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AN EXTENSION OF THE MIYANISHI-SUGIE CANCELLATION THEOREM TO DEDEKIND RINGS

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An Extension of the Miyanishi-Sugie Cancellation Theorem to Dedekind Rings

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Abstract

Let R be a Dedekind ring containing $\mathbb{Q}$. It is shown that if A is an R-algebra containing R such that $A^{[m]} \simeq_R R^{[m+2]}$ for some $m \geq 1$, then $A \simeq_R R^{[2]}$. Also an example is given of a 2-dimensional noetherian UFD R and an R-algebra A such that the above implication does not hold (for $m = 1$).
We deduce that the ring of constants of a triangular k-derivation having a slice on $k^{[4]}$ is a polynomial ring in three variables over k, in case k is a field of characteristic zero.

Introduction

Let k be a field of characteristic zero and V an algebraic variety over k. The Cancellation Problem asks if $V \times k^m \simeq k^n \times k^m$ implies that $V \simeq k^n$ (as algebraic varieties). This problem was first posed by Zariski (in a slightly different form) in 1949. The case $n = 1$, $m = 1$ was solved affirmatively by Rentschler in [15]. The general $n = 1$ case was solved affirmatively by Abhyankar, Eakin and Heinzer in [1] (even for arbitrary fields k). For $n = 2$ the case $m = 1$ was solved affirmatively by Fujita [10], in case k is algebraically closed. A little later the general $n = 2$ case, for k an algebraically closed field, was solved affirmatively by Miyanishi and Sugie in [14] (in fact their result was even more general). It was remarked by Daigle in [4], that a straightforward use of a result of Kambayashi in [13] then solves the general $n = 2$ case. The case $n \geq 3$ remains open.

In more algebraic terms the Cancellation Problem can be reformulated as follows: if A is a k-algebra such that $A[Y_1, \ldots, Y_m] \simeq_k k[t_1, \ldots, t_n, X_n, \ldots, X_m]$ does it follow that $A \simeq_k k[t_1, \ldots, t_n]$? After the solution of the $n = 1$ case given in [1] several generalisations of this result, replacing k by a more general ring where found (see [6], [7], [5], [11]). In particular it was proved by Hamann in [11] that for any $\mathbb{Q}$-algebra R and any R-algebra A containing R the R-isomorphism $A[Y_1, \ldots, Y_m] \simeq R[t_1, X_1, \ldots, X_m]$ implies that $A \simeq_R R[t_1]$.

The aim of this note is to show that in case $n = 2$ such a result does not hold for any $\mathbb{Q}$-algebra (Example 2.2) but it does hold for a Dedekind ring R containing $\mathbb{Q}$ (Theorem 2.1). The counterexample (Example 2.2) was already obtained by Hochster

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in [12]. The proof of the positive result just mentioned is based on a theorem of Bass, Connell and Wright ([3]) concerning locally polynomial algebras and a result of Sathaye ([17]) concerning two variable polynomial rings over a discrete valuation ring.

At the end of this paper we apply our main theorem (Theorem 2.1) to the study of locally nilpotent derivations. In particular we deduce that the kernel of a triangular derivation having a slice on a polynomial ring in 4 variables is a polynomial ring in 3 variables. This is particularly interesting since in [9] a triangular derivation having a slice on a 5 variable polynomial ring is given for which it is conjectured that its kernel is not a polynomial ring in 4 variables (see also [16]).

1 Preliminaries

Throughout this paper R denotes a commutative ring and n a positive integer. The polynomial ring $R[X_1, \ldots, X_n]$ over R is often denoted by $R^{[n]}$.

An R-algebra A containing R is called a *locally polynomial ring* if $A_p := R_p \otimes_R A$ is a polynomial ring over R_p, for every prime ideal p of R. Obviously a polynomial ring over R is a locally polynomial ring over R, however the converse does not always hold. To see this we recall some well-known facts concerning symmetric algebras.

Let M be an R-module. The *tensor algebra* of M over R is

$$T_R(M) := R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \ldots$$

with $(m_1 \otimes \ldots \otimes m_r) \cdot (m'_1 \otimes \ldots \otimes m'_s) := m_1 \otimes \ldots \otimes m_r \otimes m'_1 \otimes \ldots \otimes m'_s$. The *symmetric algebra* of M over R is

$$S_R(M) := T_R(M) / \langle m_1 \otimes m_2 - m_2 \otimes m_1, m_1, m_2 \in M \rangle.$$

The following proposition collects some useful properties of the symmetric algebra (see for example [5]).

