Two-dimensional projectively-tameness over Noetherian domains of dimension one

Joost Berson

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Abstract
In this paper all coordinates in two variables over a Noetherian $\mathbb{Q}$-domain of Krull dimension one are proved to be projectively tame. In order to do this, some results concerning projectively-tameness of polynomials in general are shown. Furthermore, we deduce that all automorphisms in two variables over a Noetherian reduced ring of dimension zero are tame.

1 Introduction

One of the main open problems concerning polynomial automorphisms over a field is if each such automorphism is tame. The answer is only known for any field in dimension two: this is the classical Jung -Van der Kulk Theorem ([11], [12], [9]).

Nagata in [13], 1972, constructed a candidate counterexample in dimension three which is still not known to be tame or wild in spite of the many attempts to show this (see for example [2] and [5]). However it was shown to be stably tame by M. Smith in [14]. Also in [10] and [4] it is shown that large classes of automorphisms over commutative rings are stably tame based on the idea used in [14].

More recently, the problem of (stably) tameness has been investigated by Drensky and Yu in [6] where they considered $k[z]$-automorphisms of $k[z][x, y]$. Their paper was followed by a result of Edo -Vénéréau ([7]) in which they study a special class of Nagata-like automorphisms over a UFD, which are all shown to be stably tame. In [3] the author extended the result as follows: for any commutative ring $R$ a class $\mathcal{B}(R)$ of polynomials in $R[x, y]$ is defined (containing the polynomials studied
in [7]) and it is shown that if a polynomial of this class is a coordinate then it is projectively tame, which means that it is the image of \( x \) under a tame \( R \)-automorphism of \( R[x, y, z_1, \ldots, z_m] \), where \( z_1, \ldots, z_m \) denote \( m \) new variables for some \( m \geq 1 \).

The main result of this paper uses this result, among others, to show that in case \( R \) is a Noetherian \( \mathbb{Q} \)-domain of Krull-dimension one then every coordinate in \( R[x, y] \) is projectively tame.

2 Generalities about projectively-tameness

In this section we shall state some general facts about projectively-tameness that can be found in [3]. Furthermore, some new results will be proved and together they’ll be used in the proof of our main theorem in the next section. But let us start with the definition of the most important notion of this paper, namely that of polynomial maps resp. polynomials which are projectively tame. But before that, we have to recall some definitions of certain special groups of automorphisms. For the remainder of this section, let \( R \) be a commutative ring.

Definition 2.1. For a commutative ring \( R \) let \( R[X] := R[X_1, \ldots, X_n] \) and \( \text{Aff}(R, n) := \{ F = (F_1, \ldots, F_n) \in \text{Aut}_R R[X] \mid \deg(F_i) = 1 \ \forall i \} \), the affine subgroup of the group of all \( R \)-automorphisms. Furthermore, let \( E(R, n) \) be the subgroup generated by the elementary automorphisms, i.e. the automorphisms of the form

\[
(X_1, \ldots, X_{i-1}, X_i + a(X_1, \ldots, \hat{X}_i, \ldots, X_n), X_{i+1}, \ldots, X_n)
\]

where \( a \in R[X_1, \ldots, \hat{X}_i, \ldots, X_n] \).

Finally, the tame subgroup \( T(R, n) \) will be the subgroup generated by all affine and elementary automorphisms.

Let us now turn our attention to the 2-variable case. The following important Theorem can be found in [11] and [12] (see also [9], Theorem 5.1.11).

Theorem 2.2 (Jung, van der Kulk). If \( K \) is a field, then \( \text{Aut}_K K[x, y] = T(K, 2) \). In other words, every automorphism in two variables is tame.

With all of this in mind, it is natural to come up with the concept of projectively-tameness, which we will now describe.
Definition 2.3. Let \( n, p \in \mathbb{N}^* \). A polynomial map \( F = (F_1, \ldots, F_p) \in R[X_1, \ldots, X_n]^p \) is called **projectively tame** if there exist new variables \( Y_1, \ldots, Y_m \) and certain \( G_1, \ldots, G_{n+m-p} \in R[X_1, \ldots, X_n, Y_1, \ldots, Y_m] \) such that \( (F_1, \ldots, F_p, G_1, \ldots, G_{n+m-p}) \in T(R, n + m) \). (Note that since this implies that \( R[Y_1, \ldots, Y_m] = R[G_1_{|x=0}, \ldots, G_{n+m-p}_{|x=0}] \), it follows that \( n + m - p \geq m \), so automatically we have \( p \leq n \).)

Two polynomial maps \( F^{(i)} = (F_1^{(i)}, \ldots, F_p^{(i)}) \in R[X_1, \ldots, X_n]^p \) \((i = 1, 2)\) are called **projectively tame equivalent** if there exist new variables \( Y_1, \ldots, Y_m \), certain \( G_1^{(i)}, \ldots, G_{n+m-p_i}^{(i)} \in R[X_1, \ldots, X_n, Y_1, \ldots, Y_m] \) and a \( \varphi \in T(R, n + m) \) such that 
\[
(F_1^{(2)}, \ldots, F_{p_2}^{(2)}, G_1^{(2)}, \ldots, G_{n+m-p_2}^{(2)}) = (F_1^{(1)}, \ldots, F_{p_1}^{(1)}, G_1^{(1)}, \ldots, G_{n+m-p_1}^{(1)}) \circ \varphi
\]

A polynomial \( f \in R[X_1, \ldots, X_n] \) is called a **projectively tame coordinate** if \( F := (f) \) is a projectively tame polynomial map. Furthermore, two polynomials \( f, g \in R[X_1, \ldots, X_n] \) are called **projectively tame equivalent** if \( (f) \) and \( (g) \) are projectively tame equivalent as polynomial maps. In other words, if there exists a \( \varphi \in T(R, n + m) \) for some \( m \in \mathbb{N} \) such that \( \varphi(f) = g \).

It is immediately clear that coordinates of a tame automorphism are projectively tame. Since, by Theorem 2.2, every automorphism of \( K[X, Y] \) (where \( K \) is a field) is tame, it already follows that certainly every coordinate \( f \in K[X, Y] \) is projectively tame. But if, for example, one replaces \( K \) by an integral domain \( R \), the question whether every coordinate of \( R[X, Y] \) is projectively tame is still open. Even more surprisingly, not a single non-projectively-tame coordinate over any commutative ring has yet been found.

