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Classification of cubic homogeneous polynomial maps with Jacobian matrices of rank two

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Abstract

Let $K$ be any field with $\text{char}K \neq 2, 3$. We classify all cubic homogeneous polynomial maps $H$ over $K$ with $\text{rk} JH \leq 2$. In particular, we show that, for such an $H$, if $F = x + H$ is a Keller map then $F$ is invertible, and furthermore $F$ is tame if the dimension $n \neq 4$.

1 Introduction

Let $K$ be an arbitrary field and $K[x] := K[x_1, x_2, \ldots, x_n]$ the polynomial ring in $n$ variables. For a polynomial map $F = (F_1, F_2, \ldots, F_m) \in K[x]^m$, we denote by $JF := (\frac{\partial F_i}{\partial x_j})_{m \times n}$ the Jacobian matrix of $F$ and $\deg F := \max_i \deg F_i$ the degree of $F$. A polynomial map $H \in K[x]^m$ is called homogeneous of degree $d$ if each $H_i$ is zero or homogeneous of degree $d$.

A polynomial map $F \in K[x]^n$ is called a Keller map if $\det JF \in K^*$. The Jacobian conjecture asserts that any Keller map is invertible if $\text{char}K = 0$; see [8] or [1]. It is still open for any dimension $n \geq 2$.

Following [14], we call a polynomial automorphism elementary if it is of the form $(x_1, \ldots, x_{i-1}, cx_i + a, x_{i+1}, \ldots, x_n)$, where $c \in K^*$ and $a \in K[x]$ contains no $x_i$. Furthermore, we call a polynomial automorphism tame if it is a finite composition of elementary ones. The definitions of elementary and tame may be different in other sources, but (as long as $K$ is a generalized Euclidean ring) the definitions of tame are equivalent. The Tame Generators Problem asks if every polynomial automorphism is tame. It has an affirmative answer in dimension 2.

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for arbitrary characteristic (see [10, 11]) and a negative answer in dimension 3 for the case of char$K = 0$ (see [14]), and is still open for any $n \geq 4$.

A polynomial map $F = x + H \in K[x]^n$ is called triangular if $H \in K$ and $H_i \in K[x_{i+1}, \ldots, x_n]$, $1 \leq i \leq n - 1$. A polynomial map $F$ is called linearly triangularizable if it is linearly conjugate to a triangular map, i.e., there exists an invertible linear map $T \in \text{GL}_n(K)$ such that $T^{-1}F(Tx)$ is triangular. A linearly triangularizable map is tame.

Some special polynomial maps have been investigated in the literature. For example, when char$K = 0$, a Keller map $F = x + H \in K[x]^n$ is shown to be linearly triangularizable in the cases: (1) $n = 3$ and $H$ is homogeneous of arbitrary degree $d$ (de Bondt and van den Essen [6]); (2) $n = 4$ and $H$ is quadratic homogeneous (Meisters and Olech [12]); (3) $n = 9$ and $F$ is a quadratic homogeneous quasi-translation (Sun [16]); (4) arbitrary and $H$ is quadratic with rk$JH \leq 2$ (De Bondt and Yan [7]), and to be tame in the case (5) $n = 5$ and $H$ is quadratic homogeneous (de Bondt [2] and Sun [17] independently), and to be invertible in the case (6) $n = 4$ and $H$ is cubic homogeneous (Hubbers [9]). For the case of arbitrary characteristic, de Bondt [5] described the Jacobian matrix $JH$ of rank two for any quadratic polynomial map $H$ and showed that if $JH$ is nilpotent then $JH$ is similar to a triangular one.

In this paper, we investigate cubic homogeneous polynomial maps $H$ with rk$JH \leq 2$ for any dimension $n$ when char$K \neq 2, 3$. In Section 2, we classify all such maps (Theorem 2.7). And in Section 3, we show that for such an $H$, if $F = x + H$ is a Keller map, then it is invertible and furthermore it is tame if the dimension $n \neq 4$ (Theorem 3.4).

## 2 Cubic homogeneous maps $H$ with rk$JH \leq 2$

For a polynomial map $H \in K[x]^m$, we write trdeg$K K(H)$ for the transcendence degree of $K(H)$ over $K$. It is well-known that rk$JH =$ trdeg$K K(H)$ if $K(H) \subseteq K(x)$ is separable, in particular if char$K = 0$; see [8, Proposition 1.2.9]. And for arbitrary characteristic, one has rk$JH \leq$ trdeg$K K(H)$; see [4 or 13].

It was shown in [5] that when char$K \neq 2$, for any quadratic polynomial map $H$ with rk$JH \leq 2$, one has rk$JH =$ trdeg$K K(H)$. We will show that when char$K \neq 2, 3$, for any cubic homogeneous polynomial map $H$ with rk$JH \leq 2$, one has rk$JH =$ trdeg$K K(H)$. The notation $a|_{x = c}$ below means to substitute $x$ by $c$ in $a$.

**Theorem 2.1.** Let $s \leq n$. Take

$$\tilde{x} := (x_1, x_2, \ldots, x_s) \quad \text{and} \quad L := K(x_{s+1}, x_{s+2}, \ldots, x_n).$$

To prove that for (homogeneous) polynomial maps $H \in K[x]^m$ of degree $d$,

$$\text{rk} JH = r \implies \text{trdeg} K K(H) = r, \quad \text{for every} \ r < s,$$

(2.1)
Lemma 2.2. Let $H \in K[x]^m$ be a polynomial map of degree $d$ and $r := \text{rk} JH$. Denote by $|K|$ the cardinality of $K$.

