# Classification of cubic homogeneous polynomial maps with Jacobian matrices of rank two 

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#### Abstract

Let $K$ be any field with char $K \neq 2,3$. We classify all cubic homogeneous polynomial maps $H$ over $K$ with $\operatorname{rk} J H \leq 2$. In particular, we show that, for such an $H$, if $F=x+H$ is a Keller map then $F$ is invertible, and furthermore $F$ is tame if the dimension $n \neq 4$.


## 1 Introduction

Let $K$ be an arbitrary field and $K[x]:=K\left[x_{1}, x_{2}, \ldots, x_{n}\right]$ the polynomial ring in $n$ variables. For a polynomial map $F=\left(F_{1}, F_{2}, \ldots, F_{m}\right) \in K[x]^{m}$, we denote by $\mathcal{J} F:=\left(\frac{\partial F_{i}}{\partial x_{j}}\right)_{m \times n}$ the Jacobian matrix of $F$ and $\operatorname{deg} F:=\max _{i} \operatorname{deg} F_{i}$ the degree of $F$. A polynomial map $H \in K[x]^{m}$ is called homogeneous of degree $d$ if each $H_{i}$ is zero or homogeneous of degree $d$.

A polynomial map $F \in K[x]^{n}$ is called a Keller map if $\operatorname{det} \mathcal{J} F \in K^{*}$. The Jacobian conjecture asserts that any Keller map is invertible if char $K=0$; see [8] or [1]. It is still open for any dimension $n \geq 2$.

Following [14, we call a polynomial automorphism elementary if it is of the form $\left(x_{1}, \ldots, x_{i-1}, c x_{i}+a, x_{i+1}, \ldots, x_{n}\right)$, where $c \in K^{*}$ and $a \in K[x]$ contains no $x_{i}$. Furthermore, we call a polynomial automorphism tame if it is a finite composition of elementary ones. The definitions of elementary and tame may be different in other sources, but (as long as $K$ is a generalized Euclidean ring) the definitions of tame are equivalent. The Tame Generators Problem asks if every polynomial automorphism is tame. It has an affirmative answer in dimension 2

[^0]for arbitrary characteristic (see [10, 11]) and a negative answer in dimension 3 for the case of $\operatorname{char} K=0$ (see [14]), and is still open for any $n \geq 4$.

A polynomial map $F=x+H \in K[x]^{n}$ is called triangular if $H_{n} \in K$ and $H_{i} \in K\left[x_{i+1}, \ldots, x_{n}\right], 1 \leq i \leq n-1$. A polynomial map $F$ is called linearly triangularizable if it is linearly conjugate to a triangular map, i.e., there exists an invertible linear map $T \in \mathrm{GL}_{n}(K)$ such that $T^{-1} F(T x)$ is triangular. A linearly triangularizable map is tame.

Some special polynomial maps have been investigated in the literature. For example, when char $K=0$, a Keller map $F=x+H \in K[x]^{n}$ is shown to be linearly triangularizable in the cases: (1) $n=3$ and $H$ is homogeneous of arbitrary degree $d$ (de Bondt and van den Essen 6]) ; (2) $n=4$ and $H$ is quadratic homogeneous (Meisters and Olech [12]); (3) $n=9$ and $F$ is a quadratic homogeneous quasi-translation (Sun [16]); (4) $n$ arbitrary and $H$ is quadratic with $\operatorname{rk} \mathcal{J} H \leq 2$ (De Bondt and Yan [7]), and to be tame in the case (5) $n=5$ and $H$ is quadratic homogeneous (de Bondt [2] and Sun [17] independently), and to be invertible in the case (6) $n=4$ and $H$ is cubic homogeneous (Hubbers [9]). For the case of arbitrary characteristic, de Bondt [5] described the Jacobian matrix $\mathcal{J} H$ of rank two for any quadratic polynomial map $H$ and showed that if $\mathcal{J} H$ is nilpotent then $\mathcal{J} H$ is similar to a triangular one.

In this paper, we investigate cubic homogeneous polynomial maps $H$ with $\operatorname{rk} \mathcal{J} H \leq 2$ for any dimension $n$ when $\operatorname{char} K \neq 2,3$. In Section 2 , we classify all such maps (Theorem 2.7). And in Section 3, we show that for such an $H$, if $F=x+H$ is a Keller map, then it is invertible and furthermore it is tame if the dimension $n \neq 4$ (Theorem 3.4).

## 2 Cubic homogeneous maps $H$ with rk $J H \leq 2$

For a polynomial map $H \in K[x]^{m}$, we write $\operatorname{trdeg}_{K} K(H)$ for the transcendence degree of $K(H)$ over $K$. It is well-known that $\operatorname{rk} \mathcal{J} H=\operatorname{trdeg}_{K} K(H)$ if $K(H) \subseteq$ $K(x)$ is separable, in particular if char $K=0$; see [8, Proposition 1.2.9]. And for arbitrary characteristic, one has $\operatorname{rk} \mathcal{J} H \leq \operatorname{trdeg}_{K} K(H)$; see [4] or 13].

It was shown in [5] that when char $K \neq 2$, for any quadratic polynomial map $H$ with $\operatorname{rk} \mathcal{J} H \leq 2$, one has $\operatorname{rk} \mathcal{J} H=\operatorname{trdeg}_{K} K(H)$. We will show that when char $K \neq 2,3$, for any cubic homogeneous polynomial map $H$ with rk $\mathcal{J} H \leq 2$, one has $\operatorname{rk} \mathcal{J} H=\operatorname{trdeg}_{K} K(H)$. The notation $\left.a\right|_{x=c}$ below means to substitute $x$ by $c$ in $a$.

Theorem 2.1. Let $s \leq n$. Take

$$
\tilde{x}:=\left(x_{1}, x_{2}, \ldots, x_{s}\right) \quad \text { and } \quad L:=K\left(x_{s+1}, x_{s+2}, \ldots, x_{n}\right) .
$$

To prove that for (homogeneous) polynomial maps $H \in K[x]^{m}$ of degree $d$,

$$
\begin{equation*}
\operatorname{rk} \mathcal{J} H=r \quad \text { implies } \operatorname{trdeg}_{K} K(H)=r, \quad \text { for every } r<s \tag{2.1}
\end{equation*}
$$

it suffices to show that for (homogeneous) polynomial maps $\tilde{H} \in L[\tilde{x}]^{s}$ of degree $d$,

$$
\begin{equation*}
\operatorname{trdeg}_{L} L(\widetilde{H})=s \quad \text { implies } \quad \operatorname{rk} \mathcal{J}_{\tilde{x}} \tilde{H}=s \tag{2.2}
\end{equation*}
$$

Proof. Suppose that $H \in K[x]^{m}$ is (homogeneous) of degree $d$, such that (2.1) does not hold. Then there exists an $r<s$ such that rk $\mathcal{J} H=r<\operatorname{trdeg}_{K} K(H)$. We need to show that (2.2) does not hold.

