

Contracting the indices γ and α, we obtain the contracted Gauss relation

$$ \gamma^\mu{}_\alpha\gamma^\nu{}_\beta\ {}^{(4)}R_{\mu\nu} + \gamma_{\alpha\mu}n^\nu\gamma^\rho{}_\beta n^\sigma\ {}^{(4)}R^\mu{}_{\nu\rho\sigma} = \mathscr{R}_{\alpha\beta} + KK_{\alpha\beta} - K^\mu{}_\beta K_{\alpha\mu}\,. \tag{B.25} $$

Because the right-hand side depends only on three-dimensional quantities, we can take its trace with respect to $\gamma^{\alpha\beta}$, obtaining the scalar Gauss relation

$$ {}^{(4)}R + 2\,{}^{(4)}R_{\mu\nu}n^\mu n^\nu = \mathscr{R} + K^2 - K_{ij}K^{ij}\,. \tag{B.26} $$

To project the Riemann tensor along the normal n^μ, we can pick any of its indices. Due to the symmetry properties of the tensor, this choice can be made without loss of generality because the expressions obtained that way will differ at most by a minus sign. With this in mind, we repeat the choice made in [42] to obtain the Codazzi equation,

$$ \gamma^\mu{}_\alpha\gamma^\nu{}_\beta\gamma^\gamma{}_\rho n^\sigma\ {}^{(4)}R^\rho{}_{\sigma\mu\nu} = \nabla_\alpha K^\gamma{}_\beta - \nabla_\beta K^\gamma{}_\alpha\,. \tag{B.27} $$

Finally, we project the Riemann tensor twice onto Σ_t and twice along n^μ, obtaining the Ricci equation,

$$ \gamma_{\alpha\mu}n^\rho\gamma^\nu{}_\beta n^\sigma\ {}^{(4)}R^\mu{}_{\rho\nu\sigma} = \frac{1}{N}\nabla_\alpha\nabla_\beta N + K_{\alpha\mu}K^\mu{}_\beta - \frac{1}{N}\mathscr{L}_{\vec{m}}K_{\alpha\beta}\,. \tag{B.28} $$

We use these projections to write the four-dimensional Ricci scalar ${}^{(4)}R$ in terms of three-dimensional quantities. Note that the left-hand side of the Ricci eq. (B.28) appears in the contracted Gauss relation (B.25). Making the appropriate substitutions, we obtain

$$ \gamma^\mu{}_\alpha\gamma^\nu{}_\beta\ {}^{(4)}R_{\mu\nu} = \mathscr{R}_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu}K^\mu{}_\beta + \frac{1}{N}\mathscr{L}_{\vec{m}}K_{\alpha\beta} - \frac{1}{N}\nabla_\alpha\nabla_\beta N\,. \tag{B.29} $$

Because the right-hand side of eq. (B.29) depends only on spatial quantities, we can compute its trace with respect to $\gamma^{\alpha\beta}$, yielding

$$ {}^{(4)}R + {}^{(4)}R_{\mu\nu}\, n^\mu n^\nu = \mathscr{R} + K^2 - 2K_{ij}K^{ij} + \frac{g^{ij}}{N}\mathscr{L}_{\vec{m}} K_{ij} - \frac{1}{N}\nabla_i\nabla^i N\,. \tag{B.30} $$

Using the Leibniz rule on the Lie derivative term, one obtains

$$ g^{ij}\mathscr{L}_{\vec{m}} K_{ij} = \mathscr{L}_{\vec{m}} K - K_{ij}\mathscr{L}_{\vec{m}}\, g^{ij} = \mathscr{L}_{\vec{m}} K + 2K_{ij}K^{ij}\,. \tag{B.31} $$

We can turn eq. (B.30) into the sought-for relation between the four-dimensional Ricci scalar and three-dimensional quantities using eqs. (B.31) and (B.26),

$$ {}^{(4)}R = \mathscr{R} + K^2 + K^{ij}K_{ij} + \frac{2}{N}\mathscr{L}_{\vec{m}} K - \frac{2}{N}\nabla_i\nabla^i N\,. \tag{B.32} $$

In chapter 4, we obtain spherically symmetric solutions to the λ-R model in terms of variables on Σ_t and use eq. (B.32) to show that the four-dimensional curvature of these solutions in general does not vanish.

B.2 The ADM formulation of general relativity

The ADM formulation of general relativity, named after Arnowitt, Deser, and Misner, the authors of reference [3], is a formulation of Einstein's theory of gravity in terms of the $3+1$ decomposition discussed in the previous section. In this section, we will follow reference [42], because it is more consistent with the language used in this thesis. We first write the four-dimensional metric ${}^{(4)}g_{\mu\nu}$ in terms of three-dimensional objects, show how Einstein's equations reduce to four constraints and six equations of motion when projected via the $3+1$ split and then write the Einstein-Hilbert action in this language. After that, we review the Hamiltonian formulation of general relativity with the tools presented in appendix A.

To decompose the metric, we impose coordinates adapted to the foliation as described in the previous section and write

$$ {}^{(4)}\mathbf{g} :={}^{(4)} g_{\mu\nu} dx^\mu \otimes dx^\nu\,, \qquad {}^{(4)}g_{\mu\nu} ={}^{(4)} \mathbf{g}\left(\partial_\mu, \partial_\nu\right). \tag{B.33} $$

Explicitly, ${}^{(4)}g_{00}$ and ${}^{(4)}g_{0i}$ read

$$ {}^{(4)}g_{00} ={}^{(4)} g_{\mu\nu}\left(\partial_t\right)^\mu\left(\partial_t\right)^\nu = -N^2 + N^iN_i\,, \qquad {}^{(4)}g_{0i} ={}^{(4)} \mathbf{g}\left(\partial_t, \partial_i\right) = N_j dx^j\left(\partial_i\right) = N_i\,, \tag{B.34} $$

while ${}^{(4)}g_{ij} = g_{ij}$ as noted previously. In matrix terms, this reads

$$ {}^{(4)}g_{\mu\nu} = \begin{pmatrix} -N^2 + N^iN_i & N_j \\ N_i & g_{ij} \end{pmatrix}. \tag{B.35} $$

The components of the inverse metric ${}^{(4)}g^{\mu\nu}$ are given by the matrix inverse of eq. (B.35),

$$ {}^{(4)}g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & g^{ij} - \frac{N^jN^i}{N^2} \end{pmatrix}. \tag{B.36} $$

Equivalently, we can encode the decomposition in the line element,

$$ ds^2 = -N^2dt^2 + g_{ij}\left(dx^i + N^idt\right)\left(dx^j + N^jdt\right). \tag{B.37} $$

B.2.1 Einstein-Hilbert action in the ADM formulation

Recall the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{-{}^{(4)}g} \left({}^{(4)}R - 2\Lambda \right). \tag{B.38}$$

To decompose the action (B.38), we first write $\sqrt{-{}^{(4)}g}$ in terms of three-dimensional quantities,

$$\sqrt{-{}^{(4)}g} = N\sqrt{g}. \tag{B.39}$$

Replacing the Ricci scalar ${}^{(4)}R$ as given in eq. (B.32), we obtain

$$\begin{aligned} S_{EH} =& \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} N \left(\mathscr{R} + K^2 + K^{ij}K_{ij} + \frac{2}{N}\mathscr{L}_{\vec{m}} K - \frac{2}{N}\nabla_i\nabla^i N - 2\Lambda \right) \\ =& \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} \left(N\left(\mathscr{R} + K^2 + K^{ij}K_{ij} - 2\Lambda \right) + 2\mathscr{L}_{\vec{m}} K - 2\nabla_i\nabla^i N \right) \end{aligned} \tag{B.40}$$

$$= \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} \left(N\left(\mathscr{R} + K^2 + K^{ij}K_{ij} - 2\Lambda \right) + 2\mathscr{L}_{\vec{m}} K \right) - 2\int_{t_1}^{t_2} dt \int_{\partial\Sigma_t} ds_i \sqrt{g} \nabla^i N. \tag{B.41}$$

The $\mathscr{L}_{\vec{m}} K$-term yields

$$\mathscr{L}_{\vec{m}} K = N\left(D_\mu \left(K n^\mu \right) - K^2 \right), \tag{B.42}$$

which when substituted back into the action results in

$$S_{EH} = \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} N \left(\mathscr{R} + K^{ij}K_{ij} - K^2 - 2\Lambda + 2D_\mu \left(K n^\mu \right) \right) - 2\int_{t_1}^{t_2} dt \int_{\partial\Sigma_t} ds_i \sqrt{g} \nabla^i N \tag{B.43}$$

$$= \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} N \left(\mathscr{R} + K^{ij}K_{ij} - K^2 - 2\Lambda \right) - 2\int_{t_1}^{t_2} dt \int_{\partial\Sigma_t} ds_i \sqrt{g} \left(\nabla^i N - K N^i \right), \tag{B.44}$$

where we have discarded the boundary term in the t-direction since it does not contribute to the variational problem. We leave the discussion of the boundary contributions for both the asymptotically flat and asymptotically null cases to the analogous discussion of the λ-R model in chapter 2. For now, let us assume that the hypersurfaces Σ_t are closed and compact. In this case the Einstein-Hilbert action in terms of ADM variables is given by

$$S_{EH} = \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} N \left(\mathscr{R} + K^{ij}K_{ij} - K^2 - 2\Lambda \right), \tag{B.45}$$

where the $K_{ij}K^{ij} - K^2$-term can be interpreted as the kinetic term of the action, since it contains all time derivatives of the metric g_{ij}. The $\mathscr{R} - 2\Lambda$ term, containing only spatial derivatives and a constant, can be interpreted as the potential. We can introduce a supermetric G^{ijkl}, a metric on $\mathrm{Riem}\,\Sigma$, such that the kinetic part S_K of the action becomes

$$S_K = \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} N \left(K^{ij}K_{ij} - K^2 \right) = \frac{1}{16\pi G_N} \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \sqrt{g} N\, G^{ijkl} K_{ij} K_{jk}, \tag{B.46}$$

where G^{ijkl} is the Wheeler-DeWitt metric, given by

$$G^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - g^{ij}g^{kl}. \tag{B.47}$$

Its inverse G_{ijkl} is given by

$$G_{ijkl} = \frac{1}{2}\left(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}\right). \tag{B.48}$$

Finally, note that the variations of the action (B.45) are subject to four constraints. To see this, one can take the equations of motion of the original action,

$${}^{(4)}R_{\mu\nu} - \Lambda^{(4)}g_{\mu\nu} = 0, \tag{B.49}$$

and project them in three different ways. Projecting them onto the normal vector n^μ, one obtains the so-called Hamiltonian constraint

$$\mathscr{R} - 2\Lambda + K^2 - K_{ij}K^{ij} = 0. \tag{B.50}$$

Projecting them once onto the normal and once onto Σ_t yields the three momentum constraints,

$$\nabla_j K^j{}_i - D_i K = 0. \tag{B.51}$$

Finally, projecting them twice onto Σ_t yields the six equations of motion for the metric g_{ij}, namely,

$$\mathscr{L}_{\vec{m}}K_{ij} = -\nabla_i\nabla_j N + N\left(R_{ij} + KK_{ij} - 2K_{ik}K^k_j\right). \tag{B.52}$$