(i) If $|K| > (d - 1)r$ and $JH : x = 0$, then there exist $S \in \text{GL}_m(K)$ and $T \in \text{GL}_n(K)$, such that for $\tilde{H} := SH(Tx)$,

$$
\tilde{H}|_{x = e_{r+1}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.
$$
Lemma 5.1 (ii). If \(|K| > (d-1)r + 1\) and \(\mathcal{J}H \cdot x \neq 0\), then there exist \(S \in \text{GL}_m(K)\) and \(T \in \text{GL}_n(K)\), such that for \(\tilde{H} := ST\),

\[
\tilde{H}|_{x=e_1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.
\]

Moreover, \(|K|\) may be one less (i.e. at least \((d-1)r\) and \((d-1)r+1\) respectively) if every nonzero component of \(H\) is homogeneous.

Proof. (i) Assume without loss of generality that \(a_0 := \det \mathcal{J}x_1, x_2, \ldots, x_r (H_1, H_2, \ldots, H_r) \neq 0\).

Suppose that \(|K| > (d-1)r\). It follows by [3] Lemma 5.1 (i)] that there exists a vector \(w \in K^n\) such that \(a_0(w) \neq 0\). So \(\text{rk} (\mathcal{J}H)|_{x=w} = r\). There exist \(n-r\) independent vectors \(v_{r+1}, v_{r+2}, \ldots, v_n \in K^n\), such that \(\mathcal{J}H|_{x=w} \cdot v_i = 0\) for \(i = r+1, r+2, \ldots, n\). And we may take \(v_{r+1} = w\) since

\[
(\mathcal{J}H)|_{x=w} \cdot w = (\mathcal{J}H \cdot x)|_{x=w} = 0.
\]

Take \(T = (v_1, v_2, \ldots, v_n) \in \text{GL}_n(K)\). From the chain rule, we deduce that

\[
(\mathcal{J}(H(Tx))|_{x=e_{r+1}} e_i = (\mathcal{J}H)|_{x=T e_{r+1}} T e_i = (\mathcal{J}H)|_{x=w} \cdot v_i \quad (1 \leq i \leq n).
\]

In particular, \(\text{rk} \mathcal{J}(H(Tx))|_{x=e_{r+1}} = r\) and the last \(n-r\) columns of \((\mathcal{J}(H(Tx))|_{x=e_{r+1}}\) are zero. There exists \(S \in \text{GL}_m(K)\) such that

\[
(\mathcal{J}(SH(Tx))|_{x=e_{r+1}} = S \cdot (\mathcal{J}(H(Tx))|_{x=e_{r+1}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.
\]

(2) Suppose that \(|K| > (d-1)r + 1\). Since \(\mathcal{J}H \cdot x \neq 0\), we may assume that

\[
\text{rk} \left( \mathcal{J}H \cdot x, \mathcal{J}x_2, x_3, \ldots, x_r H \right) = r,
\]

and that

\[
a_1 := \det \left( \mathcal{J}H_1, H_2, \ldots, H_r, x, \mathcal{J}x_2, x_3, \ldots, x_r (H_1, H_2, \ldots, H_r) \right) \neq 0.
\]

It follows by [3] Lemma 5.1 (i)] that there exists \(w \in K^n\) such that \(a_1(w) \neq 0\). One may observe that \(\text{rk} (\mathcal{J}H)|_{x=w} = r\) and thus there exist independent vectors \(v_{r+1}, v_{r+2}, \ldots, v_n \in K^n\), such that \(\mathcal{J}H|_{x=w} \cdot v_i = 0\) for \(i = r+1, r+2, \ldots, n\). Since \((\mathcal{J}H \cdot x)|_{x=w}\) is the first column of a full column rank matrix, we have

\[
(\mathcal{J}H)|_{x=w} \cdot w = (\mathcal{J}H \cdot x)|_{x=w} \neq 0.
\]

So \(v_1 := w\) is independent of \(v_{r+1}, v_{r+2}, \ldots, v_n\).

Take \(T = (v_1, v_2, \ldots, v_n) \in \text{GL}_n(K)\). Then

\[
(\mathcal{J}(H(Tx))|_{x=e_i} e_i = (\mathcal{J}H)|_{x=T e_1} T e_i = (\mathcal{J}H)|_{x=w} \cdot v_i \quad (1 \leq i \leq n).
\]

The rest of the proof of (ii) is similar to that of (i).

The last claim follows from [3] Lemma 5.1 (ii), as an improvement to [3] Lemma 5.1 (i)].
Proposition 2.3. Assume that char\(K \notin \{1, 2, \ldots, d\}\). Then for any cubic homogeneous polynomial map \(H \in K[x]^m\) of degree \(d\) with \(\text{rk } \mathcal{J}H \leq 1\), the components of \(H\) are linearly dependent over \(K\) in pairs, and one has \(\text{rk } \mathcal{J}H = \text{trdeg}_K K(H)\).

