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POLYNOMIAL AUTOMORPHISMS AND INVARIANTS

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Introduction

The tame generators problem asks if the automorphism group of a polynomial ring in n variables over a field k can be generated by triangular and linear automorphisms. For $n = 2$ the answer is yes; this is the well-known Jung-van der Kulk theorem (see [10], [11] and [8]). At the writing of this paper the case $n = 3$ seems to be solved in the negative by Shestakov and Umirbaev. In their preprints (under review) [16] and [17] they give an algorithm for recognizing tame automorphisms in dimension three. As a consequence they obtain that the famous Nagata automorphism σ of $k[x, y, z]$ ([15]) given by

$$\begin{aligned} \sigma(x) &= x - 2(xz + y^2)y - (xz + y^2)^2 z \\ \sigma(y) &= y + (xz + y^2)z \\ \sigma(z) &= z \end{aligned}$$

is not tame.

The first question which then comes to mind is: what could possibly be a better set of candidate generators for the automorphism group of k^3 or more generally of k^n? The aim of this paper is to discuss and study such a set of automorphisms. To motivate the idea we look again at the Nagata automorphism σ. The polynomial $\Delta := xz + y^2$ has the property that $\sigma(\Delta) = \Delta$ i.e. it is σ-invariant. Therefore σ is a $\mathbb{C}[\Delta]$-homomorphism of $\mathbb{C}[x, y, z]$, which viewed over $\mathbb{C}[\Delta]$ is linear in x, y and z. Namely,

$$\sigma(x, y, z) = (x - 2\Delta y - \Delta^2 z, y + \Delta z, z).$$

We call such a map, which is "linear" over its ring of invariants quasi-linear. In a similar way we introduce quasi-triangular maps (definition 1.6). Together with the linear automorphisms they generate the so-called quasi-tame automorphisms.

In this paper we begin to study these automorphisms. To motivate the reader we show in section two that many existing examples of exotic automorphisms constructed by various authors belong to this class. In section three we start the more systematic approach: we study quasi-triangular automorphisms of polynomial rings whose coefficients belong to a commutative $\mathbb{Q}$-algebra. It is shown that these automorphisms are exponential (theorem 3.1). Using this result we get a better understanding of the Nilpotency subgroup (which plays a central role in the study of the Jacobian Conjecture, [7], [9]). It is shown in theorem 4.2 that each such an automorphism is stably quasi-tame, but not tame in general (example 4.1).

1 Invariants and polynomial maps

Throughout this section A denotes a commutative ring and $A[x] := A[x_1, \ldots, x_n]$ the polynomial ring in n variables over A. By a polynomial map over A we mean an n-tuple $(F_1, \ldots, F_n)$ with $F_i \in A[x]$ for each i. There is a one-to-one correspondence between polynomial maps over A and A-endomorphisms of $A[x]$ given by

$$F := (F_1, \ldots, F_n) \rightsquigarrow \phi_F : g \to g(F_1, \ldots, F_n), \text{ for all } g \in A[x].$$

If $G = (G_1, \ldots, G_n)$ is also a polynomial map over A, then the composition of G and F is given by

$$G \circ F := (G_1(F_1, \ldots, F_n), \ldots, G_n(F_1, \ldots, F_n)).$$

In other words $\phi_{G \circ F} = \phi_F \circ \phi_G$.
Below we will consider subrings R of $A[x]$ containing A which may be strictly larger than A. The elements of $A[x]$ can then also be expressed as polynomials in $x_1, \ldots, x_n$ with coefficients in R, although these expressions need not be unique in general. Nevertheless we call such an n-tuple of polynomial expressions in $x_1, \ldots, x_n$ with coefficients in R a polynomial map over R! Given two such polynomial maps G and F over R we define their composition $G \circ F$ as polynomial maps over R as follows: the i-th component is by definition the polynomial obtained by replacing in the polynomial expression of G_i as a polynomial in $x_1, \ldots, x_n$ with coefficients in R only the elements x_i by F_i. Observe that this depends on the way the components of F and G are expressed as polynomials in $x_1, \ldots, x_n$ with coefficients in R.

