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RAMANUJAN-TYPE FORMULAE FOR $1/\pi$: $q$-ANALOGUES

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ABSTRACT. The hypergeometric formulae designed by Ramanujan more than a century ago for efficient approximation of $\pi$, Archimedes’ constant, remain an attractive object of arithmetic study. In this note we discuss some $q$-analogues of Ramanujan-type evaluations and of related supercongruences.

1. Introduction

Let $q$ be inside the unit disc, $|q| < 1$. In the recent joint paper [4] of one of these authors, Ramanujan’s formulas [7]

$$\sum_{n=0}^{\infty} \frac{(6n+1) \left(\frac{1}{2}\right)_n^3}{n!^3 4^n} = \frac{4}{\pi} \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n (6n+1) \frac{\left(\frac{1}{2}\right)_n^3}{n!^3 8^n} = \frac{2\sqrt{2}}{\pi}, \quad (1)$$

were supplied with $q$-analogues

$$\sum_{n=0}^{\infty} q^{n^2}[6n+1] \frac{(q; q^2)_n^3 (q^2; q^4)_n^3}{(q^4; q^4)_n^3} = \frac{(1 + q)(q^2; q^4)_\infty (q^n q^4; q^4)_\infty}{(q^4; q^4)^2_\infty}, \quad (2)$$

$$\sum_{n=0}^{\infty} (-1)^n q^{3n^2}[6n+1] \frac{(q; q^2)_n^3 (q^2; q^4)_n^3}{(q^4; q^4)_n^3} = \frac{(q^3; q^4)_\infty (q^5; q^4)_\infty}{(q^4; q^4)^2_\infty}. \quad (3)$$

Here and in what follows

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j)$$

and we use the standard notation

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad \text{and} \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}$$

for the Pochhammer symbol and its $q$-version, so that

$$(a)_n = \prod_{j=0}^{n-1} (a + j) \quad \text{and} \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$$

for positive integers $n$. We also define the $q$-numbers by $[n] = [n]_q = (1 - q^n)/(1 - q)$.

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Apart from another entry
\[
\sum_{n=0}^{\infty} (-1)^n q^{n^2} [4n + 1] (q^2; q^2)^2_n = \frac{(q; q^2)_\infty (q^3; q^2)_\infty}{(q^2; q^2)_\infty},
\]
which follows trivially from a limiting case of Jackson’s formula \cite[eq. (2.7.1)]{1} and
which represents a \(q\)-analogue of Bauer’s (Ramanujan-type) formula
\[
\sum_{n=0}^{\infty} (-1)^n (4n + 1) \frac{\left(\frac{1}{2}\right)^3}{n^{13}} = \frac{2}{\pi},
\]
the entries \((2)\) and \((3)\) provide us with the first examples of \(q\)-analogues of Ramanujan’s formulae for \(1/\pi\).

The principal ingredients in the proof of \((2)\) and \((3)\) in \cite{4} are suitable chosen \(q\)-Wilf–Zeilberger (WZ) pairs and some basic hypergeometric identities.

In this note we use the \(q\)-WZ pairs from \cite{4} and original ideas of J. Guillera from \cite{2} to give a few further \(q\)-analogues of Ramanujan’s formulae and their generalizations.

**Theorem 1.** The following identities are true:

\[
\sum_{n=0}^{\infty} (-1)^n q^{2n^2} (q^2; q^4)^2_n (q; q^2)_{2n} \left[8n + 1 + [4n + 1] \frac{q^{4n+1}}{1 + q^{4n+2}}\right] = \frac{(1 + q)(q^2; q^4)_\infty (q^6; q^4)_\infty}{(q^4; q^4)_\infty},
\]

\[
\sum_{n=0}^{\infty} (-1)^n q^{2n^2} (q^2; q^4)^2_n (q; q^2)_{2n} \left(10n + 1\right) + q^{6n+1} \frac{4n + 2}{[12n + 4]} + q^{6n+3} \frac{6n + 1}{[12n + 4][12n + 8]} = \frac{(1 + q)(q^2; q^4)_\infty (q^6; q^4)_\infty}{(q^4; q^4)_\infty},
\]

\[
\sum_{n=0}^{\infty} q^{4n^2} (q^2; q^2)^2_n (q; q^4)_{2n} \left(8n + 1\right) - q^{8n+3} \frac{[4n + 1]^2}{[8n + 4]} = \frac{(q^3; q^4)_\infty (q^5; q^4)_\infty}{(q^4; q^4)_\infty}.
\]