Proposition 1.1 (i) $S_R(R^n) \simeq_R R^{[n]}$.

(ii) *$S_R(M \oplus N) \simeq_R S_R(M) \otimes S_R(N)$ for all modules M and N.*

(iii) *Let M and N be finitely generated R-modules, then $M \simeq_R N$ if and only if $S_R(M) \simeq_R S_R(N)$.*

(iv) *$S_R(M)_p \simeq_{R_p} S_{R_p}(M_p)$ for all prime ideals p of R.*

Suppose now that M is a finitely generated projective R-module. Then M_p is a free R_p-module of finite rank, hence $S_{R_p}(M_p) \simeq_{R_p} R_p^{[n_p]}$ for some $n_p \geq 1$ (by (i)), whence so is $S_R(M)_p$ (by (iv)). Consequently $S_R(M)$ is a locally polynomial ring over R. If additionally M is not a free R-module, then $A := S_R(M)$ is not a polynomial ring (if $A \simeq R^{[n]}$ for some $n \geq 1$ then by (i) $S_R(M) \simeq_R S(R^n)$, whence by (iii) $M \simeq_R R^n$ i.e. M is free, a contradiction).

Now the point is that if A is a *finitely presented* R-algebra i.e. $A \simeq R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ for some f_i, then A is a locally polynomial ring over R if and only if $A \simeq S_R(M)$ for some finitely generated projective R-module M. More precisely we have

Theorem 1.2 (Bass, Connell, Wright, [3].) *Let A be a finitely presented R-algebra. Then there is equivalence between*

(i) $A \simeq_R S_R(M)$ *for some finitely projective R-module.*

(ii) *A is a locally polynomial ring over R.*

(iii) *For each maximal ideal m of R $A_m \simeq R_m^{[n_m]}$, for some $n_m \geq 1$.*

To conclude this section we recall a result of Sathaye concerning how to characterize polynomial rings in two variables over a discrete valuation ring.

Theorem 1.3 (Sathaye, [17].) *Let R be a discrete valuation ring containing $\mathbb{Q}$. Let m be its maximal ideal, $k := R/m$ and $K := \mathbb{Q}(R)$. Let A be a finitely generated R-domain.*
If $K \otimes_R A \simeq_K K^{[2]}$ and $k \otimes_R A \simeq k^{[2]}$, then $A \simeq_R R^{[2]}$.

2 The main result

The main result of this section is the following theorem.

Theorem 2.1 *Let R be a Dedekind ring containing $\mathbb{Q}$ and A an R-algebra containing R. If for some $m \geq 1$ $A^{[m]} \simeq_R R^{[m+2]}$, then $A \simeq_R R^{[2]}$.*

Proof. i) We may assume that $A[Y_1, \ldots, Y_m] = R[x, y, X_1, \ldots, X_m]$. It follows that A is a finitely presented R-algebra.
Claim: A is a locally polynomial algebra over R.

Let us assume this claim first. Then by Theorem 1.2 $A \simeq_R S_R(M)$ for some finitely generated projective R-module M. So by 1.1 we get

$$\begin{aligned} S_R(M \oplus R^m) &\simeq_R S_R(M) \otimes_R R^{[m]} \simeq_R A[Y_1, \ldots, Y_m] \\ &= R[x, y, X_1, \ldots, X_m] \simeq_R S_R(R^{m+2}). \end{aligned}$$