Example 1.1. Consider the polynomial map $F \in A[x, y, z]^3$ given by

$$F = (F_1, F_2, F_3) = (x - 2(xz + y^2)y - (xz + y^2)^2 z, y + (xz + y^2)z, z).$$

Let R be the subring of $A[x, y, z]$ given by $R := A[\Delta]$, where $\Delta := xz + y^2$. Then we can write F as a linear polynomial map with coefficients in R as follows

$$F = (x - 2\Delta y - \Delta^2 z, y + \Delta z, z).$$

So its components are linear expressions in x, y, z with coefficients in R. Now consider $G := (x + 2\Delta y - \Delta^2 z, y - \Delta z, z) \in A[x, y, z]^3$. This is polynomial map over A which can also be viewed as a polynomial map over R. By the definition given above the composition $G \circ F$ as polynomial maps over R is then equal to

$$(F_1 + 2\Delta F_2 - \Delta^2 F_3, F_2 - \Delta F_3, F_3) =$$

$$= (x - 2\Delta y - \Delta^2 z + 2\Delta(y + \Delta z) - \Delta^2 z, (y + \Delta z) - \Delta z, z) = (x, y, z).$$

(this is not surprising since G was chosen to be the R-linear inverse of the triangular R-linear map F).

Now the interesting point of the above example is that the composition $G \circ F$ as polynomial maps over A is the same as the composition $G \circ F$ considered as polynomial maps over R (and hence F is invertible over A!). To see this one first verifies that the element Δ is an invariant of the A-endomorphism ϕ_F i.e. $\phi_F(\Delta) = \Delta$ or equivalently $\Delta(F_1, F_2, F_3) = \Delta$. Consequently $G \circ F$ (=the composition as polynomial maps over A) is equal to

$$(F_1 + 2\Delta(F_1, F_2, F_3)F_2 - \Delta(F_1, F_2, F_3)^2 F_3, F_2 - \Delta(F_1, F_2, F_3)F_3, F_3) =$$

$$= (F_1 + 2\Delta F_2 - \Delta^2 F_3, F_2 - \Delta F_3, F_3) =$$

$$= \text{the composition } G \circ F \text{ as polynomial maps over } R!$$

Since we saw in Example 1.1 that this last composition equals (x, y, z) it follows that F is invertible over A with inverse G. This example clearly shows that it is interesting to study invariants of polynomial maps.

Definition 1.2. Let ϕ be an A-endomorphism of $A[x]$. An element $f \in A[x]$ is called an **invariant of** ϕ or a ϕ**-invariant** if $\phi(f) = f$. The set of all ϕ-invariants is an A-subalgebra of $A[x]$, denoted by $A[x]^\phi$.

Following the arguments given above we obtain the following easy, but useful result.

Proposition 1.3. *Let F and G be polynomial maps over A and let ϕ be the A-endomorphism that corresponds to F. Let R be an A-subalgebra over $A[x]^\phi$. View F and G as polynomial maps over R. Then the composition $G \circ F$ as polynomial maps over A equals the composition $G \circ F$ as polynomial maps over R. In particular if this last composition equals $(x_1, \ldots, x_n)$ i.e. the polynomial map F has a left inverse, considered as a polynomial map over R, then F is invertible considered as polynomial map over A (with left inverse G).*

Corollary 1.4. *With the notations as in Proposition 1.3. The composition of F and G as polynomial maps over R does not depend on the way the components of F and G are written as polynomials in $x_1, \ldots, x_n$ with coefficients in R.*

Now let $\phi \in End_A A[x]$ and F the corresponding polynomial map over A. Put $R := A[x]^\phi$. View F as a polynomial map over R. We say that F is **quasi-invertible** over R if there exists a polynomial map G over R such that the composition $G \circ F$ of polynomial maps over R equals $(x_1, \ldots, x_n)$. It follows from 1.4 that this definition does not depend on the way the components of F are written as polynomials over R. Consequently the second part of 1.3 implies

Corollary 1.5. *F is invertible over A iff F is quasi-invertible over R.*

The usefulness of 1.5 comes from the fact that in various cases one can verify easily that a given map ϕ is quasi-invertible over R. Some of the most practical cases are described in the following definition.