B.3 General relativity in the Hamiltonian formalism

In this section, we will review the Hamiltonian formulation of general relativity. As in the previous section, we will only discuss the case of closed and compact spatial hypersurfaces Σ_t. This case differs from the open case only in the boundary Hamiltonian. Because we derive the boundary Hamiltonian for the λ-R model in detail in chapters 2 (asymptotically flat hypersurfaces) and 4 (asymptotically null hypersurfaces with spherical symmetry), we will not repeat the discussion here. Recall that in the ADM formulation the Einstein-Hilbert action for compact and closed hypersurfaces is given by

$$S_{EH} = \frac{1}{16\pi G_N}\int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x\,\sqrt{g}N\left(\mathscr{R} + K^{ij}K_{ij} - K^2 - 2\Lambda\right). \tag{B.53}$$

We define momentum densities associated with g_{ij}, N and N^i, by

$$\pi^{ij} := \frac{\delta S}{\delta \dot{g}_{ij}} = \sqrt{g}\,G^{ijkl}K_{kl}, \tag{B.54a}$$

$$\phi := \frac{\delta S}{\delta \dot{N}} = 0, \qquad \phi_i := \frac{\delta S}{\delta \dot{N}^i} = 0. \tag{B.54b}$$

As mentioned, the four primary constraints are given by the vanishing of the momenta associated with lapse and shift. Since eq. (B.54a) for π^{ij} is invertible, these are the only primary constraints of the theory and we obtain $\dot{g}_{ij}$ in terms of canonical variables as

$$\dot{g}_{ij} = \frac{2N}{\sqrt{g}} G_{ijkl}\pi^{ij} + \nabla_i N_j + \nabla_j N_i \,. \tag{B.55}$$

This allows us to write the Hamiltonian H as

$$H = \int d^3x \left(N\mathscr{H} + N^i \mathscr{H}_i \right), \tag{B.56}$$

where $\mathscr{H}$ and $\mathscr{H}_i$ are functionals of g_{ij} and π^{ij}, given by

$$\mathscr{H} = \frac{1}{\sqrt{g}} G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g}\left(\mathscr{R} - 2\Lambda\right), \tag{B.57a}$$

$$\mathscr{H}_i = -2g_{ij}\nabla_k \pi^{jk}. \tag{B.57b}$$

Adding the primary constraints $\phi = 0$ and $\phi_i = 0$ with associated Lagrange multipliers α and α^i, we obtain the total Hamiltonian H_{tot},

$$H_{tot} = H + \int d^3x \left(\alpha\phi + \alpha^i \phi_i \right) = \int d^3x \Big(N\mathscr{H} + N^i \mathscr{H}_i + \alpha\phi + \alpha^i\phi_i \Big). \tag{B.58}$$

Following the procedure outlined in appendix A, we demand that the primary constraints should be preserved in time,

$$\dot{\phi} \approx 0 \;\Rightarrow\; \{\phi, H_t\} = -\mathscr{H} \approx 0\,, \tag{B.59a}$$

$$\dot{\phi}_i \approx 0 \;\Rightarrow\; \{\phi_i, H_t\} = -\mathscr{H}_i \approx 0\,. \tag{B.59b}$$

The constraint surface of general relativity is defined by the simultaneous vanishing of the four primary constraints (B.54b) and the four secondary constraints (B.59). The constraint obtained by imposing $\dot{\phi} = 0$ is the Hamiltonian constraint $\mathscr{H} \approx 0$, while $\mathscr{H}_i \approx 0$ are the momentum constraints. To show that no further constraints are generated, we compute their Poisson brackets, leading to the so-called Dirac algebra of constraints,

$$\left\{ \int d^3x N_1^i \mathscr{H}_i, \int d^3x N_2^j \mathscr{H}_j \right\} = \int d^3z\, \mathscr{H}_j \left(N_1^i \nabla_i N_2^j - N_2^i \nabla_i N_1^j \right), \tag{B.60a}$$

$$\left\{ \int d^3x N\mathscr{H}, \int d^3x'\, N^a \mathscr{H}_a \right\} = -\int d^3z\, \mathscr{H} N^a \nabla_a N\,, \tag{B.60b}$$

$$\left\{ \int d^3x\, \eta\mathscr{H}, \int d^3x'\, N\mathscr{H} \right\} = \int d^3z\, g^{ij}\mathscr{H}_i \left(\eta\nabla_j N - N\nabla_j \eta \right). \tag{B.60c}$$

From these relations, we can compute the time evolution of $\mathscr{H}$ and $\mathscr{H}_i$,

$$\dot{\mathscr{H}} = g^{ij}\left(2\mathscr{H}_i \nabla_j N + N\nabla_j \mathscr{H}_i \right) \approx 0\,, \tag{B.61a}$$

$$\dot{\mathscr{H}}_i = \mathscr{H}\nabla_i N + \mathscr{H}_j \nabla_i N^j + N^j \nabla_j \mathscr{H}_i + \mathscr{H}_i \nabla_j N^j \approx 0\,. \tag{B.61b}$$

Since the right-hand sides of these equations vanish weakly, no new constraints are generated and the Dirac algorithm terminates.

The Dirac algebra also shows that all secondary constraints Poisson-commute among themselves on the constraint surface. This implies that they are first-class constraints, because they depend on the metric g_{ij} and the momentum π^{ij}, and their Poisson brackets with the primary constraints vanish weakly. All eight constraints are therefore first class. Using eq. (A.43) of appendix A for the number of local physical degrees of freedom, we conclude that general relativity has two local degrees of freedom,

$$\mathscr{N} = \frac{1}{2}\left(\dim\mathscr{P} - 2\mathscr{C}_1 - \mathscr{C}_2\right) = 2. \tag{B.62}$$

C

The conformal method in general relativity

In this appendix, we review the conformal method for solving the initial value problem of general relativity (see [80] for a recent review). We will do so in the Lagrangian formalism, in which it was developed originally. This means that we will temporarily base our discussion on the extrinsic curvature tensor instead of momentum densities.

The progress in the development of the conformal method falls broadly into two parts. First, following the work of French mathematician Lichnerowicz [69], in which it was noticed that when the extrinsic curvature tensor K_{ij} is traceless and transverse with respect to the metric g_{ij}, that is,

$$g^{ij}K_{ij} = 0\,, \qquad \text{and} \qquad \nabla_i K^{ij} = 0\,, \tag{C.1}$$

the momentum constraints $\mathscr{H}_i \approx 0$ are solved. From now on, a tensor K_{ij} satisfying assumptions (C.1) will be denoted by K_{ij}^{TT}, where TT stands for transverse-traceless. There are two important properties associated with eq. (C.1). On the one hand, solving the momentum constraints in this way decouples them from the Hamiltonian constraint, which is not solved by this choice. On the other hand, a transverse-traceless tensor solving $\mathscr{H}_i \approx 0$ retains this property under conformal transformations. This allows us to cast the Hamiltonian constraint into an equation for the conformal factor of the metric, the so-called Lichnerowicz equation. As we will show below, the set of initial data for which a solution to the equation exists is restricted. The second part in the development of the conformal method consists of a generalisation of these ideas such that the set of allowed initial data becomes arbitrary.

This was achieved by York [99–101], who showed that the conformal invariance of the solution to the momentum constraints is unchanged if the extrinsic curvature tensor has not only a transverse-traceless piece but also includes a non-vanishing - albeit spatially constant - trace K. The generalisation of the Lichnerowicz equation with a non-vanishing K is the so-called Lichnerowicz-York equation and allows for an almost unrestricted choice of initial data[1].

Recall the form of the Hamiltonian and momentum constraints with $\Lambda = 0$ written in the

[1]The remaining restrictions correspond to a set of measure zero in the space of possible initial data.

Lagrangian formulation presented in eqs. (B.50) and (B.51) in appendix B,

$$\mathscr{R} - K_{ij}K^{ij} + K^2 = 0\,, \tag{C.2a}$$

$$\nabla_j \left(K^{ij} - g^{ij}K\right) = 0\,, \tag{C.2b}$$

where we have raised the free index in eq. (B.51) to obtain eq. (C.2b). Because of the Dirac algebra, if eqs. (C.2) are satisfied on an initial hypersurface Σ_0, they will be satisfied at all times. To solve them, one has to specify a spatial metric g_{ij} and a symmetric tensor K_{ij} describing the extrinsic curvature of Σ_0 in the four-dimensional manifold M.

In the remainder of this appendix, we will first address the conformal method in maximal slicing coordinates and the associated Lichnerowicz equation. To discuss the existence of solutions to the Lichnerowicz equation, we will review the Yamabe classification of Riemannian manifolds, which is an important piece of the analysis presented in chapter 3. We then present the conformal method in constant mean curvature coordinates, focusing on the Lichnerowicz-York equation and its properties.

Analogous to what we did in chapter 3, we will denote initial data by $\left(g_{ij}, K_{ij}\right)$ and constraint-solving data by $\left(\bar{g}_{ij}, \bar{K}_{ij}\right)$.

C.1 Maximal slicing and the Lichnerowicz equation

The conformal method is based on Lichnerowicz's insight [69] that choosing a transverse-traceless extrinsic curvature tensor solves the momentum constraints eq. (C.2b). Consider the following decomposition of K_{ij},

$$K_{ij} = K_{ij}^{TT} + \nabla_i v_j + \nabla_j v_i - \frac{2}{3} g_{ij} \nabla_k v^k + \frac{1}{3} g_{ij} K\,. \tag{C.3}$$

where the components of v_i describe the longitudinal components of K_{ij}. To solve the momentum constraints with an arbitrary metric g_{ij}, it is sufficient to choose a K_{ij} that is transverse and traceless, such that the only non-vanishing contribution on the right-hand side of eq. (C.3) is K_{ij}^{TT}. This reduces the Hamiltonian constraint in eq. (C.2a) to

$$\mathscr{R} - K_{ij}^{TT} K_{TT}^{ij} = 0\,. \tag{C.4}$$

Consider the conformal transformation

$$\bar{g}_{ij} = \phi^4 g_{ij}\,, \tag{C.5}$$

of the metric, where ϕ is a function on Σ_0 that is strictly positive everywhere. If the extrinsic curvature is transverse-traceless, as we are assuming, then this property is invariant under conformal transformation (C.5), provided K_{ij}^{TT} transforms as

$$\bar{K}_{ij}^{TT} = \phi^{-2} K_{ij}^{TT}\,. \tag{C.6}$$

The power of ϕ in the transformation law (C.6) is chosen such that given a K_{ij}^{TT} that is transverse-traceless with respect to g_{ij}, the transformed tensor $\bar{K}_{ij}^{TT} = \phi^n K_{ij}^{TT}$ is transverse-traceless with respect to $\bar{g}_{ij}$ given by eq. (C.5). In three dimensions, this can be shown to hold if and only if $n = -2$. Under the transformation (C.5), the Ricci scalar transforms as

$$\bar{\mathscr{R}} = \phi^{-4}\mathscr{R} - 8\,\phi^{-5}\nabla^2\phi\,. \tag{C.7}$$

This transformation behaviour was the motivation for defining eq. (C.5) with the fourth power of ϕ in the first place, since it is the only choice for which the derivative operator in eq. (C.7) is the Laplacian.

We now have two sets of conformally related data satisfying the momentum constraints, (g_{ij}, K^{TT}_{ij}) and $(\bar{g}_{ij}, \bar{K}^{TT}_{ij})$, which we call "initial data" and "constraint-solving data" respectively. Despite their name, note that $\bar{g}_{ij}$ and $\bar{K}^{TT}_{ij}$ only solve the constraints after ϕ has been determined through the Lichnerowicz or Lichnerowicz-York equation. To transform the Hamiltonian constraint into an equation for the conformal factor, we first write it in terms of barred variables, $\mathscr{H}\left[\bar{g}_{ij}, \bar{K}_{ij}\right]$, and then substitute the latter using eqs. (C.5), (C.6) and (C.7). The resulting equation for ϕ is the so-called Lichnerowicz equation,

$$8\,\nabla^2\phi = \mathscr{R}\phi - \phi^{-7}K^{ij}_{TT}K^{TT}_{ij}\,. \tag{C.8}$$

To determine the solutions of this equation, it is preferable to work with a constant Ricci scalar $\mathscr{R}$. To understand how we can generally choose an initial metric g_{ij} such that $\mathscr{R}$ is constant on Σ_0, let us briefly review some results by Yamabe [97, 98] on the conformal properties of Riemannian manifolds.