Let $s^{\prime}=\operatorname{trdeg}_{K} K(H)$. Assume without loss of generality that $H_{1}, H_{2}, \ldots, H_{s^{\prime}}$ are algebraically independent over $K$, and that the components of

$$
H^{\prime}:=\left(H_{1}, H_{2}, \ldots, H_{s^{\prime}}, x_{s^{\prime}+1}^{d}, x_{s^{\prime}+2}^{d}, \ldots, x_{s}^{d}\right)
$$

are algebraically independent over $K$ if $s^{\prime}<s$. Then

$$
\operatorname{rk} \mathcal{J} H^{\prime} \leq r+\left(s-s^{\prime}\right)<s=\operatorname{trdeg}_{K} K\left(H^{\prime}\right)
$$

For the case of $s^{\prime} \geq s$, just take $H^{\prime}=\left(H_{1}, H_{2}, \ldots, H_{s}\right)$, and we have also rk $\mathcal{J} H^{\prime} \leq r<s$.

Notice that (2.1) is also unsatisfied for $H^{\prime}$. So, replacing $H$ by $H^{\prime}$, we may assume that $H \in K[x]^{s}$ with rk $\mathcal{J} H=r<\operatorname{trdeg}_{K} K(H)=s$.

One may observe that $H_{1}\left(x_{1}, x_{1} x_{2}, x_{1} x_{3}, \ldots, x_{1} x_{n}\right)$ is algebraically independent over $K$ of $x_{2}, x_{3}, \ldots, x_{n}$. On account of the Steinitz Mac Lane exchange lemma, we may assume without loss of generality that the components of

$$
\left(H\left(x_{1}, x_{1} x_{2}, x_{1} x_{3}, \ldots, x_{1} x_{n}\right), x_{s+1}, x_{s+2}, \ldots, x_{n}\right)
$$

are algebraically independent over $K$. Then the components of $H\left(x_{1}, x_{1} x_{2}, x_{1} x_{3}\right.$, $\left.\ldots, x_{1} x_{n}\right)$ are algebraically independent over $L:=K\left(x_{s+1}, x_{s+2}, \ldots, x_{n}\right)$, and so are the components of

$$
\tilde{H}:=H\left(x_{1}, x_{2}, \ldots, x_{s}, x_{1} x_{s+1}, x_{1} x_{s+2}, \ldots, x_{1} x_{n}\right) \in L[\widetilde{x}]^{s},
$$

where $\widetilde{x}=\left(x_{1}, x_{2}, \ldots, x_{s}\right)$. That is, $\operatorname{trdeg}_{L} L(\widetilde{H})=s$.
Let $G:=\left(x_{1}, x_{2}, \ldots, x_{s}, x_{1} x_{s+1}, x_{1} x_{s+2}, \ldots, x_{1} x_{n}\right)$. Then it follows from the chain rule that

$$
\mathcal{J}_{\tilde{x}} \tilde{H}=\left.(\mathcal{J} H)\right|_{x=G} \cdot \mathcal{J}_{\tilde{x}} G,
$$

so $\operatorname{rk} \mathcal{J}_{\tilde{x}} \tilde{H} \leq\left.\operatorname{rk}(\mathcal{J} H)\right|_{x=G} \leq \operatorname{rk} \mathcal{J} H<s$. Therefore (2.2) does not hold for $\widetilde{H}$, which completes the proof.

Lemma 2.2. Let $H \in K[x]^{m}$ be a polynomial map of degree $d$ and $r:=\operatorname{rk} \mathcal{J} H$. Denote by $|K|$ the cardinality of $K$.
(i) If $|K|>(d-1) r$ and $\mathcal{J} H \cdot x=0$, then there exist $S \in \mathrm{GL}_{m}(K)$ and $T \in \mathrm{GL}_{n}(K)$, such that for $\tilde{H}:=S H(T x)$,

$$
\left.\tilde{H}\right|_{x=e_{r+1}}=\left(\begin{array}{cc}
I_{r} & 0 \\
0 & 0
\end{array}\right)
$$

(ii) If $|K|>(d-1) r+1$ and $\mathcal{J} H \cdot x \neq 0$, then there exist $S \in \mathrm{GL}_{m}(K)$ and $T \in \mathrm{GL}_{n}(K)$, such that for $\tilde{H}:=S H(T x)$,

$$
\left.\tilde{H}\right|_{x=e_{1}}=\left(\begin{array}{cc}
I_{r} & 0 \\
0 & 0
\end{array}\right)
$$

Moreover, $|K|$ may be one less (i.e. at least $(d-1) r$ and $(d-1) r+1$ respectively) if every nonzero component of $H$ is homogeneous.

Proof. (i) Assume without loss of generality that

$$
a_{0}:=\operatorname{det} \mathcal{J}_{x_{1}, x_{2}, \ldots, x_{r}}\left(H_{1}, H_{2}, \ldots, H_{r}\right) \neq 0
$$

Suppose that $|K|>(d-1) r$. It follows by [3, Lemma 5.1 (i)] that there exists a vector $w \in K^{n}$ such that $a_{0}(w) \neq 0$. So $\left.\operatorname{rk}(\mathcal{J} H)\right|_{x=w}=r$. There exist $n-r$ independent vectors $v_{r+1}, v_{r+2}, \ldots, v_{n} \in K^{n}$, such that $\left.(\mathcal{J} H)\right|_{x=w} \cdot v_{i}=0$ for $i=r+1, r+2, \ldots, n$. And we may take $v_{r+1}=w$ since

$$
\left.(\mathcal{J} H)\right|_{x=w} \cdot w=\left.(\mathcal{J} H \cdot x)\right|_{x=w}=0
$$

Take $T=\left(v_{1}, v_{2}, \cdots, v_{n}\right) \in \mathrm{GL}_{n}(K)$. From the chain rule, we deduce that

$$
\left.(\mathcal{J}(H(T x)))\right|_{x=e_{r+1}} \cdot e_{i}=\left.(\mathcal{J} H)\right|_{x=T e_{r+1}} \cdot T e_{i}=\left.(\mathcal{J} H)\right|_{x=w} \cdot v_{i} \quad(1 \leq i \leq n)
$$

In particular, rk $\left.\mathcal{J}(H(T x))\right|_{x=e_{r+1}}=r$ and the last $n-r$ columns of $\left.(\mathcal{J}(H(T x)))\right|_{x=e_{r+1}}$ are zero. There exists $S \in \mathrm{GL}_{m}(K)$ such that

$$
\left.(\mathcal{J}(S H(T x)))\right|_{x=e_{r+1}}=\left.S \cdot(\mathcal{J}(H(T x)))\right|_{x=e_{r+1}}=\left(\begin{array}{cc}
I_{r} & 0 \\
0 & 0
\end{array}\right)
$$