Definition 1.6. Let ϕ be an A-endomorphism of $A[x]$, $R := A[x]^{\phi}$. Then ϕ is called **quasi-linear** (resp. **quasi-affine**, resp. **quasi-triangular**, resp. a **quasi-translation**) if $\phi(x_i) \in Rx_1 + \ldots + Rx_n$ for all i (resp. $\phi(x_i) \in Rx_1 + \ldots + Rx_n + R$ for all i, resp. $\phi(x_i) - x_i \in R[x_{i+1}, \ldots, x_n]$ for all i, resp. $\phi(x_i) - x_i \in R$ for all i).

Corollary 1.7. *If ϕ is quasi-triangular or a quasi-translation then ϕ is invertible over A.*

In the next section we discuss various examples of the automorphisms described in 1.6. It turns out that all these examples are finite compositions of quasi-triangular maps

2 Examples

In this section we discuss all kinds of exotic polynomial automorphisms which were found in the literature by several authors. Most of these examples were constructed to solve open problems or conjectures. The point is that all these exotic examples are most probably not tame but can be written as finite compositions of quasi-triangular maps!

1.**The Drensky-Gupta automorphisms**

This class of examples was introduced by Drensky and Gupta in [2]. They studied automorphisms of the generic trace algebra T associated with two generic 2×2 matrices. By restricting such automorphisms to the center of T, which is a polynomial map in 5 variables, they obtained the following class of automorphisms of $k[x_1, x_2, x_3, y_1, y_2]$, where k is a field. In fact we can replace k by any commutative ring A.

Consider the polynomial ring $A[x, y] := A[x_1, x_2, x_3, y_1, y_2]$. In it define the polynomials $\Delta_1 := x_1x_2 - x_3^2$, $\Delta_2 := y_2^2x_1 + y_1^2x_2 - 2y_1y_2x_3$. For any $f \in A[t_1, t_2, t_3, t_4]$ define $\tilde{f} := f(y_1, y_2, \Delta_1, \Delta_2)$ and let ϕ be the A-endomorphism of $A[x, y]$ given by

$$\begin{aligned}
\phi(x_1) &= x_1 + 2y_1(y_2x_1 - y_1x_3)\tilde{f} + y_1^2\Delta_2\tilde{f}^2 \\
\phi(x_2) &= x_2 + 2y_2(y_2x_3 - y_1x_2)\tilde{f} + y_2^2\Delta_2\tilde{f}^2 \\
\phi(x_3) &= x_3 + (y_2^2x_1 - y_1^2x_2)\tilde{f} + y_1y_2\tilde{f}^2 \\
\phi(y_1) &= y_1 \\
\phi(y_2) &= y_2
\end{aligned}$$

One easily verifies that $A[y_1, y_2, \Delta_1, \Delta_2] \subset R := A[x, y]^\phi$. So $\tilde{f} \in R$ which implies that ϕ is quasi-affine. To see that ϕ is an A-automorphism of $A[x, y]$ is suffices by 1.5 to show that the quasi-linear map sending x_1, x_2, x_3 to

$$((1 + 2y_1y_2\tilde{f})x_1 - 2y_1\tilde{f}x_3, (1 - 2y_1y_2\tilde{f})x_2 + 2y_2^2\tilde{f}x_3, y_2^2\tilde{f}x_1 - y_1^2\tilde{f}x_2 + x_3)$$

is invertible over R. This in turn follows from

$$\begin{vmatrix} 1 + 2y_1y_2\tilde{f} & 0 & -2y_1^2\tilde{f} \\ 0 & 1 - 2y_1y_2\tilde{f} & 2y_2^2\tilde{f} \\ y_2^2\tilde{f} & -y_1^2\tilde{f} & 1 \end{vmatrix} = 1.$$

