Classification of cubic homogeneous polynomial maps with Jacobian matrices of rank two

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Abstract

Let $K$ be any field with $\text{char}K \neq 2, 3$. We classify all cubic homogeneous polynomial maps $H$ over $K$ with $\text{rk} JH \leq 2$. In particular, we show that, for such an $H$, if $F = x + H$ is a Keller map then $F$ is invertible, and furthermore $F$ is tame if the dimension $n \neq 4$.

1 Introduction

Let $K$ be an arbitrary field and $K[x] := K[x_1, x_2, \ldots, x_n]$ the polynomial ring in $n$ variables. For a polynomial map $F = (F_1, F_2, \ldots, F_m) \in K[x]^m$, we denote by $JF := (\frac{\partial F_i}{\partial x_j})_{m \times n}$ the Jacobian matrix of $F$ and $\text{deg} F := \max_i \text{deg} F_i$ the degree of $F$. A polynomial map $H \in K[x]^m$ is called homogeneous of degree $d$ if each $H_i$ is zero or homogeneous of degree $d$.

A polynomial map $F \in K[x]^n$ is called a Keller map if $\det JF \in K^*$. The Jacobian conjecture asserts that any Keller map is invertible if $\text{char}K = 0$; see [8] or [1]. It is still open for any dimension $n \geq 2$.

Following [13], we call a polynomial automorphism elementary if it is of the form $(x_1, \ldots, x_{i-1}, cx_i + a, x_{i+1}, \ldots, x_n)$, where $c \in K^*$ and $a \in K[x]$ contains no $x_i$. Furthermore, we call a polynomial automorphism tame if it is a finite composition of elementary ones. The definitions of elementary and tame may be different in other sources, but (as long as $K$ is a generalized Euclidean ring) the definitions of tame are equivalent. The Tame Generators Problem asks if every polynomial automorphism is tame. It has an affirmative answer in dimension 2.

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for arbitrary characteristic (see [10, 11]) and a negative answer in dimension 3 for the case of \( \text{char} K = 0 \) (see [14]), and is still open for any \( n \geq 4 \).

A polynomial map \( F = x + H \in K[x]^n \) is called triangular if \( H_i \in K[x_{i+1}, \ldots, x_n], 1 \leq i \leq n-1 \). A polynomial map \( F \) is called linearly triangularizable if it is linearly conjugate to a triangular map, i.e., there exists an invertible linear map \( T \in \text{GL}_n(K) \) such that \( T^{-1}F(Tx) \) is triangular. A linearly triangularizable map is tame.

Some special polynomial maps have been investigated in the literature. For example, when \( \text{char} K = 0 \), a Keller map \( F = x + H \in K[x]^n \) is shown to be linearly triangularizable in the cases: (1) \( n = 3 \) and \( H \) is homogeneous of arbitrary degree (de Bondt and van den Essen [6]); (2) \( n = 4 \) and \( H \) is quadratic homogeneous (Meisters and Olech [12]); (3) \( n = 9 \) and \( F \) is a quadratic homogeneous quasi-translation (Sun [16]); (4) \( n \) arbitrary and \( H \) is quadratic with \( \text{rk} J_H \leq 2 \) (De Bondt and Yan [7]), and to be tame in the case (5) \( n = 5 \) and \( H \) is quadratic homogeneous (de Bondt [2] and Sun [17] independently), and to be invertible in the case (6) \( n = 4 \) and \( H \) is cubic homogeneous (Hubbers [9]). For the case of arbitrary characteristic, de Bondt [5] described the Jacobian matrix \( J_H \) of rank two for any quadratic polynomial map \( H \) and showed that if \( J_H \) is nilpotent then \( J_H \) is similar to a triangular one.

In this paper, we investigate cubic homogeneous polynomial maps \( H \) with \( \text{rk} J_H \leq 2 \) for any dimension \( n \) when \( \text{char} K \neq 2, 3 \). In Section 2, we classify all such maps (Theorem 2.7). And in Section 3, we show that for such an \( H \), if \( F = x + H \) is a Keller map, then it is invertible and furthermore it is tame if the dimension \( n \neq 4 \) (Theorem 3.4).

## 2 Cubic homogeneous maps \( H \) with \( \text{rk} J_H \leq 2 \)

For a polynomial map \( H \in K[x]^m \), we write \( \text{trdeg}_K K(H) \) for the transcendence degree of \( K(H) \) over \( K \). It is well-known that \( \text{rk} J_H = \text{trdeg}_K K(H) \) if \( K(H) \subseteq K(x) \) is separable, in particular if \( \text{char} K = 0 \); see [8, Proposition 1.2.9]. And for arbitrary characteristic, one has \( \text{rk} J_H \leq \text{trdeg}_K K(H) \); see [4] or [13].

It was shown in [5] that when \( \text{char} K \neq 2 \), for any quadratic polynomial map \( H \) with \( \text{rk} J_H \leq 2 \), one has \( \text{rk} J_H = \text{trdeg}_K K(H) \). We will show that when \( \text{char} K \neq 2, 3 \), for any cubic homogeneous polynomial map \( H \) with \( \text{rk} J_H \leq 2 \), one has \( \text{rk} J_H = \text{trdeg}_K K(H) \). The notation \( a|_{x=c} \) below means to substitute \( x \) by \( c \) in \( a \).