Proof. The case \(\text{rk } \mathcal{J}H = 0\) is obvious, so let \(\text{rk } \mathcal{J}H = 1\). On account of Lemma 2.2 we may assume that \(\mathcal{J}H|_{x = e_1} = E_{11}\). Let \(j \geq 2\). Since \(\text{deg}_{x_1} H_j < d\), we infer that either \(H_j = 0\), or \(\text{deg} \frac{\partial}{\partial x_j} H_j < \text{deg} \frac{\partial}{\partial x_1} H_j\) for some \(i \geq 2\), where \(\text{deg}_{x_1} 0 = -\infty\). The latter is impossible due to \(\text{rk } \mathcal{J}H = 1\), so \(H_j = 0\). This holds for all \(j \geq 2\), which yields the desired results.

Lemma 2.4. Let \(H = (h, x^2_1x_2, x^2_2x_3)\) or \((h, x^2_1x_2, x^2_2x_3) \in K[x_1, x_2, x_3]^3\), where \(h\) is cubic homogeneous, and assume that \(\text{char } K \neq 2, 3\). Then \(\text{rk } \mathcal{J}H = \text{trdeg}_K K(H)\).

Proof. It suffices to consider the case of \(\text{rk } \mathcal{J}H = 2\). Define a derivation \(D\) on \(A = K[x_1, x_2, x_3]\) as follows: for any \(f \in A\),

\[
D(f) = \frac{x_1x_2x_3}{H_2H_3} \det \mathcal{J}H.
\]

In the case \(H = (h, x^2_1x_2, x^2_2x_3)\), an easy calculation shows that \(D = x_1 \partial_{x_1} - 2x_2 \partial_{x_2} + 4x_3 \partial_{x_3}\). Then for any term \(u = x_1^{d_1}x_2^{d_2}x_3^{d_3} \in A\), \(D(u) = (d_1 - 2d_2 + 4d_3)u\). And thus \(\ker D := \{g \in A \mid D(g) = 0\}\), the kernel of \(D\), is linearly spanned by all terms \(u\) with \(d_1 - 2d_2 + 4d_3 = 0\). So the only cubic terms in \(\ker D\) are \(x_1^2x_2\) and \(x_2^2x_3\). Since \(\text{rk } \mathcal{J}H = 2\), we have \(\det \mathcal{J}H = 0\) and thus \(h \in \ker D\), which implies that \(h\) is a linear combinations of \(x_1^2x_2\) and \(x_2^2x_3\). Thus \(\text{trdeg}_K K(H) = 2\).

In the case \(H = (h, x^2_1x_2, x^2_2x_3)\), one may verify that \(x_1^2x_2, x_1x_2x_3\) and \(x_2^2x_3\) are the only cubic terms in \(\ker D\). The conclusion follows similarly.

Theorem 2.5. Assume that \(\text{char } K \neq 2, 3\). Then for any cubic homogeneous polynomial map \(H \in K[x]^m\) with \(\text{rk } \mathcal{J}H \leq 2\), one has \(\text{rk } \mathcal{J}H = \text{trdeg}_K K(H)\).

Proof. Due to Theorem 2.1 and replacing \(L\) there by \(K\), we may assume that \(H \in K[x_1, x_2, x_3]^3\), and it suffices to show that

\[
\text{trdeg}_K K(H) = 3 \implies \text{rk } \mathcal{J}H = 3,
\]

or equivalently,

\[
\det \mathcal{J}H = 0 \implies \text{trdeg}_K K(H) < 3. \quad (2.3)
\]

So assume that \(\det \mathcal{J}H = 0\). Since we may replace \(K\) by an extension field to make it large enough, it follows by Lemma 2.2 that we may assume that \((\mathcal{J}H)|_{x = e_1} = E_{11} + E_{22}\). Then \(\mathcal{J}H\) is of the form

\[
\begin{pmatrix}
  x^2_1 + \ast & \ast & \ast \\
  \ast & x^2_2 + \ast & \ast \\
  \ast & \ast & \frac{\partial H_j}{\partial x_3}
\end{pmatrix},
\]

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Moreover, if \( c \) we see that we can combine with \( c \)\( x \) where the \( x_1 \)-degree of each element \( * \) is less than 2. Observing the terms with \( x_1 \)-degree \( \geq 5 \) in \( \det JH \), we have that \( \frac{\partial H_1}{\partial x_3} \in K[x_2, x_3] \). Notice that \( H_2 \) and \( H_3 \) are of the form:

\[
H_2 = x_1^2 x_2 + b_{10} x_1 x_3^2 + b_{11} x_1 x_2 x_3 + b_{12} x_1 x_2^2 + b_0(x_2, x_3);
\]

\[
H_3 = c_{12} x_1 x_2^2 + c_{00} x_3^3 + c_{01} x_2 x_3^2 + c_{02} x_2^2 x_3 + c_{03} x_3^3.
\]

We shall show that \( x_2^2 \mid H_3 \), i.e., \( c_{00} = c_{01} = 0 \).