2. Automorphisms associated with the class $H(n, A)$

In [5] the first author and Hubbers introduced a large class of polynomial automorphisms. More precisely they introduced a class, denoted by $H(n, A)$, consisting of polynomial maps $H = (H_1, \ldots, H_n) \in A[x]^n$ with the property that their Jacobian matrix JH is nilpotent. It was shown that the corresponding polynomial maps $F = x + H$ are all invertible over A. In the case $n = 2$ these automorphisms are all of the form

$$(x_1 + a_2f(a_1x_1 + a_2x_2) + c_1, x_2 - a_1f(a_1x_1 + a_2x_2) + c_2),$$

for some a_i, c_i in A and $f(t) \in A[t]$. Automorphisms of this type were used in [1] to give counterexamples in all dimensions ≥ 3 to the Markus-Yamabe Conjecture (see [14]) and the LaSalle Problem (see [13]). Also other conjectures were shown to be false by using these automorphisms ([6]). It was shown that in case A is a $\mathbb{Q}$-algebra the automorphisms of the form $x + H$ with $H \in H(n, A)$ are all stably tame by showing that they can be written as finite compositions of $\exp D$'s where each D is a so-called nice A-derivation on $A[x]$ (see [8], chapter 7). Being nice implies in particular that $D^2x_i = 0$ for each i ([8], Proposition 7.3.14). From this we deduce

Proposition 2.1. *Let $H \in H(n, A)$. Put $F := x + H$. Then F is a finite composition of quasi-translations.*

Proof.
As observed above, F is a finite composition of $\exp D$'s where each D satisfies $D^2x_i = 0$ for all i. Then the desired result follows from the following lemma. $\diamond$

Lemma 2.2. *Let D be any locally nilpotent A-derivation on $A[x]$ such that $D^2x_i = 0$ for all i. Then* $\exp D$ *is a quasi-translation.*

Proof.
Put $\phi := \exp D$. It is not difficult to show that $R := A[x]^{\phi} = \ker D$ (see 3.6 below). Since $D(Dx_i) = D^2x_i = 0$ it follows that $Dx_i \in R$ for each i. Furthermore, since $D^2x_i = 0$ for each i we have

$$(\exp\ D(x_1), \ldots, \exp\ D(x_n)) = (x_1 + D(x_1), \ldots, x_n + D(x_n)).$$

This, together with $Dx_i \in R$ for each i implies that ϕ is a quasi-translation. $\diamond$

3. **Edo-Vénéreau variables**

In [4] an interesting class of new variables was introduced by Edo and Vénéreau. For simplicity we restrict ourselves here to just a very special case of their construction (A more thorough investigation of the Edo-Vénéreau variables will be carried out in a subsequent paper).

Let A be a commutative ring and $g(y,z) \in A[y,z]$. Define $f_2 := z^2x + y + zy^2g(y,z) \in A[x,y,z]$. We will show that f_2 is a component of a quasi-triangular A-automorphism of $A[x,y,z]$ (hence f_2 is a variable over A). To see this we first write $f_2 = y + z\Delta$ where $\Delta := zx + y^2g(y,z)$. Put $h(y) := y^2g(y,0)$. Then $\Delta \equiv h(y) \pmod{z}$ in $A[x,y,z]$. Consequently, since $f_2 \equiv y \pmod{z}$ we get

$$\Delta - f_2^2 g(f_2, z) \equiv \Delta - y^2g(y,0) \equiv \Delta - h(y) \equiv 0 \pmod{z}.$$

Hence
$$f_1 := \frac{1}{z}(\Delta - f_2^2 g(f_2,z)) \in A[x,y,z].$$

Proposition 2.3. *$F = (f_1, f_2, z)$ is a quasi-triangular automorphism of $A[x,y,z]$.*

Proof.
(i) Obviously z is an invariant of F. Now we show that also Δ is an invariant of F, namely

$$\Delta(f_1,f_2,z) = zf_1 + f_2^2g(f_2,z) = z\cdot\frac{1}{z}(\Delta - f_2^2g(f_2,z)) + f_2^2g(f_2,z) = \Delta.$$