As pointed out to us by C. Krattenthaler, the formulae \((2)\) and \((3)\) can be alternatively proved through an intelligent use of quadratic transformations from \cite{1} Section 3.8; more details of this machinery appear in \cite{5}. We adapt the related technique here to establish a new \(q\)-analogue of Ramanujan’s formula as follows.

**Theorem 2.** We have

\[
\sum_{n=0}^{\infty} q^{2n^2} (q^2; q^2)^2_n (q; q^2)_{2n} [8n + 1] = \frac{(q^3; q^4)_\infty (q^5; q^4)_\infty}{(q^2; q^4)_\infty (q^6; q^4)_\infty}.
\]
Observe that equations (4) and (7) are $q$-analogues of Ramanujan’s

$$\sum_{n=0}^{\infty} \frac{(-1)^n(q^{(1)}_n)(q^{(1)}_n(q^{(1)}_n)(q^{(1)}_n(2n)))}{n!^3 4^n} (2n) = \sum_{n=0}^{\infty} \frac{(-1)^n(4n)(2n)^2}{2^{10n}} (2n + 3) = \frac{8}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(q^{(1)}_n(q^{(1)}_n(q^{(1)}_n(q^{(1)}_n(3n)))}{n!^3 9^n} (8n + 1) = \sum_{n=0}^{\infty} \frac{(2n)(2n)^2}{2^{8n} 3^{2n}} (8n + 1) = \frac{2\sqrt{3}}{\pi},$$

while the $q \to 1$ cases of (5) and (6) read

$$\sum_{n=0}^{\infty} \frac{(-1)^n(6n)(4n)(2n)}{2^{12n}} \frac{576n^3 + 624n^2 + 190n + 15}{(3n + 1)(3n + 2)} = \frac{16}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(2n)^2(2n)}{2^{12n}} \frac{48n^2 + 32n + 3}{2n + 1} = \frac{8\sqrt{2}}{\pi}.$$}

The details of our proofs of Theorems 1 and 2 are given in Sections 2 and 3, respectively. In the final section we highlight some connections of our findings with $q$-analogues of Ramanujan-type supercongruences, which were an original source of the proofs of (2) and (3) in [4]. A simple look of (some) $q$-analogues of Ramanujan-type formulae for $1/\pi$ and certain similarity with the Rogers–Ramanujan identities makes the former a plausible candidate for combinatorial explorations.

As hypergeometric summation and transformation formulae occasionally possess several (sometimes very different!) $q$-analogues, we cannot exclude a possibility that there are multiple $q$-analogues of some Ramanujan-type formulae for $1/\pi$. Another source of such multiple $q$-entries can be caused by using different $q$-WZ pairs: the phenomenon of their existence has been recorded in the non-$q$-settings in Guillera’s PhD thesis; see [3, Sections 1.3, 1.4].

We thank Jesús Guillera for drawing our attention to some parts of his PhD thesis [3] and Christian Krattenthaler for giving us details of his hypergeometric proof of (2).