**Theorem 2.1.** Let \( s \leq n \). Take

\[
\tilde{x} := (x_1, x_2, \ldots, x_s) \quad \text{and} \quad L := K(x_{s+1}, x_{s+2}, \ldots, x_n).
\]

To prove that for (homogeneous) polynomial maps \( H \in K[x]^m \) of degree \( d \),

\[
\text{rk} J_H = r \quad \text{implies} \quad \text{trdeg}_K K(H) = r, \quad \text{for every} \ r < s,
\]

(2.1)
Lemma 2.2. Let it suffices to show that for (homogeneous) polynomial maps $\tilde{H} \in L[\bar{x}]^{s}$ of degree $d$,
\[
\text{trdeg}_{L}(\tilde{H}) = s \implies \text{rk} \mathcal{J}_{\bar{x}} \tilde{H} = s. \tag{2.2}
\]

Proof. Suppose that $H \in K[x]^{m}$ is (homogeneous) of degree $d$, such that (2.1) does not hold. Then there exists an $r < s$ such that $\text{rk} \mathcal{J}H = r < \text{trdeg}_{K} K(H)$. We need to show that (2.2) does not hold.

Let $s' = \text{trdeg}_{K} K(H)$. Assume without loss of generality that $H_{1}, H_{2}, \ldots, H_{s'}$ are algebraically independent over $K$, and that the components of
\[
H' := (H_{1}, H_{2}, \ldots, H_{s'}, x_{s'+1}^{d}, x_{s'+2}^{d}, \ldots, x_{s}^{d})
\]
are algebraically independent over $K$ if $s' < s$. Then
\[
\text{rk} \mathcal{J}H' \leq r + (s - s') < s = \text{trdeg}_{K} K(H').
\]

For the case of $s' \geq s$, just take $H' = (H_{1}, H_{2}, \ldots, H_{s})$, and we have also $\text{rk} \mathcal{J}H' \leq r < s$.

Notice that (2.1) is also unsatisfied for $H'$. So, replacing $H$ by $H'$, we may assume that $H \in K[x]^{s}$ with $\text{rk} \mathcal{J}H = r < \text{trdeg}_{K} K(H) = s$.

One may observe that $H_{1}(x_{1}, x_{1}x_{2}, x_{1}x_{3}, \ldots, x_{1}x_{n})$ is algebraically independent over $K$ of $x_{2}, x_{3}, \ldots, x_{n}$. On account of the Steinitz Mac Lane exchange lemma, we may assume without loss of generality that the components of
\[
(H(x_{1}, x_{1}x_{2}, x_{1}x_{3}, \ldots, x_{1}x_{n}), x_{s+1}, x_{s+2}, \ldots, x_{n})
\]
are algebraically independent over $K$. Then the components of $H(x_{1}, x_{1}x_{2}, x_{1}x_{3}, \ldots, x_{1}x_{n})$ are algebraically independent over $L := K(x_{s+1}, x_{s+2}, \ldots, x_{n})$, and so are the components of
\[
\tilde{H} := H(x_{1}, x_{2}, \ldots, x_{s}, x_{1}x_{s+1}, x_{1}x_{s+2}, \ldots, x_{1}x_{n}) \in L[\bar{x}]^{s},
\]
where $\bar{x} = (x_{1}, x_{2}, \ldots, x_{s})$. That is, $\text{trdeg}_{L}(\tilde{H}) = s$.

Let $G := (x_{1}, x_{2}, \ldots, x_{s}, x_{1}x_{s+1}, x_{1}x_{s+2}, \ldots, x_{1}x_{n})$. Then it follows from the chain rule that
\[
\mathcal{J}_{\bar{x}} \tilde{H} = (\mathcal{J}H)|_{x=G} \cdot \mathcal{J}_{\bar{x}} G,
\]
so $\text{rk} \mathcal{J}_{\bar{x}} \tilde{H} \leq \text{rk}(\mathcal{J}H)|_{x=G} \leq \text{rk} \mathcal{J}H < s$. Therefore (2.2) does not hold for $\tilde{H}$, which completes the proof. \hfill \Box

Lemma 2.2. Let $H \in K[x]^{m}$ be a polynomial map of degree $d$ and $r := \text{rk} \mathcal{J}H$. Denote by $|K|$ the cardinality of $K$.

(i) If $|K| > (d - 1)r$ and $\mathcal{J}H : x = 0$, then there exist $S \in \text{GL}_{m}(K)$ and $T \in \text{GL}_{n}(K)$, such that for $\tilde{H} := SH(Tx)$,
\[
\tilde{H}|_{x=e_{r+1}} = \begin{pmatrix} I_{r} & 0 \\ 0 & 0 \end{pmatrix}.
\]
Lemma 5.1 (i). The rest of the proof of (ii) is similar to that of (i). So

\[ H|_{x=e_1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \]

Moreover, \(|K|\) may be one less (i.e. at least \((d-1)r\) and \((d-1)r+1\) respectively) if every nonzero component of \(H\) is homogeneous.

**Proof.** (i) Assume without loss of generality that

\[ a_0 := \det \mathcal{J}_{x_1, x_2, \ldots, x_r}(H_1, H_2, \ldots, H_r) \neq 0. \]

Suppose that \(|K| > (d-1)r\). It follows by [3] Lemma 5.1 (i)] that there exists a vector \(w \in K^n\) such that \(a_0(w) \neq 0\). So \(\text{rk} (\mathcal{J}H)|_{x=w} = r\). There exist \(n-r\) independent vectors \(v_{r+1}, v_{r+2}, \ldots, v_n \in K^n\), such that \(\mathcal{J}H|_{x=w} \cdot v_i = 0\) for \(i = r+1, r+2, \ldots, n\). And we may take \(v_{r+1} = w\) since

\[ (\mathcal{J}H)|_{x=w} \cdot w = (\mathcal{J}H \cdot x)|_{x=w} = 0. \]