In two variables some progress has been made, though. In the paper [3] a certain class of polynomials defined over any commutative ring \( R \) (denoted \( \mathcal{B}(R) = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n(R) \)) was introduced which in case \( R \) is a field exactly describes all coordinates in two variables. The main theorem in that paper stated, that all coordinates in \( \mathcal{B}(R) \) are projectively tame (where \( R \) is an arbitrary commutative ring), although in that paper the term ”stably tame” was used instead of ”projectively tame”. Theorem 3.6 in that paper, which was very crucial for the proof of its main theorem, will prove to be very useful in the next section. It states the following:

**Theorem 2.4 (Berson).** Suppose \( f \in R[X] := R[X_1, \ldots, X_r] \) has the following property : there exist \( p_1, \ldots, p_n \in R \) such that \( \overline{f} \in R/(p_i)[X] \)
is a projectively tame coordinate for each $p_i$. Then $p_1 \cdots p_n Z + f$ is a projectively tame coordinate, where $Z$ is a new variable.

Since we shall be using some facts about the class $\mathcal{B}(R)$ in the next section, it may be wise to recall the definition.

**Definition 2.5.** Let $R$ be a commutative ring and take $g_0, p_1, p_2, \ldots \in R$, $p_0 \in R^*$ and $G_1(Y), G_2(Y), \ldots \in R[Y]$. Then we define the following polynomials in $R[X,Y]$:

- $F_0 = p_0 Y + g_0$
- $F_1 = p_1 X + G_1(Y)$
- $F_2 = p_2 Y + G_2(p_1 X + G_1(Y))$
- $F_n = p_n F_{n-2} + G_n(F_{n-1})$ (for all $n \geq 3$)

Now $\mathcal{B}_n(R) := \{ F_n | g_0, p_1, p_2, \ldots \in R, p_0 \in R^*, G_1(Y), G_2(Y), \ldots \in R[Y] \}$. Furthermore, let $\mathcal{B}(R) = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n(R)$.

The following property of the polynomials of the class just defined may be worth mentioning, for we shall use it later on.

**Lemma 2.6.** $\det J(F_n, F_{n-1}) = (-1)^{n+1} p_1 \cdots p_n$ for all $n \geq 2$.

**Proof.** $\det J(F_2, F_1) = \det J(p_2 Y + G_2(p_1 X + G_1(Y)), p_1 X + G_1(Y)) = \det J(p_2 Y, p_1 X + G_1(Y)) = p_2 \det J(Y, p_1 X) = -p_1 p_2$.

Now assume, that the statement is true for all $k < n$. Then we have:

$\det J(F_n, F_{n-1}) = \det J(p_n F_{n-2} + G_n(F_{n-1}), F_{n-1}) = p_n \det J(F_{n-2}, F_{n-1}) = -p_n \det J(F_{n-1}, F_{n-2}) = -p_n((-1)^n p_1 \cdots p_{n-1}) = (-1)^{n+1} p_1 \cdots p_n$. $\square$

As remarked earlier, every coordinate in two variables over a field $K$ is an element of the class $\mathcal{B}(K)$. It is still not certain whether every tame coordinate in two variables over $R$ is in my class, but at least we can prove this for all coordinates of a special class of automorphisms, the so-called *strongly tame* automorphisms.

**Definition 2.7.** Let $a_1, a_2, a_3, a_4 \in R$, then $F = (a_1 X + a_2 Y, a_3 X + a_4 Y) \in Aff(R, 2)$ is called *simply linear* if there is an $i$ such that $a_i \in R^*$. Let $SAff(R, 2)$ be the collection of all simply linear automorphisms. Then by a *strongly tame* automorphism we mean an element of the automorphism subgroup $ST(R, 2) := \langle SAff(R, 2), E(R, 2) \rangle$ of $Aut_R R[X,Y]$. 

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Lemma 2.8. Let $R$ be a commutative ring and $\sigma \in ST(R, 2)$. Then there exist $p_0 \in R^*, p_1, \ldots, p_n, g_0 \in R$ and $G_1(Y), \ldots, G_{n+1}(Y) \in R[Y]$ such that $\sigma(X) = F_{n+1}$ and $\sigma(Y) = F_n$, as in Definition 2.5. We may even assume, that $p_i \in R^*$ for all $i$.

Proof. Write $\sigma = \tau_n \lambda_n \cdots \tau_1 \lambda_1 \tau_0$, with $n \in \mathbb{N}^*$, $\tau_i$ elementary and $\lambda_i$ simply linear for all $i$. By using automorphisms which are elementary and simply linear to simplify most of the automorphisms in the composition of $\sigma$, beginning with the leftmost one, and by inserting the identity multiple times if necessary, we may assume the following:

1. for $i \geq 0$ $\tau_i$ is of the form $\tau_i = (X + h_i(Y), Y)$, where $h_i(Y) \in R[Y]$
2. for $i \geq 1$ $\lambda_i$ is of the form $\lambda_i = (a_iX + b_iY, d_iX + e_iY)$, where $a_i, b_i, e_i \in R, d_i \in R^*$

To prove the statement, we will use induction with respect to $n$:

- The case $n = 0$ : by assumption 1., $\tau_0$ can be written as $\tau_0 = (p_1X + G_1(Y), p_0Y + g_0)$ (with $p_1, p_0 \in R^*, G_1(Y) \in R[Y]$ and $g_0 \in R$)
- If $n \geq 1$, then write $\tau_{n-1} \lambda_{n-1} \cdots \tau_1 \lambda_1 \tau_0 = (F_n, F_{n-1})$

Then $\sigma = (X + h_n(Y), Y)(a_nX + b_nY, d_nX + e_nY)(F_n, F_{n-1})$, and writing $F_n = p_nF_{n-2} + G_n(F_{n-1})$ (where, in case $n \leq 2$, $F_0 := Y$ and $F_{-1} := X$) and $\alpha := a_ne_n - b_nd_n$, we get

\[
\begin{align*}
\sigma(Y) &= d_n F_n + e_n F_{n-1} = d_n p_n F_{n-2} + d_n G_n(F_{n-1}) + e_n F_{n-1} \\
\sigma(X) &= a_n F_n + b_n F_{n-1} + h_n(\sigma(Y)) \\
&= \frac{1}{d_n}(a_n d_n F_n + b_n d_n F_{n-1}) + h_n(\sigma(Y)) \\
&= \frac{1}{d_n}(a_n d_n F_n + (a_n e_n - \alpha) F_{n-1}) + h_n(\sigma(Y)) \\
&= \frac{a_n}{d_n} \sigma(Y) - \frac{\alpha}{d_n} F_{n-1} + h_n(\sigma(Y))
\end{align*}
\]

