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# NOTE ON THE PROPORTIONS OF FINANCIAL ASSETS WITH DEPENDENT DISTRIBUTIONS IN OPTIMAL PORTFOLIOS

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# Note on the proportions of financial assets with dependent distributions in optimal portfolios[1]

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**Abstract**

In this paper we shall study the proportions of the financial assets in optimal portfolios, where the portfolio is optimized by the maximization of its expected utility with respect to a given utility function. Our main goal is to investigate the magnitude of the proportions of the assets in optimal portfolios provided that the assets' distributions display a certain type of stochastic dominance, which means that one asset is better in a certain sense than the other. Our main question is to understand why people buy more of an asset than of another one. For this, we introduce and study new notions of stochastic dominance which can be appropriate candidates for the study of the proportions. Based on these notions we derive new versions (and a generalization) of the results of Hadar and Seo [1], where, in case of independent asset returns, the stochastic dominance of the returns implies that the dominant asset has a larger proportion in the optimal portfolio. Our results apply to not necessarily independent returns as well. We give several realistic examples in which these new types of stochastic dominance are fulfilled.

**Keywords.** Optimal portfolio, utility function, risk aversion, stochastic dominance, multidimensional distributions for the rate of returns of financial assets.

## 1 Introduction and notations

A widely studied area of econometrics is the problem of finding optimal portfolios under uncertainty. The basic setup is the following: we are given a market with financial assets and a certain capital to be invested. Now, many approaches are known on how to invest our capital. In this paper we study the expected utility

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approach, i.e. the possible portfolios are ordered according to the expected utility of their future value, and thus an optimal portfolio ('best allocation' of our money) can be taken.

The different notions of stochastic dominance provide tools to compare the future value and the riskiness of financial assets. Now, one can formulate the main questions of the paper generally as follows: "What conditions lead the investors to invest more in an asset than in another one?" and "How are the proportions of the financial assets in the optimal portfolios related to the riskiness of the assets?"

To put the questions in a more precise way, we first summarize some notations and fundamental results in literature.

*Securities market.* Let us consider a securities market where the individuals are trying to invest their money and thus to create their portfolio by allocating their money among the different financial assets available in the market. We shall suppose that the number of assets is finite, say $n$. Now the market is modeled by a set

$$\{r_1, r_2, \ldots, r_n\},$$

where $r_i$ is a random variable, with property $\mathbb{P}(-1 \leq r_i) = 1$ $(i = 1, \ldots, n)$, representing the rate of return of stock $i$ at some future time point $T$. We assume that the individual does not intend to reallocate the portfolio before $T$, hence the $r_i$'s contain all the information available on the market at the time of the investment decision. Put $X_i = 1 + r_i$ for $i = 1, \ldots, n$.

*Portfolios.* A portfolio will be denoted by

$$\pi = (\beta_1, \beta_2, \ldots, \beta_n), \quad (\beta_i \in \mathbb{R}),$$

where $\beta_i$ is the amount of money invested in asset $i$. Let $X_0 > 0$ be the initial capital to be invested by the individual. Denote the value of portfolio $\pi$ at time $T$ by $X_T^\pi$. Thus,

$$X_T^\pi = \sum_{i=1}^{n} \beta_i (1 + r_i) = \sum_{i=1}^{n} \beta_i X_i,$$

where $\sum_{i=1}^{n} \beta_i = X_0$.

Now, the individuals are supposed to perform rationally in the market and thus to choose the optimal portfolio according to their preferences. The individual's preferences shall be given by his or her utility function $U$. Thus, we shall call a portfolio optimal if it is promising the largest possibly expected utility. Here we mention that it is just one of the several definitions for optimality of the portfolio known in literature. (For this and other approaches see e.g. [2], [3] or [5].)

In other words, we shall call a portfolio optimal and denote it by

$$\pi_U^* = (\beta_{1,U}^*, \beta_{2,U}^*, \ldots, \beta_{n,U}^*) \tag{1}$$

if

$$\mathbb{E}\, U\left(X_T^{\pi_U^*}\right) = \sup_{\pi \in C_{X_0}} \mathbb{E}\, U\left(X_T^\pi\right), \tag{2}$$

where $C_{X_0}$ is the set of portfolios which can be set up from initial capital $X_0$, i.e.

$$C_{X_0} = \left\{ \pi \;\middle|\; \pi = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n, \; \sum_{i=1}^{n} \beta_i = X_0 \right\}.$$

Note that no selling or buying restrictions are imposed ($\pi \in \mathbb{R}^n$). If it does not cause any misunderstanding, we will omit to indicate the utility function $U$ in the solution (1).