(ii) To see that F is quasi-triangular it suffices to show that $f_1 - x \in A[x,\Delta][y]$ (since $f_2 - y = z\Delta \in A[z,\Delta]$). Therefore write $g(y,z) = g(y,0) + z\tilde{g}(y,z)$. So $\Delta = zx + h(y) + zy^2\tilde{g}(y,z)$. Hence

$$f_1 = \frac{1}{z}\left(zx + zy^2\tilde{g}(y,z) + h(y) - (y+z\Delta)^2[g(y+z\Delta,0) + z\tilde{g}(y+z\Delta,z)]\right) =$$

$$= x + y^2\tilde{g}(y,z) + \frac{1}{z}(h(y) - h(y+z\Delta)) - \tilde{g}(y+z\Delta,z).$$

So indeed $f_1 - x \in A[z,\Delta][y]$.

(iii) Since F is quasi-triangular it follows from 1.7 that F is invertible over A. $\diamond$

Remark 2.4. In [12] the automorphism f_t given by

$$f_t(u,v) = (u - 3v^2(ut+v^3) - 3vt(ut+v^3)^2 - t^2(ut+v^3)^3, v + t(ut+v^3))$$

was used to construct a counter-example to a question posed by Drensky and Yu in [3], concerning the linear orthogonal group of $\mathbb{C}^3$. The automorphism f_t is just a special case of the construction above: namely take $g = y$ i.e. $f_2 = z^2x + y + zy^3$ and rename the variables $x \to u$, $y \to v$, $z \to t$. Then $f_t = (f_1(u,v,t), f_2(u,v,t))$.

3 Quasi-triangular automorphisms

All the examples discussed in the previous section were shown to be finite compositions of quasi-triangular maps. Therefore we study these maps in more detail in this section. Throughout this section A will denote a commutative $\mathbb{Q}$-algebra. The main result asserts that every quasi-triangular A-automorphism of $A[x]$ is a so-called exponential map. More precisely

Theorem 3.1. *Let $\phi \in End_A\, A[x]$ be quasi-triangular. Then $\phi = \exp D$ for some locally nilpotent A-derivation D on $A[x]$.*

Before we prove this theorem, we recall some results of [8], chapter 2. Let B be any commutative A-algebra and $f \in End_A\, B$. To such a map we associate the map $E := f - 1_B$. This is a so-called **f-derivation** on B i.e. it satisfies

$$E(ab) = E(a)f(b) + aE(b) \text{ for all } a, b \in B.$$

Using induction on m one easily deduces that

Lemma 3.2.

$$E^m(ab) = \sum_{k=0}^{m} \binom{m}{k} E^k(a) f^k(E^{m-k}(b)) \text{ for all } a, b \in B.$$

The map E is called **locally nilpotent** if for each $b \in B$ there exists $m \in \mathbb{Z}^+$ such that $E^m(b) = 0$. Suppose now that B is a finitely generated A-algebra, say generated by $b_1, \ldots, b_q$. Then it follows from 3.2 that E is locally nilpotent iff for each b_i there exists m_i such that $E^{m_i}(b_i) = 0$. The following result ([8], Proposition 2.1.3) will be crucial in the proof of 3.1.

Proposition 3.3. *Let $f \in End_A\, B$ and $E := f - 1_B$. Then f is an exponential automorphism iff E is locally nilpotent. Furthermore, if E is locally nilpotent then* *the map $D: B \to B$ defined by*

$$D(b) = \sum_{i \geq 1} (-1)^{i+1} \frac{E^i(b)}{i}, \text{ for all } b \in B$$

is a locally nilpotent derivation on B and $f = \exp D$.

A proof of theorem 3.1

(i) We apply 3.3 to $B := A[x] = A[x_1, \ldots, x_n]$. So it suffices to show that E is locally nilpotent on $A[x]$, where $E := \phi - 1_{A[x]}$.

(ii) First, if $a \in R := A[x]^\phi$ then $E(a) = \phi(a) - a = a - a = 0$.