Take \(T = (v_1, v_2, \ldots, v_n) \in \text{GL}_n(K)\). From the chain rule, we deduce that

\[ (\mathcal{J}(H(Tx)))|_{x=e_{r+1}} \cdot e_i = (\mathcal{J}H)|_{x=Tv_{r+1}} \cdot Te_i = (\mathcal{J}H)|_{x=w} \cdot v_i \quad (1 \leq i \leq n). \]

In particular, \(\text{rk} \mathcal{J}(H(Tx))|_{x=e_{r+1}} = r\) and the last \(n-r\) columns of \(\mathcal{J}(H(Tx))|_{x=e_{r+1}}\) are zero. There exists \(S \in \text{GL}_m(K)\) such that

\[ (\mathcal{J}(SH(Tx)))|_{x=e_{r+1}} = S \cdot (\mathcal{J}(H(Tx)))|_{x=e_{r+1}} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \]

(2) Suppose that \(|K| > (d-1)r+1\). Since \(\mathcal{J}H \cdot x \neq 0\), we may assume that

\[ \text{rk} \left( \mathcal{J}H \cdot x, \mathcal{J}x_2, x_3, \ldots, x_r, H \right) = r, \]

and that

\[ a_1 := \det \left( \mathcal{J}(H_1, H_2, \ldots, H_r) \cdot x, \mathcal{J}x_2, x_3, \ldots, x_r, (H_1, H_2, \ldots, H_r) \right) \neq 0. \]

It follows by [3] Lemma 5.1 (i)] that there exists \(w \in K^n\) such that \(a_1(w) \neq 0\). One may observe that \(\text{rk} (\mathcal{J}H)|_{x=w} = r\) and thus there exist independent vectors \(v_{r+1}, v_{r+2}, \ldots, v_n \in K^n\), such that \(\mathcal{J}H|_{x=w} \cdot v_i = 0\) for \(i = r+1, r+2, \ldots, n\). Since \(\mathcal{J}H \cdot x|_{x=w}\) is the first column of a full column rank matrix, we have

\[ (\mathcal{J}H)|_{x=w} \cdot w = (\mathcal{J}H \cdot x)|_{x=w} \neq 0. \]

So \(v_1 := w\) is independent of \(v_{r+1}, v_{r+2}, \ldots, v_n\).

Take \(T = (v_1, v_2, \ldots, v_n) \in \text{GL}_n(K)\). Then

\[ (\mathcal{J}(H(Tx)))|_{x=e_1} \cdot e_i = (\mathcal{J}H)|_{x=Tv_1} \cdot Te_i = (\mathcal{J}H)|_{x=w} \cdot v_i \quad (1 \leq i \leq n). \]

The rest of the proof of (ii) is similar to that of (i).

The last claim follows from [3] Lemma 5.1 (ii), as an improvement to [3] Lemma 5.1 (i)].
Proposition 2.3. Assume that char\(K \notin \{1, 2, \ldots, d\}\). Then for any cubic homogeneous polynomial map \(H \in K[x]^m\) of degree \(d\) with \(\text{rk} \mathcal{J}H \leq 1\), the components of \(H\) are linearly dependent over \(K\) in pairs, and one has \(\text{rk} \mathcal{J}H = \text{trdeg}_K K(H)\).

Proof. The case \(\text{rk} \mathcal{J}H = 0\) is obvious, so let \(\text{rk} \mathcal{J}H = 1\). On account of Lemma \ref{lem:2.2}, we may assume that \(\mathcal{J}H|_{x = e_1} = E_{11}\). Let \(j \geq 2\). Since \(\deg_{x_1} H_j < d\), we infer that either \(H_j = 0\), or \(\deg_{x_1} \frac{\partial}{\partial x_1} H_j < \deg_{x_1} \frac{\partial}{\partial x_1} H_j\) for some \(i \geq 2\), where \(\deg_{x_1} 0 = -\infty\). The latter is impossible due to \(\text{rk} \mathcal{J}H = 1\), so \(H_j = 0\). This holds for all \(j \geq 2\), which yields the desired results.

Lemma 2.4. Let \(H = (h, x_1^2 x_2, x_2^2 x_3)\) or \((h, x_1^2 x_3, x_2^2 x_3) \in K[x_1, x_2, x_3]^3\), where \(h\) is cubic homogeneous, and assume that \(\text{char} K \neq 2, 3\). Then \(\text{rk} \mathcal{J}H = \text{trdeg}_K K(H)\).

Proof. It suffices to consider the case of \(\text{rk} \mathcal{J}H = 2\). Define a derivation \(D\) on \(A = K[x_1, x_2, x_3]\) as follows: for any \(f \in A\),

\[
D(f) = \frac{x_1 x_2 x_3}{H_2 H_3} \det \mathcal{J}H.
\]

In the case \(H = (h, x_1^2 x_2, x_2^2 x_3)\), an easy calculation shows that \(D = x_1 \partial_{x_1} - 2x_2 \partial_{x_2} + 4x_3 \partial_{x_3}\). Then for any term \(u = x_1^{d_1} x_2^{d_2} x_3^{d_3} \in A\), \(D(u) = (d_1 - 2d_2 + 4d_3)u\). And thus \(\ker D := \{g \in A \mid D(g) = 0\}\), the kernel of \(D\), is linearly spanned by all terms \(u\) with \(d_1 - 2d_2 + 4d_3 = 0\). So the only cubic terms in \(\ker D\) are \(x_1^2 x_2\) and \(x_2^2 x_3\). Since \(\text{rk} \mathcal{J}H = 2\), we have \(\det \mathcal{J}H = 0\) and thus \(h \in \ker D\), which implies that \(h\) is a linear combinations of \(x_1^2 x_2\) and \(x_2^2 x_3\). Thus \(\text{trdeg}_K K(H) = 2\).