We shall suppose in the paper that $X_0 = 1$. In fact, this assumption does not lead to any loss of generality. Indeed, given $X_0$ and a utility function $U$, one can reformulate the problem of maximizing $\mathbb{E}U(X_T^\pi)$ as follows. Define $\bar{U}(x) = U(X_0 x)$ ($x \in \mathbb{R}$) and then maximize $\mathbb{E}\bar{U}(X_T^\pi)$ provided that the initial capital is 1. If we have a solution $(\beta^*_{1,\bar{U}}, \ldots, \beta^*_{n,\bar{U}})$ of this problem then $(X_0\beta^*_{1,\bar{U}}, \ldots, X_0\beta^*_{n,\bar{U}})$ is a solution of the original problem.

Hence, given the assumption $X_0 = 1$ we can consider the value $\beta_i$ as the proportion of asset $i$ in the portfolio.

*Stochastic dominance versus proportions.* In this paper our main focus is on investigating the features of securities which lead the individuals to invest more in an asset than in another one.

For this we discuss the relations between stochastic dominance and the proportions of the asset. The notion of stochastic dominance between the returns of the financial assets shall be used to express that an asset is better or less risky —in a certain sense— than another one. Now, it would be natural to claim risk averse investors to invest more of the less risky asset. However, easy counter-examples can be given (see e.g. [1], [2]). Therefore, our main purpose is to derive conditions —especially for dependently distributed asset price returns— under which more money will be invested in the less risky asset, indeed.

Take first the example where in a two-securities market $\{r_1, r_2\}$, the distribution of the rate of return of the first asset displays first order stochastic dominance over the second one. It is known that under this condition an individual —even if he is risk averse, e.g. he has concave utility function (see [2])— shall not necessarily invest more in asset one than in asset two (see [1]). Hadar and Seo have shown (Theorem 4, [1]) that for risk averse individuals with nondecreasing utility function $U$ the following two statements are equivalent:

**(1)** $\beta_1^* \geq \beta_2^*$ for each independent $r_1$ and $r_2$ with $r_1 \succcurlyeq_{FSD} r_2$,

**(2)** the function $x \mapsto xU'(x)$ is nondecreasing over its domain,

where $r_1 \succcurlyeq_{FSD} r_2$ denotes that $r_1$ displays first order stochastic dominance over $r_2$.

There remains, however, the question of the dependent case in the problem considered by Hadar and Seo which is the subject of our next results. For this, first we introduce a new notion of stochastic dominance which is one way to represent the dependence of the rates of return of the assets.

We shall also consider the problem at issue for other types of stochastic dominance and for the case of more than two assets.

## 2 A strong version of the first order stochastic dominance

Next, we introduce a new notion of stochastic dominance.

Given a random variable $\xi$, the measure $\mathbb{P}_\xi$ will denote its distribution.

**Definition 3** *The random variable $\xi$ is said to display strong first order stochastic dominance (SFSD) over the random variable $\eta$, which is denoted by $\xi \succcurlyeq_{SFSD} \eta$, if $F_{\xi|\eta}(x|y) \leq F_\eta(x)$ for all $x \in \mathbb{R}$ and $\mathbb{P}_\eta$-a.e. $y \in \mathbb{R}$, where $F_\eta$ is the distribution function of $\eta$ and $F_{\xi|\eta}$ is a regular*[2] *conditional distribution function of $\xi$ given $\eta$.*

*Secondly, strong second order stochastic dominance (SSSD) of $\xi$ over $\eta$ is defined by $\int_{-\infty}^{x} \left[ F_{\xi|\eta}(u|y) - F_\eta(u)\right] du \leq 0$ for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Notation: $\xi \succcurlyeq_{SSSD} \eta$.*