Let (Σ, g) be a Riemannian manifold, either compact or asymptotically flat, of dimension $d \geq 3$. Then, there always exists a conformal transformation taking g_{ij} to $\tilde{g}_{ij}$ such that the Ricci scalar associated with $\tilde{g}_{ij}$ is constant. Moreover, there is a conformally invariant constant, since dubbed the "Yamabe constant" Y, which is defined by

$$Y := \inf_{\theta} \frac{\int d^3x\,\sqrt{g}\left(\mathscr{R}\theta^2 + 8\,(\nabla\theta)^2\right)}{\left(\int d^3x\,\sqrt{g}\,\theta^6\right)^{1/3}}, \tag{C.9}$$

where the infimum is taken over smooth functions θ for compact Σ and smooth functions θ of compact support in the asymptotically flat case. The value of Y defines a conformal equivalence class of metrics. When the minimising function θ itself is used as a conformal factor, $\phi = \theta$, Σ is mapped onto a manifold Σ_1 of constant curvature $\mathscr{R}_1$. Once the transformation has been performed, the same invariant Y can be computed with $\theta = 1$ as a minimising function, yielding

$$Y = \mathscr{R}_1 V_1^{2/3}, \tag{C.10}$$

where $V_1 = \int d^3x\sqrt{\bar{g}}$ is the volume of Σ_1. The sign of Y tells us that the manifold can be conformally mapped to another one with constant curvature of the same sign. This splits all metrics into three Yamabe classes, defined by having positive, negative, or vanishing Yamabe constants.

Consider for a moment a different notation where the transverse-traceless initial data is written as $(\tilde{g}_{ij}, \tilde{K}^{TT}_{ij})$. One then computes the Yamabe constant of this data and performs a conformal transformation by $\theta(x)$ to a set (g_{ij}, K^{TT}_{ij}) whose scalar curvature is constant.

From here, one proceeds as described above and obtains the Lichnerowicz equation with a constant Ricci scalar.

To avoid adding yet another notation, we will keep denoting initial data by undecorated variables $\left(g_{ij}, K_{ij}^{TT}, K\right)$ and tacitly assume that it has a constant scalar curvature. We should nevertheless keep in mind that in terms of uniqueness and existence of solutions, we are dealing with a whole class of initial data.

Returning to the Lichnerowicz equation (C.8), we integrate both sides over Σ_0. Due to Stokes' theorem, the left-hand side must vanish,

$$8\int d^3x\sqrt{g}\,\nabla^2\phi = 0\,, \tag{C.11}$$

implying that the same must be true for the integrated right-hand side, leading to the condition

$$\int d^3x\sqrt{g}\,\phi\mathscr{R} = \int d^3x\sqrt{g}\,\phi^{-7}K^{ij}_{TT}K^{TT}_{ij}\,. \tag{C.12}$$

Given that $\phi > 0$, $K_{ij}^{TT}K_{TT}^{ij} \geq 0$, and $\mathscr{R}$ is a constant, we see that only manifolds belonging to the positive Yamabe class will admit a solution to the Lichnerowicz equation.

C.2 The Lichnerowicz-York equation

We now review how to generalise the conformal method outlined above such that the restriction of the initial data in terms of its Yamabe class is lifted [82, 83]. As shown by York, to remove this restriction, one should choose a K_{ij} with a non-vanishing constant trace K.

We again impose the constant mean curvature condition $\nabla_i K = \partial_i K = 0$, but instead of choosing K_{ij} to be a symmetric transverse-traceless tensor, we allow for a non-vanishing constant-trace term

$$K_{ij} = K_{ij}^{TT} + \frac{1}{3}g_{ij}K\,. \tag{C.13}$$

Since $\nabla_i K = 0$, the trace drops out of the momentum constraints (C.2b) and therefore eq. (C.13) is also a solution to the constraints. Analogous to the requirement that the transverse-traceless property of K_{ij}^{TT} be preserved by conformal transformations, we also demand that the trace remains constant on Σ_0. The set of constraint-solving data is now related to the initial data by

$$\bar{g}_{ij} = \phi^4 g_{ij}\,, \qquad \bar{K}_{ij}^{TT} = \phi^{-2}K_{ij}^{TT}\,, \qquad \bar{K} = K\,, \tag{C.14}$$

which implies that the extrinsic curvature on the left-hand side of eq. (C.13) does not transform homogeneously under ϕ. In this version, the initial data is not a pair of a symmetric extrinsic curvature tensor and a Riemannian three-dimensional metric, but rather a trio consisting of a metric g_{ij}, a symmetric transverse-traceless tensor K_{ij}^{TT}, and a constant scalar K. All three are to be specified independently.

Evaluating the Hamiltonian constraint in eq. (C.2a) for the constraint-solving data,

$$\bar{\mathscr{R}} - \bar{K}_{ij}^{TT}\bar{K}_{TT}^{ij} + \frac{2}{3}\bar{K}^2 = 0, \tag{C.15}$$

and substituting the expressions (C.14) into eq. (C.15) yields the Lichnerowicz-York equation

$$8\nabla^2\phi = \mathscr{R}\phi - \phi^{-7}K^{ij}_{TT}K^{TT}_{ij} + \frac{2}{3}\phi^5 K^2. \tag{C.16}$$

To discuss the existence and uniqueness of solutions of eq. (C.16), it is useful to think of the right-hand side as a polynomial in ϕ,

$$P(\phi) := \mathscr{R}\phi - \phi^{-7}K^{ij}_{TT}K^{TT}_{ij} + \frac{2}{3}\phi^5 K^2. \tag{C.17}$$

Note that the coefficient of the ϕ^{-7}-term, $K^{TT}_{ij}K^{ij}_{TT} \geq 0$, depends also on $x \in \Sigma_0$, whereas $\mathscr{R}$ and K are purely time-dependent. As before, the integral of the left-hand side of eq. (C.16) must vanish, implying

$$\int d^3x \sqrt{g}\, P(\phi) = 0\,, \tag{C.18}$$

which leads to the conclusion that $P(\phi)$, viewed as a real function, must have at least one zero. Provided $K^{TT}_{ij}K^{ij}_{TT} \neq 0$, the asymptotic behaviour of $P(\phi)$ guarantees that the polynomial vanishes at some point,

$$\lim_{\phi \to 0^+} P(\phi) = -\infty\,, \qquad \lim_{\phi \to +\infty} P(\phi) = +\infty\,. \tag{C.19}$$

Therefore, unlike in the $K = 0$ case, the Yamabe class of (Σ_0, g) is not relevant for the existence of at least one zero, although the mere existence of such a zero is not enough to guarantee the existence of a solution. In reference [82], York and Ó Murchadha prove two theorems that not only guarantee the existence of a solution, but also show that it is almost always unique. By "almost always", we mean that the cases for which the solution is not unique have measure zero in the set of all initial data. For completeness, let us state the two theorems without proofs, starting with the existence theorem.

Theorem 3. *The equation $\nabla^2\phi = \frac{P(\phi)}{8}$ has a positive bounded solution ϕ if there exist two positive constants $\phi_- < \phi_+$ such that*

$$\left.\begin{aligned} P(\phi_-) &< 0 \\ P(\phi_+) &> 0 \end{aligned}\right\} \forall x \in \Sigma\,. \tag{C.20}$$

The solution lies in the interval (ϕ_-, ϕ_+).

Moreover, it is shown in reference [82] that $P(\phi)$ always has a single zero. Because $\mathscr{R}$ and K are spatial constants, eq. (C.19) shows that the polynomial behaves as required by eq. (C.20) around this zero, going from negative to positive values. In other words, Theorem 3 is satisfied as long as $K^{ij}_{TT}K^{TT}_{ij}$ is bounded, which is a reasonable condition since $K^{ij}_{TT}K^{TT}_{ij} \to \infty$ would be unphysical. The uniqueness theorem proved in reference [82] states that

Theorem 4. *On a closed manifold, any positive bounded solution to* (C.16) *is unique except in the trivial case of $K^{TT}_{ij}K^{ij}_{TT} = K = 0$ everywhere.*

All statements provided in chapter 3 regarding the existence and uniqueness of solutions to the generalised Lichnerowicz-York equation describing the λ-R model rely on both Theorem 3 and Theorem 4.

D

The constraint algebra of the λ-R model with matter

In this appendix, we generalise the results on the constraint algebra of the λ-R model from chapter 2 to include matter. The way in which we introduce matter follows closely reference [47]. Consider the total Hamiltonian

$$H_{tot} = \int d^3x \left(N \left(\mathscr{H}_\lambda + \mathscr{H}^{(M)} \right) + N^i \left(\mathscr{H}_i + \mathscr{H}_i^{(M)} \right) + \alpha\phi + \alpha^i \phi_i \right), \tag{D.1a}$$

$$:= \int d^3x \left(N \mathscr{H}^{\lambda,(M)} + N^i \mathscr{H}_i^{\lambda,(M)} + \alpha\phi + \alpha^i \phi_i \right). \tag{D.1b}$$

As in the purely gravitational case, the total Hamiltonian is linear in the lapse and the shift. The Hamiltonian constraint $\mathscr{H}^{\lambda,(M)}$ is a sum of the gravitational Hamiltonian $\mathscr{H}_\lambda$ and the matter Hamiltonian $\mathscr{H}^{(M)}$, while the total momentum constraint $\mathscr{H}_i^{\lambda,(M)}$ is a sum of the gravitational momentum $\mathscr{H}_i$ and the matter momentum $\mathscr{H}_i^{(M)}$. We describe the matter fields collectively by ψ_A, where the range of the index A depends on the matter content considered. The momentum associated with ψ_A is denoted by π^A. The ansatz (D.1a) for the total Hamiltonian assumes that the matter momenta are invertible functions of ψ_A and its time derivatives. The only primary constraints of the theory are therefore $\phi = 0$ and $\phi_i = 0$.