In the case \(H = (h, x_1^2 x_3, x_2^2 x_3)\), one may verify that \(x_1^2 x_3, x_1 x_2 x_3\) and \(x_2^2 x_3\) are the only cubic terms in \(\ker D\). The conclusion follows similarly.

Theorem 2.5. Assume that \(\text{char} K \neq 2, 3\). Then for any cubic homogeneous polynomial map \(H \in K[x]^m\) with \(\text{rk} \mathcal{J}H \leq 2\), one has \(\text{rk} \mathcal{J}H = \text{trdeg}_K K(H)\).

Proof. Due to Theorem \ref{thm:2.1} and replacing \(L\) there by \(K\), we may assume that \(H \in K[x_1, x_2, x_3]^3\), and it suffices to show that

\[
\text{trdeg}_K K(H) = 3 \text{ implies } \text{rk} \mathcal{J}H = 3,
\]

or equivalently,

\[
\det \mathcal{J}H = 0 \text{ implies } \text{trdeg}_K K(H) < 3. \tag{2.3}
\]

So assume that \(\det \mathcal{J}H = 0\). Since we may replace \(K\) by an extension field to make it large enough, it follows by Lemma \ref{lem:2.2} that we may assume that \(\mathcal{J}(H)|_{x = e_1} = E_{11} + E_{22}\). Then \(\mathcal{J}H\) is of the form

\[
\begin{pmatrix}
  x_1^2 + * & * & * \\
  * & x_1^2 + * & * \\
  * & * & \frac{\partial H_j}{\partial x_3}
\end{pmatrix},
\]
where the $x_1$-degree of each element $*$ is less than 2. Observing the terms with $x_1$-degree $\geq 5$ in $\det JH$, we have that $\frac{\partial H_k}{\partial x_3} \in K[x_2, x_3]$. Notice that $H_2$ and $H_3$ are of the form:

$$H_2 = x_1^2 x_2 + b_1 x_1 x_3^2 + b_1 x_1 x_2 x_3 + b_1 x_2 x_3^2 + b_1(x_2, x_3);$$

$$H_3 = c_0 x_1 x_2^2 + c_0 x_3^3 + c_0 x_2 x_3^2 + c_0 x_3 x_2^2 + c_0 x_3^3.$$ 

We shall show that $x_2^2 \mid H_3$, i.e., $c_0 = c_1 = 0$.

Noticing that the part of $x_1$-degree 4 of $\det JH$ is $(\frac{\partial H_1}{\partial x_3} - \frac{\partial H_2}{\partial x_1} - \frac{\partial H_3}{\partial x_1}) x_1^4$, we see that $\frac{\partial H_1}{\partial x_3} - \frac{\partial H_2}{\partial x_1} - \frac{\partial H_3}{\partial x_1} = 0$. Consequently,

$$(3c_0 x_3^2 + 2c_0 x_2 x_3 + c_0 x_2^2) = (2b_{10} x_3 + b_{11} x_2)(2c_{12} x_2)$$

so

$$c_0 = 0, \quad c_1 = 2b_{10} c_{12}, \quad c_0 = 2b_{11} c_{12}.$$ 

One may observe that the coefficient of $x_1^3 x_3^2$ in $\det JH$ is $2c_0 b_{10} = 0$, which we can combine with $c_0 = 2b_{10} c_{12}$ to obtain $c_0 = 0$. Therefore,

$$H_3 = (c_1 x_1 + c_0 x_2 + c_0 x_3) x_2^2.$$ 

Moreover, if $c_{12} = 0$ then $c_0 = 2b_{11} c_{12} = 0$ and thus $H_3 = c_0 x_3 x_2^3$.

We distinguish two cases.

- **Case 1:** $c_{12} \neq 0$ and $c_{12} x_1 + c_0 x_2 + c_0 x_3 \mid H_i$ for some $i$.

Then $H_3$ is the product of two linear forms, of which two are distinct. Hence we can compose $H$ with invertible linear maps on both sides, to obtain a map $H'$ for which $H'_2 = x_1^2 x_2$, and $x_2 \mid H'_1$.

Notice that $H'_1(1, 0, t) \neq 0$. As $K$ has at least 5 elements, it follows from [3, Lemma 5.1 (i)] that there exists a $\lambda \in K$, such that $H'_1(1, 0, \lambda) \neq 0$. Hence the coefficient of $x_1^3$ in $H'_1(x_1, x_2, x_3 + \lambda x_1)$ is nonzero. Furthermore, $H'_2(x_1, x_2, x_3 + \lambda x_1) = x_1^2 x_2$.

Replacing $H'$ by $H'(x_1, x_2, x_3 + \lambda x_1)$, we may assume that $H'_2 = x_1^2 x_2$ and that $H'_1$ contains $x_1^3$ as a term. We may even assume that the coefficient of $x_1^3$ in $H'_1$ equals 1. Then $\det JH'|_{x_3 = e_1}$ is of the form

$$\begin{pmatrix}
1 & * & a \\
0 & 1 & 0 \\
* & * & *
\end{pmatrix},$$

and has rank 2. Furthermore, $v_3 = (-a, 0, 1)^t$ belongs to its null space. We may apply the proof of Lemma 2.2 on $H'$ by taking $T = (e_1, e_2, v_3)$ and taking an appropriate $S \in \text{GL}_3(K)$ such that $\tilde{H} := SH'(Tx)$ satisfies $\det \tilde{H}|_{x = e_1} = S \det JH'|_{x = T e_1} = E_{11} + E_{22}$. Notice that $Tx$ is of the form $(L_1, x_2, L_3)$, and observing the form of $\det JH'|_{x = e_1}$ one may also choose $S x$ to be of the form $(*, x_2, *)$. Then $\tilde{H}_2 = L_1^2 x_2$. 