*Given random variables $\xi_1, \xi_2, \ldots, \xi_n$, $n > 2$, we say that $\xi_1$ conditionally dominates $\xi_2$ in the sense of strong first order stochastic dominance, and we write $\xi_1|\xi_3, \ldots, \xi_n \succcurlyeq_{SFSD} \xi_2|\xi_3, \ldots, \xi_n$, if $F_{\xi_1|\xi_2,\ldots,\xi_n}(x|x_2, \ldots, x_n) \leq F_{\xi_2|\xi_3,\ldots,\xi_n}(x|x_3, \ldots, x_n)$ for all $x \in \mathbb{R}$ and $\mathbb{P}_{\xi_2,\ldots,\xi_n}$-a.e. $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, where $F_{\xi_1|\xi_2,\ldots,\xi_n}$ and $F_{\xi_2|\xi_3,\ldots,\xi_n}$ are regular conditional distribution functions.*

Now, we summarize some easy properties of SFSD.

**Theorem 1 (Some features of SFSD)**

**(i)** *For any r.v.'s $\xi$ and $\eta$, $\xi \succcurlyeq_{SFSD} \eta$ implies $\xi \succcurlyeq_{FSD} \eta$.*

**(ii)** *If $\xi$ and $\eta$ are independent r.v.'s then $\xi \succcurlyeq_{SFSD} \eta$ is equivalent with $\xi \succcurlyeq_{FSD} \eta$.*

**(iii)** *For any random variables $\xi$ and $\eta$, $\xi \succcurlyeq_{SFSD} \eta$ holds if and only if we have $\mathbb{E}\left(g(\xi)|\eta\right) \geq \mathbb{E}g(\eta)$ a.s. for all nondecreasing function $g : \mathbb{R} \to \mathbb{R}$.*

**Proof.**

**(i)** Let $\mathbb{P}_\eta$ be the distribution of $\eta$. Now, notice that for any $x \in \mathbb{R}$

$$\int_{\mathbb{R}} F_{\xi|\eta}(x|y)\ \mathbb{P}_\eta(dy) = \int_{\mathbb{R}} \mathbb{P}\left(\xi < x|\eta = y\right)\ \mathbb{P}_\eta(dy) = \mathbb{P}(\xi < x) = F_\xi(x). \tag{4}$$

[2] See e.g. [7] for the definition of regularity.

Hence, $F_{\xi|\eta}(x|y) \leq F_\eta(x)$, $x \in \mathbb{R}$, $\mathbb{P}_\eta$-a.s. $y \in \mathbb{R}$, together with (4) implies $F_\xi(x) \leq F_\eta(x)$, which is equivalent with $\xi \succcurlyeq_{FSD} \eta$ (see 1.A.1. in [6]).

**(ii)** By the independence of the r.v.'s. we have $F_{\xi|\eta}(x|y) = F_\xi(x)$ for $x \in \mathbb{R}$ and $\mathbb{P}_\eta$-a.s. $y \in \mathbb{R}$.

**(iii)** Fix $y \in \mathbb{R}$ and take a r.v. $\zeta$ with cdf $F_\zeta(x) = F_{\xi|\eta}(x|y)$, $x \in \mathbb{R}$. The SFSD property implies $\zeta \succcurlyeq_{FSD} \eta$. Secondly, recall that $\zeta \succcurlyeq_{FSD} \eta$ if and only if $\mathbb{E}g(\zeta) \geq \mathbb{E}g(\eta)$ for all non-decreasing function $g : \mathbb{R} \to \mathbb{R}$ (see 1.A.1. in [6]). $\square$

Next, we collect some basic facts on the SSSD property.

**Theorem 2 (Some features of SSSD)**

- **(i)** *For any r.v.'s $\xi$ and $\eta$, $\xi \succcurlyeq_{SSSD} \eta$ implies $\xi \succcurlyeq_{SSD} \eta$.*
- **(ii)** *If $\xi$ and $\eta$ are independent r.v.'s then $\xi \succcurlyeq_{SSSD} \eta$ is equivalent with $\xi \succcurlyeq_{SSD} \eta$.*
- **(iii)** *For any r.v.'s $\xi$ and $\eta$, $\xi \succcurlyeq_{SSSD} \eta$ holds if and only if we have $\mathbb{E}\left(g(\xi)|\eta\right) \geq \mathbb{E}g(\eta)$ a.s. for all nondecreasing, concave function $g : \mathbb{R} \to \mathbb{R}$.*

**Proof.** The proof of this theorem is fairly analogous to the proof of Theorem 1. The description of SSD (see Chapter 5 in [3]) gives further hint for proving statement (iii).