We assume that both the matter Hamiltonian $\mathscr{H}^{(M)}$ and momentum $\mathscr{H}_i^{(M)}$ are functionals of ψ_A and π^A. Furthermore, the matter momentum $\mathscr{H}_i^{(M)}$ is independent of the metric and its momenta, while the matter Hamiltonian $\mathscr{H}^{(M)}$ has an ultra-local dependence on the metric g_{ij}. Under these assumptions, the secondary constraints are given by

$$\mathscr{H}^{\lambda,(M)} = \mathscr{H}_\lambda + \mathscr{H}^{(M)} \approx 0, \tag{D.2}$$

$$\mathscr{H}_i^{\lambda,(M)} = \mathscr{H}_i + \mathscr{H}_i^{(M)} \approx 0. \tag{D.3}$$

We must compute the Poisson brackets between the matter and gravitational pieces of the secondary constraints to determine their evolution in time. The Poisson bracket between the matter Hamiltonian and the gravitational momentum can be readily computed and yields

$$\left\{\int d^3x N\mathscr{H}^{(M)}, \int d^3\, yN^i\mathscr{H}_i\right\} = -2\int d^3z\, N^i g_{ik}\nabla_l\left(N\frac{\partial\mathscr{H}^{(M)}}{\partial g_{kl}}\right). \tag{D.4}$$

Because the λ-R model is invariant under spatial diffeomorphisms, we want to preserve this part of the Dirac algebra, and we impose that

$$\left\{\int d^3x N^i\mathscr{H}_i^{\lambda,(M)}, \int d^3y\tilde{N}^j\mathscr{H}_j^{\lambda,(M)}\right\} = \int d^3z\mathscr{H}_j^{\lambda,(M)}\left(N^i\nabla_i\tilde{N}^j - \tilde{N}^i\nabla_i N^j\right), \tag{D.5a}$$

$$\left\{\int d^3x N\mathscr{H}^{\lambda,(M)}, \int d^3y N^i\mathscr{H}_i^{\lambda,(M)}\right\} = -\int d^3z\mathscr{H}^{\lambda,(M)}N^i\nabla_i N. \tag{D.5b}$$

Eqs. (D.5) are indeed satisfied when the Poisson brackets between the matter Hamiltonian $\mathscr{H}^{(M)}$ and the matter momentum $\mathscr{H}_i^{(M)}$ are given by

$$\left\{\int d^3x N^i\mathscr{H}_i^{(M)}, \int d^3y\tilde{N}^j\mathscr{H}_j^{(M)}\right\} = \int d^3z\,\mathscr{H}_j^{(M)}\left(N^i\nabla_i\tilde{N}^j - \tilde{N}^i\nabla_i N^j\right), \tag{D.6a}$$

$$\left\{\int d^3x N\mathscr{H}^{(M)}, \int d^3y N^i\mathscr{H}_i^{(M)}\right\} = -\int d^3z\left[\mathscr{H}^{(M)}N^i\nabla_i N - 2N^i g_{ik}\nabla_l\left(N\frac{\partial\mathscr{H}^{(M)}}{\partial g_{kl}}\right)\right], \tag{D.6b}$$

$$\left\{\int d^3x N\mathscr{H}^{(M)}, \int d^3y\tilde{N}\mathscr{H}^{(M)}\right\} = \int d^3z\, g^{ij}\mathscr{H}_i^{(M)}\left(N\nabla_j\tilde{N} - \tilde{N}\nabla_j N\right), \tag{D.6c}$$

where the second term of eq. (D.6b) was introduced to cancel the right-hand side of eq. (D.4).

The time derivatives of the momentum constraints $\mathscr{H}_i^{\lambda,(M)}$ therefore vanish on the constraint surface. In order to evaluate the time derivative of the Hamiltonian constraint $\mathscr{H}^{\lambda,(M)}$, we must compute the Poisson bracket of the Hamiltonian constraint $\mathscr{H}^{\lambda,(M)}$ with itself, which we decompose in terms of Poisson brackets involving the matter and gravitational Hamiltonians,

$$\left\{\mathscr{H}^{\lambda,(M)}, \mathscr{H}^{\lambda,(M)}\right\} = \{\mathscr{H}_\lambda, \mathscr{H}_\lambda\} + \left\{\mathscr{H}^{(M)}, \mathscr{H}_\lambda\right\} + \left\{\mathscr{H}_\lambda, \mathscr{H}^{(M)}\right\} + \left\{\mathscr{H}^{(M)}, \mathscr{H}^{(M)}\right\} \tag{D.7a}$$

$$\approx \{\mathscr{H}_\lambda, \mathscr{H}_\lambda\} + \left\{\mathscr{H}^{(M)}, \mathscr{H}_\lambda\right\} + \left\{\mathscr{H}_\lambda, \mathscr{H}^{(M)}\right\}, \tag{D.7b}$$

where we have discarded the Poisson bracket of the matter Hamiltonian with itself because it vanishes on the constraint surface by construction (D.6c). In the absence of matter, the first Poisson bracket on the right-hand side of eq. (D.7b) yields the general relativistic result and an additional term proportional to $(1-\lambda)$, which leads to the tertiary constraint $\omega \approx 0$ in the λ-R model. The cross terms only depend on the variation of $\mathscr{H}_\lambda$ with respect to π^{ij} and on the variation of $\mathscr{H}^{(M)}$ with respect to g_{ij}. Since both functionals are local in these fields, the cross terms cancel each other. Hence, $\mathscr{H}^{\lambda,(M)} \approx 0$ is preserved in time whenever

$$\begin{aligned}\dot{\mathscr{H}}^{\lambda,(M)} =& g^{ij}\left(2\mathscr{H}_i^{\lambda,(M)}\nabla_j N + N\nabla_j\mathscr{H}_i^{\lambda,(M)}\right) - \mathscr{H}^{\lambda,(M)}N^i\nabla_i N \\ &+ 2\frac{\lambda-1}{3\lambda-1}g^{ij}\left(2\nabla_i\pi\nabla_j N + N\nabla_i\nabla_j\pi\right) \approx 0,\end{aligned} \tag{D.8}$$

which is satisfied when $\omega \approx 0$. The next step in the Dirac algorithm is to demand that the constant mean curvature condition $\omega \approx 0$ is preserved in time, which in the absence of matter yields a lapse-fixing equation. Because ω does not depend on the matter fields and $\mathscr{H}_i^{(M)}$ does not depend on the metric and its momenta, it still holds that

$$\left\{\int d^3x\eta\omega, \int d^3yN^i\mathscr{H}_i^{\lambda,(M)}\right\} = \int d^3z\,\eta\,\omega\nabla_i N^i \approx 0, \tag{D.9}$$

and that

$$\left\{\int d^3x\eta\omega, \int d^3yN\mathscr{H}^{\lambda,(M)}\right\} = \left\{\int d^3x\eta\omega, \int d^3yN\mathscr{H}_\lambda\right\} - \int d^3zNg_{ij}\frac{\partial \mathscr{H}^{(M)}}{\partial g_{ij}}. \tag{D.10}$$

Using eq. (2.37), we see that the lapse-fixing equation becomes

$$\mathscr{M} = D_{\lambda,(M)}N - \frac{\sqrt{g}}{V}\int d^3xD_{\lambda,(M)}N \approx 0, \tag{D.11}$$

where $D_{\lambda,(M)}$ is given by

$$D_{\lambda,(M)} =: \sqrt{g}\left(R - 3\Lambda + \frac{a^2}{2(3\lambda-1)} - \frac{3}{4}\frac{\mathscr{H}^{(M)}}{\sqrt{g}} - \frac{1}{2}\frac{g_{ij}}{\sqrt{g}}\frac{\partial \mathscr{H}^{(M)}}{\partial g_{ij}} - \nabla^2\right). \tag{D.12}$$

The presence of matter therefore modifies the lapse-fixing equation by adding two terms to the differential operator $D_{\lambda,(M)}$. As in chapter 2, the Dirac algorithm finishes when we impose that the lapse-fixing equation $\mathscr{M} \approx 0$ is preserved in time. As we did when studying the λ-R model, we redefine the momentum constraints to explicitly show that they are first class,

$$\tilde{\mathscr{H}}_i^{\lambda,(M)} = \tilde{\mathscr{H}}_i + \mathscr{H}_i^{(M)}. \tag{D.13}$$

The Poisson bracket between the redefined momentum constraints and the quaternary constraint $\mathscr{M} \approx 0$ therefore yields

$$\left\{\int d^3x\eta\mathscr{M}, \int d^3yN^i\tilde{\mathscr{H}}_i^{\lambda,(M)}\right\} = \int d^3z\mathscr{M}N^i\nabla_i\eta, \tag{D.14}$$

where we have used that

$$\left\{\int d^3xNg_{ab}\frac{\partial \mathscr{H}^{(M)}}{\partial g_{ab}}, \int d^3yN^i\mathscr{H}_i\right\} = -2\int d^3z\,N^ig_{ik}\nabla_l N\left(\frac{\partial \mathscr{H}^{(M)}}{g_{kl}} + g_{ab}\frac{\partial^2 \mathscr{H}^{(M)}}{\partial g_{kl}\partial g_{ab}}\right), \tag{D.15}$$

and we have assumed that the matter Hamiltonian $\mathscr{H}^{(M)}$ and matter momentum $\mathscr{H}_i^{(M)}$ are such that

$$\begin{aligned}\left\{\int d^3xNg_{ab}\frac{\partial \mathscr{H}^{(M)}}{\partial g_{ab}}, \int d^3yN^i\mathscr{H}_i^{(M)}\right\} = -\int d^3z\Bigg\{ & g_{ab}\frac{\partial \mathscr{H}^{(M)}}{\partial g_{ab}}N^i\nabla_i N \\ & -2N^ig_{ik}\nabla_l\left(N\frac{\partial}{\partial g_{jk}}\left(g_{ab}\frac{\partial \mathscr{H}^{(M)}}{\partial g_{ab}}\right)\right)\Bigg\}.\end{aligned} \tag{D.16}$$

This assumption is necessary to explicitly exhibit the first-class nature of the momentum constraints, as we did in chapter 2.

The remainder of the Poisson bracket between $\mathscr{M}$ and the total Hamiltonian yields a shift-independent equation for α of the general form

$$\tilde{F} + D_{\lambda,(M)}\alpha - \frac{\sqrt{g}}{V}\int d^3x \left(\tilde{F} + D_{\lambda}\alpha\right) \approx 0, \tag{D.17}$$

where $\tilde{F}$ is a scalar density, which is the sum of the density F of eq. (2.43) and new terms from both the matter Hamiltonian and the matter dependence of $\mathscr{M} \approx 0$. It is important to note that if we set $\lambda = 1$ and impose the constant mean curvature as a gauge condition, we would obtain the $\lambda = 1$ versions of eqs. (D.14) and (D.17). Moreover, none of the matter-dependent contributions to these equations appears due to the presence of λ.

In conclusion, when we add matter to the λ-R model under the assumptions introduced in reference [47], we obtain the constraint structure of the λ-R model without matter with matter-dependent terms that are the same as those in the constant mean curvature version of general relativity with matter.

E

Examples of simple solutions of the λ-R model

In this appendix, we study two types of solutions of the λ-R model, which turn out to agree with their general relativistic counterparts, namely, linear perturbations around Minkowski space and the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes.

E.1 Linear perturbations around Minkowski spacetime

We begin with linear perturbations around Minkowski space. In this section, we set $\Lambda = 0$ and parametrise the perturbations by a small real number η. The metric, its inverse, momentum tensor, lapse, and shift are given by

$$g_{ij} = \delta_{ij} + \eta h_{ij}, \qquad g^{ij} = \delta^{ij} - \eta h^{ij} \tag{E.1a}$$

$$\pi^{ij} = \eta p^{ij}, \tag{E.1b}$$

$$N = 1 + \eta n, \tag{E.1c}$$

$$N^i = \eta n^i. \tag{E.1d}$$

We also expand the Lagrange multiplier α as $\alpha = \eta \tilde{\alpha}$. If we substitute eq. (E.1) into the constraints and keep only terms up to first order in the parameter η, we obtain expressions that coincide with the general relativistic ones for first-order perturbations in the constant mean curvature gauge. The tertiary constraint $\omega \approx 0$ of eq. (2.36), after expanding the time-dependent function $a(t)$ as $a(t) = \eta \tilde{a}(t)$, reads

$$p = \tilde{a}(t). \tag{E.2}$$

Up to first order in η, the only remaining term in the Hamiltonian constraint comes from the Ricci scalar. Therefore, λ drops out of the constraint and we have

$$\mathscr{H}_\lambda = \eta \left(\partial_i \partial_j h^{ij} - \delta^{ij} \partial_i \partial_j h \right) \approx 0. \tag{E.3}$$

The momentum constraints (2.23b) become $\partial_i p^{ij} = 0$ and can therefore be solved by choosing the momentum p^{ij} as we did for π^{ij} in chapter 3,

$$p^{ij} = p^{ij}_{TT} + \frac{\delta^{ij}}{3} p, \tag{E.4}$$

where p^{ij}_{TT} is a symmetric transverse-traceless tensor with respect to δ_{ij}, that is,

$$\partial_i p^{ij}_{TT} = 0, \qquad \delta_{ij} p^{ij}_{TT} = 0, \tag{E.5}$$

and p is a function of time, as established in eq. (E.2).