6
So we can compose $\tilde{H}$ with an invertible linear map on the right, to obtain a map $\tilde{H}'$ for which $\tilde{H}'_2 = x_1^2 x_2$ and $\tilde{H}'_3 = x_2^3 L'$ for some linear form $L'$.

Suppose first that $L'$ is a linear combination of $x_1$ and $x_2$. If $\tilde{H}'_1 \in K[x_1, x_2]$, then we are done. Otherwise, we have $\det \mathcal{J}_{x_1 x_2}(\tilde{H}'_2, \tilde{H}'_3) = 0$, and then by Proposition 2.3 \( \text{trdeg}_K K(\tilde{H}'_2, \tilde{H}'_3) < 2 \).

Suppose next that $L'$ is not a linear combination of $x_1$ and $x_2$. Then we may assume that $\tilde{H}'_3 = x_2^2 x_3$. By Lemma 2.4 \( \text{(i)} \), \( \text{trdeg}_K K(\tilde{H}') < 3 \).

- **Case 2:** $c_{12} = 0$ or $c_{12} x_1 + c_{03} x_2 + c_{02} x_3 \mid H_i$ for all $i$.

Since $x_2^2 \mid H_3$, we can compose $H$ with invertible linear maps on both sides, to obtain a map $H'$ for which $H'_1 \in \{x_1^3, x_1^2 x_2\}$. After a possible interchange of $H'_2$ and $H'_3$, the first two rows of $\mathcal{J} H'$ are independent. Now we may apply the proof of Lemma 2.2 to $H'$, more precisely, there exist $S, T \in \text{GL}_3(K)$ such that $\tilde{H} := S \mathcal{H}'(T x)$ satisfies $\mathcal{J} \tilde{H}|_{x=x_1} = E_{11} + E_{22}$.

If we choose $w$ such that first two rows of $(\mathcal{J} H')_{x=w}$ are independent, then we can take $S$ such that $S x = (f_1 x_1 + f_2 x_2, g_1 x_1 + g_2 x_2, \ast)$. By repeating the discussion for $\tilde{H}$ as for $H$ above, we may assume that $\tilde{H}_3 = L x_2^2$ for some linear form $L$.

Let $T x = (L_1, L_2, L_3)$. Notice that $H'_1(T x) \in \{L_1^3, L_1^2 L_2\}$ and that $H'_1(T x)$ is a linear combination of $\tilde{H}_1$ and $\tilde{H}_2$. Hence we can compose $\tilde{H}$ with a linear map on the left, to obtain a map $\tilde{H}'$ for which $H'_2 \in \{L_1^3, L_1^2 L_2\}$ and $H'_3 = L x_2^2$.

Suppose first that $\tilde{H}'_2 = L_1^2 L_2$. Then $c_{12} \neq 0$, so $c_{12} x_1 + c_{03} x_2 + c_{02} x_3 \mid H_i$ for all $i$. From this, we infer that $L_2 \mid \tilde{H}_i$ and $L_2 \mid \tilde{H}'_i$ for all $i$. As $x_2 \not\mid \tilde{H}_1$, we deduce that $L$ and $L_2$ are dependent linear forms, which are independent of $x_2$. If $L$ and $L_2$ are linear combinations of $L_1$ and $x_2$, then we can reduce to Proposition 2.3, and otherwise we can reduce to Lemma 2.4 \( \text{(ii)} \).

Suppose next that $\tilde{H}'_2 = L_1^3$. If $L$, $L_1$ and $x_2$ are linearly dependent over $K$, then we can reduce to Proposition 2.3. Otherwise, $\tilde{H}$ is as $H$ in the previous case.

**Remark 2.6.** Inspired by Lemma 2.4, we investigated maps $H$ of which the components are terms, and searched for $H$ with algebraically independent components for which $\det \mathcal{J} H = 0$. One can infer that $H$ is as such, if and only if the matrix with entries $\deg_{x_i} H_j$ has determinant zero over $K$, but not over $\mathbb{Z}$.

We found the following non-homogeneous $H$ as above over fields of characteristic $5$:

$$(x_1^3 x_2, x_1 x_2^2), \quad (x_1^2 x_2, x_1 x_2^2, x_2 x_3)$$

with the following homogenizations respectively:

$$(x_1^3 x_2, x_1 x_2^2 x_3, x_3^4), \quad (x_1^2 x_2, x_1 x_2^2, x_2 x_3 x_4, x_3^3)$$
Besides these homogenizations, we found the following homogeneous $H$ over fields of characteristic 5:

$$(x_1^2x_3^3, x_1x_2^3, x_2x_3^2), \quad (x_4x_1^2, x_1x_2^2, x_2x_3^2, x_3x_4^2)$$

We conclude with a homogeneous $H$ over fields of characteristic 7, and a homogeneous $H$ over any characteristic $p \in \{1, 2, \ldots, d\}$ respectively:

$$(x_3x_1^3, x_1x_2^3, x_2x_3^2), \quad (x_4^d, x_1^{d-r}x_2^r)$$

These examples show that the conditions in Proposition 2.3 and Theorem 2.5 cannot be relaxed.

**Theorem 2.7.** Suppose that $\text{char} K \neq 2, 3$ and let $H \in K[x]^m$ be cubic homogeneous. Let $r := \text{rk} JH$ and suppose that $r \leq 2$. Then there exist $S \in \text{GL}_m(K)$ and $T \in \text{GL}_n(K)$, such that for $\tilde{H} := SH(Tx)$, one of the following statements holds:

1. $\tilde{H}_{r+1} = \tilde{H}_{r+2} = \cdots = \tilde{H}_m = 0$;
2. $r = 2$ and $\tilde{H} \in K[x_1, x_2]^m$;
3. $r = 2$ and $K\tilde{H}_1 + K\tilde{H}_2 + \cdots + K\tilde{H}_m = Kx_3x_1^2 \oplus Kx_3x_1x_2 \oplus Kx_3x_2^2$.