# 3 Generalization of the theorem of Hadar and Seo

**Theorem 3** *Let $U : \mathbb{R} \mapsto \mathbb{R}$ be differentiable, concave and nondecreasing. Then the following statements are equivalent:*

- **(i)** *$\beta_1^* \geq \beta_2^*$ in any two-securities market $\{r_1, r_2\}$ with $r_1 \succcurlyeq_{SFSD} r_2$,*
- **(ii)** *the function $x \mapsto xU'(x)$ is nondecreasing.*

**Proof.** **(1) $\Longrightarrow$ (2)** Take markets $\{r_1, r_2\}$ with independent returns $r_1$, $r_2$. Now statement (ii) in Theorem 1 together with a theorem of Hadar and Seo (Theorem 4. in [1]) directly implies (2).

**(2) $\Longleftarrow$ (1)** As in the proof of Theorem 4 in of [1], we can show $\beta_1^* \geq \beta_2^*$ by proving that

$$\left.\frac{\partial \mathbb{E}\, X_T^\pi}{\partial \beta_1}\right|_{\beta_1 = 1/2} = \mathbb{E}\,(X_1 - X_2)U'\left(\frac{X_1 + X_2}{2}\right) \geq 0 \tag{5}$$

(which is equivalent with $\beta_1^* \geq \beta_2^*$). To check this first write

$$\iint (x-y)U'\left(\frac{x+y}{2}\right)\, \mathbb{P}_{X_1,X_2}(dx,dy) =$$

$$\iint xU'\left(\frac{x+y}{2}\right)\, \mathbb{P}_{X_1|X_2}(dx|y)\, \mathbb{P}_{X_2}(dy) - \int x \int U'\left(\frac{x+y}{2}\right)\, \mathbb{P}_{X_1|X_2}(dy|x)\, \mathbb{P}_{X_2}(dx). \tag{6}$$

Now, note that both the function $x \mapsto -U'(x)$ and the function $x \mapsto xU'\left(\frac{x+y}{2}\right)$ for $y \geq 0$ are nondecreasing and hence (iii) of Theorem 1 can be applied to both terms of the second line in (6) to get for

$$\int xU'\left(\frac{x+y}{2}\right)\, \mathbb{P}_{X_1|X_2}(dx|y) \geq \int xU'\left(\frac{x+y}{2}\right)\, \mathbb{P}_{X_2}(dx) \tag{7}$$

for $\mathbb{P}_{X_2}$-a.e. $y \in \mathbb{R}$ and

$$\int U'\left(\frac{x+y}{2}\right)\, \mathbb{P}_{X_1|X_2}(dy|x) \leq \int U'\left(\frac{x+y}{2}\right)\, \mathbb{P}_{X_2}(dy) \tag{8}$$

for $\mathbb{P}_{X_2}$-a.e. $x \in \mathbb{R}$. Finally, (5) follows directly from the combination of (6) with (7) and (8). $\square$

Theorem 3 can be literally rewritten for the case of second order stochastic dominance and thus we get Theorem 4. One can easily generalize the statement of Theorem 3 for $n$-securities market as well (Theorem 5).

**Theorem 4** *Let $U : \mathbb{R} \mapsto \mathbb{R}$ be differentiable, concave and nondecreasing. Then the following statements are equivalent:*

- **(i)** $\beta_1^* \geq \beta_2^*$ *in any two-securities market* $\{r_1, r_2\}$ *with* $r_1 \succcurlyeq_{SSSD} r_2$,
- **(ii)** *the function* $x \mapsto xU'(x)$ *is concave and nondecreasing.*

Since the proof of this theorem is analogous to the proof of Theorem 3, we omit its proof here.