So $\sigma(Y) = \tilde{F}_n$ and $\sigma(X) = \tilde{F}_{n+1}$, where $\tilde{F}_n$ and $\tilde{F}_{n+1}$ are in our class of Definition 2.5 defined by $\tilde{p}_1, \ldots, \tilde{p}_{n+1}$ and $\tilde{G}_1(Y), \ldots, \tilde{G}_{n+1}(Y)$, with $\tilde{p}_i = p_i$ and $\tilde{G}_i(Y) = G_i(Y)$ if $i < n$, $\tilde{p}_n = d_n p_n \in R^*, \tilde{G}_n(Y) = d_n G_n(Y) + e_n Y, \tilde{p}_{n+1} = -\frac{\alpha}{d_n} \in R^*$ and $\tilde{G}_{n+1}(Y) = h_n(Y) + \frac{a_n}{d_n} Y$. 

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The reason why Theorem 2.4 turned out to be very useful in the proof of the main theorem in [3] may become clear when considering the following proposition, which can be found as Proposition 3.2 in [3]. Of course, by the ‘$F_n$’ in the statement the ‘$F_n$’ in Definition 2.5 is meant.

**Proposition 2.9 (Berson).** For every $n \geq 2$, $F_n$ is projectively tame equivalent to $p_1 \cdots p_n Z + F_n \in R[X,Y,Z]$.

**Remark 2.10.** The proof of Proposition 2.9 gives us certain tame $\varphi_{n-1} \in \text{Aut}_R R[X,Y,Z_2,\ldots,Z_{n+1}]$ for new variables $Z_2,\ldots,Z_{n+1}$ such that $\varphi_{n-1}^{-1} \varphi_n(F_n) = F_n$ and $\varphi_{n-1}^{-1} \varphi_n(F_{n-1}) = p_1 \cdots p_n Z_{n+1} + F_{n-1}$ (where $\varphi_{n-1}$ is extended by $\varphi_{n-1}(Z_{n+1}) := Z_{n+1}$). We shall use these facts in the proof of our main theorem.

In the remainder of this section we shall discuss some new results concerning projectively-tameness. Apparently, as we shall see, something can be said about the projectively-tameness of a polynomial when viewing the coefficients modulo some ideal. A good example of this is Theorem 2.14 below. But we first need a rather technical proposition.

**Proposition 2.11.** Suppose $F = (X_1+H_1(X),\ldots,X_n+H_n(X)) \in R[X]^n$ $(X := (X_1,\ldots,X_n))$ where $R$ is a commutative ring with ideal $a$ and $H_i(X) \in a[X]$ for all $i$. Write $H_i(X) = \sum_{j \in \mathbb{N}} c_j^{(i)} X_1^j \cdots X_n^j$ for certain $c_j^{(i)} \in a$. Define $J := \{j \in \mathbb{N}^n \mid \exists i \in \{1,\ldots,n\} \mid c_j^{(i)} \neq 0 \}$. Let $Y_j$ be a new variable for every $j \in J$. Then $F$ is projectively tame equivalent to a polynomial map of the form

$$(X_1 + \tilde{H}_1(X,Y),\ldots,X_n + \tilde{H}_n(X,Y), (Y_j + \tilde{H}_j(X,Y))_{(j \in J)})$$

where $\tilde{H}_1,\ldots,\tilde{H}_n \in a^2[X,Y]$ and also $\tilde{H}_j \in a^2[X,Y]$ for every $j \in J$.

**Proof.** Let $m$ be the number of new variables $Y_j$. Choose for every $j \in J$ a polynomial $P_j(X,Y) \in a[X,Y]$ (we will specify them later). Then let’s define $\widetilde{F} \in R[X]^{n+m}$ by

$$\widetilde{F} = (X_1 + H_1(X),\ldots,X_n + H_n(X), (P_j(X,Y))_{(j \in J)})$$

Now let $\overline{R} := R/a^2$. We will show, that the $P_j$ can be chosen in such a way that $\widetilde{F} \in E(\overline{R}, n + m)$ and then we are done.
Define $\varphi_1 \in E(R, n+m)$ by $\varphi_1(X_i) = X_i + \sum_j c_j^{(i)} Y_j$ for all $i$ and $\varphi_1(Y_j) = Y_j$ for all $j$. Then we have (since $\bar{a}^2 = (0)$)

$$F \circ \varphi_1 = (X_i + \sum_{j \in J} c_j^{(i)} Y_j + H_i(X))_{(i)}, \ldots)$$

(The other components will be specified later.)

Furthermore, define $\varphi_2 \in E(R, n+m)$ by $\varphi_2(X_i) = X_i$ for all $i$ and $\varphi_2(Y_j) = Y_j - X^j$ for all $j$ (where $X^j := X_1^j \cdots X_n^j$). This gives us (since $H_i(X) = \sum_j c_j^{(i)} X^j$)

$$F \circ \varphi_1 \circ \varphi_2 = (X_i + \sum_{j \in J} c_j^{(i)} Y_j)_{(i)}, \ldots)$$

Composing this with $\varphi_1^{-1}$, we get

$$F \circ \varphi_1 \circ \varphi_2 \circ \varphi_1^{-1} = (X_1, \ldots, X_n, \ldots) \quad (1)$$

Now we will take a look at the effect of the $\varphi_i$ on the $P_j$. Since $\varphi_1$ is the identity modulo $a$, we have

$$F \circ \varphi_1 \circ \varphi_2 \circ \varphi_1^{-1} \circ \varphi_2^{-1} \equiv (\ldots, (P_j(X,Y))_{(j \in J)}) \quad (\text{mod } a)$$

which implies that we can choose every $P_j$ of the form $P_j(X,Y) = Y_j + Q_j(X,Y)$ with $Q_j(X,Y) \in a[X,Y]$ and such that

$$F \circ \varphi_1 \circ \varphi_2 \circ \varphi_1^{-1} \circ \varphi_2^{-1} = (\ldots, (Y_j)_{(j \in J)}) \quad (2)$$

Equation (1) gives (since $\varphi_2(X_i) = X_i$ for all $i$)

$$F \circ \varphi_1 \circ \varphi_2 \circ \varphi_1^{-1} \circ \varphi_2^{-1} = (X_1, \ldots, X_n, \ldots)$$

So together with equation (2) we deduce

$$F \circ \varphi_1 \circ \varphi_2 \circ \varphi_1^{-1} \circ \varphi_2^{-1} = (X_1, \ldots, X_n, (Y_j)_{(j \in J)})$$

This implies that $F \varphi_1 \varphi_2 \varphi_1^{-1} \varphi_2^{-1} = (X, (Y_j)_{(j \in J)}) \in a^2[X,Y]^{m+n}$, which completes the proof.