(2) Suppose that $|K|>(d-1) r+1$. Since $\mathcal{J} H \cdot x \neq 0$, we may assume that

$$
\operatorname{rk}\left(\mathcal{J} H \cdot x, \mathcal{J}_{x_{2}, x_{3}, \ldots, x_{r}} H\right)=r
$$

and that

$$
a_{1}:=\operatorname{det}\left(\mathcal{J}\left(H_{1}, H_{2}, \ldots, H_{r}\right) \cdot x, \mathcal{J}_{x_{2}, x_{3}, \ldots, x_{r}}\left(H_{1}, H_{2}, \ldots, H_{r}\right)\right) \neq 0
$$

It follows by [3, Lemma 5.1 (i)] that there exists $w \in K^{n}$ such that $a_{1}(w) \neq$ 0 . One may observe that $\left.\operatorname{rk}(\mathcal{J} H)\right|_{x=w}=r$ and thus there exist independent vectors $v_{r+1}, v_{r+2}, \ldots, v_{n} \in K^{n}$, such that $\left.(\mathcal{J} H)\right|_{x=w} \cdot v_{i}=0$ for $i=r+1$, $r+2, \ldots, n$. Since $\left.(\mathcal{J} H \cdot x)\right|_{x=w}$ is the first column of a full column rank matrix, we have

$$
\left.(\mathcal{J} H)\right|_{x=w} \cdot w=\left.(\mathcal{J} H \cdot x)\right|_{x=w} \neq 0
$$

So $v_{1}:=w$ is independent of $v_{r+1}, v_{r+2}, \ldots, v_{n}$.
Take $T=\left(v_{1}, v_{2}, \cdots, v_{n}\right) \in \operatorname{GL}_{n}(K)$. Then

$$
\left.(\mathcal{J}(H(T x)))\right|_{x=e_{1}} \cdot e_{i}=\left.(\mathcal{J} H)\right|_{x=T e_{1}} \cdot T e_{i}=\left.(\mathcal{J} H)\right|_{x=w} \cdot v_{i} \quad(1 \leq i \leq n)
$$

The rest of the proof of (ii) is similar to that of (i).
The last claim follows from [3, Lemma 5.1 (ii)], as an improvement to [3, Lemma 5.1 (i)].

Proposition 2.3. Assume that char $K \notin\{1,2, \ldots, d\}$. Then for any cubic homogeneous polynomial map $H \in K[x]^{m}$ of degree $d$ with $\operatorname{rk} \mathcal{J} H \leq 1$, the components of $H$ are linearly dependent over $K$ in pairs, and one has $\operatorname{rk} \mathcal{J} H=$ $\operatorname{trdeg}_{K} K(H)$.

Proof. The case $\operatorname{rk} \mathcal{J} H=0$ is obvious, so let $\operatorname{rk} \mathcal{J} H=1$. On account of Lemma 2.2, we may assume that $\left.\mathcal{J} H\right|_{x=e_{1}}=E_{11}$. Let $j \geq 2$. Since $\operatorname{deg}_{x_{1}} H_{j}<d$, we infer that either $H_{j}=0$, or $\operatorname{deg}_{x_{1}} \frac{\partial}{\partial x_{1}} H_{j}<\operatorname{deg}_{x_{1}} \frac{\partial}{\partial x_{i}} H_{j}$ for some $i \geq 2$, where $\operatorname{deg}_{x_{1}} 0=-\infty$. The latter is impossible due to $\operatorname{rk} \mathcal{J} H=1$, so $H_{j}=0$. This holds for all $j \geq 2$, which yields the desired results.

Lemma 2.4. Let $H=\left(h, x_{1}^{2} x_{2}, x_{2}^{2} x_{3}\right)$ or $\left(h, x_{1}^{2} x_{3}, x_{2}^{2} x_{3}\right) \in K\left[x_{1}, x_{2}, x_{3}\right]^{3}$, where $h$ is cubic homogeneous, and assume that char $K \neq 2,3$. Then $\operatorname{rk} \mathcal{J} H=$ $\operatorname{trdeg}_{K} K(H)$.

Proof. It suffices to consider the case of $\operatorname{rk} \mathcal{J} H=2$. Define a derivation $D$ on $A=K\left[x_{1}, x_{2}, x_{3}\right]$ as follows: for any $f \in A$,

$$
D(f)=\frac{x_{1} x_{2} x_{3}}{H_{2} H_{3}} \operatorname{det} \mathcal{J} H
$$

In the case $H=\left(h, x_{1}^{2} x_{2}, x_{2}^{2} x_{3}\right)$, an easy calculation shows that $D=x_{1} \partial_{x_{1}}-$ $2 x_{2} \partial_{x_{2}}+4 x_{3} \partial_{x_{3}}$. Then for any term $u=x_{1}^{d_{1}} x_{2}^{d_{2}} x_{3}^{d_{3}} \in A, D(u)=\left(d_{1}-2 d_{2}+\right.$ $\left.4 d_{3}\right) u$. And thus $\operatorname{ker} D:=\{g \in A \mid D(g)=0\}$, the kernel of $D$, is linearly spanned by all terms $u$ with $d_{1}-2 d_{2}+4 d_{3}=0$. So the only cubic terms in $\operatorname{ker} D$ are $x_{1}^{2} x_{2}$ and $x_{2}^{2} x_{3}$. Since $\operatorname{rk} \mathcal{J} H=2$, we have $\operatorname{det} \mathcal{J} H=0$ and thus $h \in \operatorname{ker} D$, which implies that $h$ is a linear combinations of $x_{1}^{2} x_{2}$ and $x_{2}^{2} x_{3}$. Thus $\operatorname{trdeg}_{K} K(H)=2$.

In the case $H=\left(h, x_{1}^{2} x_{3}, x_{2}^{2} x_{3}\right)$, one may verify that $x_{1}^{2} x_{3}, x_{1} x_{2} x_{3}$ and $x_{2}^{2} x_{3}$ are the only cubic terms in $\operatorname{ker} D$. The conclusion follows similarly.

Theorem 2.5. Assume that char $K \neq 2,3$. Then for any cubic homogeneous polynomial map $H \in K[x]^{m}$ with $\operatorname{rk} \mathcal{J} H \leq 2$, one has $\operatorname{rk} \mathcal{J} H=\operatorname{trdeg}_{K} K(H)$.