Furthermore, we may take $S = T^{-1}$ if $m = n$.

**Proof.** By Theorem 2.3, $\text{trdeg}_K K[H] = \text{rk} JH = r \leq 2$. Since $H$ is homogeneous, we have $\text{trdeg}_K K(tH) = r$ as well, where $t$ is a new variable.

Suppose first that $r \leq 1$. It follows by [4] Theorem 2.7 that we may take $\tilde{H}$ as in (1).

Suppose next that $r = 2$. By [4] Theorem 2.7, $H$ is of the form $g \cdot h(p, q)$, such that $g, h$ and $(p, q)$ are homogeneous and $\deg g + \deg h \cdot \deg(p, q) = 3$.

If $\deg h \leq 1$, then every triple of components of $h$ is linearly dependent over $K$, and thus we may take $\tilde{H}$ as in (1). If $\deg h = 3$, then $\deg(p, q) = 1$ and $\deg g = 0$, whence we may take $\tilde{H}$ as in (2).

So assume that $\deg h = 2$. Then $\deg(p, q) = 1$ and $\deg g = 1$. If $g$ is a linear combination of $p$ and $q$, then we may take $\tilde{H}$ as in (2). If $g$ is not a linear combination of $p$ and $q$, then we may take $\tilde{H}$ as in (3) or (1).

Finally, if $m = n$ and $\tilde{H} = SH(Tx)$ is as in (1), then $SH(S^{-1}x) = \tilde{H}(T^{-1}S^{-1}x)$ is still as in (1). So we may take $S = T^{-1}$. If $m = n$ and $\tilde{H} = SH(Tx)$ is as in (2) or (3), then $T^{-1}H(Tx) = T^{-1}S^{-1}H$ is still as in (2) or (3), whence we may also take $S = T^{-1}$. \qed

### 3 Cubic homogeneous Keller maps $x + H$ with $\text{rk} JH \leq 2$

For two matrices $M, N \in \text{Mat}_n(K[x])$, we say that $M$ is similar over $K$ to $N$, if there exists $T \in \text{GL}_n(K)$ such that $N = T^{-1}MT$. 

8
Theorem 3.1. Let $F = x + H \in K[x]^n$ be a Keller map with $\text{trdeg}_K K(H) = 1$. Then $JH$ is similar over $K$ to a triangular matrix, and the following statements are equivalent:

1. $\det JF = 1$;
2. $JH$ is nilpotent;
3. $(JH) \cdot (JH)|_{x=y} = 0$, where $y = (y_1, y_2, \ldots, y_n)$ are $n$ new variables.

Proof. Since $\text{trdeg}_K K(H) = 1$, by [4, Corollary 3.2] there exists a polynomial $p \in K[x]$ such that $H_i \in K[p]$ for each $i$. Say that $H_i = h_i(p)$, where $h_i \in K[t]$ for each $i$. Write $h'_i = \frac{\partial h_i}{\partial t}$, then

$$JH = h'_i(p) \cdot Jp. \quad (3.1)$$

Assume without loss of generality that $h'_1 = h'_2 = \cdots = h'_s = 0$, and that

$$0 \leq \deg h'_{s+1} < \deg h'_{s+2} < \cdots < \deg h'_n.$$

For $s < i < n$,

$$\deg h'_i(p) = \deg h'_i \cdot \deg p \leq (\deg h'_{i+1} - 1) \cdot \deg p = \deg h'_{i+1}(p) - \deg p.$$

Since the degrees of the entries of $Jp$ are less than $\deg p$, we deduce from (3.1) that the nonzero entries on the diagonal of $JH$ have different degrees in increasing order. Furthermore, the nonzero entries beyond the $(s+1)$th entry on the diagonal of $JH$ have positive degrees.

By (3.1), $\text{rk}(-JH) \leq 1$, and thus $n-1$ eigenvalues of $-JH$ are zero. It follows that the trailing degree of the characteristic polynomial of $-JH$ is at least $n-1$. More precisely,

$$\det(tI_n + JH) = t^n - \text{tr}(-JH) \cdot t^{n-1},$$

and thus

$$\det JF = (t^n - \text{tr}(-JH) \cdot t^{n-1})|_{t=1} = 1 + \text{tr} JH.$$

Observe that the diagonal of $JH$ is totally zero, except maybe the $(s+1)$th entry, which is a constant.

So $\frac{\partial}{\partial x} p = 0$ for all $i > s + 1$, and $JH$ is lower triangular. If the $(s+1)$th entry on the diagonal of $JH$ is nonzero, then (1), (2) and (3) do not hold. If the $(s+1)$th entry on the diagonal of $JH$ is zero, then $\frac{\partial}{\partial x} p = 0$ for all $i > s$, whence (1), (2) and (3) hold. \qed

Let $H \in K[x]^n$ be homogeneous of degree $d \geq 2$. Then $x + H$ is a Keller map if and only if $JH$ is nilpotent; see for example [3 Lemma 6.2.11]. So we first investigate nilpotent matrices over $K[x]$. 9
Lemma 3.2. Let $N \in \text{Mat}_2(K[x])$ such that $N$ is nilpotent. Then there exist $a, b, c \in K[x]$ such that

$$N = c \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix}.$$  

Furthermore, $N$ is similar over $K$ to a triangular matrix if and only if $a$ and $b$ are linearly dependent over $K$.