**Theorem 2.12.** Let $R$ be a commutative ring and $\eta$ its nilradical. Every map of the form $F = (X_1 + H_1(X), \ldots, X_n + H_n(X))$, where $H_i(X) \in \eta[X]$ for all $i$, is projectively tame.

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Proof. First, suppose that the statement is true as soon as the coefficient ring is Noetherian. Now let $R$ be an arbitrary commutative ring and $F \in R[X]^n$ as in the statement of this theorem. Let $I$ be the collection of all coefficients of $F$. Then the subring $R' := \mathbb{Z}[I]$ of $R$ generated by the elements of $I$ is Noetherian and $F \in R'[X]^n$, which implies that $F$ is projectively tame over $R'$, so it is certainly projectively tame over $R$.

So we may assume $R$ to be Noetherian. But then there exists a $p \in \mathbb{N}^*$ such that $\eta^{2p} = (0)$. If we now apply Proposition 2.11 several times, by taking for $a$ consecutively $\eta, \eta^2, \ldots, \eta^{2p-1}$, we conclude that $F$ is projectively tame.

Corollary 2.13. Let $F \in R[X]^n$ be a projectively tame polynomial automorphism and let $H \in \eta[X]^n$. Then $F + H$ is also projectively tame.

Proof. Let $G \in R[X]^n$ be the inverse of $F$. Then we have the identity $F + H = (X + H(G)) \circ F$. Since $X + H(G)$ is projectively tame by Theorem 2.12 and $F$ is projectively tame by assumption, it follows easily that also $F + H$ is projectively tame.

This result has its consequence in the following theorem which tells us something about the projectively-tameness of a polynomial when viewing it modulo the nilradical.

Theorem 2.14. Let $R$ be a commutative ring and $f \in R[X] = R[X_1, \ldots, X_n]$. Suppose that $\overline{f} \in \overline{R}[X]$ (where $\overline{R} := R/\eta$) is projectively tame. Then $f$ is already projectively tame.

Proof. By assumption we can choose new variables $Y := (Y_1, \ldots, Y_m)$ and $G_1, \ldots, G_{m+n-1} \in R[X,Y]$ such that $\psi := (f, G_1, \ldots, G_{m+n-1}) \in T(\overline{R}, m+n-1)$, say $\psi = \psi_1 \circ \cdots \circ \psi_k$ where the $\psi_i$ are either affine or elementary automorphisms. If a given $\psi_i$ is elementary, then obviously there exists an elementary $\Psi_i \in Aut_{R[R,X,Y]}$ such that $\overline{\psi}_i = \psi_i$. If $\psi_i$ is affine, then there also exists an affine $\Psi_i \in Aut_{R[R,X,Y]}$ such that $\overline{\psi}_i = \psi_i$. Furthermore, since $\det J\overline{\psi}_i = \det J\psi_i \in (\overline{R})^*$, it follows that $\det J\Psi_i \in R^*$, which means that $\Psi_i$ is invertible. Now let $\Psi = (f, G_1, \ldots, G_{m+n-1})$, then we have $\Psi_1 \cdots \Psi_k - \Psi \in \eta[X,Y]^{n+m}$, so by Corollary 2.13 it follows, that $\Psi$ is a projectively tame map. This implies that $f$ is a projectively tame coordinate.
So to check if a polynomial is projectively tame one may reduce it modulo nilpotent elements. From this it already follows that all one-dimensional coordinates are projectively tame, as is shown in the following example.

Example 2.15. Let \( f \in R[X_1] \) be a coordinate in 1 variable. Take a prime ideal \( p \) and consider \( \overline{f} \in R/p[X_1] \). Because \( \overline{f} \) is a coordinate in 1 variable and \( R/p \) is an integral domain, \( \overline{f} \) must be linear.

So \( f \) is linear modulo every prime ideal of \( R \), which means that \( f \) is of the form \( f = a_0 + a_1X_1 + \cdots + a_nX_1^n \) for some \( a_i \in R \), where \( a_i \in \eta \) whenever \( i \geq 2 \). Consequently, \( f \) is a linear coordinate when viewed modulo the nilradical, so the previous theorem implies that \( f \) is projectively tame.

To conclude this section we introduce another criterion for projectively tameness which will be used in the next section. It is a consequence of Theorem 2.16 below, which proof gives in fact a method to construct a decomposition of a tame automorphism modulo the product of 2 comaximal ideals, by only using the decompositions modulo each of the two ideals.

Theorem 2.16. Let \( R \) be a commutative ring and \( a_1, a_2 \subseteq R \) two ideals such that \( a_1 + a_2 = (1) \). If \( F \in R[X]^n := R[X_1, \ldots, X_n] \) is a tame automorphism when viewed modulo \( a_1 \) as well as when viewed modulo \( a_2 \), then it is also tame when viewed modulo \( a_1a_2 \).

Even more generally, if \( f \in R[X] \) is a tame coordinate when viewed modulo \( a_1 \) as well as when viewed modulo \( a_2 \), then it is also a tame coordinate when viewed modulo \( a_1a_2 \).