Proof. Due to Theorem 2.1, and replacing $L$ there by $K$, we may assume that $H \in K\left[x_{1}, x_{2}, x_{3}\right]^{3}$, and it suffices to show that

$$
\operatorname{trdeg}_{K} K(H)=3 \text { implies rk } \mathcal{J} H=3
$$

or equivalently,

$$
\begin{equation*}
\operatorname{det} \mathcal{J} H=0 \text { implies } \operatorname{trdeg}_{K} K(H)<3 \tag{2.3}
\end{equation*}
$$

So assume that $\operatorname{det} \mathcal{J} H=0$. Since we may replace $K$ by an extension field to make it large enough, it follows by Lemma 2.2 that we may assume that $\left.(\mathcal{J} H)\right|_{x=e_{1}}=E_{11}+E_{22}$. Then $\mathcal{J} H$ is of the form

$$
\left(\begin{array}{ccc}
x_{1}^{2}+* & * & * \\
* & x_{1}^{2}+* & * \\
* & * & \frac{\partial H_{3}}{\partial x_{3}}
\end{array}\right)
$$

where the $x_{1}$-degree of each element $*$ is less than 2 . Observing the terms with $x_{1}$-degree $\geq 5$ in $\operatorname{det} \mathcal{J} H$, we have that $\frac{\partial H_{3}}{\partial x_{3}} \in K\left[x_{2}, x_{3}\right]$. Notice that $H_{2}$ and $H_{3}$ are of the form:

$$
\begin{aligned}
& H_{2}=x_{1}^{2} x_{2}+b_{10} x_{1} x_{3}^{2}+b_{11} x_{1} x_{2} x_{3}+b_{12} x_{1} x_{2}^{2}+b_{0}\left(x_{2}, x_{3}\right) ; \\
& H_{3}=c_{12} x_{1} x_{2}^{2}+c_{00} x_{3}^{3}+c_{01} x_{2} x_{3}^{2}+c_{02} x_{2}^{2} x_{3}+c_{03} x_{2}^{3} .
\end{aligned}
$$

We shall show that $x_{2}^{2} \mid H_{3}$, i.e., $c_{00}=c_{01}=0$.
Noticing that the part of $x_{1}$-degree 4 of $\operatorname{det} \mathcal{J} H$ is $\left(\frac{\partial H_{3}}{\partial x_{3}}-\frac{\partial H_{2}}{\partial x_{1} \partial x_{3}} \frac{\partial H_{3}}{\partial x_{1} \partial x_{2}}\right) x_{1}^{4}$, we see that $\frac{\partial H_{3}}{\partial x_{3}}-\frac{\partial H_{2}}{\partial x_{1} \partial x_{3}} \frac{\partial H_{3}}{\partial x_{1} \partial x_{2}}=0$. Consequently,

$$
\left(3 c_{00} x_{3}^{2}+2 c_{01} x_{2} x_{3}+c_{02} x_{2}^{2}\right)=\left(2 b_{10} x_{3}+b_{11} x_{2}\right)\left(2 c_{12} x_{2}\right)
$$

so

$$
c_{00}=0 \quad c_{01}=2 b_{10} c_{12} \quad c_{02}=2 b_{11} c_{12}
$$

One may observe that the coefficient of $x_{1}^{3} x_{3}^{3}$ in $\operatorname{det} \mathcal{J} H$ is $2 c_{01} b_{10}=0$, which we can combine with $c_{01}=2 b_{10} c_{12}$ to obtain $c_{01}=0$. Therefore,

$$
H_{3}=\left(c_{12} x_{1}+c_{03} x_{2}+c_{02} x_{3}\right) x_{2}^{2}
$$

Moreover, if $c_{12}=0$ then $c_{02}=2 b_{11} c_{12}=0$ and thus $H_{3}=c_{03} x_{2}^{3}$.
We distinguish two cases.

- Case 1: $c_{12} \neq 0$ and $c_{12} x_{1}+c_{03} x_{2}+c_{02} x_{3} \nmid H_{i}$ for some $i$.

Then $H_{3}$ is the product of two linear forms, of which two are distinct. Hence we can compose $H$ with invertible linear maps on both sides, to obtain a map $H^{\prime}$ for which $H_{2}^{\prime}=x_{1}^{2} x_{2}$, and $x_{2} \nmid H_{1}^{\prime}$.
Notice that $H_{1}^{\prime}(1,0, t) \neq 0$. As $K$ has at least 5 elements, it follows from [3, Lemma 5.1 (i)] that there exists a $\lambda \in K$, such that $H_{1}^{\prime}(1,0, \lambda) \neq 0$. Hence the coefficient of $x_{1}^{3}$ in $H_{1}^{\prime}\left(x_{1}, x_{2}, x_{3}+\lambda x_{1}\right)$ is nonzero. Furthermore, $H_{2}^{\prime}\left(x_{1}, x_{2}, x_{3}+\lambda x_{1}\right)=x_{1}^{2} x_{2}$.
Replacing $H^{\prime}$ by $H^{\prime}\left(x_{1}, x_{2}, x_{3}+\lambda x_{1}\right)$, we may assume that $H_{2}^{\prime}=x_{1}^{2} x_{2}$ and that $H_{1}^{\prime}$ contains $x_{1}^{3}$ as a term. We may even assume that the coefficient of $x_{1}^{3}$ in $H_{1}^{\prime}$ equals 1. Then $\left.\mathcal{J} H^{\prime}\right|_{x=e_{1}}$ is of the form

$$
\left(\begin{array}{lll}
1 & * & a \\
0 & 1 & 0 \\
* & * & *
\end{array}\right)
$$

and has rank 2. Furthermore, $v_{3}=(-a, 0,1)^{t}$ belongs to its null space. We may apply the proof of Lemma 2.2 on $H^{\prime}$ by taking $T=\left(e_{1}, e_{2}, v_{3}\right)$ and taking an appropriate $S \in \mathrm{GL}_{3}(K)$ such that $\widetilde{H}:=S H^{\prime}(T x)$ satisfies $\left.\mathcal{J} \widetilde{H}\right|_{x=e_{1}}=\left.S \mathcal{J} H^{\prime}\right|_{x=T e_{1}} T=E_{11}+E_{22}$. Notice that $T x$ is of the form $\left(L_{1}, x_{2}, L_{3}\right)$, and observing the form of $\left.\mathcal{J} H^{\prime}\right|_{x=e_{1}}$ one may also choose $S x$ to be of the form $\left(*, x_{2}, *\right)$. Then $\widetilde{H}_{2}=L_{1}^{2} x_{2}$.

So we can compose $\widetilde{H}$ with an invertible linear map on the right, to obtain a map $\widetilde{H}^{\prime}$ for which $\widetilde{H}_{2}^{\prime}=x_{1}^{2} x_{2}$ and $\widetilde{H}_{3}^{\prime}=x_{2}^{2} L^{\prime}$ for some linear form $L^{\prime}$.
Suppose first that $L^{\prime}$ is a linear combination of $x_{1}$ and $x_{2}$. If $\tilde{H}_{1}^{\prime} \in$ $K\left[x_{1}, x_{2}\right]$, then we are done. Otherwise, we have $\operatorname{det} \mathcal{J}_{x_{1}, x_{2}}\left(\widetilde{H}_{2}^{\prime}, \widetilde{H}_{3}^{\prime}\right)=0$, and then by Proposition 2.3, $\operatorname{trdeg}_{K} K\left(H_{2}^{\prime}, H_{3}^{\prime}\right)<2$.
Suppose next that $L^{\prime}$ is not a linear combination of $x_{1}$ and $x_{2}$. Then we may assume that $\widetilde{H}_{3}^{\prime}=x_{2}^{2} x_{3}$. By Lemma $2.4(\mathrm{i}), \operatorname{trdeg}_{K} K\left(\widetilde{H}^{\prime}\right)<3$.