In the lapse-fixing equation (2.40), the λ-dependent term is quadratic in the trace of the momentum π and therefore drops out. The only two terms in eq. (2.40) that are linear in the perturbations are the second-order derivative of the lapse and the first-order contribution from the Ricci scalar. Recall that the latter must vanish by itself because of the Hamiltonian constraint (E.3). Hence, the lapse-fixing equation reduces to

$$\partial^2 n = 0, \tag{E.6}$$

where the contribution from the right-hand side of eq. (2.40) vanishes due to Stokes' theorem. Similarly, the equation for the expanded Lagrange multiplier $\tilde{\alpha}$ becomes

$$\partial^2 \tilde{\alpha} = 0. \tag{E.7}$$

From eqs. (E.2), (E.3), (E.6), and (E.7), we conclude that the constraint algebra of the linearised λ-R model must obey the same conditions as the linearised version of general relativity in the constant mean curvature gauge. However, there is a λ-dependent equation of motion, namely, eq. (2.63a), which reads

$$\dot{h}_{ij} = 2\left(p^{TT}_{ij} - \frac{\delta_{ij}\tilde{a}}{3(3\lambda - 1)} \right) + \partial_i n_j + \partial_j n_i, \tag{E.8}$$

after substituting eq. (E.1). We now turn to the extrinsic curvature, which we also expand as a perturbation around Minkowski space,

$$K_{ij} = \eta k_{ij}. \tag{E.9}$$

From eq. (2.20), we obtain $p = (1 - 3\lambda) k$ and from the Legendre transformation (2.19a),

$$p^{ij}_{TT} = k^{ij} - \frac{\delta^{ij}}{3} k \Rightarrow p^{ij}_{TT} := k^{ij}_{TT}, \tag{E.10}$$

where we have also chosen, without loss of generality, the extrinsic curvature perturbation k^{ij} to be of the form

$$k^{ij} = k^{ij}_{TT} + \frac{g^{ij}}{3} k. \tag{E.11}$$

Inserting $p = (1 - 3\lambda) k$ and eq. (E.10) into eq. (E.8), it simply yields the usual definition of the extrinsic curvature

$$k_{ij} = \frac{1}{2} \left(\dot{h}_{ij} - \partial_i n_j - \partial_j n_i \right), \tag{E.12}$$

while the equation for $\dot{p}^{ij}$ (2.63b) reads

$$\dot{k}_{ij}^{TT} = -\frac{1}{2}\left(\partial_k\partial_i h_j^k + \partial_k\partial_j h_i^k - \partial^2 h_{ij} - \partial_i\partial_j h\right) + \partial_i\partial_j n + (3\lambda - 1)\frac{\delta_{ij}}{3}\dot{k}. \tag{E.13}$$

This equation can be interpreted as the general relativistic equation of motion for the fluctuation of the transverse-traceless components of the extrinsic curvature in constant mean curvature coordinates with an effective trace k_{eff} given by

$$k_{eff} = \frac{3\lambda - 1}{2}k. \tag{E.14}$$

Notice that by reconstructing the tensor k^{ij} with which we started (E.11), we would not obtain a general relativistic solution. The general relativistic solution is the one with linearised extrinsic curvature tensor k_{eff}^{ij} given by

$$k_{eff}^{ij} = k_{TT}^{ij} + \frac{g^{ij}}{3}k. \tag{E.15}$$

The reason why we are able to absorb the λ-dependence in the equation of motion is that by considering only linear perturbations, all other λ-dependent terms vanish immediately. Had we kept fluctuations up to second order in η, the Hamiltonian constraint, lapse-fixing equation and equation for α would have remained λ-dependent. In that case, there would have been $\frac{k^2}{3\lambda-1}$-dependent terms and it would not be possible to absorb λ in the way we did in eq. (E.14).

E.2 The FLRW metric

We consider a homogeneous and isotropic open spatial hypersurface Σ_t with coordinates x^i and use the FLRW metric as an ansatz for the four-dimensional metric [7]. In the ADM decomposition, this implies

$$g_{ij} = a_F^2(t)\gamma_{ij} = a_F^2(t)\left(\delta_{ij} + \sigma\frac{x_i x_j}{1 - \sigma x^2}\right), \qquad N = 1, \qquad N^i = 0, \tag{E.16}$$

where a_F is the scale factor and $\sigma \in \mathbb{R}$ is related to the Ricci curvature of Σ_t via

$$\mathscr{R} = 6\frac{\sigma}{a_F^2}. \tag{E.17}$$

Note that, like in chapter 4, assuming that the homogeneous and isotropic spatial slices coincide with those of the preferred foliation cannot be done without loss of generality, as is the case in general relativity.

Inserting eq. (E.16) into the Legendre transformation (2.19a), we obtain

$$\pi^{ij} = (1 - 3\lambda)\sqrt{\gamma}\dot{a}_F\,\gamma^{ij} \Rightarrow \pi = 3(1 - 3\lambda)\sqrt{\gamma}a_F^2\,\dot{a}_F. \tag{E.18}$$

Hence, the momentum tensor π^{ij} consists only of its trace contribution, that is,

$$\pi^{ij} = \frac{g^{ij}}{3}\pi, \tag{E.19}$$

where π satisfies the constant mean curvature condition, $\nabla_i \pi = 0$. The momentum constraints are therefore satisfied. We can separately obtain the general equations of motion for π^{ij}_{TT} and π,

$$\dot{\pi}^{ij}_{TT} = \frac{N}{\sqrt{g}}\left(\frac{2}{3\,(3\lambda-1)}\pi^{ij}_{TT}\pi - 2g_{kl}\pi^{ik}_{TT}\pi^{jl}_{TT}\right) - N\sqrt{g}\left(R^{ij} - \frac{1}{3}g^{ij}R\right) \tag{E.20}$$

$$-\sqrt{g}\left(g^{ik}g^{jl} - \frac{1}{3}g^{ij}g^{kl}\right)\nabla_k\nabla_l N, \tag{E.21}$$

$$\dot{\pi} = 2\sqrt{g}\left(R - 3\Lambda - \nabla^2\right)N. \tag{E.22}$$

The λ-dependent momentum of the FLRW metric (E.18) satisfies eq. (E.19) with $\pi^{ij}_{TT} = 0$. Hence, the momentum and tertiary constraints are solved trivially by the FLRW metric. The Hamiltonian constraint yields a λ-dependent Friedmann equation,

$$\left(\frac{\dot{a}_F}{a_F}\right)^2 = \frac{2}{3\lambda-1}\left(\frac{\Lambda}{3} - \frac{\sigma}{a_F^2}\right), \tag{E.23}$$

while the lapse-fixing equation (2.40) for non-compact hypersurfaces is the same as eq. (E.22). Therefore, the constraints are either trivially satisfied or, in the case of $\mathscr{H}_\lambda \approx 0$, yield the modified Friedmann equation. The equations of motion for the metric g_{ij} and the momentum π^{ij}_{TT} are also satisfied immediately. After some algebraic manipulations on the constraint surface, the equation of motion for the trace π (E.22) yields a modified second Friedmann equation

$$\frac{\ddot{a}_F}{a_F} = \frac{2}{3\lambda-1}\frac{\Lambda}{3}. \tag{E.24}$$

We see that we can absorb λ by redefining Λ and σ. The same holds when we include matter. Consider the energy-momentum tensor $T^{\mu\nu}$ of the perfect fluid

$$T^{\mu\nu} = (\rho + P)\,U^\mu U^\nu - P g^{\mu\nu}, \tag{E.25}$$

where U^μ is the four-velocity of the fluid, P the isotropic pressure and ρ the energy density. Because this energy-momentum tensor is ultra-local in the metric, it does not change the constraint algebra of the model (see appendix D). Restoring Newton's constant G_N, the two λ-dependent Friedmann equations become

$$\left(\frac{\dot{a}_F}{a_F}\right)^2 = \frac{2}{3\lambda-1}\left(\frac{\Lambda}{3} + \frac{8\pi G_N}{3}\rho - \frac{\sigma}{a_F^2}\right), \tag{E.26a}$$

$$\frac{\ddot{a}_F}{a_F} = \frac{2}{3\lambda-1}\left(\frac{\Lambda}{3} - \frac{4\pi G_N}{3}\left(\rho - 3P\right)\right). \tag{E.26b}$$

The constants Λ and G_N come from the action of the λ-R model and σ from the ansatz for the spatial metric. We can scale all of them by the same λ-dependent factor, defining

$$G'_N = \frac{2}{3\lambda-1}G_N, \qquad \Lambda' = \frac{2}{3\lambda-1}\Lambda, \qquad \sigma' = \frac{2}{3\lambda-1}\sigma \tag{E.27}$$

which turns eqs. (E.28) into the usual Friedmann equations

$$\left(\frac{\dot{a}_F}{a_F}\right)^2 = \frac{\Lambda'}{3} + \frac{8\pi G'_N}{3}\rho - \frac{\sigma'}{a_F^2}, \tag{E.28a}$$

$$\frac{\ddot{a}_F}{a_F} = \frac{\Lambda'}{3} - \frac{4\pi G'_N}{3}(\rho - 3P), \tag{E.28b}$$

with effective Newton's constant G'_N, cosmological constant Λ' and parameter σ'. Therefore, we have shown that the λ-dependence of the FRLW metric is unphysical in the sense that it is possible to redefine parameters of the reduced model such that it reproduces its general relativistic counterpart, as was the case with the linearised theory.

Summary

Throughout history, science and its applications have changed both the way in which we see the world and the world itself. Whenever a new theory emerges and consequently alters the way certain phenomena are explained, that is not necessarily a permanent change, even if in general it will not be a reversible one. A particularly relevant example in our context is the evolution of our understanding of gravity. Established in the seventeenth century, Newton's law of universal gravitation was accepted until the twentieth century, when it was replaced by Einstein's general relativity. The latter not only changed the way we understand gravitational interactions, but also how we think of concepts as fundamental as space and time - a shift that had begun a few years prior with special relativity. Despite the exceptional success of general relativity, there is good reason to presume that it does not constitute the final chapter in this history. It is, however, very likely that we will not return to Newton's gravity.

If, just for a moment, we imagine that general relativity is free from any issues, there still stands an argument to justify the study of alternatives to it. A physical theory is not just a set of predictions, there is a whole underlying mathematical structure that, more often than not, encompasses many unphysical pieces that do not correspond to anything in the real world. Thus, by investigating the physical properties of an alternative to an accepted theory, it is possible to establish just how much of that theory's structure is strictly necessary for its predictions to match observations. It is also a procedure that can benefit further developments in the field as it generates knowledge about a larger class of theories. Now that the moment in which we imagined that the theory had no issues is over, it is time to point out that they exist and that their existence is yet another good reason to consider alternatives to general relativity.