2. WZ machinery and Guillera’s invention

The heart of the proof of (2) in [4] is the $q$-WZ pair

$$F(n, k) = F(n, k; q) = \frac{q^{(n-k)^2(6n - 2k + 1)}(q^2; q^4)_n(q; q^2)_{n-k}(q; q^2)_{n+k}}{(q^4; q^4)_n(q^4; q^4)_{n-k}(q^2; q^4)_k},$$

$$G(n, k) = G(n, k; q) = \frac{q^{(n-k)^2(q^2; q^4)_n(q; q^2)_{n-k}(q; q^2)_{n+k-1}}}{(1 - q)(q^4; q^4)_n(q^4; q^4)_{n-k}(q^2; q^4)_k},$$

where it is set that $1/(q^4; q^4)_m = 0$ for any negative integer $m$, which satisfies

$$F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k).$$ (9)
Summing the both sides of (9) over \( n = 0, 1, \ldots, m - 1 \) and, afterwards, over \( k = 1, \ldots, m - 1 \) we get
\[
\sum_{n=0}^{m-1} F(n, 0) = \sum_{k=1}^{m} G(m, k),
\]
equivalently,
\[
\sum_{n=0}^{m-1} q^{n^2}(6n + 1)[(q; q^2)_n^2(q^2; q^4)_m/(q^4; q^4)_n^3] = \sum_{k=1}^{m} \frac{q^{(m-k)^2}(q^2; q^4)_m(q^2; q^4)_{m-k}(q^2; q^4)_{m+k-1}}{(1 - q)(q^4; q^4)_m(q^4; q^4)_{m-k}}
\]
(changing the summation index from \( k \) to \( m - k \))
\[
= \frac{(q^2; q^4)_m}{(1 - q)(q^4; q^4)_m} \sum_{k=0}^{m-1} q^{k^2}(q^2; q^4)_k(q^2; q^4)_{2m-k-1}
\]
Now letting \( m \to \infty \) and using
\[
\sum_{k=0}^{\infty} q^{k^2} (q^2; q^4)_k/(q^4; q^4)_k = (q^2; q^4)_\infty/(q; q^2)_\infty,
\]
due to Slater, we arrive at (2).

If we denote \( \tilde{F}(n, k) = F(n, -k) \) and \( \tilde{G}(n, k) = G(n, -k) \) then relation (9) becomes a ‘standard’ form of the WZ relation
\[
\tilde{F}(n, k + 1) - \tilde{F}(n, k) = \tilde{G}(n + 1, k) - \tilde{G}(n, k).
\]
(10)

It was pointed out by Guillera in [2] that we can iterate a WZ pair \((\tilde{F}_1, \tilde{G}_1)\) satisfying (10) by taking
\[
\tilde{F}_2(n, k) = \tilde{F}_1(n, n + k) + \tilde{G}_1(n + 1, n + k) \quad \text{and} \quad \tilde{G}_2(n, k) = \tilde{G}_1(n, n + k).
\]
Indeed,
\[
\tilde{F}_2(n, k + 1) - \tilde{F}_2(n, k) = (\tilde{F}_1(n, n + k + 1) + \tilde{G}_1(n + 1, n + k + 1))
\]
\[
- (\tilde{F}_1(n, n + k) + \tilde{G}_1(n + 1, n + k))
\]
\[
= \tilde{G}_1(n + 1, n + k) - \tilde{G}_1(n, n + k)
\]
\[
+ \tilde{G}_1(n + 1, n + k + 1) - \tilde{G}_1(n + 1, n + k)
\]
\[
= \tilde{G}_2(n + 1, k) - \tilde{G}_2(n, k).
\]
He also observes in [2] that the sums
\[
\sum_{n=0}^{\infty} \tilde{F}_1(n, k) = C
\]
do not depend on \( k = 0, 1, 2, \ldots \) (and, in most of the cases, even on \( k > -1 \) real) thanks to Carlson’s theorem — this allows one to compute \( C \) by choosing an
appropriate real value for \( k \) and that
\[
\sum_{n=0}^{\infty} \tilde{F}_2(n, k) = C
\]
as well, the same right-hand side.