- Case 2: $c_{12}=0$ or $c_{12} x_{1}+c_{03} x_{2}+c_{02} x_{3} \mid H_{i}$ for all $i$.

Since $x_{2}^{2} \mid H_{3}$, we can compose $H$ with invertible linear maps on both sides, to obtain a map $H^{\prime}$ for which $H_{1}^{\prime} \in\left\{x_{1}^{3}, x_{1}^{2} x_{2}\right\}$. After a possible interchange of $H_{2}^{\prime}$ and $H_{3}^{\prime}$, the first two rows of $\mathcal{J} H^{\prime}$ are independent. Now we may apply the proof of Lemma 2.2 to $H^{\prime}$, more precisely, there exist $S, T \in \mathrm{GL}_{3}(K)$ such that $\widetilde{H}:=S H^{\prime}(T x)$ satisfies $\left.\mathcal{J} \widetilde{H}\right|_{x=e_{1}}=E_{11}+E_{22}$. If we choose $w$ such that first two rows of $\left(\mathcal{J} H^{\prime}\right)_{x=w}$ are independent, then we can take $S$ such that $S x=\left(f_{1} x_{1}+f_{2} x_{2}, g_{1} x_{1}+g_{2} x_{2}, *\right)$. By repeating the discussion for $\widetilde{H}$ as for $H$ above, we may assume that $\widetilde{H}_{3}=L x_{2}^{2}$ for some linear form $L$.
Let $T x=\left(L_{1}, L_{2}, L_{3}\right)$. Notice that $H_{1}^{\prime}(T x) \in\left\{L_{1}^{3}, L_{1}^{2} L_{2}\right\}$ and that $H_{1}^{\prime}(T x)$ is a linear combination of $\widetilde{H}_{1}$ and $\widetilde{H}_{2}$. Hence we can compose $\widetilde{H}$ with a linear map on the left, to obtain a map $\widetilde{H}^{\prime}$ for which $\widetilde{H}_{2}^{\prime} \in\left\{L_{1}^{3}, L_{1}^{2} L_{2}\right\}$ and $\widetilde{H}_{3}^{\prime}=L x_{2}^{2}$.
Suppose first that $\tilde{H}_{2}^{\prime}=L_{1}^{2} L_{2}$. Then $c_{12} \neq 0$, so $c_{12} x_{1}+c_{03} x_{2}+c_{02} x_{3} \mid H_{i}$ for all $i$. From this, we infer that $L_{2} \mid \widetilde{H}_{i}$ and $L_{2} \mid \widetilde{H}_{i}^{\prime}$ for all $i$. As $x_{2} \nmid \widetilde{H}_{1}$, we deduce that $L$ and $L_{2}$ are dependent linear forms, which are independent of $x_{2}$. If $L$ and $L_{2}$ are linear combinations of $L_{1}$ and $x_{2}$, then we can reduce to Proposition 2.3, and otherwise we can reduce to Lemma 2.4 (ii).

Suppose next that $\widetilde{H}_{2}^{\prime}=L_{1}^{3}$. If $L, L_{1}$ and $x_{2}$ are linearly dependent over $K$, then we can reduce to Proposition 2.3, Otherwise, $\tilde{H}$ is as $H$ in the previous case.

Remark 2.6. Inspired by Lemma 2.4 we investigated maps $H$ of which the components are terms, and searched for $H$ with algebraically independent components for which $\operatorname{det} \mathcal{J} H=0$. One can infer that $H$ is as such, if and only if the matrix with entries $\operatorname{deg}_{x_{i}} H_{j}$ has determinant zero over $K$, but not over $\mathbb{Z}$.

We found the following non-homogeneous $H$ as above over fields of characteristic 5:

$$
\left(x_{1}^{3} x_{2}, x_{1} x_{2}^{2}\right), \quad\left(x_{1}^{2} x_{2}, x_{1} x_{3}^{2}, x_{2} x_{3}\right)
$$

with the following homogenizations respectively:

$$
\left(x_{1}^{3} x_{2}, x_{1} x_{2}^{2} x_{3}, x_{3}^{4}\right), \quad\left(x_{1}^{2} x_{2}, x_{1} x_{3}^{2}, x_{2} x_{3} x_{4}, x_{4}^{3}\right)
$$

Besides these homogenizations, we found the following homogeneous $H$ over fields of characteristic 5:

$$
\left(x_{1}^{2} x_{3}^{2}, x_{1} x_{2}^{3}, x_{2} x_{3}^{3}\right), \quad\left(x_{4} x_{1}^{2}, x_{1} x_{2}^{2}, x_{2} x_{3}^{2}, x_{3} x_{4}^{2}\right)
$$

We conclude with a homogeneous $H$ over fields of characteristic 7, and a homogeneous $H$ over any characteristic $p \in\{1,2, \ldots, d\}$ respectively:

$$
\left(x_{3} x_{1}^{3}, x_{1} x_{2}^{3}, x_{2} x_{3}^{3}\right), \quad\left(x_{1}^{d}, x_{1}^{d-p} x_{2}^{p}\right)
$$

These examples show that the conditions in Proposition 2.3 and Theorem 2.5 cannot be relaxed.

Theorem 2.7. Suppose that char $K \neq 2,3$ and let $H \in K[x]^{m}$ be cubic homogeneous. Let $r:=\operatorname{rk} \mathcal{J} H$ and suppose that $r \leq 2$. Then there exist $S \in \mathrm{GL}_{m}(K)$ and $T \in \mathrm{GL}_{n}(K)$, such that for $\tilde{H}:=S H(\overline{T x})$, one of the following statements holds:
(1) $\tilde{H}_{r+1}=\tilde{H}_{r+2}=\cdots=\tilde{H}_{m}=0$;
(2) $r=2$ and $\tilde{H} \in K\left[x_{1}, x_{2}\right]^{m}$;
(3) $r=2$ and $K \tilde{H}_{1}+K \tilde{H}_{2}+\cdots+K \tilde{H}_{m}=K x_{3} x_{1}^{2} \oplus K x_{3} x_{1} x_{2} \oplus K x_{3} x_{2}^{2}$.

Furthermore, we may take $S=T^{-1}$ if $m=n$.
Proof. By Theorem 2.5, $\operatorname{trdeg}_{K} K(H)=\operatorname{rk} \mathcal{J} H=r \leq 2$. Since $H$ is homogeneous, we have $\operatorname{trdeg}_{K} K(t H)=r$ as well, where $t$ is a new variable.

Suppose first that $r \leq 1$. It follows by [4, Theorem 2.7] that we may take $\tilde{H}$ as in (1).

Suppose next that $r=2$. By [4, Theorem 2.7], $H$ is of the form $g \cdot h(p, q)$, such that $g, h$ and $(p, q)$ are homogeneous and $\operatorname{deg} g+\operatorname{deg} h \cdot \operatorname{deg}(p, q)=3$.