Proof. Since \( f \) is a tame coordinate when viewed modulo \( a_i \) (\( i = 1, 2 \)), there exist \( g_1^{(i)}, \ldots, g_{n-1}^{(i)} \in R[X] \) with \( (\overline{f}, \overline{g_1^{(i)}}, \ldots, \overline{g_{n-1}^{(i)}}) \in T(\overline{R}, n) \) (where \( \overline{R} = R/a_i \)). According to the Chinese Remainder Theorem, there exist \( c_1 \in a_2 \) and \( c_2 \in a_1 \) such that \( c_1 \equiv 1 \pmod{a_1} \) and \( c_2 \equiv 1 \pmod{a_2} \). So \( F := (f, c_1g_1^{(1)} + c_2g_1^{(2)}, \ldots, c_1g_{n-1}^{(1)} + c_2g_{n-1}^{(2)}) \) is tame when viewed modulo \( a_1 \) as well as when viewed modulo \( a_2 \). We can write \( F = \phi_1 \cdots \phi_{m_1} \) (over \( R/a_1 \)), where the \( \phi_i \) are either linear or elementary automorphisms over \( R/a_1 \). Likewise, we can write \( F = \psi_1 \cdots \psi_{m_2} \) (over \( R/a_2 \)), where the \( \psi_i \) are either linear or elementary automorphisms over \( R/a_2 \). Furthermore, we may assume that all mentioned elementary automorphisms are of the form

\[
(X_1 + H(X_2, \ldots, X_n), X_2, \ldots, X_n)
\]
Similarly, in 

Observe that we constructed \( I \) as in equation (3) resp. linear. Let 

Choose \( \Phi_i \) with 

Consequently, 

So we may conclude, that by putting 

A similar argument can be held about \( P_{m_1+i} \) for \( i = 1, \ldots, m_2 \).

So we may conclude, that by putting \( \overline{R} := R/a_1a_2 \), we have 

Corollary 2.17. Let \( R, a_1, a_2 \) be as in Theorem 2.16. If \( f \in R[X_1, \ldots, X_n] \) is a projectively tame coordinate both modulo \( a_1 \) and modulo \( a_2 \), then it is also projectively tame modulo \( a_1a_2 \).

Proof. By assumption, there exist new variables \( Y := (Y_1, \ldots, Y_m) \) and 

where \( \overline{R} = R/a_1 \). So \( f \), as an element of \( R[X, Y] \), is a tame coordinate when viewed modulo \( a_1 \) as well as when viewed modulo \( a_2 \). By Theorem 2.16, \( f \in R[X, Y] \) is a tame coordinate when viewed modulo \( a_1a_2 \). Consequently, \( f \in R[X] \) is projectively tame modulo \( a_1a_2 \).
3 The Kdim 1 case

Now we can use the projectively-tameness tools which we obtained in the previous section to our advantage. We will prove the main theorem of this paper, which is stated below.

**Theorem 3.1.** Let \( R \) be a Noetherian \( \mathbb{Q} \)-domain with \( \dim(R) = 1 \). Then every coordinate \( F \in R[X,Y] \) is projectively tame.

A very important ingredient of the proof of this theorem is Theorem 3.3, which tells us something about the leading coefficients of coordinates. In the proof of this theorem we will use the result of the following lemma. It is an almost direct consequence of the well-known Abhyankar-Moh Theorem, which can be found in [1].

**Lemma 3.2.** Let \( R \) be a domain with characteristic \( 0 \). Suppose \((F,G) \in \text{Aut}_R R[X,Y]\). Then \( \deg_X(F) \mid \deg_X(G) \) or \( \deg_X(G) \mid \deg_X(F) \). (And then, of course, the same holds with \( X \) replaced by \( Y \)).

**Proof.** Substituting \( Y := 0 \) in \( R[X,Y] = R[F,G] \), we see that \( R[X] = R[F(Y = 0), G(Y = 0)] \). Since the coefficients of the highest powers of \( X \) in \( F \) and \( G \) are constants (by Corollary 3.3.7 in [9]), \( \deg_X(F) = \deg_X(F(Y = 0)) \) and \( \deg_X(G) = \deg_X(G(Y = 0)) \). Now let \( K \) be the quotient field of \( R \). Since \( K[X] = K[F(Y = 0), G(Y = 0)] \), we can use the Abhyankar-Moh Theorem in [1] to conclude, that \( \deg_X(F) \mid \deg_X(G) \) or \( \deg_X(G) \mid \deg_X(F) \).

**Theorem 3.3.** Let \( R \) be \( \mathbb{Q} \)-domain and suppose \((F,G) \in \text{Aut}_R R[X,Y]\) satisfies \( \deg_X(F) > \deg_X(G) \geq 1 \) and \( G(0,0) = 0 \). Following Lemma 3.2, write \((F,G)\) as

\[
(a_{mn}X^{mn} + a_{mn-1}(Y)X^{mn-1} + \cdots + a_{0}(Y), b_{n}X^n + b_{n-1}(Y)X^{n-1} + \cdots + b_{0}(Y))
\]

with \( m, n \in \mathbb{N}^*, \ m > 1 \) and \( b_{0}(0) = 0 \).

Then for \( k = 1, \ldots, n \) we have \( b_{n} \mid a_{mn}^{k}b_{n-k}(Y) \).

**Proof.** For the case \( n = 1 \) we can skip the first and second part of the proof; so in the first two parts we may assume, that \( n \geq 2 \).

i) \((\tilde{F},G) := (F - \frac{a_{mn}}{(b_{0})^{n}}G^{m},G) \in \text{Aut}_K K[X,Y] \) (with \( K := Q(R) \)), so again by Lemma 3.2 we may conclude, that \( \deg_X(\tilde{F}) \mid \deg_X(G) \)
or \( \deg_X(G) \mid \deg_X(\tilde{F}) \). In any case, we must have \( \deg_X(H(X,Y)) = \deg_X(F) \leq (m-1)n \), where \( H(X,Y) := b_n^mF - a_{mn}G^m \).

So we’ll be looking at the coefficients of \( X^{mn-1}, \ldots, X^{mn-(n-1)} \) in \( H(X,Y) \), all of which must be equal to zero. Take \( k \in \{1, \ldots, n-1\} \) and assume that the statement in the Theorem is true for all positive natural numbers smaller than \( k \). We shall prove that the statement is true for \( k \). After that, the case \( k = n \) shall be dealt with.

ii) First, let’s define the following index set:
\[
I := \{(x_0, \ldots, x_n) \in \mathbb{N}^{n+1} | x_0 + \cdots + x_n = m, x_1 + 2x_2 + \cdots + nx_n = mn - k\}.
\]

The coefficient of \( X^{mn-k} \) in \( H(X,Y) \) now looks like
\[
b_n^m a_{mn-k}(Y) - a_{mn} \sum_{(e_0, \ldots, e_n) \in I} \frac{m!}{e_n! \cdots e_0!} b_{n-1}^e(Y)^{e_{n-1}} \cdots b_0(Y)^{e_0}
\]
and this expression must equal zero.