**Proof of (4) and (5).** The WZ pair \((F, G)\) satisfying (3) transforms into
\[
\tilde{F}_1(n, k) = \frac{(-1)^k q^{n^2+2nk-k^2} [6n + 2k + 1](q^2; q^4)_n(q; q^2)^{n+k}(q; q^2)_{n-k}(q^2; q^4)_k}{(q^4; q^4)_n^2(q^4; q^4)_{n+k}},
\]
\[
\tilde{G}_1(n, k) = \frac{(-1)^k q^{n^2+2nk-k^2} (q^2; q^4)_n(q; q^2)^{n+k}(q; q^2)_{n-k-1}(q^2; q^4)_k}{(1-q)(q^4; q^4)_n^{2n}(q^4; q^4)_{n+k}},
\]
whose subsequent iterations are
\[
\tilde{F}_2(n, k) = \frac{(-1)^n q^{2n^2}(q^2; q^4)_n(q; q^2)^{2n+k}(q; q^2)_{2n+k}(q^2; q^4)_k}{(q^4; q^4)_n^2(q^4; q^4)_{2n+k}} \times \left( [8n + 2k + 1] + \frac{q^{4n+2k+1}(1-q^{4n+2})(1-q^{4n+2k+1})}{(1-q)(1-q^{8n+4k+4})} \right),
\]
\[
\tilde{G}_2(n, k) = \frac{(-1)^n q^{2n^2+2k+1}(q^2; q^4)_n(q; q^2)^{2n+k}(q; q^2)_{2n+k}(q^2; q^4)_k}{(1-q)(q^4; q^4)_n^{2n+1}(q^4; q^4)_{2n+k}(q; q^2)_{k+1}},
\]
and
\[
\tilde{F}_3(n, k) = \frac{(-1)^n q^{2n^2}(q^2; q^4)_n(q; q^2)^{3n+k}(q; q^2)_{3n+k}(q^2; q^4)_k}{(q^4; q^4)_n^2(q^4; q^4)_{3n+k}} \times \left( [10n + 2k + 1] + \frac{q^{6n+2k+1}(1-q^{6n+2})(1-q^{6n+2k+1})}{(1-q)(1-q^{12n+4k+4})} \right.
\]
\[
+ \frac{q^{6n+2k+3}(1-q^{6n+2})(1-q^{6n+2k+3})(1-q^{6n+2k+4})}{(1-q)(1-q^{12n+4k+8})(1-q^{12n+4k+8})},
\]
\[
\tilde{G}_3(n, k) = \frac{(-1)^n q^{2n^2+2n+k+1}(q^2; q^4)_n(q; q^2)^{2n+k}(q; q^2)_{2n+k}(q^2; q^4)_k}{(1-q)(q^4; q^4)_n^{2n+2}(q^4; q^4)_{2n+k}(q; q^2)_{n+k+1}},
\]
It follows (2) that in this case
\[
C = \sum_{n=0}^{\infty} \tilde{F}_1(n, k) = \sum_{n=0}^{\infty} \tilde{F}_1(n, 0) = \frac{(1+q)(q^2; q^4)_\infty(q^6; q^4)_\infty}{(q^4; q^4)_\infty^2},
\]
hence also
\[
\sum_{n=0}^{\infty} \tilde{F}_2(n, 0) = C \quad \text{and} \quad \sum_{n=0}^{\infty} \tilde{F}_3(n, 0) = C,
\]
which are precisely formulae (4) and (5).
Proof of (3). The main part in proving (3) is a WZ pair, whose ‘standard’ form is
\[
\tilde{F}_1(n, k) = \tilde{F}_1(n, k; q) = (-1)^{n+k} \frac{[6n + 2k + 1](q; q^2)_{n-k}(q; q^2)_{n+k}}{(q^4; q^4)^2_n(q^4; q^4)_{n+k}},
\]
\[
\tilde{G}_1(n, k) = \tilde{G}_1(n, k; q) = \frac{(-1)^{n+k}(q; q^2)_{n-k-1}(q; q^2)_{n+k}}{(1 - q)(q^4; q^4)_n(q^4; q^4)_{n+k}},
\]
with further iteration
\[
\tilde{F}_2(n, k; q) = \frac{q^{k^2}(q; q^2)^2_{2n+k}}{(q^4; q^4)_n(q^4; q^4)_{2n+k}(q; q^2)_k}[8n + 2k + 1] - \frac{[4n + 2k + 1]^2}{[8n + 4k + 4]},
\]
\[
\tilde{G}_2(n, k; q) = -\frac{q^{(k+1)^2}(q; q^2)^2_{2n+k}}{(1 - q)(q^4; q^4)_{n-1}(q^4; q^4)_{2n+k}(q; q^2)_{k+1}}.
\]
The identity (3) reads
\[
C = \sum_{n=0}^{\infty} \tilde{F}_1(n, 0; q^{-1}) = \frac{(q^3; q^4)_{\infty}(q^5; q^4)_{\infty}}{(q^4; q^4)^2_{\infty}}
\]
and leads to
\[
\sum_{n=0}^{\infty} \tilde{F}_2(n, 0; q^{-1}) = C,
\]
which is the desired relation (6). □