(iii) Furthermore $E(x_n) = \phi(x_n) - x_n \in R$ (since ϕ is quasi-triangular), so by (ii) $E^2(x_n) = E(E(x_n)) = 0$. Consequently E is locally nilpotent on the finitely generated R-algebra $R[x_n]$.

(iv) Now $E(x_{n-1}) = \phi(x_{n-1}) - x_{n-1} \in R[x_n]$ (ϕ is quasi-triangular). Since by (iii) E is locally nilpotent on $R[x_n]$ it follows that for some m $E^m(E(x_{n-1})) = 0$ i.e. $E^{m+1}(x_{n-1}) = 0$. Consequently E is locally nilpotent on $R[x_n, x_{n-1}]$.

(v) Continuing in this way we obtain that E is locally nilpotent on $R[x_n, \ldots, x_1] = A[x]$, which completes the proof. $\diamond$

As a consequence of this theorem we characterize all the quasi-triangular maps in one variable, a result which will be used in the next section.

Proposition 3.4. *Let $A[y]$ be a univariate polynomial ring over A and $\phi \in End_A\, A[y]$. There is an equivalence between*
i) ϕ is quasi-triangular.
ii) ϕ is a quasi-translation.
iii) $\phi = \exp D$ for some A-derivation D on $A[y]$ with $D^2(y) = 0$.

To prove this result we need

Lemma 3.5. *Let B be any $\mathbb{Q}$-algebra and D a derivation on B. If $b \in B$ and $m \geq 1$ satisfy $D^{m+1}b = 0$ and $\lambda_1 Db + \lambda_2 D^2 b + \ldots + \lambda_m D^m b = 0$ for some $\lambda_i \in \mathbb{Q}^\times$, then $Db = 0$.*

Proof.
By induction on m. If $m = 1$ we are done, so let $m \geq 2$. Applying D^{m-1} to the equation gives $\lambda_1 D^m b = 0$, so $D^m b = 0$. Consequently $\lambda_1 Db + \ldots + \lambda_{m-1} D^{m-1} b = 0$. Then $Db = 0$ follows from the induction hypothesis. $\diamond$

Corollary 3.6. *Let B be any $\mathbb{Q}$-algebra, D a locally nilpotent derivation on B and $\phi = \exp D$. Then $B^\phi = \ker D$.*

A proof of Proposition 3.4

The implications $iii) \to ii) \to i)$ are obvious. It remains to prove $i) \to iii)$. So assume that ϕ is quasi-triangular. Then by 3.1 $\phi = \exp D$ for some locally nilpotent A-derivation D on $A[y]$. Since D is locally nilpotent there exists $m \geq 1$ such that $D^{m+1}y = 0$. Hence

$$\exp D(y) = y + D(y) + \frac{1}{2!}D^2(y) + \ldots + \frac{1}{m!}D^m(y) := y + a(y).$$

Since ϕ is quasi-triangular $a(y) \in A[y]^\phi = \ker D$, by 3.6. So $Da(y) = 0$, hence

$$D(D(y)) + \frac{1}{2!}D^2(D(y)) + \ldots + \frac{1}{m!}D^m(D(y)) = 0.$$

Consequently $D(D(y)) = 0$ by 3.5, which completes the proof. $\diamond$

4 The nilpotency subgroup

As before A denotes a commutative $\mathbb{Q}$-algebra and $A[x] := A[x_1, \ldots, x_n]$. The nilpotency subgroup $N(A,n)$ of $Aut_A\, A[x]$ (see [8], chapter 2) consists of all F of the form

$$F = (x_1 + g_1, \ldots, x_n + g_n),$$

where each g_i is a nilpotent element of $A[x]$ or equivalently belongs to $\eta A[x]$, where η denotes the nil radical of A. It was shown in [8], 2.1.3 that each element of $N(A,n)$ is an exponential automorphism. In light of 3.1 it is therefore natural to ask if each such an element is quasi-triangular or (weaker) a finite composition of linear automorphisms and quasi-triangular automorphisms. We will show that the answer to both questions is negative in general. On the other hand we prove (Theorem 4.2) that each $F \in N(A,n)$ is stably a finite composition of linear and quasi-triangular ones! Let's start with