In both gravitational theories alluded to above, observational data served as a guide in their construction. That is, there were phenomena that were either unexplained by any known theory or which contradicted the then accepted theory. With general relativity, there is a broad range of scales in which there is no direct contradiction between the theory and observation. However, there are issues such as the generic presence of singularities (spacetime points in which the theory loses its predictive power), the seeming necessity of introducing

the concepts of dark energy and dark matter in standard cosmology to account for observational data, and the lack of a theory of quantum gravity. With regard to singularities, one might say that they merely indicate that at very high energies / small distances, general relativity is no longer valid and a new description of gravitational phenomena is required, often thought to be a theory of quantum gravity. Such a theory of quantum gravity had been a goal of the physics community for over fifty years, so far with no single candidate passing enough consistency requirements (predictive power, mathematical consistency, and correct low energy limit) so as to be considered a valid theory. While discrepancies between general relativity and a theory of quantum gravity are expected to be mostly relevant for the shortest of distances and extremely high energies, regimes that are quite difficult to access experimentally, dark matter and dark energy were introduced to explain data pertaining to very large distances. As with quantum gravity, there are many attempts to provide a theoretical explanation for their presence. One possibility which further motivates the study of alternatives to general relativity is that the latter is simply not the appropriate description of gravity at those scales.

In this thesis, we studied a modified theory of gravity called the λ-R model. It can be seen as a one-parameter family of gravitational theories, that parameter being λ, which includes general relativity for a particular value (λ=1), but otherwise modifies it. It appeared independently in two different contexts, first in an attempt to scrutinise the role played by mathematical structures that, broadly speaking, are associated with spatial geometries and their evolution in time, and later as a possible classical limit of a candidate theory of quantum gravity known as Hořava-Lifshitz gravity.

Introduced in 2009, Hořava-Lifshitz gravity is a theory which postulates that, at very high energies, the symmetries between space and time that characterise general relativity are not valid and that the universe behaves as if it has a preferred foliation of spacetime by spatial leaves of constant time. A useful analogy to illustrate the concept of foliation is to imagine spacetime as an uncut loaf of bread which can be sliced along any direction. A preferred foliation of spacetime is then analogous to a preferred direction along which to slice the loaf of bread. This feature is absent from general relativity. There, a foliation is possible under general and physically reasonable conditions, but all directions along which to foliate are physically equivalent. Models of Hořava-Lifshitz gravity usually include many additional parameters in comparison to general relativity. However, one can argue that most of these parameters do not play a role in the theory's low energy physics. Because there are several versions of the overall theory, its low energy description changes from version to version. One of those possibilities precisely corresponds to the λ-R model.

We split the analysis of the λ-R model into three main and three secondary chapters. In the first main one, we introduce and define the model, further determining its constraint structure. This is a procedure in which the symmetries of the theory give rise to conditions on an initial spatial slice, which once satisfied there are guaranteed to hold for all times. We show that the introduction of the additional parameter forces the spatial slices to curve in a specific manner along the time direction. In the second chapter, we address the initial value

formulation of the model, that is, we determine which quantities are fixed by the aforementioned conditions on the initial hypersurface and which can be freely specified. We were able to show that it is necessary to specify the same data in both general relativity and the λ-R model, which allowed us to make a precise comparison between the solutions of both theories. Using both the constraint structure and the time evolution equations determined in the previous chapter, we were thus able to show that the solutions of both theories are not equivalent in general as well as quantify some of their differences. Finally, the third chapter deals with the specific case of solutions with spherical symmetry, which in general relativity correspond to the outside of a star or a black hole. We determined a class of solutions in which the spherical symmetry is aligned with the preferred foliation and showed that, for each value of λ, they correspond to a one-function family of generalisations of the general relativistic solution. One particularly interesting feature of these solutions is that their space-time curvature, an important observable in Einstein's theory which is only non-zero in the presence of matter, can attain a non-zero value in the λ-R model that depends exclusively on its geometry. The secondary chapters include one containing a general introduction and outlook of the thesis, one with comments regarding classical models of Hořava-Lifshitz gravity in the context of what we learned about the λ-R model, and finally one where, based on the work presented in the main chapters, we characterise the role of λ in the λ-R model and present our conclusions.

Samenvatting

Door de geschiedenis heen hebben de wetenschap en haar toepassingen zowel de manier waarop we de wereld zien als de wereld zelf veranderd. Telkens wanneer een nieuwe theorie naar voren komt en de manier waarop bepaalde verschijnselen worden verklaard verandert, is dat niet per se een permanente verandering, ook al is deze over het algemeen niet omkeerbaar. Een bijzonder relevant voorbeeld in onze context is de evolutie van ons begrip van de zwaartekracht. Gevestigd in de zeventiende eeuw, werd Newtons wet van universele zwaartekracht aanvaard tot de twintigste eeuw, waarin deze werd vervangen door Einsteins algemene relativiteitstheorie. Dit laatste veranderde niet alleen de manier waarop we interacties met de zwaartekracht begrijpen, maar ook hoe we fundamentele begrippen zoals ruimte en tijd beschouwen - een verschuiving die al een paar jaar eerder in gang werd gezet door speciale relativiteitstheorie. Ondanks het uitzonderlijke succes van algemene relativiteitstheorie, is er goede reden om aan te nemen dat dit niet het laatste hoofdstuk in deze geschiedenis is. Het is echter zeer waarschijnlijk dat we niet terug zullen keren naar iets dat lijkt op de zwaartekracht van Newton.

Als we ons voor een moment voorstellen dat de algemene relativiteitstheorie vrij is van problemen, is er nog steeds voldoende reden om de studie naar alternatieven te rechtvaardigen. Een theorie in de fysica is niet alleen een reeks voorspellingen maar bevat een onderliggende wiskundige structuur die, vaker wel dan niet, vele niet-fysische onderdelen bevat die niet overeenkomen met de realiteit. Dus, door de fysische eigenschappen van een alternatief voor een geaccepteerde theorie te onderzoeken, is het mogelijk om vast te stellen hoeveel van de structuur van die theorie strikt noodzakelijk is zodat voorspellingen overeenkomen met waarnemingen. Het is ook een procedure die de verdere ontwikkelingen op dit gebied ten goede kan komen, aangezien het kennis verschaft over een grotere klasse van theorieën. Nu het moment waarop we ons voorstelden dat de theorie geen problemen had voorbij is, wordt het tijd om erop te wijzen dat ze wel degelijk bestaan en dat het bestaan hiervan nog een goede reden is om alternatieven voor de algemene relativiteitstheorie te overwegen.

In beide hiervoor genoemde zwaartekrachttheorieën, dienden waarnemingen als richtlijn tijdens de totstandkoming. Dat wil zeggen, er waren verschijnselen die ofwel niet verklaard werden door bekende theorieÃ«n of die de toendertijd geaccepteerde theorie tegenspraken. Algemene relativiteitstheorie beschikt over een breed scala aan schalen waarin geen directe tegenspraak is tussen de theorie en waarnemingen. Er zijn echter knelpunten zoals de aanwezigheid van singulariteiten (ruimtetijdspunten waarin de theorie zijn voorspellende kracht verliest), de schijnbare noodzaak om donkere energie en donkere materie in standaard kosmologie te introduceren om observationele gegevens te kunnen verklaren en het gebrek aan een theorie van kwantumzwaartekracht. Met betrekking tot singulariteiten zou men kunnen zeggen dat ze slechts aangeven dat voor hoge energieÃ«n / kleine afstanden de algemene relativiteitstheorie niet langer geldig is en dat een nieuwe beschrijving van zwaartekrachtverschijnselen een vereiste is. Een populaire optie hiervoor is een theorie van kwantumzwaartekracht. Zo'n theorie van kwantumzwaartekracht is al meer dan vijftig jaar een doel van de fysische gemeenschap, tot nu toe met geen enkele kandidaat die verscheidene consistentie testen met succes doorstaat (voorspellende kracht, wiskundige consistentie en het correcte lage energielimiet) om als een geldige theorie te worden beschouwd. Hoewel naar verwachting de verschillen tussen de algemene relativiteitstheorie en een theorie van de kwantumzwaartekracht vooral relevant zijn voor de kortste afstanden en extreem hoge energieÃ«n, regimes die vanuit een experimenteel oogpunt moeilijk toegankelijk zijn, werden donkere materie en donkere energie geÃ¯ntroduceerd om gegevens met betrekking tot zeer grote afstanden te verklaren. Vergelijkbaar met kwantumzwaartekracht, zijn er veel pogingen om een theoretische verklaring voor hun aanwezigheid te vinden. Één van de mogelijkheden die de studie van alternatieven voor de algemene relativiteitstheorie verder motiveert, is dat deze eenvoudigweg niet de juiste beschrijving van de zwaartekracht op die schalen is.

In dit proefschrift hebben we een gemodificeerde zwaartekrachttheorie bestudeerd, het zogenoemde λ-R model. Het kan worden gezien als een familie van zwaartekrachttheorieën met één parameter, de parameter λ, die voor een bepaalde waarde ($\lambda = 1$) de algemene relativiteitstheorie bevat, maar deze anderszins modificeert. Het verscheen onafhankelijk in twee verschillende contexten, eerst in een poging om de rol gespeeld door wiskundige structuren die in grote lijnen worden geassocieerd met ruimtelijke geometrieÃ«n en hun evolutie in tijd te bestuderen en later als een mogelijk klassiek limiet van een kandidaat theorie van kwantumzwaartekracht, bekend als Hořava-Lifshitz zwaartekracht.

GeÃ¯ntroduceerd in 2009, stelt de zwaartekrachttheorie van Hořava-Lifshitz dat, bij zeer hoge energieÃ«n, de symmetrieÃ«n tussen ruimte en tijd die kenmerkend zijn voor de algemene relativiteitstheorie niet geldig zijn en dat het universum zich gedraagt alsof het een foliatie van ruimtetijd door ruimtelijke, constante tijd bladen prefereert. Een analogie om het concept van foliÃ«ren te illustreren, is om de ruimtetijd te zien als een ongesneden brood dat in elke richting kan worden gesneden. Een geprefereerde foliatie van ruimtetijd is dan analoog aan een voorkeursrichting waarlangs het brood wordt gesneden. Deze eigenschap is niet aanwezig in de algemene relativiteitstheorie. Daar is een foliatie mogelijk onder algemene en fysisch gemotiveerde omstandigheden, maar alle richtingen waarlangs wordt gefolieerd zijn fysisch gelijk. Hořava-Lifshitz zwaartekrachtmodellen bevatten meestal veel extra parameters in vergelijking met algemene relativiteitstheorie. Men kan echter beargumenteren dat de meeste van deze parameters geen rol spelen in de beschrijving van de lage energiefysica van Hořava-Lifshitz zwaartekracht. Omdat er verschillende versies van de algehele theorie zijn, verandert de beschrijving van de lage energie tussen versies. Één van deze versies komt precies overeen met het λ-R-model.