**Theorem 5** *Let $U : \mathbb{R} \mapsto \mathbb{R}$ be differentiable, concave and nondecreasing. Then the following statements are equivalent:*

- **(i)** $\beta_1^* \geq \beta_2^*$ *in any $n$-securities market* $\{r_1, \ldots, r_n\}$ *with* $r_1|r_3, \ldots, r_n \succcurlyeq_{SFSD} r_2|r_3, \ldots, r_n$,
- **(ii)** *the function* $x \mapsto xU'(x)$ *is nondecreasing.*

**Proof.** First we mention that instead of (5) this time it is sufficient to show that for all $(\beta_3, \ldots, \beta_n) \in \mathbb{R}^{n-2}$ we have

$$0 \leq \mathbb{E}(X_1 - X_2)U'\left(\beta X_1 + \beta X_2 + \sum_{i=3}^{n} \beta_i X_i\right),$$

where $\beta = 1 - \sum_{i=3}^{n} \beta_i/2$. One can write

$$\mathbb{E}(X_1 - X_2)U'(\beta X_1 + \beta X_2 + \sum_{i=3}^{n} \beta_i X_i) =$$

$$\int \cdots \int \Bigg[ \int\int (x_1 - x_2)U'(\beta x_1 + \beta x_2 + \sum_{i=3}^{n} \beta_i x_i)$$

$$\mathbb{P}_{X_1|X_2,\ldots,X_n}(dx_3, \ldots, dx_n)\mathbb{P}_{X_2|X_3,\ldots,X_n}(dx_3, \ldots, dx_n)\Bigg] \mathbb{P}_{X_3,\ldots,X_n}(dx_3, \ldots, dx_n).$$

The remaining part of the proof can be carried out in a similar way as in the proof of Theorem 3, thus we omit the details here. □

We can see that having proved Theorem 3, it was easy to find its 'multi-security' version, namely, Theorem 5. One could, of course, define the conditional strong second order stochastic dominance in the way the first order one was defined and then the multi-security case of Theorem 4 could also be written immediately, but we shall not consider this case in this paper.

In the Introduction we assumed that there are no trading constraints and hence the proportions in the portfolios can take negative values as well. Hence, one needs utility functions defined on the whole real line in such case since $X_T^\pi$ can be negative. One can, of course, handle also the case where the utility function is defined only on $[0, \infty)$ or $(0, \infty)$. In the first case, we have to assume furthermore that there are trading constraints, namely, let $\beta_i \geq 0$ $(i = 1, \ldots, n)$. The later case can also be handled by assuming the same trading constraints and, furthermore, the positivity of the asset values: $\mathbb{P}(X_i > 0) = 1$ $(i = 1, \ldots, n)$. Thus Theorem 3 can be literally rewritten for $U$ defined on $[0, \infty)$ or $(0, \infty)$.

In the theorems of Hadar and Seo and also in our theorems the monotonicity of $xU'(x)$ turned out to be a crucial property. Hence, it should be mentioned that commonly used utility functions have such a property. Here are some of them: logarithmic utility ($U(x) = \log(x)$, $x > 0$); exponential ($U(x) = c\exp(dx)$ for $x \geq 0$ with $c, d < 0$); power type (also known as Cobb-Douglas utility function) ($U(x) = x^\alpha$, $x > 0$, $0 < \alpha \leq 1$).

# 4 Examples for the SFSD property

In this section we give several examples where the SFSD property is fulfilled and examine some commonly used families of distributions as well. We try to study distributions which seem to be realistic to play the role of the rates of returns ($r_i$'s) or of the future market prices ($X_i = (r_i + 1)$'s). For this, we note that the SFSD (or SSSD) property is preserved if $r_1$ and $r_2$ are shifted by a constant $c$, i.e. when $X_i$ is replaced by $X_i + c$ for all $i$, where we claim $X_i + c \geq 0$ for all $i$. Thus, in the following examples one can choose an appropriate value for the constant $c$ to make the market more realistic.

Just to show that SFSD can easily occur, first we give a fairly simple example.

**Example 1** Consider a two-securities market $\{r_1, r_2\}$ where both of the market prices concentrate on two atoms, 0 and 1, as follows. Take an $\varepsilon \in [-\frac{1}{12}, \frac{1}{12}]$ and put

$$p_{1,1} = \frac{2}{6} + \varepsilon, \quad p_{1,0} = \frac{2}{6} - \varepsilon, \quad p_{0,1} = \frac{1}{6} - \varepsilon, \quad p_{0,0} = \frac{1}{6} + \varepsilon,$$

where $p_{i,j} = \mathbb{P}(X_1 = i,\ X_2 = j)$.