Proof. Since $\det N = 0$, we may write

$$N = c \cdot \begin{pmatrix} b \\ a \end{pmatrix} \cdot \begin{pmatrix} a & -\tilde{b} \end{pmatrix},$$

where $a, b \in K[x]$ and $\tilde{b}, c \in K(x)$. Since $\text{tr} N = 0$, we have $\tilde{b} = b$. If we choose $a$ and $b$ to be relatively prime, then $c \in K[x]$ as well.

Furthermore, $a$ and $b$ are linearly dependent over $K$ if and only if the rows of $N$ are linearly dependent over $K$, if and only if $N$ is similar over $K$ to a triangular matrix. \qed

Lemma 3.3. Let $H \in K[x]^2$ be cubic homogeneous, such that $J_{x_1,x_2}H$ is nilpotent. Then there exists $T \in \text{GL}_2(K)$ such that for $\tilde{H} := T^{-1}H(T(x_1,x_2), x_3, x_4, \ldots, x_n)$, one of the following statements holds:

1. $J_{x_1,x_2}\tilde{H}$ is a triangular matrix;
2. there are independent linear forms $a, b \in K[x]$, such that

$$J_{x_1,x_2}\tilde{H} = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \quad \text{and} \quad J_{x_1,x_2} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

3. $\text{char} K = 3$ and there are independent linear forms $a, b \in K[x]$, such that

$$J_{x_1,x_2}\tilde{H} = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \quad \text{and} \quad J_{x_1,x_2} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Suppose that (1) does not hold. By Lemma 3.2 there are $a, b, c \in K[x]$, such that

$$J_{x_1,x_2}H = c \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix}$$

where $a$ and $b$ are linearly independent over $K$. As $H$ is cubic homogeneous, the entries of $J_{x_1,x_2}H$ are quadratic homogeneous, so $c \in K$ and $a$ and $b$ are independent linear forms.

If we take

$$T = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{then} \quad J_{x_1,x_2}\tilde{H} = \begin{pmatrix} \tilde{a}b & -\tilde{b}^2 \\ \tilde{a}^2 & -\tilde{a}b \end{pmatrix},$$

where $\tilde{a} = c \cdot a|_{x_1= cx_1}$ and $\tilde{b} = c^{-1} \cdot b|_{x_1= cx_1}$.\]
We claim that the coefficient $k_2$ of $x_2$ in $\tilde{b}$ is 0. Suppose conversely that $k_2 \neq 0$. Then the coefficient of $x_2^2$ in 

$$3\tilde{H}_1 = \mathcal{J}_{x_1,x_2}\tilde{H}_1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tilde{b}(x_1\tilde{a} - x_2\tilde{b})$$

is nonzero. In particular, char $K \neq 3$. One may verify that 

$$\mathcal{J}_{x_1,x_2}(H_1 + \frac{1}{2}k_2^{-1}\tilde{b}^3) = (\tilde{b}, 0),$$

where $\tilde{c} := \tilde{a} + k_2^{-1} \tilde{b}(\frac{\partial}{\partial x_1}\tilde{b})$. As a consequence, $\frac{\partial}{\partial x_1}(\tilde{b}) = \frac{\partial}{\partial x_1}0 = 0$. Furthermore, $\tilde{c}$ and $\tilde{b}$ are independent, just like $\tilde{a}$ and $\tilde{b}$. By $\frac{\partial}{\partial x_2}(\tilde{b}) = 0$, we have $\tilde{c} \tilde{b} \in K[x_1, x_3, x_4, \ldots, x_n]$ if char $K \neq 2$. Since $\tilde{c}$ and $\tilde{b}$ are independent, we deduce that if char $K = 2$ then $\tilde{c} \tilde{b} \in K[x_1, x_3, x_4, \ldots, x_n]$ as well. Since the coefficient $\lambda$ of $x_3$ in $\tilde{b}$ is nonzero, we have $\tilde{c} = 0$, a contradiction.

So the coefficient of $x_2$ in $\tilde{b}$ is 0. Similarly, the coefficient of $x_1$ in $\tilde{a}$ is 0. Consequently, 

$$\mathcal{J}_{x_1,x_2} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix},$$

where $\lambda, \mu \in K$. Therefore 

$$\mathcal{J}_{x_1,x_2}\tilde{H} = \begin{pmatrix} (\lambda x_2 + \cdots)(\mu x_1 + \cdots) & -(\lambda x_2 + \cdots)(\mu x_1 + \cdots) \\ (\lambda x_2 + \cdots)^2 & -(\mu x_1 + \cdots)^2 \end{pmatrix}.$$ 

So the coefficient of $x_1^2 x_2$ in $2\tilde{H}_1$ is equal to both $\lambda \mu$ and $-2\mu^2$. Similarly, the coefficient of $x_1 x_2^2$ in $2\tilde{H}_2$ is equal to both $\lambda \mu$ and $-2\lambda^2$. It follows that either $\lambda = \mu = 0$ or $0 \neq \lambda = -2\mu = 4\lambda$. In the former case, $\tilde{H}$ satisfies (2). In the latter case, char $K = 3$ and $\lambda = \mu$. Replacing $\tilde{H}$ by $\lambda \tilde{H}(\lambda^{-1}(x_1, x_2), x_3, x_4, \ldots, x_n)$, we have that $\tilde{H}$ satisfies (3). 

\begin{theorem}
Suppose that char $K \neq 2, 3$. Let $H \in K[x]^n$ be cubic homogeneous such that $x + H$ is a Keller map, i.e., $\mathcal{J}H$ is nilpotent.