Noticing that the part of \( x_1 \)-degree 4 of \( \det JH \) is \( \left( \frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_1} \right) x_1^4 \), we see that \( \frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_1} = 0 \). Consequently,

\[
(3c_{00} x_3^2 + 2c_{01} x_2 x_3 + c_{02} x_2^2) = (2b_{10} x_3 + b_{11} x_2)(2c_{12} x_2)
\]

so

\[
c_{00} = 0 \quad c_{01} = 2b_{10} c_{12} \quad c_{02} = 2b_{11} c_{12}
\]

One may observe that the coefficient of \( x_1^2 x_3^2 \) in \( \det JH \) is \( 2c_{01} b_{10} = 0 \), which we can combine with \( c_{01} = 2b_{10} c_{12} \) to obtain \( c_{01} = 0 \). Therefore,

\[
H_3 = (c_{12} x_1 + c_{03} x_2 + c_{02} x_3) x_2^2.
\]

Moreover, if \( c_{12} = 0 \) then \( c_{02} = 2b_{11} c_{12} = 0 \) and thus \( H_3 = c_{03} x_2^3 \).

We distinguish two cases.

- **Case 1**: \( c_{12} \neq 0 \) and \( c_{12} x_1 + c_{03} x_2 + c_{02} x_3 \nmid H_i \) for some \( i \).

Then \( H_3 \) is the product of two linear forms, of which two are distinct. Hence we can compose \( H \) with invertible linear maps on both sides, to obtain a map \( H' \) for which \( H'_2 = x_1^2 x_2 \) and \( x_2 \nmid H'_1 \).

Notice that \( H'_1(1, 0, t) \neq 0 \). As \( K \) has at least 5 elements, it follows from [3, Lemma 5.1 (i)] that there exists a \( \lambda \in K \), such that \( H'_1(1, 0, \lambda) \neq 0 \).

Hence the coefficient of \( x_1^3 \) in \( H'_1(x_1, x_2, x_3 + \lambda x_1) \) is nonzero. Furthermore, \( H'_2(x_1, x_2, x_3 + \lambda x_1) = x_1^2 x_2 \).

Replacing \( H' \) by \( H'(x_1, x_2, x_3 + \lambda x_1) \), we may assume that \( H'_2 = x_1^2 x_2 \) and that \( H'_1 \) contains \( x_1^3 \) as a term. We may even assume that the coefficient of \( x_1^3 \) in \( H'_1 \) equals 1. Then \( JH'|_{x=x_1} \) is of the form

\[
\begin{pmatrix}
1 & * & a \\
0 & 1 & 0 \\
* & * & *
\end{pmatrix},
\]

and has rank 2. Furthermore, \( v_3 = (-a, 0, 1)^t \) belongs to its null space.

We may apply the proof of Lemma 2.2 on \( H' \) by taking \( T = (e_1, e_2, v_3) \) and taking an appropriate \( S \in GL_3(K) \) such that \( \tilde{H} := SH'(Tx) \) satisfies \( \tilde{JH}|_{x=x_1} = S JH'|_{x=x_1} T = E_{11} + E_{22} \). Notice that \( Tx \) is of the form \( (L_1, x_2, L_3) \), and observing the form of \( JH'|_{x=x_1} \) one may also choose \( Sx \) to be of the form \((*, x_2, *)\). Then \( \tilde{H}_2 = L_1^2 x_2 \).
So we can compose \( \tilde{H} \) with an invertible linear map on the right, to obtain a map \( \tilde{H}' \) for which \( \tilde{H}'_2 = x_1^2 x_2 \) and \( \tilde{H}'_3 = x_2^2 L' \) for some linear form \( L' \).

Suppose first that \( L' \) is a linear combination of \( x_1 \) and \( x_2 \). If \( \tilde{H}'_1 \in K[x_1, x_2] \), then we are done. Otherwise, we have \( \det \mathcal{J}_{x_1, x_2}(H'_2, H'_3) = 0 \), and then by Proposition 2.3 \( \text{trdeg}_K K(H'_2, H'_3) < 2 \).

Suppose next that \( L' \) is not a linear combination of \( x_1 \) and \( x_2 \). Then we may assume that \( \tilde{H}'_3 = x_2^2 x_3 \). By Lemma 2.4 (i), \( \text{trdeg}_K K(\tilde{H}') < 3 \).

- **Case 2:** \( c_{12} = 0 \) or \( c_{12} x_1 + c_{03} x_2 + c_{02} x_3 \mid H_i \) for all \( i \).

Since \( x_2^2 \mid H_3 \), we can compose \( H \) with invertible linear maps on both sides, to obtain a map \( H' \) for which \( H'_1 \in \{x_1^3, x_1^2 x_2\} \). After a possible interchange of \( H'_2 \) and \( H'_3 \), the first two rows of \( \mathcal{J} H' \) are independent. Now we may apply the proof of Lemma 2.2 to \( H' \), more precisely, there exist \( S, T \in \text{GL}_3(K) \) such that \( \tilde{H} := SH'(Tx) \) satisfies \( \mathcal{J} \tilde{H}|_{x=1} = E_{11} + E_{22} \).

If we choose \( w \) such that first two rows of \( (\mathcal{J} H')_{x=w} \) are independent, then we can take \( S \) such that \( Sx = (f_1 x_1 + f_2 x_2, g_1 x_1 + g_2 x_2, \ast) \). By repeating the discussion for \( H \) as for \( H' \) above, we may assume that \( \tilde{H}'_3 = L x_2^2 \) for some linear form \( L \).