Clearly, independence of the two rates of returns occur if and only if $\varepsilon = 0$. Note, moreover, that $X_1 \succcurlyeq_{FSD} X_2$ since we have $\mathbb{P}(X_1 = 0) = \frac{1}{3} < \mathbb{P}(X_2 = 0) = \frac{1}{2}$.

Furthermore, we can easily check that $X_1 \succcurlyeq_{SFSD} X_2$. Indeed, we have

$$F_{X_1|X_2}(x|y) = \frac{1/6 + \varepsilon}{1/2} \leq F_{x_2}(x) = \frac{1}{2},$$

for $0 \leq y < 1$ and $0 < x \leq 1$, and

$$F_{X_1|X_2}(x|y) = \frac{1/6 - \varepsilon}{1/2} \leq F_{x_2}(x) = \frac{1}{2}$$

for $1 \leq y$ and $0 < x \leq 1$. The remaining cases are trivial.

The following example shows that one can find the SFSD property among the absolutely continuous distributions as well.

**Example 2** Let $\varepsilon$ be a constant in $[-1/2, (\sqrt{3}-1)/2]$ and take a two-securities market $\{r_1, r_2\}$, where the joint density function of $X_1$ and $X_2$ is

$$f(x,y) = \begin{cases} x + 2\varepsilon y + \frac{1}{2} - \varepsilon, & \text{if} \quad (x,y) \in [0,1] \times [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then the cdf of $X_1$ is $F_{X_1}(x) = x(x+1)/2$ for $x \in [0,1]$, whereas the cdf of $X_2$ is $F_{X_2}(x) = \varepsilon x^2 + (1-\varepsilon)x$ over $[0,1]$ and hence $X_1$ displays first order stochastic dominance over $X_2$ due to $F_{X_1}(x) \leq F_{X_2}(x)$ for $x \in \mathbb{R}$. Note that the case $\varepsilon = 0$ is equivalent again with the independence of the two returns.

Turning to the verification of the SFSD property notice that a conditional distribution function of $X_1$ given $X_2$ is

$$F_{X_1|X_2}(x|y) = \begin{cases} \frac{\frac{1}{2}x(x+1) + \varepsilon(2xy - x)}{1 + \varepsilon(2y-1)}, & \text{if} \quad (x,y) \in [0,1] \times [0,1] \\ 1, & \text{if} \quad x > 1,\ y \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Thus, it remains to check the following inequality

$$F_{X_1|X_2}(x|y) \leq F_{X_2}(x), \qquad \forall x, y \in [0,1]. \tag{10}$$

It's easy to see that (10) is equivalent with

$$\varepsilon^2(2y-1) + \varepsilon - \frac{1}{2} \leq 0 \qquad \text{for} \quad y \in [0,1],$$

which is fulfilled due to the choice of $\varepsilon$.

# 5 Preferred stocks

In the previous section we have shown examples for the SFSD property introduced in Definition 3. It should be mentioned that the way we gave a modification for the first order stochastic dominance in Definition 3 in order to keep the statement of the theorem of Hadar and Seo is not necessarily the only possibility. Although it was easy to construct examples to fulfill the definition, many of the commonly used classical two-dimensional distributions cannot satisfy the required property or lead to the independent case. For instance, taking a two-dimensional exponential distribution defined by the survival function

$$\mathbb{P}(X_1 > x, X_2 > y) = \bar{F}(x,y) = \begin{cases} \exp(-\lambda_1 x - \lambda_2 y - \lambda_{1,2}\max(x,y)) & \text{for} \quad x, y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

($\lambda_1, \lambda_2 > 0$, $\lambda_{1,2} \geq 0$), one can check that we have $X_1 \succcurlyeq_{SFSD} X_2$ if and only if $\lambda_{1,2} = 0$ and $\lambda_1 \geq \lambda_2$. However, these two conditions are met if and only if the coordinates are independent and $X_2$ is dominated by $X_1$ in the sense of first order stochastic dominance. Thus, it is a case for which the Definition 3 is not fruitful.

Therefore, still having in mind the purpose to understand the reasons that could lead someone to prefer one asset to another one, we go on seeking for another possible notion of stochastic dominance.