(i) If $\text{rk } \mathcal{J}H = 1$, then there exists $T \in \text{GL}_n(K)$ such that for $\tilde{H} := T^{-1}H(Tx)$,

$$\tilde{H}_1 \in K[x_2, x_3, x_4, \ldots, x_n],$$

$$\tilde{H}_2 = \tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0.$$ 

(ii) If $\text{rk } \mathcal{J}H = 2$, then either $H$ is linearly triangularizable or there exists $T \in \text{GL}_n(K)$ such that for $\tilde{H} := T^{-1}H(Tx)$,

$$\tilde{H}_1 - (x_1 x_3 x_4 - x_2 x_4^2) \in K[x_3, x_4, \ldots, x_n],$$

$$\tilde{H}_2 - (x_1 x_3^2 - x_2 x_3 x_4) \in K[x_3, x_4, \ldots, x_n],$$

$$\tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0.$$

\end{theorem}
Furthermore, $x + tH$ is invertible over $K[t]$ if $\text{rk} JH \leq 2$, where $t$ is a new variable. Moreover, $x + tH$ is even tame over $K[t]$ if either $\text{rk} JH = 1$ or $\text{rk} JH = 2$ and $n \neq 4$. In particular, $x + \lambda H$ is invertible and tame under the above condition respectively for every $\lambda \in K$.

**Proof.** We may take $\tilde{H}$ as in (1), (2) or (3) of Theorem 2.7. If $\text{rk} JH = 1$, then $\tilde{H}$ is as in (1) of Theorem 2.7 i.e., $\tilde{H}_i = 0, 2 \leq i \leq n$, whence (i) holds because $\text{tr} J\tilde{H} = 0$. So assume that $\text{rk} J\tilde{H} = 2$. Notice that $J\tilde{H}$ is nilpotent.

If $\tilde{H}$ is as in (1) or (2) of Theorem 2.7 i.e., $\tilde{H}_i = 0, 3 \leq i \leq n$ or $\tilde{H} \in K[x_1, x_2]^n$, then $J_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is nilpotent.

If $\tilde{H}$ is as in (3) of Theorem 2.7 i.e., $K\tilde{H}_1 + K\tilde{H}_2 + \cdots + K\tilde{H}_n = Kx_3x_1^2 \oplus Kx_3x_1x_2 \oplus Kx_3^2x_2$, then $\tilde{H}_3 = 0$, because $x_3^1\tilde{H}_3$ is the constant part with respect to $x_3$ of $\text{tr} J\tilde{H} = 0$. So $J_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is nilpotent in any case.

One may observe that, in all the cases (1), (2) and (3) of Theorem 2.7 if $J_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is similar over $K$ to a triangular matrix, then $J\tilde{H}$ is similar over $K$ to a triangular matrix, and so is $JH$, and thus $H$ is linearly triangularizable.

Now suppose $J_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ is not similar over $K$ to a triangular matrix. Noticing that $\text{char} K \neq 2, 3$, $J_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$ must be as in (2) of Lemma 3.5 i.e.,

$$J_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2) = \begin{pmatrix} ab & -b^2 & -ab \\ a^2 & -b^2 & -ab \end{pmatrix} \quad \text{and} \quad J_{x_1, x_2} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $a, b$ are linearly independent linear forms.

If $\tilde{H}_1 \in K[x_1, x_2, x_3], \text{then } a, b \in k[x_3], \text{a contradiction. So } \tilde{H} \text{ is not as in (2) or (3) of Theorem 2.7 and thus is as in (1) of Theorem 2.7 i.e., } \tilde{H}_3 = \tilde{H}_4 = \cdots = \tilde{H}_n = 0. \text{ Consequently, by linear coordinate transformation, we may take } \tilde{H} \text{ such that } a = x_3 \text{ and } b = x_4. \text{ So (ii) holds.}$

For the last claim, when $\text{rk} JH = 1$, $\tilde{H}$ is of the form in (i), whence $x + t\tilde{H}$ is elementary and thus tame. When $\text{rk} JH = 2$, $\tilde{H}$ is of the form in (ii), and it suffices to show the following automorphism

$$F = (x_1 + tx_4(x_3x_1 - x_4x_2), x_2 + tx_3(x_3x_1 - x_4x_2), x_3, x_4, x_5)$$

is tame over $K[t]$.

For that purpose, let $w = t(x_3x_1 - x_4x_2)$ and let $D := x_4\partial_{x_1} + x_3\partial_{x_2}$ be a derivation of $K[t][x_1, x_2, x_3, x_4]$. Observe that $D$ is triangular and $w \in \ker D$, and that $F = (\exp(wD), x_5)$. Therefore $F$ is tame over $K[t]$ due to the following Lemma 3.5.

Recall that a derivation $D$ of $K[x]$ is called locally nilpotent if for every $f \in K[x]$ there exists an $m$ such that $D^m(f) = 0$. For such a derivation, $\exp D := \sum_{i=0}^{\infty} \frac{1}{i!} D^i$ is a polynomial automorphism of $K[x]$. A derivation $D$ of $K[x]$ is called triangular if $D(x_i) \in K[x_i+1, \ldots, x_n]$ for $i = 1, 2, \ldots, n-1$ and $D(x_n) \in K$. A triangular derivation is locally nilpotent.

**Lemma 3.5.** Let $D$ be a triangular derivation of $K[t][x]$ and $w \in \ker D$ i.e. $D(w) = 0$. Then $(\exp(wD), x_{n+1})$ is tame over $K[t]$.
Proof. From [15, Corollary], it follows that there exists a $k$ such that $(\exp(wD), x_{n+1}, x_{n+2}, \ldots, x_{n+k})$ is tame over $K(t)$. Inspecting the proof of [15, Corollary] yields that $(\exp(wD), x_{n+1})$ is tame over $K[t]$.

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References