If deg $h \leq 1$, then every triple of components of $h$ is linearly dependent over $K$, and thus we may take $\tilde{H}$ as in (1). If $\operatorname{deg} h=3$, then $\operatorname{deg}(p, q)=1$ and $\operatorname{deg} g=0$, whence we may take $\tilde{H}$ as in (2).

So assume that $\operatorname{deg} h=2$. Then $\operatorname{deg}(p, q)=1$ and $\operatorname{deg} g=1$. If $g$ is a linear combination of $p$ and $q$, then we may take $\tilde{H}$ as in (2). If $g$ is not a linear combination of $p$ and $q$, then we may take $\tilde{H}$ as in (3) or (1).

Finally, if $m=n$ and $\tilde{H}=S H(T x)$ is as in (1), then $S H\left(S^{-1} x\right)=$ $\tilde{H}\left(T^{-1} S^{-1} x\right)$ is still as in (1). So we may take $S=T^{-1}$. If $m=n$ and $\tilde{H}=S H(T x)$ is as in (2) or (3), then $T^{-1} H(T x)=T^{-1} S^{-1} \tilde{H}$ is still as in (2) or (3), whence we may also take $S=T^{-1}$.

## 3 Cubic homogeneous Keller maps $x+H$ with $\operatorname{rk} J H \leq 2$

For two matrices $M, N \in \operatorname{Mat}_{n}(K[x])$, we say that $M$ is similar over $K$ to $N$, if there exists $T \in \mathrm{GL}_{n}(K)$ such that $N=T^{-1} M T$.

Theorem 3.1. Let $F=x+H \in K[x]^{n}$ be a Keller map with $\operatorname{trdeg}_{K} K(H)=1$. Then $\mathcal{J} H$ is similar over $K$ to a triangular matrix, and the following statements are equivalent:
(1) $\operatorname{det} \mathcal{J} F=1$;
(2) $\mathcal{J} H$ is nilpotent;
(3) $\left.(\mathcal{J} H) \cdot(\mathcal{J} H)\right|_{x=y}=0$, where $y=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ are $n$ new variables.

Proof. Since $\operatorname{trdeg}_{K} K(H)=1$, by [4, Corollary 3.2] there exists a polynomial $p \in K[x]$ such that $H_{i} \in K[p]$ for each $i$. Say that $H_{i}=h_{i}(p)$, where $h_{i} \in K[t]$ for each $i$. Write $h_{i}^{\prime}=\frac{\partial}{\partial t} h_{i}$, then

$$
\begin{equation*}
\mathcal{J} H=h^{\prime}(p) \cdot \mathcal{J} p \tag{3.1}
\end{equation*}
$$

Assume without loss of generality that

$$
h_{1}^{\prime}=h_{2}^{\prime}=\cdots=h_{s}^{\prime}=0
$$

and that

$$
0 \leq \operatorname{deg} h_{s+1}^{\prime}<\operatorname{deg} h_{s+2}^{\prime}<\cdots<\operatorname{deg} h_{n}^{\prime}
$$

For $s<i<n$,

$$
\operatorname{deg} h_{i}^{\prime}(p)=\operatorname{deg} h_{i}^{\prime} \cdot \operatorname{deg} p \leq\left(\operatorname{deg} h_{i+1}^{\prime}-1\right) \cdot \operatorname{deg} p=\operatorname{deg} h_{i+1}^{\prime}(p)-\operatorname{deg} p
$$

Since the degrees of the entries of $\mathcal{J} p$ are less than $\operatorname{deg} p$, we deduce from (3.1) that the nonzero entries on the diagonal of $\mathcal{J} H$ have different degrees in increasing order. Furthermore, the nonzero entries beyond the $(s+1)$ th entry on the diagonal of $\mathcal{J} H$ have positive degrees.

By (3.1), $\operatorname{rk}(-\mathcal{J} H) \leq 1$, and thus $n-1$ eigenvalues of $-\mathcal{J} H$ are zero. It follows that the trailing degree of the characteristic polynomial of $-\mathcal{J} H$ is at least $n-1$. More precisely,

$$
\operatorname{det}\left(t I_{n}+\mathcal{J} H\right)=t^{n}-\operatorname{tr}(-\mathcal{J} H) \cdot t^{n-1}
$$

and thus

$$
\operatorname{det} \mathcal{J} F=\left.\left(t^{n}-\operatorname{tr}(-\mathcal{J} H) \cdot t^{n-1}\right)\right|_{t=1}=1+\operatorname{tr} \mathcal{J} H
$$

Observe that the diagonal of $\mathcal{J} H$ is totally zero, except maybe the $(s+1)$ th entry, which is a constant.

So $\frac{\partial}{\partial x_{i}} p=0$ for all $i>s+1$, and $\mathcal{J} H$ is lower triangular. If the $(s+1)$ th entry on the diagonal of $\mathcal{J} H$ is nonzero, then (1), (2) and (3) do not hold. If the $(s+1)$ th entry on the diagonal of $\mathcal{J} H$ is zero, then $\frac{\partial}{\partial x_{i}} p=0$ for all $i>s$, whence (1), (2) and (3) hold.

Let $H \in K[x]^{n}$ be homogeneous of degree $d \geq 2$. Then $x+H$ is a Keller map if and only if $\mathcal{J} H$ is nilpotent; see for example [8, Lemma 6.2.11]. So we first investigate nilpotent matrices over $K[x]$.

Lemma 3.2. Let $N \in \operatorname{Mat}_{2}(K[x])$ such that $N$ is nilpotent. Then there exist $a, b, c \in K[x]$ such that

$$
N=c\left(\begin{array}{cc}
a b & -b^{2} \\
a^{2} & -a b
\end{array}\right)
$$

Furthermore, $N$ is similar over $K$ to a triangular matrix if and only if $a$ and $b$ are linearly dependent over $K$.

Proof. Since $\operatorname{det} N=0$, we may write $N$ in the form

$$
N=c \cdot\binom{b}{a} \cdot\left(\begin{array}{ll}
a & -\tilde{b}
\end{array}\right),
$$

where $a, b \in K[x]$ and $\tilde{b}, c \in K(x)$. Since $\operatorname{tr} N=0$, we have $\tilde{b}=b$. If we choose $a$ and $b$ to be relatively prime, then $c \in K[x]$ as well.

Furthermore, $a$ and $b$ are linearly dependent over $K$ if and only if the rows of $N$ are linearly dependent over $K$, if and only if $N$ is similar over $K$ to a triangular matrix.