Let \( Tx = (L_1, L_2, L_3) \). Notice that \( H'_1(Tx) \in \{L_1^3, L_1^2 L_2\} \) and that \( H'_1(Tx) \) is a linear combination of \( \tilde{H}_1 \) and \( \tilde{H}_2 \). Hence we can compose \( H \) with a linear map on the left, to obtain a map \( \tilde{H}' \) for which \( H'_2 \in \{L_1^3, L_1^2 L_2\} \) and \( H'_3 = L x_2^2 \).

Suppose first that \( \tilde{H}'_2 = L_1^2 L_2 \). Then \( c_{12} \neq 0 \), so \( c_{12} x_1 + c_{03} x_2 + c_{02} x_3 \mid H_i \) for all \( i \). From this, we infer that \( L_2 \mid \tilde{H}_i \) and \( L_2 \mid \tilde{H}'_i \) for all \( i \). As \( x_2 \nmid \tilde{H}_1 \), we deduce that \( L \) and \( L_2 \) are dependent linear forms, which are independent of \( x_2 \). If \( L \) and \( L_2 \) are linear combinations of \( L_1 \) and \( x_2 \), then we can reduce to Proposition 2.3, and otherwise we can reduce to Lemma 2.4 (ii).

Suppose next that \( \tilde{H}'_2 = L_1^3 \). If \( L \), \( L_1 \) and \( x_2 \) are linearly dependent over \( K \), then we can reduce to Proposition 2.3. Otherwise, \( H \) is as \( H \) in the previous case.\( \square \)

**Remark 2.6.** Inspired by Lemma 2.4 we investigated maps \( H \) of which the components are terms, and searched for \( H \) with algebraically independent components for which \( \det \mathcal{J} H = 0 \). One can infer that \( H \) is as such, if and only if the matrix with entries \( \text{deg}_{x_i} H_j \) has determinant zero over \( K \), but not over \( \mathbb{Z} \).

We found the following non-homogeneous \( H \) as above over fields of characteristic 5:

\[
(x_1^3 x_2, x_1 x_2^3, x_2), \quad (x_1^2 x_2, x_1 x_3^3, x_2 x_3)
\]

with the following homogenizations respectively:

\[
(x_1^3 x_2, x_1 x_2^2 x_3, x_3^4), \quad (x_1^2 x_2, x_1 x_3^2, x_2 x_3^4, x_4^3)
\]
Besides these homogenizations, we found the following homogeneous $H$ over fields of characteristic 5:

$$(x_1^2 x_3^2, x_1 x_2^2, x_2 x_3^2), \quad (x_4 x_1^2, x_1 x_2^2, x_2 x_3^2, x_3 x_4^2)$$

We conclude with a homogeneous $H$ over fields of characteristic 7, and a homogeneous $H$ over any characteristic $p \in \{1, 2, \ldots, d\}$ respectively:

$$(x_3 x_1^3, x_1 x_2^3, x_2 x_3^3), \quad (x_1^d, x_1^{d-r} x_2^p)$$

These examples show that the conditions in Proposition 2.3 and Theorem 2.5 cannot be relaxed.

**Theorem 2.7.** Suppose that $\text{char} K \neq 2, 3$ and let $H \in K[x]^m$ be cubic homogeneous. Let $r := \text{rk} JH$ and suppose that $r \leq 2$. Then there exist $S \in \text{GL}_m(K)$ and $T \in \text{GL}_n(K)$, such that for $\tilde{H} := SH(Tx)$, one of the following statements holds:

1. $\tilde{H}_{r+1} = \tilde{H}_{r+2} = \cdots = \tilde{H}_m = 0$;
2. $r = 2$ and $\tilde{H} \in K[x_1, x_2]^m$;
3. $r = 2$ and $K \tilde{H}_1 + K \tilde{H}_2 + \cdots + K \tilde{H}_m = K x_3 x_1^2 \oplus K x_3 x_1 x_2 \oplus K x_3 x_2^2$.

Furthermore, we may take $S = T^{-1}$ if $m = n$.

**Proof.** By Theorem 2.5, $\text{trdeg}_K K(\tilde{H}) = \text{rk} JH = r \leq 2$. Since $H$ is homogeneous, we have $\text{trdeg}_K K(tH) = r$ as well, where $t$ is a new variable.

Suppose first that $r \leq 1$. It follows by Theorem 2.7 that we may take $\tilde{H}$ as in (1).

Suppose next that $r = 2$. By Theorem 2.7, $H$ is of the form $g \cdot h(p, q)$, such that $g, h$ and $(p, q)$ are homogeneous and $\deg g + \deg h \cdot \deg(p, q) = 3$.

If $\deg h \leq 1$, then every triple of components of $h$ is linearly dependent over $K$, and thus we may take $\tilde{H}$ as in (1). If $\deg h = 3$, then $\deg(p, q) = 1$ and $\deg g = 0$, whence we may take $\tilde{H}$ as in (2).

So assume that $\deg h = 2$. Then $\deg(p, q) = 1$ and $\deg g = 1$. If $g$ is a linear combination of $p$ and $q$, then we may take $\tilde{H}$ as in (2). If $g$ is not a linear combination of $p$ and $q$, then we may take $\tilde{H}$ as in (3) or (1).