3. A QUADRATIC TRANSFORMATION AND A CUBIC TRANSFORMATION

This is the shortest section of this note.

New proof of (2) and (3). The formula [6 eq. (4.6)] reads
\[
\sum_{n=0}^{\infty} \frac{(a; q)_n(1 - aq^{3n})(d; q)_n(q/d; q)_n(b; q^2)_n}{(q^2; q^2)_n(1 - a)(aq^2/d; q^2)_n(adq^2; q^2)_n(aq/b; q)_n} a^n q^{(n^2 + 1)/2} b^n
\]
\[
= \frac{(aq; q^2)_{\infty}(aq^2; q^2)_{\infty}(adq/b; q^2)_{\infty}(aq^2/bd; q^2)_{\infty}}{(aq/b; q^2)_{\infty}(aq^2/b; q^2)_{\infty}(aq^2/d; q^2)_{\infty}(adq; q^2)_{\infty}}.
\]
(11)
Letting \( q \to q^2 \) and taking \( a = d = q \) and \( b = q^2 \), we are led to (2). Similarly, letting \( q \to q^2 \) and \( b \to \infty \), and taking \( a = d = q \), we conclude with (3). □

Proof of (7). Letting \( d \to 0 \) in the cubic transformation [11 eq. (3.8.18)] we arrive at the formula
\[
\sum_{n=0}^{\infty} \frac{(1 - acq^{4n})(a; q)_n(q/a; q)_n(ac; q)_{2n}}{(1 - ac)(cq^3; q^3)_n(aq^2c^2; q^3)_n(q; q)_{2n}} q^{n^2}
\]
\[
= \frac{(aq^2; q^3)_{\infty}(aq^3; q^3)_{\infty}(aq; q^3)_{\infty}(q^2/a; q^3)_{\infty}}{(q; q^3)_{\infty}(q^2; q^3)_{\infty}(aq^2c^2; q^3)_{\infty}(cq^3; q^3)_{\infty}}.
\]
Now replace \( q \) with \( q^2 \), and take \( a = q \) and \( c = 1 \). The resulting identity is (7). □
4. \(q\)-Supercongruences

In this section we briefly highlight some links of our results with the supercongruences of ‘Ramanujan type’. The general pattern for them in [8] predicts, for example, that the congruence counterpart of Ramanujan’s formula (8) is

\[ \sum_{k=0}^{p-1} \frac{\binom{1}{k}(\frac{3}{4})_k^{3} (\frac{3}{4})_k^{2}}{k!^3 9^k} (8k + 1) \equiv p \left( \frac{-3}{p} \right) \quad (\text{mod } p^3) \quad \text{for } p > 3 \text{ prime}, \]

where the Jacobi–Kronecker symbol \((-3/p)\) ‘replaces’ the square root of 3. For this particular entry one also observes experimentally that

\[ \sum_{k=0}^{(p-1)/2} \frac{\binom{1}{k}(\frac{3}{4})_k^{3} (\frac{3}{4})_k^{2}}{k!^3 9^k} (8k + 1) \equiv p \left( \frac{-3}{p} \right) \quad (\text{mod } p^3) \quad \text{for } p > 3 \text{ prime}, \]

when the sum on the left-hand side is shorter.