Example 4.1. Let $A = \mathbb{C}[t]/(t^3)$ and put $\epsilon := \bar{t}$. So $A = \mathbb{C}[\epsilon]$. Let $f := y + \epsilon y^2$. Then $f \in N(A,1)$. We show that f is not a finite composition of linear and quasi-triangular automorphisms: therefore observe that by 3.4 $g \in A[y]$ is quasi-triangular iff $g = \exp D$ for some $D = a(y)\partial_y$ satisfying $D^2 y = 0$ i.e. $a(y)a'(y) = 0$. It is an easy computation to verify that this last equation implies that $a(y) \equiv \alpha_0 + \alpha_1\epsilon \,(\bmod\ \epsilon^2 A[y])$ for some $\alpha_0, \alpha_1 \in \mathbb{C}$. So if g is quasi-triangular then $g = y + a(y) \equiv y + \alpha_0 + \alpha_1\epsilon \,(\bmod\ \epsilon^2)$. In particular if we write $\overline{g}$ for the residue class $\bmod\ \epsilon^2$ we obtain that $\deg_y \overline{g} = 1$. Consequently if $y + \epsilon y^2$ is a finite composition of linear and quasi-triangular maps then, working modulo ϵ^2, we obtain that $y + \overline{\epsilon} y^2$ has y-degree 1, a contradiction since $\overline{\epsilon} \neq 0$ ($\epsilon^2 \neq 0$!).

Although the above example shows that elements of $N(A,n)$ need not be finite compositions of linear and quasi-triangular maps we have

Theorem 4.2. *Let $F \in N(A,n)$. Then F is stably quasi-tame i.e. there exist new variables $t_1,\ldots,t_m$ such that $(F,t_1,\ldots,t_m)$ is a finite composition of linear and quasi-triangular automorphisms of $Aut_A\, A[x,t_1,\ldots,t_m]$.*

To prove this result we first study the case $n=1$. Therefore we consider the univariate polynomial ring $A[y]$ and introduce one new variable t. The crucial result is

Lemma 4.3. *Let $d \geq 2$, $d \in \mathbb{Z}^+$ and $c \in \eta$, the nilradical of A. Then there exists an A-derivation D of $A[y,t]$ of the form $a(y,t)\partial_y + \partial_t$ such that $D^2 y = 0$ and*

$$a(y,t) = cy^d + \sum_{i=1}^{m} a_{d+i} y^{d+i},$$

for some $m \geq 1$ with $a_j \in \eta^2 A[t]$ for all j. In particular

$$\exp\, D = (y + cy^d + a_{d+1}y^{d+1} + \ldots + a_{d+m}y^{d+m}, t+1)$$

is quasi-triangular.

Proof.
We have to solve $a \in A[y,t]$ from $D^2y = 0$ i.e. from $aa_y + a_t = 0$ satisfying the additional condition that $a = cy^d + \ldots$. So if we write $a = y^d b$ then b has to satisfy the equation

$$y^d bb_y + dy^{d-1}b^2 + b_t = 0, \qquad (*)$$

with the additional assumption that $b(y=0) = c$.
Claim: there exist $q_i \in \mathbb{Q}$ with $q_0 = 1$ such that $b = c\sum_{i=0}^{\infty} q_i (tcy^{d-1})^i$ satisfies (*).
Observe that if this claim is proved we are done, for the sum is finite since c is nilpotent and for each $i \geq 1$ the coefficient of y^{d+i} in $a = y^d b$ clearly belongs to $\eta^2 A[t]$ (since $c^2 \in \eta^2$). To see the claim first put $w := tcy^{d-1}$. Then it is an easy computation to show that it suffices to find $q_i \in \mathbb{Q}$, $q_0 = 1$ such that

$$(d-1)(\sum_{i=0}^{\infty} q_i w^i)(\sum_{i=0}^{\infty} iq_i w^i) + d(\sum_{i=0}^{\infty} q_i w^i)^2 + \sum_{i=0}^{\infty} iq_i w^{i-1} = 0.$$