Finally, if $m = n$ and $\tilde{H} = SH(Tx)$ is as in (1), then $SH(S^{-1}x) = \tilde{H}(T^{-1}S^{-1}x)$ is still as in (1). So we may take $S = T^{-1}$. If $m = n$ and $\tilde{H} = SH(Tx)$ is as in (2) or (3), then $T^{-1}H(Tx) = T^{-1}S^{-1} \tilde{H}$ is still as in (2) or (3), whence we may also take $S = T^{-1}$.

**3 Cubic homogeneous Keller maps $x + H$ with $\text{rk} JH \leq 2$**

For two matrices $M, N \in \text{Mat}_n(K[x])$, we say that $M$ is similar over $K$ to $N$, if there exists $T \in \text{GL}_n(K)$ such that $N = T^{-1}MT$. 

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Theorem 3.1. Let $F = x + H \in K[x]^n$ be a Keller map with $\text{trdeg}_K K(H) = 1$. Then $\mathcal{J}H$ is similar over $K$ to a triangular matrix, and the following statements are equivalent:

1. $\det \mathcal{J}F = 1$;
2. $\mathcal{J}H$ is nilpotent;
3. $(\mathcal{J}H) \cdot (\mathcal{J}H)|_{x=y} = 0$, where $y = (y_1, y_2, \ldots, y_n)$ are $n$ new variables.

Proof. Since $\text{trdeg}_K K(H) = 1$, by [4, Corollary 3.2] there exists a polynomial $p \in K[x]$ such that $H_i \in K[p]$ for each $i$. Say that $H_i = h_i(p)$, where $h_i \in K[t]$ for each $i$. Write $h_i' = \frac{\partial}{\partial t} h_i$, then

$$\mathcal{J}H = h'(p) \cdot \mathcal{J}p.$$ (3.1)

Assume without loss of generality that $h'_1 = h'_2 = \cdots = h'_s = 0$, and that

$$0 \leq \deg h'_{s+1} < \deg h'_{s+2} < \cdots < \deg h'_n.$$

For $s < i < n$,

$$\deg h'_i(p) = \deg h'_i \cdot \deg p \leq (\deg h'_{i+1} - 1) \cdot \deg p = \deg h'_{i+1}(p) - \deg p.$$

Since the degrees of the entries of $\mathcal{J}p$ are less than $\deg p$, we deduce from (3.1) that the nonzero entries on the diagonal of $\mathcal{J}H$ have different degrees in increasing order. Furthermore, the nonzero entries beyond the $(s+1)$th entry on the diagonal of $\mathcal{J}H$ have positive degrees.

By (3.1), $\text{rk}(-\mathcal{J}H) \leq 1$, and thus $n-1$ eigenvalues of $-\mathcal{J}H$ are zero. It follows that the trailing degree of the characteristic polynomial of $-\mathcal{J}H$ is at least $n-1$. More precisely,

$$\det(tI_n + \mathcal{J}H) = t^n - \text{tr}(-\mathcal{J}H) \cdot t^{n-1},$$

and thus

$$\det \mathcal{J}F = (t^n - \text{tr}(-\mathcal{J}H) \cdot t^{n-1})|_{t=1} = 1 + \text{tr} \mathcal{J}H.$$

Observe that the diagonal of $\mathcal{J}H$ is totally zero, except maybe the $(s+1)$th entry, which is a constant.

So $\frac{\partial}{\partial x_i} p = 0$ for all $i > s + 1$, and $\mathcal{J}H$ is lower triangular. If the $(s+1)$th entry on the diagonal of $\mathcal{J}H$ is nonzero, then (1), (2) and (3) do not hold. If the $(s+1)$th entry on the diagonal of $\mathcal{J}H$ is zero, then $\frac{\partial}{\partial x_i} p = 0$ for all $i > s$, whence (1), (2) and (3) hold.

Let $H \in K[x]^n$ be homogeneous of degree $d \geq 2$. Then $x + H$ is a Keller map if and only if $\mathcal{J}H$ is nilpotent; see for example [3, Lemma 6.2.11]. So we first investigate nilpotent matrices over $K[x]$. 

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Lemma 3.2. Let $N \in \text{Mat}_2(K[x])$ such that $N$ is nilpotent. Then there exist $a, b, c \in K[x]$ such that
$$N = c \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix}.$$
Furthermore, $N$ is similar over $K$ to a triangular matrix if and only if $a$ and $b$ are linearly dependent over $K$.

Proof. Since $\det N = 0$, we may write $N$ in the form
$$N = c \cdot \begin{pmatrix} b \\ a \end{pmatrix} \cdot (a - \tilde{b}),$$
where $a, b \in K[x]$ and $\tilde{b}, c \in K(x)$. Since $\text{tr} N = 0$, we have $\tilde{b} = b$. If we choose $a$ and $b$ to be relatively prime, then $c \in K[x]$ as well.