The \(q\)-WZ pairs used in the proofs of Section 2 were originally designed to verify \(q\)-analogues of the supercongruences corresponding to the identities in (1). As suggested by [8], one can turn the argument into the opposite direction, to provide \(q\)-analogues of Ramanujan-type supercongruences from Ramanujan-type identities. Our experimental observations include the \(q\)-analogues

\[ \sum_{k=0}^{(n-1)/2} \frac{(q^2;q^2)_k^2 (q^2; q^2)_k (q^4; q^4)_k^2}{(q^2; q^2)_k^2 q^{2k}} [8k + 1] q^{2k^2} \equiv q^{-(n-1)/2} [n] \left( \frac{-3}{n} \right) \left( \text{mod } n \Phi_n(q)^2 \right), \]

\[ \sum_{k=0}^{n-1} \frac{(q^2;q^2)_k^2 (q^2; q^2)_k (q^4; q^4)_k^2}{(q^2; q^2)_k^2 q^{2k}} [8k + 1] q^{2k^2} \equiv q^{-(n-1)/2} [n] \left( \frac{-3}{n} \right) \left( \text{mod } n \Phi_n(q)^2 \right) \]

for positive \(n\) coprime with 6, of the two congruences above, where \(\Phi_n(q)\) denotes the \(n\)-th cyclotomic polynomial.

Similar congruences, in a weaker form, seem to occur for related identities not of Ramanujan type. For example, replacing \(q\) with \(q^2\) in (11), then choosing \(a = q\) and \(d = -q\), and finally taking \(b = q^2\) or \(b \to \infty\), respectively, we obtain

\[ \sum_{n=0}^{\infty} \frac{(q^2; q^4)_n^2 (-q; q^2)_n^2}{(q^2; q^4)_n^2 (-q^4; q^4)_n^2} [6n + 1] q^{n^2} = \frac{(-q^2; q^4)_\infty^2}{(1 - q)(-q^4; q^4)_\infty^2}, \]

\[ \sum_{n=0}^{\infty} (-1)^n \frac{(q^2; q^2)_n^2 (-q; q^2)_n^2}{(q^2; q^4)_n^2 (-q^4; q^4)_n^2} [6n + 1] q^{3n^2} = \frac{(q^3; q^\infty^2)(q^5; q^4)_\infty^2}{(-q^4; q^4)_\infty^2}. \]

Numerical experiment suggests the following congruences for the truncated sums:

\[ \sum_{k=0}^{(n-1)/2} \frac{(q^2; q^4)_k^2 (-q; q^2)_k^2}{(q^4; q^4)_k^2 (-q^4; q^4)_k^2} [6k + 1] q^{k^2} \equiv 0 \quad (\text{mod } [n]), \]

\[ \sum_{k=0}^{n-1} \frac{(q^2; q^4)_k^2 (-q; q^2)_k^2}{(q^4; q^4)_k^2 (-q^4; q^4)_k^2} [6k + 1] q^{k^2} \equiv 0 \quad (\text{mod } [n]), \]
\[
\sum_{k=0}^{(n-1)/2} (-1)^k \frac{(q; q^2)_k(-q^2 q^2)_k}{(q^4; q^4)_k(-q^4 q^4)_k} [6k + 1] q^{3k^2} \equiv 0 \pmod{[n]},
\]

\[
\sum_{k=0}^{n-1} (-1)^k \frac{(q; q^2)_k(-q^2 q^2)_k}{(q^4; q^4)_k(-q^4 q^4)_k} [6k + 1] q^{3k^2} \equiv 0 \pmod{[n]}
\]

for all positive odd integers \(n\). We plan to discuss these and other instances of such ‘\(q\)-supercongruences’ in a forthcoming project.

References


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