Suppose a term appears in the above sum for which \( e_i > 0 \) for some \( i \leq n-k \). Then, if we want to optimize the sum \( ne_n + (n-1)e_{n-1} + \cdots + e_1 \), we must choose \( e_n \) equal to \( m - 1 \) (as \( e_0 + \cdots + e_n = m \)). So we have
\[
\begin{align*}
    nm - k &= ne_n + (n-1)e_{n-1} + \cdots + e_1 \\
          &\leq n(m - 1) + 0 + \cdots + 0 + i \cdot 1 + 0 + \cdots + 0 \\
          &\leq n(m - 1) + (n-k) \cdot 1 \\
          &= nm - k
\end{align*}
\]
So we must have equality at all places. This implies, that \( e_i = 0 \) for \( i < n-k \). We may furthermore deduce from the above, that only one term appears in the sum for which \( e_{n-k} > 0 \), namely the case \( e_n = m-1 \) and \( e_{n-k} = 1 \) (and \( e_i = 0 \) for all other \( i \)). So the equation becomes

\[
b_n^m a_{mn-k}(Y) = ma_{mn}b_{n-1}^{m-1}b_{n-k}(Y) + \sum_{(e_{n-(k-1)}, \ldots, e_n) \in I_k} a_{mn} \frac{m!}{e_n! \cdots e_{n-(k-1)}!} b_{n-1}^e(Y)^{e_{n-1}} \cdots b_{-(k-1)}(Y)^{e_{n-(k-1)}}
\]

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where the index set $I_k$ is defined by

$$I_k := \{(x_1, \ldots, x_k) \in \mathbb{N}^k | x_1 + \cdots + x_k = m, (n-(k-1))x_1 + \cdots + nx_k = mn-k\}$$

Note that, for every choice of the $e_i$, we have

$$k = mn - (mn - k)$$

$$= (e_n + \cdots + e_{n-(k-1)})n - (ne_n + \cdots + (n-(k-1))e_{n-(k-1)})$$

$$= e_{n-1} + 2e_{n-2} + \cdots + (k-1)e_{n-(k-1)}$$

So if we multiply equation (4) by $a_{nn}^{-1}$, we get

$$a_{nn}^{-1}b_n a_{mn-k}(Y) = ma_{mn}^{-1}b_{n-k}(Y) +$$

$$\sum_{(e_{n-(k-1)}, \ldots, e_n) \in I_k} \frac{m!}{e_n! \cdots e_{n-(k-1)!}} b_n^e (a_{mn} b_{n-1} (Y))^{e_{n-1}} \cdots (a_{mn} b_{n-(k-1)} (Y))^{e_{n-(k-1)}}$$

By the induction hypothesis, $b_n | a_{mn}^q b_{n-q} (Y)$ for all $q < k$, which implies that all terms of the above sum are divisible by

$$b_n^e b_{n-1} \cdots b_{n-(k-1)} = b_n^m$$

Since the lefthandside of equation (5) is also divisible by $b_n^m$, it follows that $b_n | ma_{mn}^{-1} b_{n-k}(Y)$, and since $b_n \neq 0$, $b_n | a_{mn} b_{n-k}(Y)$. So we proved that $b_n | a_{mn}^{-1} b_{n-k}(Y)$ for $k = 1, \ldots, n-1$.

iii) But the case $k = n$ still remains. From our last observation it follows, that $b_n | a_{mn}^{-1} b_{n-k}(Y)$ for $k = 1, \ldots, n-1$. We can take this into account while taking a look at

$$\det J(F, G) = (mn a_{mn} X^{mn-1} + \cdots + a_1 (Y))(b_{n-1} (Y) X^{n-1} + \cdots + b_1 (Y))$$

$$- (nb_n X^{n-1} + \cdots + b_1 (Y))(a_{mn-1} (Y) X^{mn-1} + \cdots + a_0 (Y)) \in \mathbb{R}$$

(6)

The coefficient of $X^{mn-1}$ in this equation equals zero, so we may conclude that

$$mn a_{mn} b_0 (Y) \in (b_1 (Y), \ldots, b_{n-1} (Y), b_1 (Y), \ldots, b_{n-1} (Y), b_n)$$

(7)

Since $a_{mn}^{-1} b_{n-k} (Y) = a_{mn}^{-1-k}(a_{mn}^{-1} b_{n-k} (Y)) \in (b_n)$ for $k = 1, \ldots, n-1$ (and, analogously, $a_{mn}^{-1} b_{n-k}' (Y) \in (b_n)$), we may conclude from equation (7) (multiplying it by $a_{mn}^{-1}$), that $mn a_{mn} b_0 (Y) \in (b_n)$, and since $b_0 (0) = 0$, $a_{mn} b_0 (Y) \in (b_n)$. So $b_n | a_{mn}^{-1} b_{n-k} (Y) \in (b_n)$ for $k = 1, \ldots, n$. □
Corollary 3.4. In the situation of Theorem 3.3, we have \( b_n | a_{mn}^n \).

Proof. \( \det J(F,G)|_{X=0} = a_1(Y)b_0'(Y) - a_0'(Y)b_1(Y) = \lambda \in R^* \), which implies that \( a_{mn}^n(a_1(Y)b_0'(Y) - \lambda) = a_0'(Y)a_{mn}^n b_1(Y) \in (b_n) \) (according to Theorem 3.3), and since also \( a_{mn}^n b_0'(Y) \in (b_n) \) (according to Theorem 3.3), we may conclude that \( a_{mn}^n \lambda \in (b_n) \). Thus, \( b_n | a_{mn}^n \). \( \square \)

Corollary 3.5. Look again at the situation of Theorem 3.3. Assume that \( a_{mn}^n \in R^* \). Then also \( b_n \in R^* \).

Consequently, \((\tilde{F},G) \in Aut_R R[X,Y]\) (using notations as in the first part of the proof of Theorem 3.3).