Furthermore, $a$ and $b$ are linearly dependent over $K$ if and only if the rows of $N$ are linearly dependent over $K$, if and only if $N$ is similar over $K$ to a triangular matrix. \(\square\)

Lemma 3.3. Let $H \in K[x]^2$ be cubic homogeneous, such that $J_{x_1,x_2}H$ is nilpotent. Then there exists $T \in \text{GL}_2(K)$ such that for $\tilde{H} := T^{-1}H(T(x_1,x_2), x_3, x_4, \ldots, x_n)$, one of the following statements holds:

1. $J_{x_1,x_2}\tilde{H}$ is a triangular matrix;
2. there are independent linear forms $a, b \in K[x]$, such that
$$J_{x_1,x_2}\tilde{H} = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \quad \text{and} \quad J_{x_1,x_2} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$
3. $\text{char}K = 3$ and there are independent linear forms $a, b \in K[x]$, such that
$$J_{x_1,x_2}\tilde{H} = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \quad \text{and} \quad J_{x_1,x_2} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Suppose that (1) does not hold. By Lemma 3.2, there are $a, b, c \in K[x]$, such that
$$J_{x_1,x_2}H = c \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix}$$
where $a$ and $b$ are linearly independent over $K$. As $H$ is cubic homogeneous, the entries of $J_{x_1,x_2}H$ are quadratic homogeneous, so $c \in K$ and $a$ and $b$ are independent linear forms.

If we take
$$T = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{then} \quad J_{x_1,x_2}\tilde{H} = \begin{pmatrix} \tilde{a}b & -\tilde{b}^2 \\ \tilde{a}^2 & -\tilde{a}b \end{pmatrix},$$
where $\tilde{a} = c \cdot a|_{x_1 = cx_1}$ and $\tilde{b} = c^{-1} \cdot b|_{x_1 = cx_1}.$
We claim that the coefficient $k_2$ of $x_2$ in $\tilde{b}$ is 0. Suppose conversely that $k_2 \neq 0$. Then the coefficient of $x_2^2$ in

$$3\tilde{H}_1 = J_{x_1,x_2}\tilde{H}_1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tilde{b}(x_1\tilde{a} - x_2\tilde{b})$$

is nonzero. In particular, $\text{char } K \neq 3$. One may verify that

$$J_{x_1,x_2}(H_1 + \frac{k_2}{2} \tilde{b}^3) = (\tilde{c}b, 0),$$

where $\tilde{c} := \tilde{a} + k_2^{-1}\tilde{b}\left(\frac{\partial}{\partial x_1}\tilde{b}\right)$. As a consequence, $\frac{\partial}{\partial x_2}(\tilde{c}b) = \frac{\partial}{\partial x_1} 0 = 0$. Furthermore, $\tilde{c}$ and $\tilde{b}$ are independent, just like $\tilde{a}$ and $\tilde{b}$. By $\frac{\partial}{\partial x_2}(\tilde{c}b) = 0$, we have $\tilde{c}b \in K[x_1, x_3, x_4, \ldots, x_n]$ if $\text{char } K \neq 2$. Since $\tilde{c}$ and $\tilde{b}$ are independent, we deduce that if $\text{char } K = 2$ then $\tilde{c}b \in K[x_1, x_3, x_4, \ldots, x_n]$ as well. Since the coefficient $\lambda$ of $x_2$ in $\tilde{b}$ is nonzero, we have $\tilde{c} = 0$, a contradiction.

So the coefficient of $x_2$ in $\tilde{b}$ is 0. Similarly, the coefficient of $x_1$ in $\tilde{a}$ is 0. Consequently,

$$J_{x_1,x_2} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ -\mu & 0 \end{pmatrix},$$

where $\lambda, \mu \in K$. Therefore

$$J_{x_1,x_2}\tilde{H} = \begin{pmatrix} (\lambda x_2 + \cdots)(\mu x_1 + \cdots) & -(\mu x_1 + \cdots)^2 \\ (\lambda x_2 + \cdots)^2 & -(\lambda x_2 + \cdots)(\mu x_1 + \cdots) \end{pmatrix}.$$

So the coefficient of $x_1^2 x_2$ in $2\tilde{H}_1$ is equal to both $\lambda \mu$ and $-2\mu^2$. Similarly, the coefficient of $x_1 x_2^2$ in $2\tilde{H}_2$ is equal to both $\lambda \mu$ and $-2\lambda^2$. It follows that either $\lambda = \mu = 0$ or $0 \neq \lambda = -2\mu = 4\lambda$. In the former case, $\tilde{H}$ satisfies (2). In the latter case, $\text{char } K = 3$ and $\lambda = \mu$. Replacing $\tilde{H}$ by $\lambda \tilde{H} \left(\lambda^{-1}(x_1, x_2, x_3, x_4, \ldots, x_n)\right)$, we have that $\tilde{H}$ satisfies (3).

**Theorem 3.4.** Suppose that $\text{char } K \neq 2, 3$. Let $H \in K[x]^n$ be cubic homogeneous such that $x + H$ is a Keller map, i.e., $JH$ is nilpotent.

(i) If $\text{rk } JH = 1$, then there exists $T \in \text{GL}_n(K)$ such that for $\tilde{H} := T^{-1}H(Tx)$,

$$\tilde{H}_1 \in K[x_2, x_3, x_4, \ldots, x_n],$$

$$\tilde{H}_2 = \tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0.$$  

(ii) If $\text{rk } JH = 2$, then either $H$ is linearly triangularizable or there exists $T \in \text{GL}_n(K)$ such that for $\tilde{H} := T^{-1}H(Tx)$,

$$\tilde{H}_1 - (x_1 x_3 x_4 - x_2 x_4^2) \in K[x_3, x_4, \ldots, x_n],$$

$$\tilde{H}_2 - (x_1 x_3^2 - x_2 x_3 x_4) \in K[x_3, x_4, \ldots, x_n],$$

$$\tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0.$$
Furthermore, $x + tH$ is invertible over $K[t]$ if $\text{rk} \mathcal{J}H \leq 2$, where $t$ is a new variable. Moreover, $x + tH$ is even tame over $K[t]$ if either $\text{rk} \mathcal{J}H = 1$ or $\text{rk} \mathcal{J}H = 2$ and $n \neq 4$. In particular, $x + \lambda H$ is invertible and tame under the above condition respectively for every $\lambda \in K$.