Then looking at the coefficient of w^n we get the recursive equation

$$(d-1)\sum_{i+j=n, 0\leq i,j\leq n} q_i j q_j + d \sum_{i+j=n, 0\leq i,j\leq n} q_i q_j + (n+1)q_{n+1} = 0,$$

which shows that one can indeed find q_i as desired. $\diamond$

Corollary 4.4. *If $f \in N(A,1)$ then there exist $t_1, \ldots, t_m$ such that $(f, t_1, \ldots, t_m)$ is a finite composition of linear automorphisms and quasi-translations.*

Proof.
Since f has only a finite number of non-zero coefficients we may assume that $\eta^e = 0$ for some $e \geq 1$. Furthermore we may assume that $f = y + cy^d + \ldots$ with $d \geq 2$ and $c \in \eta$. Then introduce a new variable t and apply 4.3 with $-c$ instead of c. This gives a quasi-translation $g = \exp D$ of the form

$$g = (y - cy^d + a_{d+1}y^{d+1} + \ldots + a_{d+m}y^{d+m}, t+1),$$

for some $m \geq 1$ and with $a_{d+i} \in \eta^2$. Then

$$(f,t) \circ g = (y + b_{d+2}y^{d+2} + \ldots + b_{d+N}y^{d+N}, t+1),$$

for some $N \geq 1$ and $b_j \in \eta^2 A[t]$ for all j. Hence

$$(f,t) \circ g \circ (y, t-1) = (y + \tilde{b}_{d+2}y^{d+2} + \ldots + \tilde{b}_{d+N}y^{d+N}, t) = (\tilde{f}, t),$$

with $\tilde{b}_j \in \eta^2 A[t]$ for all j. This new map is better in the sense that all the elements $\tilde{b}_j$ belong to $\eta^2 A[t]$. So again we apply 4.3 but now to the ring $A[t]$ and we introduce again a new variable t_2 i.e. we consider $\tilde{f} \in A[t][y]$ and compose $(\tilde{f}, t_2)$ with a new quasi-translation etc... .
The map we finally get after composing $(f, t, t_2, \ldots, t_{e-1})$ with the quasi-translations obtained in $e-1$ such steps has the property that all the coefficients of the non-linear y terms belong to $\eta^e A[t, t_2, \ldots, t_{e-1}]$, i.e. the final map is just $(y, t, t_2, \ldots, t_{e-1})$. In other words taking the inverses of all maps appearing in the composition, we get that $(f, t, t_2, \ldots, t_{e-1})$ is a finite composition of linear automorphisms and quasi-translations. $\diamond$

A proof of theorem 4.2

Replacing F by $F - F(0)$ and then by $JF(0)^{-1}F$ we may assume that $F(0) = 0$ and that the linear part of F equals x. We use induction on n. The case $n = 1$ follows from 4.4. So let $n \geq 2$. Put $f := F_n$ and view it in $B[x_n]$ where $B = A[x_1, \ldots, x_{n-1}]$. So $f \in N(B,1)$. Then by 4.4 there exist variables $t_1, \ldots, t_s$ and $q_1, \ldots, q_m$ which are either linear or quasi-triangular B-automorphisms of $B[x_n, t_1, \ldots, t_s]$ such that $(f, t_1, \ldots, t_s) = q_1 \circ \ldots \circ q_m$. So if we write x' instead of $(x_1, \ldots, x_{n-1})$ we get that each $Q_i := (x', q_i)$ is either a linear or a quasi-triangular automorphism of $A[x,t] := A[x, t_1, \ldots, t_s]$ (and hence so is $Q_i^{-1} = (x', q_i^{-1})$). Furthermore

$$(F,t) \circ Q_m^{-1} \circ \ldots \circ Q_1^{-1} = (f_1, \ldots, f_{n-1}, x_n, t),$$

where $(f_1, \ldots, f_{n-1}) \in N(A[x_n, t], n-1)$. Then the result follows from the induction hypothesis applied to the ring $A[x_n, t]$. $\diamond$

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