Proof. Since \( b_n | a_{mn}^n \) by Corollary 3.4, we must have \( b_n \in R^* \). \( \square \)

Corollary 3.6. In the situation of Corollary 3.5 : \((F,G) \in ST(R,2)!\)

Also : there exist certain \( k \in N^*, p_0, p_1, \ldots, p_{k+1} \in R^*, g_0 \in R \) and \( G_1(Y), \ldots, G_{k+1}(Y) \in R[Y] \) such that \( F = F_{k+1} \) and \( G = G_k \), as in Definition 2.5.

Proof. By applying Corollary 3.5 repeatedly, we see that \((F,G)\) is of the form \((F,G) = \tau_1 \cdots \tau_k \pi^\varepsilon \lambda\), where \( \tau_1, \ldots, \tau_k \) are elementary, \( \pi := (X,Y) \), \( \varepsilon \in \{0,1\} \) and \( \lambda = (aX, bY) \) for some \( a, b \in R^* \). So obviously \((F,G)\) is strongly tame. The second statement then follows from Lemma 2.8. \( \square \)

Remark 3.7. If \( R \) is not a domain, then in general Theorem 3.3 is false. Even Corollary 3.5 doesn’t hold anymore in this case, as can be seen from the following example. Let \( K \) be a field, \( R := K[T]/(T^3 + T) \), \( a := T \) and \( b := a^2 + 1 \). Then \( F \) in \((F,G) := ((X^2 + bY)^2 + X + aY, b(X^2 + bY) + aX) \in Aut_R R[X,Y]\) is monic in \( X \), but nevertheless \( b \notin R^* \) (\( b \) is even a zerodivisor : \( ab = 0 \)).

The following lemma will be very useful for the proof of our main theorem. It gives an explicit description of polynomials in our special class of Definition 2.5 over a localization of a domain.

Lemma 3.8. Let \( R \) be a domain and \( S \subseteq R \) a multiplicatively closed subset. Then for every \( F \in B_n(S^{-1}R) \) there exists an \( s \in S \) such that \( \tilde{F} := sF \in B_n(R) \).
Proof. We will prove by induction on \( n \), that for every \( F_1, \ldots, F_n \in \mathcal{B}(S^{-1}R) \) such that \( F_k = p_k F_{k-2} + G_k(F_{k-1}) \) (for \( 2 \leq k \leq n, F_0 := Y \) ), there exist \( \tilde{F}_1, \ldots, \tilde{F}_n \in \mathcal{B}(R) \) such that \( \tilde{F}_k = \tilde{p}_k \tilde{F}_{k-2} + \tilde{G}_k(\tilde{F}_{k-1}) \) (for \( 2 \leq k \leq n, \tilde{F}_0 := y ) \) and \( s_1, \ldots, s_n \in S \) satisfying \( \tilde{F}_k = s_k F_k \) for \( k = 1, \ldots, n \). The case \( n = 1 \) is easy. So assume \( n \geq 2 \) and that the statement is true for positive integers smaller than \( n \).

Now let \( F_1, \ldots, F_n \in \mathcal{B}(S^{-1}R) \) such that \( F_k = p_k F_{k-2} + G_k(F_{k-1}) \) (for \( 2 \leq k \leq n, F_0 := y ) \). The induction hypothesis gives \( s_1, \ldots, s_{n-1} \in S \) and \( F_1, \ldots, \tilde{F}_{n-1} \in \mathcal{B}(R) \) such that \( \tilde{F}_k = \tilde{p}_k \tilde{F}_{k-2} + \tilde{G}_k(\tilde{F}_{k-1}) \) (for \( 2 \leq k \leq n-1, F_0 := y ) \), satisfying \( \tilde{F}_k = s_k F_k \) for \( k = 1, \ldots, n - 1 \). This implies, that \( F_n = p_n(\frac{1}{s_{n-2}} \tilde{F}_{n-2}) + G_n(\frac{1}{s_{n-1}} \tilde{F}_{n-1}) = \tilde{p}_n \tilde{F}_{n-2} + \tilde{G}_n(\tilde{F}_{n-1}) \), where \( \tilde{p}_n := \frac{p_n}{s_{n-2}} \) and \( \tilde{G}_n(Y) := G_n(\frac{1}{s_{n-1}} Y) \). Let \( s_n \in S \) such that \( s_n(\tilde{p}_n X + \tilde{G}_n(Y)) = \tilde{p}_n X + \tilde{G}_n(Y) \) for certain \( \tilde{p}_n \in R \) and \( \tilde{G}_n(Y) \in R[Y] \). Then we have \( \tilde{F}_n := s_n F_n = \tilde{p}_n \tilde{F}_{n-2} + \tilde{G}_n(\tilde{F}_{n-1}) \), as desired. \( \square \)

Proof of Theorem 3.1

Let \( F \in R[X,Y] \) be a coordinate. Let \( G \in R[X,Y] \) be a mate for \( F \) with \( \det J(F,G) = 1 \). We may assume, that \( \deg_X(G) > 0 \) (if \( G \in R[Y] \) then one can deduce from \( \det J(F,G) = 1 \) that \( F \) is a tame coordinate).

By Corollary 3.3.7 in [9], the leading coefficient of \( F \) with respect to \( X \) is a constant \( a \in R \setminus \{0\} \). So there exists an \( h(T) \in R_a[T] \) such that \( g := G - h(F) \in R_a[X,Y] \) satisfies \( \deg_X(g) < \deg_X(F) \) (in the remainder of this proof we will use the letters \( F, G, H, \ldots \) for polynomials over \( R \) and \( f, g, h, \ldots \) for polynomials over \( R_a \)). But then Lemma 3.2 tells us that \( \deg_X(g) \mid \deg_X(F) \). By Corollary 3.6, \( (F,g) \in ST(R_a, 2) \). But then also \( (F,G) = (X,Y + h(X)) \circ (F,g) \in ST(R_a, 2) \). By Lemma 2.8 there exist certain \( n \in \mathbb{N}^*, p_0, p_1, \ldots, p_{n+1} \in \mathcal{B}(R_a), g_0 \in R_a \) and \( g_1(Y), \ldots, g_{n+1}(Y) \in R_a[Y] \) such that \( F = f_{n+1} \) and \( G = f_n \), as in Definition 2.5. From Lemma 3.8 it follows (taking \( S := \{1, a, a^2, \ldots\} \) ), that \( F_n := a^k G \in \mathcal{B}_n(R) \) for some \( k \in \mathbb{N} \). By definition this gives an \( F_{n-1} \in \mathcal{B}_{n-1}(R) \) satisfying \( \det J(F_{n-1}, F_n) = (-1)^n p_1 \cdots p_n \) (by Lemma 2.6).