Lemma 3.3. Let $H \in K[x]^{2}$ be cubic homogeneous, such that $\mathcal{J}_{x_{1}, x_{2}} H$ is nilpotent. Then there exists $T \in \mathrm{GL}_{2}(K)$ such that for $\tilde{H}:=T^{-1} H\left(T\left(x_{1}, x_{2}\right), x_{3}\right.$, $\left.x_{4}, \ldots, x_{n}\right)$, one of the following statements holds:
(1) $\mathcal{J}_{x_{1}, x_{2}} \tilde{H}$ is a triangular matrix;
(2) there are independent linear forms $a, b \in K[x]$, such that

$$
\mathcal{J}_{x_{1}, x_{2}} \tilde{H}=\left(\begin{array}{cc}
a b & -b^{2} \\
a^{2} & -a b
\end{array}\right) \quad \text { and } \quad \mathcal{J}_{x_{1}, x_{2}}\binom{a}{b}=\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)
$$

(3) $\operatorname{char} K=3$ and there are independent linear forms $a, b \in K[x]$, such that

$$
\mathcal{J}_{x_{1}, x_{2}} \tilde{H}=\left(\begin{array}{cc}
a b & -b^{2} \\
a^{2} & -a b
\end{array}\right) \quad \text { and } \quad \mathcal{J}_{x_{1}, x_{2}}\binom{a}{b}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) .
$$

Proof. Suppose that (1) does not hold. By Lemma 3.2, there are $a, b, c \in K[x]$, such that

$$
\mathcal{J}_{x_{1}, x_{2}} H=c\left(\begin{array}{cc}
a b & -b^{2} \\
a^{2} & -a b
\end{array}\right)
$$

where $a$ and $b$ are linearly independent over $K$. As $H$ is cubic homogeneous, the entries of $\mathcal{J}_{x_{1}, x_{2}} H$ are quadratic homogeneous, so $c \in K$ and $a$ and $b$ are independent linear forms.

If we take

$$
T=\left(\begin{array}{cc}
c & 0 \\
0 & 1
\end{array}\right), \quad \text { then } \quad \mathcal{J}_{x_{1}, x_{2}} \tilde{H}=\left(\begin{array}{cc}
\tilde{a} \tilde{b} & -\tilde{b}^{2} \\
\tilde{a}^{2} & -\tilde{a} \tilde{b}
\end{array}\right)
$$

where $\tilde{a}=\left.c \cdot a\right|_{x_{1}=c x_{1}}$ and $\tilde{b}=\left.c^{-1} \cdot b\right|_{x_{1}=c x_{1}}$.

We claim that the coefficient $k_{2}$ of $x_{2}$ in $\tilde{b}$ is 0 . Suppose conversely that $k_{2} \neq 0$. Then the coefficient of $x_{2}^{3}$ in

$$
3 \tilde{H}_{1}=\mathcal{J}_{x_{1}, x_{2}} \tilde{H}_{1} \cdot\binom{x_{1}}{x_{2}}=\tilde{b}\left(x_{1} \tilde{a}-x_{2} \tilde{b}\right)
$$

is nonzero. In particular, char $K \neq 3$. One may verify that

$$
\mathcal{J}_{x_{1}, x_{2}}\left(\tilde{H}_{1}+\frac{1}{3} k_{2}^{-1} \tilde{b}^{3}\right)=(\tilde{c} \tilde{b}, 0)
$$

where $\tilde{c}:=\tilde{a}+k_{2}^{-1} \tilde{b}\left(\frac{\partial}{\partial x_{1}} \tilde{b}\right)$. As a consequence, $\frac{\partial}{\partial x_{2}}(\tilde{c} \tilde{b})=\frac{\partial}{\partial x_{1}} 0=0$. Furthermore, $\tilde{c}$ and $\tilde{b}$ are independent, just like $\tilde{a}$ and $\tilde{b}$. By $\frac{\partial}{\partial x_{2}}(\tilde{c} \tilde{b})=0$, we have $\tilde{c} \tilde{b} \in$ $K\left[x_{1}, x_{3}, x_{4}, \ldots, x_{n}\right]$ if $\operatorname{char} K \neq 2$. Since $\tilde{c}$ and $\tilde{b}$ are independent, we deduce that if $\operatorname{char} K=2$ then $\tilde{c} \tilde{b} \in K\left[x_{1}, x_{3}, x_{4}, \ldots, x_{n}\right]$ as well. Since the coefficient $\lambda$ of $x_{2}$ in $\tilde{b}$ is nonzero, we have $\tilde{c}_{\tilde{b}}=0$, a contradiction.

So the coefficient of $x_{2}$ in $\tilde{b}$ is 0 . Similarly, the coefficient of $x_{1}$ in $\tilde{a}$ is 0 . Consequently,

$$
\mathcal{J}_{x_{1}, x_{2}}\binom{\tilde{a}}{\tilde{b}}=\left(\begin{array}{cc}
0 & \lambda \\
\mu & 0
\end{array}\right),
$$

where $\lambda, \mu \in K$. Therefore

$$
\mathcal{J}_{x_{1}, x_{2}} \tilde{H}=\left(\begin{array}{cc}
\left(\lambda x_{2}+\cdots\right)\left(\mu x_{1}+\cdots\right) & -\left(\mu x_{1}+\cdots\right)^{2} \\
\left(\lambda x_{2}+\cdots\right)^{2} & -\left(\lambda x_{2}+\cdots\right)\left(\mu x_{1}+\cdots\right)
\end{array}\right) .
$$

So the coefficient of $x_{1}^{2} x_{2}$ in $2 \tilde{H}_{1}$ is equal to both $\lambda \mu$ and $-2 \mu^{2}$. Similarly, the coefficient of $x_{1} x_{2}^{2}$ in $2 \tilde{H}_{2}$ is equal to both $\lambda \mu$ and $-2 \lambda^{2}$. It follows that either $\lambda=\mu=0$ or $0 \neq \lambda=-2 \mu=4 \lambda$. In the former case, $\widetilde{H}$ satisfies (2). In the latter case, char $K=3$ and $\lambda=\mu$. Replacing $\tilde{H}$ by $\lambda \tilde{H}\left(\lambda^{-1}\left(x_{1}, x_{2}\right), x_{3}, x_{4}, \ldots, x_{n}\right)$, we have that $\widetilde{H}$ satisfies (3).

Theorem 3.4. Suppose that char $K \neq 2,3$. Let $H \in K[x]^{n}$ be cubic homogeneous such that $x+H$ is a Keller map, i.e., $\mathcal{J} H$ is nilpotent.
(i) If $\operatorname{rk} \mathcal{J} H=1$, then there exists $T \in \mathrm{GL}_{n}(K)$ such that for $\tilde{H}:=T^{-1} H(T x)$,

$$
\begin{aligned}
& \tilde{H}_{1} \in K\left[x_{2}, x_{3}, x_{4}, \ldots, x_{n}\right] \\
& \tilde{H}_{2}=\tilde{H}_{3}=\tilde{H}_{4}=\cdots=\tilde{H}_{n}=0 .
\end{aligned}
$$

(ii) If $\operatorname{rk} \mathcal{J} H=2$, then either $H$ is linearly triangularizable or there exists $T \in \mathrm{GL}_{n}(K)$ such that for $\tilde{H}:=T^{-1} H(T x)$,

$$
\begin{aligned}
& \tilde{H}_{1}-\left(x_{1} x_{3} x_{4}-x_{2} x_{4}^{2}\right) \in K\left[x_{3}, x_{4}, \ldots, x_{n}\right], \\
& \tilde{H}_{2}-\left(x_{1} x_{3}^{2}-x_{2} x_{3} x_{4}\right) \in K\left[x_{3}, x_{4}, \ldots, x_{n}\right], \\
& \tilde{H}_{3}=\tilde{H}_{4}=\cdots=\tilde{H}_{n}=0
\end{aligned}
$$

Furthermore, $x+t H$ is invertible over $K[t]$ if $\operatorname{rk} \mathcal{J} H \leq 2$, where $t$ is a new variable. Moreover, $x+t H$ is even tame over $K[t]$ if either $\operatorname{rk} \mathcal{J} H=1$ or $\operatorname{rk} \mathcal{J} H=2$ and $n \neq 4$. In particular, $x+\lambda H$ is invertible and tame under the above condition respectively for every $\lambda \in K$.