First, we introduce a new notion and then we show that it is also appropriate for our purposes and finally we give examples of two-dimensional distributions with the new property. We show that in these examples the new type of stochastic dominance is fulfilled, although the strong first order dominance is not satisfied in most of the cases, that is, further two-dimensional distributions can be studied in our portfolio problems by the aid of the new notion.

**Definition 12** *Given two nonnegative random variables, $X$ and $Y$, we say that $X$ is preferred to $Y$ if*

$$\mathbb{E}(X-Y)f(X+Y) \geq 0 \tag{13}$$

*for all decreasing differentiable function $f : [0,\infty) \to [0,\infty)$. If $X$ is preferred to $Y$ then we write $X \succcurlyeq_{PR} Y$.*

In Definition 12 the condition taken on the derivative of the function $f$ seems to be unnecessary, since in (13) no term involves the derivative. Therefore one could claim less in the definition. However, as we shall see, for our purposes the way we chose seems to be the most appropriate one.

Note that $X \succcurlyeq_{PR} Y$ implies $X + c \succcurlyeq_{PR} Y + c$ for $c \in \mathbb{R}$ provided that $X + c$ and $Y + c$ remain nonnegative.

**Theorem 6** *Let $X$ and $Y$ be two random variables such that*

$$X := \frac{V+W}{2} \qquad \text{and} \qquad Y := \frac{V-W}{2},$$

*where, given $A > 0$ and $-A \leq B < C \leq A$, $V$ and $W$ are independent random variables with*

$$\mathbb{P}(V \geq A) = \mathbb{P}(W \in [B, C]) = 1.$$

*Then we have the following statements:*
**(a)** *$X$ and $Y$ are nonnegative.*
**(b)** *$X \succcurlyeq_{PR} Y \iff \mathbb{E}W \geq 0 \iff \mathbb{E}X \geq \mathbb{E}Y$.*
**(c)** *If, furthermore, $V$ is unbounded (i.e., $\mathbb{P}(V \geq x) > 0$ for all $x \in \mathbb{R}$) then $X \not\succcurlyeq_{SFSD} Y$.*

**Proof.**

**(a)** Since $\mathbb{P}((V, W) \in [A, \infty) \times [B, C]) = 1$, we have $\frac{V+W}{2} \geq \frac{A+B}{2} \geq 0$ and $\frac{V-W}{2} \geq \frac{A-C}{2} \geq 0$.

**(b)** If $f : [0, \infty) \to [0, \infty)$ then by the independence of $V$ and $W$ we obtain

$$\mathbb{E}\,(X - Y) f\left(\frac{X+Y}{2}\right) = \mathbb{E}(X - Y)\mathbb{E} f\left(\frac{X+Y}{2}\right).$$

Since $\mathbb{E} f\left(\frac{X+Y}{2}\right) \geq 0$ for nonnegative functions $f$ we have $X \succcurlyeq_{PR} Y$ if and only if $\mathbb{E}(X - Y) \geq 0$.

**(c)** Due to the construction $F_X(x) < 1$ and $F_Y(x) < 1$ for all $x \in \mathbb{R}$. However, fixing $y \in \mathbb{R}_+$, to satisfy both $x + y \in [A, \infty)$ and $x - y \in [B, C]$ the value of $x$ must lie in $[B + y, C + y]$. Hence, the conditional distribution of $X$ given $\{Y = y\}$ is concentrated in a bounded interval which means that $F_{X|Y}(x|y) = 1$ if $x$ is large enough. For such $x$ we have $F_{X|Y}(x|y) > F_Y(x)$. $\square$

**Theorem 7** *Given a two-securities market $\{r_1, r_2\}$ with*

$$1 + r_1 \succcurlyeq_{PR} 1 + r_2$$

*we have*

$$\beta^*_{1,U} \geq \beta^*_{2,U}$$

*for all nondecreasing, concave and twice differentiable utility functions, where $(\beta^*_{1,U}, \beta^*_{2,U})$ denotes the optimal portfolio with respect to the utility function $U$.*

**Proof.** Given a nondecreasing, concave and twice differentiable utility function $U$, we have

$$\mathbb{E}\,(X_1 - X_2)U'\left(\frac{X_1 + X_2}{2}\right) \geq 0$$

by the Definition 12. Hence, the statement is immediate (Hadar and Seo [1]). $\square$

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