So now we have \( \det J((-1)^n a^k F_{n-1}, G) = 1 \) and also \( \det J(F,G) = 1 \), which implies, that by putting \( D := \det J(-, G) : R_a[X,Y] \to R_a[X,Y] \),
we have \( F - \left( \frac{-1}{p_1 \cdots p_n} \right)^k F_{n-1} \in R_a[X, Y]^D = R_a[F_n] \) (since \( R_a[F_n] = R_a[G] \)),

say \( F = \left( \frac{-1}{p_1 \cdots p_n} \right)^k F_{n-1} + \tilde{h}(F_n) \) with \( \tilde{h}(T) \in R_a[T] \).

From the proof of Proposition 3.2 in [3] (which we stated earlier as Proposition 2.9) we get (for some new variables \( Z_2, \ldots, Z_{n+1} \)) certain tame \( \varphi_{n-1}, \varphi_n \in \text{Aut}_R R[X, Y, Z_2, \ldots, Z_{n+1}] \) satisfying \( \varphi_{n-1}^{-1} \varphi_n(F_n) = F_n \) and \( \varphi_{n-1}^{-1} \varphi_n(F_{n-1}) = p_1 \cdots p_n Z_{n+1} + F_{n-1} \). This implies that

\[
\varphi_{n-1}^{-1} \varphi_n(F) = \left( -\frac{1}{p_1 \cdots p_n} \right)^k (p_1 \cdots p_n Z_{n+1} + F_{n-1}) + \tilde{h}(F_n) = \left( -\frac{1}{p_1 \cdots p_n} \right)^k Z_{n+1} + F
\]

So \( F \) is projectively tame when \( \left( -\frac{1}{p_1 \cdots p_n} \right)^k Z_{n+1} + F \) is, and by Theorem 2.4, \( \left( -\frac{1}{p_1 \cdots p_n} \right)^k Z_{n+1} + F \in R[X, Y] \) is projectively tame as soon as \( \overline{F} \in \overline{R}[X, Y] \) is projectively tame, where \( \overline{R} := R/(a^k) \). Since \( a^k \neq 0 \) and \( \text{dim}(R) = 1 \), we have \( \text{dim}(\overline{R}) = 0 \). So \( \overline{F} \) is projectively tame by Proposition 3.9 and we are done.

**Proposition 3.9.** Let \( R \) be a Noetherian ring with \( \text{dim}(R) = 0 \) and let \( F \in R[X, Y] \) be a coordinate. Then \( F \) is projectively tame.

**Proof.** Since \( R \) is Noetherian, there exist \( p_1, \ldots, p_n \in \text{Spec}(R) \) such that \( \eta = p_1 \cap \cdots \cap p_n \). We may of course assume, that \( p_i \neq p_j \) if \( i \neq j \). Since \( \text{dim}(R) = 0 \), every \( p_i \) is maximal, so we certainly have \( p_i + p_j = (1) \) whenever \( i \neq j \).

Furthermore, \( R/p_i \) is a field, so by Theorem 2.2, \( \overline{F} \in R/p_i[X, Y] \) is a tame coordinate, so \( \overline{F} \) is certainly projectively tame. By applying Corollary 2.17 repeatedly, we deduce that \( F \) is projectively tame modulo \( p_1 \cdots p_n \). This implies, that also \( \overline{F} \in R/\eta[X, Y] = R/p_1 \cap \cdots \cap p_n[X, Y] \) is projectively tame. By Theorem 2.14, \( F \in R[X, Y] \) is projectively tame.

The perceptive reader may have noticed in the proof of the proposition above that the information we have regarding \( F \) modulo a prime ideal (being a tame coordinate) is much stronger than the result which we extract from it (the projectively-tameness). Indeed, if we make the additional assumption that \( R \) is reduced, then we get a quick proof of the following result, which can also be found as Corollary 0.6 in [13].

**Proposition 3.10.** Let \( R \) be a Noetherian reduced ring with \( \text{dim}(R) = 0 \). Let \( \varphi \in \text{Aut}_R R[X, Y] \). Then \( \varphi \) is tame.
Proof. Since $R$ is Noetherian, there exist $p_1, \ldots, p_n \in \text{Spec}(R)$ such that $(0) = \eta = p_1 \cap \cdots \cap p_n$. We may assume, that $p_i \neq p_j$ if $i \neq j$. Since $\dim(R) = 0$, every $p_i$ is maximal, so we have $p_i + p_j = (1)$ when $i \neq j$. Furthermore, $R/p_i$ is a field, so by Theorem 2.2, $\varphi \in R/p_i[X,Y]^2$ is a tame automorphism. By applying Theorem 2.16 repeatedly, we deduce, that $\varphi \in R/p_1 \cdots p_n[X,Y]^2$ is tame. But then we also have, that $\varphi \in R/p_1 \cap \cdots \cap p_n[X,Y]^2 = R/\eta[X,Y]^2$ is tame. Since $R$ is reduced, we may conclude, that $\varphi$ is tame. \qed

Remark 3.11. The assumption in the previous proposition that $R$ is reduced is necessary, as can be seen from the following. Let $R = \mathbb{C}[X]/(X^2)$ and $\varepsilon := X$. Then $R$ is a Noetherian ring (being a finitely generated $\mathbb{C}$-algebra) with only one prime ideal, namely $(\varepsilon)$. So $\dim(R) = 0$ and if we take $\varphi := X + \varepsilon X^2$, then $\varphi$ is not tame since $\det J\varphi = 1 + 2\varepsilon X \in R[X]^* \setminus R^*$ (whereas $\det J\psi \in R^*$ for all automorphisms $\psi$ which are tame). For the same reason we may conclude, that also $(X + \varepsilon X^2, Y) \in \text{Aut}_R R[X,Y]$ is not tame.

References


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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld 1
6525 ED Nijmegen
The Netherlands

Email : berson@sci.kun.nl