Proof. We may take $\tilde{H}$ as in (1), (2) or (3) of Theorem 2.7. If rk $\mathcal{J} H=1$, then $\tilde{H}$ is as in (1) of Theorem 2.7, i.e., $\tilde{H}_{i}=0,2 \leq i \leq n$, whence (i) holds because $\operatorname{tr} \mathcal{J} \tilde{H}=0$. So assume that $\operatorname{rk} \mathcal{J} H=2$. Notice that $\mathcal{J} H$ is nilpotent.

If $\tilde{H}$ is as in (1) or (2) of Theorem 2.7, i.e., $\tilde{H}_{i}=0,3 \leq i \leq n$ or $\tilde{H} \in$ $K\left[x_{1}, x_{2}\right]^{n}$, then $\mathcal{J}_{x_{1}, x_{2}}\left(\tilde{H}_{1}, \tilde{H}_{2}\right)$ is nilpotent.

If $\tilde{H}$ is as in (3) of Theorem 2.7, i.e., $K \tilde{H}_{1}+K \tilde{H}_{2}+\cdots+K \tilde{H}_{n}=K x_{3} x_{1}^{2} \oplus$ $K x_{3} x_{1} x_{2} \oplus K x_{3} x_{2}^{2}$, then $\tilde{H}_{3}=0$, because $x_{3}^{-1} \tilde{H}_{3}$ is the constant part with respect to $x_{3}$ of $\operatorname{tr} \mathcal{J} \tilde{H}=0$. So $\mathcal{J}_{x_{1}, x_{2}}\left(\tilde{H}_{1}, \tilde{H}_{2}\right)$ is nilpotent in any case.

One may observe that, in all the cases (1), (2) and (3) of Theorem 2.7 if $\mathcal{J}_{x_{1}, x_{2}}\left(\tilde{H}_{1}, \tilde{H}_{2}\right)$ is similar over $K$ to a triangular matrix, then $\mathcal{J} \tilde{H}$ is similar over $K$ to a triangular matrix, and so is $\mathcal{J} H$, and thus $H$ is linearly triangularizable.

Now suppose $\mathcal{J}_{x_{1}, x_{2}}\left(\tilde{H}_{1}, \tilde{H}_{2}\right)$ is not similar over $K$ to a triangular matrix. Noticing that char $K \neq 2,3, \mathcal{J}_{x_{1}, x_{2}}\left(\tilde{H}_{1}, \tilde{H}_{2}\right)$ must be as in (2) of Lemma 3.3, i.e.,

$$
\mathcal{J}_{x_{1}, x_{2}} \tilde{H}=\left(\begin{array}{cc}
a b & -b^{2} \\
a^{2} & -a b
\end{array}\right) \quad \text { and } \quad \mathcal{J}_{x_{1}, x_{2}}\binom{a}{b}=\left(\begin{array}{cc}
0 & 0 \\
0 & 0
\end{array}\right)
$$

where $a, b$ are linearly independent linear forms.
If $\tilde{H}_{1} \in K\left[x_{1}, x_{2}, x_{3}\right]$, then $a, b \in k\left[x_{3}\right]$, a contradiction. So $\tilde{H}$ is not as in (2) or (3) of Theorem 2.7, and thus is as in (1) of Theorem[2.7, i.e., $\tilde{H}_{3}=\tilde{H}_{4}=$ $\cdots=\tilde{H}_{n}=0$. Consequently, by linear coordinate transformation, we may take $\tilde{H}$ such that $a=x_{3}$ and $b=x_{4}$. So (ii) holds.

For the last claim, when $\operatorname{rk} \mathcal{J} H=1, \widetilde{H}$ is of the form in (i), whence $x+t \widetilde{H}$ is elementary and thus tame. When $\operatorname{rk} \mathcal{J} H=2, \widetilde{H}$ is of the form in (ii), and it suffices to show the following automorphism

$$
F=\left(x_{1}+t x_{4}\left(x_{3} x_{1}-x_{4} x_{2}\right), x_{2}+t x_{3}\left(x_{3} x_{1}-x_{4} x_{2}\right), x_{3}, x_{4}, x_{5}\right)
$$

is tame over $K[t]$.
For that purpose, let $w=t\left(x_{3} x_{1}-x_{4} x_{2}\right)$ and let $D:=x_{4} \partial_{x_{1}}+x_{3} \partial_{x_{2}}$ be a derivation of $K[t]\left[x_{1}, x_{2}, x_{3}, x_{4}\right]$. Observe that $D$ is triangular and $w \in \operatorname{ker} D$, and that $F=\left(\exp (w D), x_{5}\right)$. Therefore $F$ is tame over $K[t]$ due to the following Lemma 3.5.

Recall that a derivation $D$ of $K[x]$ is called locally nilpotent if for every $f \in K[x]$ there exists an $m$ such that $D^{m}(f)=0$. For such a derivation, $\exp D:=\sum_{i=0}^{\infty} \frac{1}{i!} D^{i}$ is a polynomial automorphism of $K[x]$. A derivation $D$ of $K[x]$ is called triangular if $D\left(x_{i}\right) \in K\left[x_{i+1}, \ldots, x_{n}\right]$ for $i=1,2, \ldots, n-1$ and $D\left(x_{n}\right) \in K$. A triangular derivation is locally nilpotent.

Lemma 3.5. Let $D$ be a triangular derivation of $K[t][x]$ and $w \in \operatorname{ker} D$ i.e. $D(w)=0$. Then $\left(\exp (w D), x_{n+1}\right)$ is tame over $K[t]$.

Proof. From [15, Corollary], it follows that there exists a $k$ such that $(\exp (w D)$, $\left.x_{n+1}, x_{n+2}, \ldots, x_{n+k}\right)$ is tame over $K(t)$. Inspecting the proof of [15, Corollary] yields that $\left(\exp (w D), x_{n+1}\right)$ is tame over $K[t]$.

Acknowledgments The first author has been supported by the Netherlands Organisation of Scientific research (NWO). The second author has been partially supported by the NSF of China (grant no. 11771176 and 11601146).

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