We splitsen de analyse van het λ-R-model in drie hoofdstukken. In het eerste hoofdstuk introduceren en definiÃ«ren we het model, verder beschrijven we de voorwaardenstructuur. Dit is een procedure waarbij de symmetrieÃ«n van de theorie resulteren in condities op een initieel ruimtelijk deel. Als eenmaal aan deze condities voldaan is, dan gelden ze gegarandeerd op elk moment in tijd. We laten zien dat de ruimtelijke sneden gedwongen worden om op een specifieke manier langs de tijdsrichting te buigen wanneer men extra parameters introduceert. In het tweede hoofdstuk bespreken we de initiële waarde formulering van het model, dat wil zeggen we bepalen welke parameters door de initiÃ«le condities worden vastgesteld en welke vrij kunnen worden gekozen. We konden aantonen dat het noodzakelijk is om in zowel algemene relativiteitstheorie als het λ-R-model dezelfde gegevens te specificeren, waardoor we een nauwkeurige vergelijking konden maken tussen de oplossingen van beide theorieÃ«n. Met behulp van zowel de voorwaarden-structuur als de tijdsevolutie vergelijkingen die in het vorige hoofdstuk werden bepaald, konden we aantonen dat de oplossingen van beide theorieÃ«n in het algemeen niet equivalent zijn. Ten slotte behandelt het derde hoofdstuk het specifieke geval waarbij de oplossingen sferisch symmetrisch zijn. Over het algemeen komen deze oplossingen overeen met het uitwendige van een ster of zwart gat. We hebben een klasse van oplossingen gevonden waarin de sferische symmetrie is uitgelijnd met de geprefereerde foliatie en hebben aangetoond dat ze voor elke waarde van λ overeenkomen met een familie van generalisaties van de oplossingen van de algemene relativiteitstheorie. Een bijzonder interessant kenmerk van deze oplossingen is dat hun ruimtetijdkromming, een belangrijke observabele in de theorie van Einstein die alleen ongelijk aan nul is in de aanwezigheid van materie, een nul waarde anders dan nul kan aannemen die uitsluitend afhankelijk is van geometrische grootheden in het λ-R-model.

Acknowledgements

First and foremost, I would like to thank my supervisor, professor Renate Loll. I will always be grateful for the many opportunities I have had throughout this period. These include the opportunity to work within the wonderful research groups you have established, first in Utrecht and now in Nijmegen, the opportunity to go to work in Waterloo's Perimeter Institute twice, and the many opportunities for meeting and learning from other great physicists. Moreover, I am grateful for the many, many physics discussions we had throughout these years. One of the most important things I have learned during this period is just how much can come from a simple, well-put question, and I did so by witnessing how much of a driving force behind my progress your inquiries during our meetings were. Finally, thank you for the opportunity of co-authoring papers with you and for painstakingly having gone through my thesis, both processes through which I have learned much, not only about physics but also about writing in general.

Secondly, I must thank my family for the encouragement and support throughout the years, particularly my father. I am truly sorry for all the unanswered calls and sincerely hope to do better from now on.

I must also thank all those with whom I discussed with and learned physics from, among them Niall Ó Murchadha, who hosted me in Cork for three of the most inspiring days of my career. I would also like to thank others that I have met along the way and that took some of their time to talk to and teach me. These include, but are not limited to, professors Jan Ambjørn, Steve Carlip, Bianca Dittrich, Domenico Giulini, Ted Jacobson, Claus Kiefer, Frank Saueressig, Thomas Sotiriou, and Matt Visser.

I have also had the luck of sharing a workplace with many fantastic individuals over these years, to the point where it was commonplace for a new member of the department to remark that the group's atmosphere was uncannily pleasant. Thus, I am grateful to have met and spent time with Adriano, Chris, Daniel, Fleur (who helped a huge deal with the "Samenvatting"), Francesca, Giulio, Jeff, Jimmy, Joshua, Lisa, Marcus, Merce, Natalia, Omar, and Sean.

Finally, to all those friends, either in Amsterdam (Franz, Val, Adel, and Nadica), Nijmegen (O, T, J, R, JB, S), Portugal (SQ and too many others to name, you know who you are), and elsewhere in the world (Jogis) that became an important part of my life whilst enduring this seemingly never-ending quest, thank you so much for your patience and companionship.

Curriculum Vitae

Luís was born on May 8, 1988, in Oeiras, a town in the outskirts of Lisbon, where he lived until he was fifteen. There, he completed his pre-university education, graduating from high school in 2006. That summer he enrolled at Instituto Superior Técnico (IST, Lisbon) on the Engineering Physics integrated master's, a five year program comprising both BSc and MSc levels. There, he received a merit diploma after his first year and was awarded an integration into research grant, which translated into a six months project at the Laboratory of Instrumentation and Experimental Particle Physics (LIP, Lisbon) as part of the Pierre Auger experiment.

Having realised that his passion resided in the theoretical side of physics, he decided to pursue a master programme in theoretical physics, which he did at Utrecht University, in the Netherlands. Two and a half years later, in December 2011, he graduated from the program with *summa cum laude*, having done his thesis on classical Hořava-Lifshitz gravity under the supervision of professor Renate Loll. In 2013, after a period of stop-start activity for his PhD that took most of 2012, he joined Radboud University in Nijmegen to pursue a doctorate degree in theoretical physics, again under the supervision of professor Renate Loll. During the period that followed, he was named the Nijmegen representative in the DRSTP PhD council, a consultative organ of the Dutch Research School of Theoretical Physics.

Bibliography

[1] J. Ambjørn, A. Görlich, J. Jurkiewicz, and R. Loll: *Nonperturbative quantum gravity*, Phys. Rept. **519** (2012) 127-210, arXiv:1203.3591 [hep-th].

[2] J. L. Anderson, and P. G. Bergmann: *Constraints on covariant field theories*, Phys. Rev. **83** (1951) 1018-1025.

[3] R. Arnowitt, S. Deser, and C.W. Misner: *The dynamics of general relativity*, in "Gravitation: an introduction to current research", edited by L. Witten (Wiley, New York, 1962), arXiv:gr-qc/0405109v1.

[4] ATLAS Collaboration: *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, Phys. Lett. B**716** (2012) 1-29, arXiv:1207.7214v2 [hep-ex].

[5] E. Barausse, T. Jacobson, and T. Sotiriou: *Black holes in Einstein-aether and Hořava-Lifshitz gravity*, Phys. Rev. D**83** (2011) 124043, arXiv:1104.2889v2 [gr-qc].

[6] A. O. Barvinsky, D. Blas, M. Herrera-Valea, S.M. Sibiryakov, and C.F. Steinwachs: *Renormalization of Hořava gravity*, Phys. Rev. D**93**, (2016) 064022, arXiv:1512.02250v1 [hep-th].

[7] D. Baumann: *Cosmology*, lecture notes http://www.damtp.cam.ac.uk/user/db275/Cosmology2015.pdf.

[8] J. Bellorín, and A. Restuccia: *Closure of the algebra of constraints for a nonprojectable Hořava model*, Phys. Rev. D**83** (2011) 044003, arXiv:1010.5531v3 [hep-th].

[9] J. Bellorín, and A. Restuccia: *On the consistency of the Hořava theory*, Int. J. Mod. Phys. D**21** (2012) 1250029, arXiv:1004.0055v3 [hep-th].

[10] J. Bellorín, and A. Restuccia: *Consistency of the Hamiltonian formulation of the lowest-order effective action of the complete Hořava theory*, Phys. Rev. D**84** (2011) 104037, arXiv:1106.5766v3 [hep-th].

[11] J. Bellorín, A. Restuccia, and A. Sotomayor: *Consistent Hořava gravity without extra modes and equivalent to general relativity at the linearized level*, Phys. Rev. D**87** (2013) 084020, arXiv:1302.1357v3 [hep-th].

[12] P. G. Bergmann: *Non-linear field theories*, Phys. Rev. **75** (1949) 680-685.

[13] P. G. Bergmann, and J. Brunings: *Non-linear field theories II. Canonical equations and quantization*, Rev. Mod. Phys. **21**, (1949) 480.

[14] P. G. Bergmann, R. Penfield, R. Schiller, and H. Zatzkis: *The Hamiltonian of general relativity with electromagnetic field*, Phys. Rev. **80** (1951) 81.

[15] G. D. Birkhoff: *Relativity and modern physics* (Harvard University Press, Cambridge, MA, 1923).

[16] D. Blas, O. Pujolàs, and S. Sibiryakov: *On the extra mode and inconsistency of Hořava gravity*, JHEP **0910** (2009) 029, arXiv:0906.3046v2 [hep-th].

[17] D. Blas, O. Pujolàs, and S. Sibiryakov: *A healthy extension of Hořava gravity*, Phys. Rev. Lett. **104** (2010) 181302, arXiv:0909.3525v1 [hep-th].

[18] D. Blas, and S. Sibiryakov: *Hořava gravity versus thermodynamics: the black hole case*, Phys. Rev. D**84** (2011) 124043, arXiv:1110.2195v2 [hep-th].

[19] D. R. Brill, J. M. Cavallo, and J. A. Isenberg: *K - surfaces in the Schwarzschild space-time and the construction of lattice cosmologies*, J. Math. Phys. **21** (1980) 2789 .

[20] C. P. Burgess, and G. D. Moore: *The standard model: a primer* (Cambridge University Press, Cambridge, 2006).

[21] D. Capasso, and A. P. Polychronakos: *General static spherically symmetric solutions in Hořava gravity*, Phys. Rev. D**81** (2010) 084009, arXiv:0911.1535v3 [hep-th].

[22] D. Capasso, and A. P. Polychronakos: *Particle kinematics in Hořava-Lifshitz gravity*, JHEP **1002** (2010) 068, arXiv:0909.5405v2 [hep-th].

[23] S. Carlip: *Quantum gravity: a progress report*, Rept. Prog. Phys. **64** (2001) 885, arXiv:gr-qc/0108040v1.

[24] C. Charmousis, G. Niz, A. Padilla, and P.M. Saffin: *Strong coupling in Hořava gravity*, JHEP **0908** (2009) 070, arXiv:0905.2579v2 [hep-th].

[25] Y. Choquet-Bruhat: *General relativity and the Einstein equations*, Oxford Mathematical Monograph (Oxford University Press, Oxford, 2009).

[26] C. A. Clarkson, and R.K. Barrett: *Covariant perturbations of Schwarzschild black holes*, Class. Quant. Grav. **20** (2003) 3855-3884, arXiv:gr-qc/0209051v3.

[27] A. Coates, M. Colombo, A. E. Gumrukcuoglu, and T. P. Sotiriou: *Uninvited guest in mixed derivatives Hořava gravity*, Phys. Rev. D**94** (2016) 8, 084014, arXiv:1604.04215v1 [hep-th].

[28] CMS Collaboration: *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, Phys. Lett. B**716** (2012) 30, arXiv:1207.7235v2 [hep-ex].

[29] M. Colombo, A. E. Gumrukcuoglu, and T. P. Sotiriou: *Hořava gravity with mixed derivative terms*, Phys. Rev. D**91** (2015) 4, 044021, arXiv:1410.6360v1 [hep-th].

[30] M. Colombo, A. E. Gumrukcuoglu, and T. P. Sotiriou: *Hořava gravity with mixed derivative terms: power counting renormalizability with lower order dispersions*, Phys. Rev. D**92** (2015) 6, 064037, arXiv:1503.07544v3 [hep-th].

[31] A. Contillo, S. Rechenberger, and F. Saueressig: *Renormalization group flow of Hořava-Lifshitz gravity at low energies*, JHEP **1312** (2013) 017, arXiv:1309.7273v1 [hep-th].

[32] B. S. DeWitt: *Quantum theory of gravity. I. The canonical theory*, Phys. Rev. **160** (1967) 1113-1148.

[33] B. S. DeWitt, and G. Esposito: *An introduction to quantum gravity*, Int. J. Geom. Meth. Mod. Phys. **5** (2008) 101-156, arXiv:0711.2445v1 [hep-th].

[34] P. A. M. Dirac: *Lectures on quantum mechanics* (Dover Publications, New York, 2001).

[35] W. Donnelly, and T. Jacobson: *Hamiltonian structure of Hořava gravity*, Phys. Rev. D**84** (2011) 104019, arXiv:1106.2131v3 [hep-th].

[36] A. Einstein: *Die Grundlage der allgemeinen Relativitätstheorie*, Annalen der Physik 354 (1916) 769-822.

[37] G. F. R. Ellis, and R. Goswami: *Variations on Birkhoff's theorem*, Gen. Rel. Grav. **45** (2013) 2123-2142, arXiv:1304.3253v1 [gr-qc].