Proof. We may take $\tilde{H}$ as in (1), (2) or (3) of Theorem 2.7 if $\text{rk} \mathcal{J}H = 1$, then $\tilde{H}$ is as in (1) of Theorem 2.7 i.e. $\tilde{H}_i = 0, 2 \leq i \leq n$, whence (i) holds because $\text{tr} \mathcal{J}H = 0$. So assume that $\text{rk} \mathcal{J}H = 2$. Notice that $\mathcal{J}H$ is nilpotent.

If $\tilde{H}$ is as in (1) or (2) of Theorem 2.7 i.e., $\tilde{H}_i = 0, 3 \leq i \leq n$ or $\tilde{H} \in K[x_1, x_2]^n$, then $\mathcal{J}_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is nilpotent.

If $\tilde{H}$ is as in (3) of Theorem 2.7 i.e., $K\tilde{H}_1 + K\tilde{H}_2 + \cdots + K\tilde{H}_n = Kx_3x_1^2 \oplus Kx_3x_1x_2 \oplus Kx_3^2x_2$, then $H_3 = 0$, because $x_3^1\tilde{H}_3$ is the constant part with respect to $x_3$ of $\text{tr} \mathcal{J}H = 0$. So $\mathcal{J}_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is nilpotent in any case.

One may observe that, in all the cases (1), (2) and (3) of Theorem 2.7 if $\mathcal{J}_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is similar over $K$ to a triangular matrix, then $\mathcal{J}H$ is similar over $K$ to a triangular matrix, and so is $\mathcal{J}H$, and thus $H$ is linearly triangularizable.

Now suppose $\mathcal{J}_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is not similar over $K$ to a triangular matrix. Noticing that $\text{char} \nmid 2, 3$, $\mathcal{J}_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ must be as in (2) of Lemma 3.3 i.e.,

$$
\mathcal{J}_{x_1, x_2} \tilde{H} = \begin{pmatrix}
ab & -b^2 \\
ab & -ab
\end{pmatrix} \quad \text{and} \quad \mathcal{J}_{x_1, x_2} \begin{pmatrix}
a \\
b
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
$$

where $a, b$ are linearly independent linear forms.

If $\tilde{H}_1 \in K[x_1, x_2, x_3]$, then $a, b \in k[x_3]$, a contradiction. So $\tilde{H}$ is not as in (2) or (3) of Theorem 2.7 and thus as in (1) of Theorem 2.7 i.e., $\tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0$. Consequently, by linear coordinate transformation, we may take $\tilde{H}$ such that $a = x_3$ and $b = x_4$. So (ii) holds.

For the last claim, when $\text{rk} \mathcal{J}H = 1$, $\tilde{H}$ is of the form in (i), whence $x + t\tilde{H}$ is elementary and thus tame. When $\text{rk} \mathcal{J}H = 2$, $\tilde{H}$ is of the form in (ii), and it suffices to show the following automorphism

$$
F = (x_1 + tx_4(x_3x_1 - x_4x_2), x_2 + tx_3(x_3x_1 - x_4x_2), x_3, x_4, x_5)
$$
is tame over $K[t]$.

For that purpose, let $w = t(x_3x_1 - x_4x_2)$ and let $D := x_4\partial_{x_4} + x_3\partial_{x_2}$ be a derivation of $K[t][x_1, x_2, x_3, x_4]$. Observe that $D$ is triangular and $w \in \ker D$, and that $F = (\exp(wD), x_5)$. Therefore $F$ is tame over $K[t]$ due to the following Lemma 3.5.

Recall that a derivation $D$ of $K[x]$ is called locally nilpotent if for every $f \in K[x]$ there exists an $m$ such that $D^m(f) = 0$. For such a derivation, $\exp D := \sum_{i=0}^{\infty} \frac{1}{i!} D^i$ is a polynomial automorphism of $K[x]$. A derivation $D$ of $K[x]$ is called triangular if $D(x_i) \in K[x_{i+1}, \ldots, x_n]$ for $i = 1, 2, \ldots, n - 1$ and $D(x_n) \in K$. A triangular derivation is locally nilpotent.

**Lemma 3.5.** Let $D$ be a triangular derivation of $K[t][x]$ and $w \in \ker D$ i.e. $D(w) = 0$. Then $(\exp(wD), x_{n+1})$ is tame over $K[t]$. 

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Proof. From [15, Corollary], it follows that there exists a $k$ such that $(\exp(wD), x_{n+1}, x_{n+2}, \ldots, x_{n+k})$ is tame over $K(t)$. Inspecting the proof of [15, Corollary] yields that $(\exp(wD), x_{n+1})$ is tame over $K[t]$.

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References