So by 1.1(iii) we get $M \oplus R^m = R^{m+2}$.
Since R is a noetherian ring of dimension 1 it follows from Bass' Cancellation Theorem for stably free modules ([2], Theorem V.3.2 or [18], Theorem 1.3) that $M \simeq_R R^2$. So $A \simeq_R S(M) \simeq R^{[2]}$.
ii) So it remains to prove the claim above. Again by 1.2 it suffices to prove that for each maximal ideal m of R we have $A_m \simeq_{R_m} R_m^{[n_m]}$ for some $n_m \geq 1$. So let m be a maximal ideal of R. The hypothesis implies that

$$(*) \qquad A_m^{[m]} \simeq_{R_m} R_m^{[m+2]}$$

Now observe that R_m is a discrete valuation ring since R is a Dedekind ring. Denote by K (resp. k) the quotient field (resp. the residue field) of R_m. We obtain from $(*)$ by tensoring with K (resp. k) that

$$(K \otimes_{R_m} A_m)^{[m]} \simeq_K K^{[m+2]} \text{ (resp. } (k \otimes_{R_m} A_m)^{[m]} \simeq_k k^{[m+2]}).$$

Since both fields K and k contain $\mathbb{Q}$ (since R does!) it follows from the Miyanishi-Sugie Cancellation Theorem applied to K resp. k that the R_m-algebra A_m satisfies the hypothesis of 1.3, whence $A_m \simeq_{R_m} R_m^{[2]}$. This concludes the proof of the claim and hence of the theorem. $\square$

To show that the statement of Theorem 2.1 does not hold for arbitrary rings R in case $m = 1$ we look for an R-module M such that $S(M)^{[1]} \simeq_R R^{[3]}$ and $S(M) \not\simeq_R R^{[2]}$ or equivalently $S(M \oplus R) \simeq_R S(R^3)$ and $S(M) \not\simeq_R S(R^2)$ i.e. we must find R and M such that $M \oplus R \simeq R^3$ and $M \not\simeq R^2$.
The following well-known result gives such a pair.

Example 2.2 (Hochster, [12]). Let $R := \mathbb{R}[x, y, z]/(x^2+y^2+z^2-1)$ and $\varphi : R^3 \to R$ given by $\varphi(r_1, r_2, r_3) = r_1\overline{x} + r_2\overline{y} + r_3\overline{z}$. Then it is well-known that $M := \ker\varphi$ satisfies $M \oplus R \simeq R^3$ but $M \not\simeq R^2$. Consequently $A := S_R(M)$ satisfies $A^{[1]} \simeq_R R^{[3]}$ but $A \not\simeq_R R^{[2]}$.

3 Applications to locally nilpotent derivations

Let B be a commutative ring. A derivation D on B is called *locally nilpotent* if for every b in B there exists a positive integer m such that $D^m(b) = 0$. An element s in B is called a *slice* of D if $D(s) = 1$. The following result is well-known (see [19], or [8], Prop. 1.3.21).

Proposition 3.1 *Let B be a $\mathbb{Q}$-algebra and D a locally nilpotent derivation on B which has a slice s in B. Then $B = B^D[s]$ a polynomial ring in s over B^D* ($:= \ker D$).

As a consequence of Theorem 2.1 we get

Proposition 3.2 *Let R be a Dedekind ring containing $\mathbb{Q}$ and D a locally nilpotent R-derivation having a slice on $R[X, Y, Z]$. Then $R[X, Y, Z]^D \simeq_R R^{[2]}$.*

Proof. Put $A := R[X, Y, Z]^D$. Then by 3.1 $R[X, Y, Z] = A[s]$, so $A^{[1]} \simeq_R R[X, Y]^{[1]}$, where $A \simeq_R R^{[2]}$ by Theorem 2.1. $\square$

Corollary 3.3 *Let k be a field of characteristic zero and*

$$D = a(X, Y, Z, W)\partial_X + b(X, Y, Z, W)\partial_Y + c(X, Y, Z, W)\partial_Z + d(W)\partial_W$$

a locally nilpotent k-derivation on $B := k[X, Y, Z, W]$ having a slice. Then $B^D \simeq_k k^{[3]}$. In particular this holds for all triangular k-derivations on B.

Proof. If $d(W) \neq 0$ then $d := d(W) \in k^*$ (since D is locally nilpotent the restriction of D to $k[W]$, which equals $d(W)\partial_W$, is locally nilpotent what readily implies that $d \in k^*$). Consequently $s := d^{-1}W$ is a slice of D. So, using that $B = B^D[s]$ we obtain that $B^D \simeq B/(s) \simeq k[X, Y, Z] \simeq k^{[3]}$. Finally, if $d(W) = 0$ then apply Proposition 3.2 with $R = k[W]$. $\square$

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