[38] D. Giulini: *The superspace of geometrodynamics*, Gen. Rel. Grav. **41** (2009) 785-815, arXiv:0902.3923v1 [gr-qc].

[39] D. Giulini: *Dynamical and Hamiltonian formulation of general relativity*, in Chap. 17 of "Springer Handbook of Spacetime", edited by A. Ashtekar, and V. Petkov (Springer Verlag, Dordrecht, 2014), arXiv:1505.01403 [gr-qc].

[40] D. Giulini, and C. Kiefer: *Wheeler-DeWitt metric and the attractivity of gravity*, Phys. Lett. A**193** (1994) 21-24, arXiv:9405040v2 [gr-qc].

[41] R. Goswami, and G. F. R. Ellis: *Almost Birkhoff theorem in general relativity*, Gen. Rel. Grav. **43** (2011) 2157-2170, arXiv:1101.4520v1 [gr-qc].

[42] E. Gourgoulhon: *3+1 formalism and bases of numerical relativity*, Lecture notes, arXiv:gr-qc/0703035v1.

[43] T. Harko, Z. Kovács, and F. S. N. Lobo: *Solar system tests of Hořava-Lifshitz gravity*, Proc. Roy. Soc. Lond. A**467** (2011) 1390-1407, arXiv:0908.2874v2 [gr-qc].

[44] S. W. Hawking, and R. Penrose: *The singularities of gravitational collapse and cosmology*, Proc. R. Soc. Lond. A**314** (1970) 529.

[45] M. Henneaux, and C. Teitelboim: *Quantization of gauge systems* (Princeton University Press, Princeton, 1992).

[46] M. Henneaux, A. Kleinschmidt, and G. L. Gómez: *A dynamical inconsistency of Hořava-Lifshitz gravity*, Phys. Rev. D**81** (2010) 064002, arXiv:0912.0399v2 [hep-th].

[47] S. A. Hojman, K. Kuchař, and C. Teitelboim: *Geometrodynamics regained*, Annals Phys. **96** (1976) 88.

[48] G. 't Hooft, and M. J. G. Veltman: *One loop divergencies in the theory of gravitation*, Ann. Inst. H. Poincare Phys. Theor. A**20** (1974) 69-94.

[49] P. Hořava: *Membranes at quantum criticality*, JHEP **0903** (2009) 020, arXiv:0812.4287v3 [hep-th].

[50] P. Hořava: *Quantum gravity at a Lifshitz point*, Phys. Rev. D**79** (2009) 084008, arXiv:0901.3775v2 [hep-th].

[51] P. Hořava: *General covariance in gravity at a Lifshitz point*, Class. Quant. Grav. **28** (2011) 114012, arXiv:1101.1081v1 [hep-th].

[52] P. Hořava, and C. M. Melby-Thompson: *General covariance in quantum gravity at a Lifshitz point*, Phys. Rev. D**82** (2010) 064027, arXiv:1007.2410v2 [hep-th].

[53] C. Isham: *Canonical quantum gravity and the problem of time*, Lond. Imp. Coll. (1992) ICTP-91-92-25, arXiv:gr-qc/9210011v1 .

[54] C. Isham: *Prima facie questions in quantum gravity*, Springer-Verlag, Lect. Notes Phys. **434** (1994) 1-21, arXiv:gr-qc/9310031v1.

[55] T. Jacobson, and D. Mattingly: *Gravity with a dynamical preferred frame*, Phys. Rev. D**64** (2001) 024028, arXiv:gr-qc/0007031v4.

[56] T. Jacobson: *Extended Hřava gravity and Einstein-aether theory*, Phys. Rev. D **82** (2010) 129901, arXiv:1001.4823v3 [hep-th].

[57] T. Jacobson, private communication.

[58] J. T. Jebsen: *Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum*, Reprinted as a Golden Oldie, Ark. Mat. Astron. Fys. 15,18.

[59] A. Kehagias, and K. Sfetsos: *The black hole and FRW geometries of non-relativistic gravity*, Phys. Lett. B **678** (2009) 123-126, arXiv:0905.0477v1 [hep-th].

[60] C. Kiefer: *Quantum gravity*, Int. Ser. Monogr. Phys. **124** (2004) 1-308 (Oxford University Press, Oxford).

[61] C. Kiefer: *Conceptual problems in quantum gravity and quantum cosmology*, ISRN Math. Phys. **2013** (2013) 509316, arXiv:1401.3578v1 [gr-qc].

[62] E. Kiritsis: *Spherically symmetric solutions in modified Hořava-Lifshitz gravity*, Phys. Rev. D **81** (2010) 044009, arXiv:0911.3164v3 [hep-th].

[63] E. B. Kiritsis, and G. Kofinas: *On Hořava-Lifshitz black holes*, JHEP **1001** (2010) 122, arXiv:0910.5487v1 [hep-th].

[64] R. A. Konoplya: *Towards constraining of the Hořava-Lifshitz gravities*, Phys. Lett. B**679** (2009) 499-503, arXiv:0905.1523v5 [hep-th].

[65] K. V. Kuchař: *Kinematics of tensor fields in hyperspace. II*, J. Math. Phys. **17** (1976) 792-800.

[66] K. V. Kuchař: *Geometrodynamics of Schwarzschild black holes*, Phys. Rev. D**50** (1994) 3961-3981, arXiv:gr-qc/9403003v1.

[67] K. W. Lee: *Dirichlet problem for the constant mean curvature equation and CMC foliation in the extended Schwarzschild spacetime*, Class. Quant. Grav. **33** (2016) 18, 185001, arXiv:1601.03811v1.

[68] K. W. Lee: *Existence of constant mean curvature foliation in the extended Schwarzschild spacetime*, Phys. Rev. D**94** (2016) 084038, arXiv:1607.07569v1 [math.DG].

[69] A. Lichnerowicz: *L'intégration des équations de la gravitation relativiste et le problème des n corps*, J. Math. Pures Appl. **23**(1944) 37.

[70] LIGO Scientific Collaboration, Virgo Collaboration: *Observation of gravitational waves from a binary black hole merger*, Phys. Rev. Lett. **116** (2016) 061102, arXiv:1602.03837v1 [gr-qc].

[71] K. Lin, O. Goldoni, M. F. da Silva, and A. Wang: *New look at black holes: existence of universal horizons*, Phys. Rev. D**91** (2015) 024047, arXiv:1410.6678v3 [gr-qc].

[72] F. S. N. Lobo, T. Harko, and Z. Kovács: *Solar systems tests of Hořava-Lifshitz black holes*, Proc. Roy. Soc. Lond. A**467** (2011) 1390-1407, arXiv:0908.2874v2 [gr-qc].

[73] R. Loll and L. Pires: *Role of the extra coupling in the kinetic term in Hořava-Lifshitz gravity*, Phys. Rev. D**90** (2014) 124050, arXiv:1407.1259v1.

[74] R. Loll and L. Pires: *Spherically symmetric solutions of the λ-R model*, Phys. Rev. D**96** (2017) 044030, arXiv:1702.08362v1 [gr-qc].

[75] H. Lü, J. Mei, and C. N. Pope: *Solutions to Hořava gravity*, Phys. Rev. Lett. **103** (2009) 091301, arXiv:0904.1595v4 [hep-th].

[76] E. Malec, and N. Ó Murchadha: *Constant mean curvature slices in the extended Schwarzschild solution and collapse of the lapse. Part I*, Phys. Rev. D **68** (2003) 124019, arXiv:gr-qc/0307046v1.

[77] E. Malec, and N. Ó Murchadha: *Constant mean curvature slices in the extended Schwarzschild solution and collapse of the lapse. Part II*, arXiv:gr-qc/0307047v1.

[78] E. Malec, and N. Ó Murchada: *The general spherically symmetric constant mean curvature foliations of the Schwarzschild solution*, Phys. Rev. D**80** (2009) 024017, arXiv:0903.4779v2.

[79] C. W. Misner, K. S. Thorne, and J. A. Wheeler: *Gravitation*, Freeman, San Francisco (1973).

[80] N. Ó Murchadha: *Readings of the Lichnerowicz-York equation*, Acta Phys. Polon. B**36** (2005) 109-120, arXiv:gr-qc/0502055v1.

[81] N. Ó Murchadha, and L. Pires: *Initial value formulation for the λ-R model*, in preparation.

[82] N. Ó Murchadha, and J. York: *Existence and uniqueness of solutions of the Hamiltonian constraint of general relativity on compact manifolds*, J. Math. Phys. **14** (1973) 1551-1557.

[83] N. Ó Murchadha, and J. York: *Initial-value problem of general relativity. I. General formulation and physical interpretation*, Phys. Rev. D**10** (1974) 428-436.

[84] R. S. Palais: *The principle of symmetric criticality*, Commun. Math. Phys. **69** (1979) 19-30.

[85] M. Park: *A test of Hořava gravity: the dark energy*, JCAP **1001** (2010) 001, arXiv:0906.4275v5 [hep-th].

[86] T. Regge, and C. Teitelboim: *Role of surface integrals in the Hamiltonian formulation of general relativity*, Annals Phys. **88** (1974) 286.

[87] T. Sotiriou: *Hořava-Lifshitz gravity: a status report*, J. Phys. Conf. Ser. **283** (2011) 012034, arXiv:1010.3218v2 [hep-th].

[88] T. Sotiriou, and V. Faraoni: *$f(R)$ theories of gravity*, Rev. Mod. Phys. **82** (2010) 451-497, arXiv:0805.1726v4 [gr-qc].

[89] T. Sotiriou, M. Visser, and S. Weinfurtner: *Quantum gravity without Lorentz invariance*, JHEP **0910** (2009) 033.

[90] T. Sotiriou, M. Visser, and S. Weinfurtner: *Phenomenologically viable Lorentz-violating quantum gravity*, Phys. Rev. Lett. **102** (2009) 251601, arXiv:0904.4464v3 [hep-th].

[91] T. Sotiriou, M. Visser, and S. Weinfurtner: *Projectable Hořava-Lifshitz gravity in a nutshell*, J. Phys. Conf. Ser. **222** (2010) 012054.

[92] K. Sundermeyer: *Constrained dynamics, with applications to Yang-Mills theory, general relativity, classical spin, dual string model*, Springer-Verlag, Lect. Notes Phys. **169** (1982) 1-318.

[93] P. Tuite, and N. Ó Murchadha: *Constant mean curvature slices of the Reissner-Nordström spacetime* (2013), arXiv:1307.4657v1 [gr-qc].

[94] A. Wang: *Hořava gravity at a Lifshitz point: a progress report*, Int. J. Mod. Phys. D**26** (2017) 1730014, arXiv:1701.06087v3 [gr-qc].

[95] C. M. Will: *The confrontation between general relativity and experiment*, Living Rev. Relativity **17** (2014) 4, arXiv:1403.7377v1 [gr-qc].

[96] S. Weinberg: *The quantum theory of fields. Vol. 1: Foundations* (Cambridge University Press, Cambridge, 2005).

[97] H. Yamabe: *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960) 21.

[98] R. Schoen: *Conformal Deformation of a Riemannian Metric to Constant Scalar Curvature*, J. Diff. Geom. **20** (1984) 479.

[99] J. York: *Gravitational degrees of freedom and the initial-value problem*, Phys. Rev. Lett. **26** (1971) 1656-1658.

[100] J. York: *Role of conformal three-geometry in the dynamics of gravitation*, Phys. Rev. **28** (1972) 1082-1085.

[101] J. York: *Mapping onto solutions of the gravitational initial value problem*, J. Math. Phys. **13** (1972) 125-130.