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A NEW CHARACTERIZATION OF THE UNIT BALL OF $H^2$

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Abstract
We derive a new expression for the norm of $H^2$ functions; we present some well-known results in a different setting.

Introduction
In 1915, Pick [3] proved the following result

**Theorem 1** Let $g$ be an analytic function on the unit disc $\Delta$ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in $\Delta$ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k z_l} \lambda_k \overline{\lambda_l} \geq 0.$$  

(1)

Ahlfors [1] page 3, gives an elegant proof of this characterization of the unit ball of $H^\infty$.

In this note we shall present a characterization of the unit ball of $H^2$. Our main tool will be an explicit solution of the “Minimal Interpolation Problem” for $H^2$. [2] page 141. As a byproduct we obtain a new proof of Pick’s theorem.

Description of the main result
Let $z_1, z_2, \ldots, z_n$ be a sequence in $\Delta$, and let $b$ be the Blaschke product generated by this sequence:

$$b(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - z_j z}.$$  

(2)

We shall prove that the following conditions are equivalent for continuous functions $f$ on $\Delta$.

1) $f$ lies in the unit ball of $H^2$.

2) for every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k) \cdot \overline{f(z_l)}}{1 - z_k z_l} \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$  

(3)
Preliminaries

For mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define

$$\Lambda = \{ f \in H^2 : f(z_j) = w_j, \; j = 1, 2, \ldots, n \}.$$ 

$\Lambda$ is not empty; it contains the Lagrange interpolation polynomial

$$\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) \cdot b'(z_k)},$$

where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of $H^p$ spaces it is more natural to work with the Blaschke interpolation function

$$\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{z}_k z}{z - z_k} \cdot \frac{w_k}{b'(z_k)(1 - |z_k|^2)},$$

with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with

$$\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) b'(z_k)}.$$  \hfill (4)

$\varphi \in \Lambda$, and $\varphi$ is analytic on some neighbourhood of $\Delta$. $\Lambda$ is a hyperplane in $H^2$.

With $\varphi$ and $b$ defined as in (4) and (2) we have

$$\Lambda = \{ \varphi + b \cdot g : g \in H^2 \}.$$

**Theorem 2** $\varphi$ is the unique solution of the “Minimal Interpolation Problem”, i.e. for every $f \in \Lambda \setminus \{ \varphi \}$ we have $\|f\|_2 > \|\varphi\|_2$.

**Proof:** It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2$).

From the decomposition $f = \varphi + b \cdot g$ we have

$$\langle f - \varphi, \varphi \rangle = \langle b \cdot g, \varphi \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it}) g(e^{it}) \overline{\varphi(e^{it})} dt = \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it}) g(e^{it}) \sum_{k=1}^{n} \frac{w_k}{(e^{-it} - z_k) \cdot b'(z_k)} dt.$$  

Note that $|b(e^{it})|^2 = 1$. Thus

$$\langle f - \varphi, \varphi \rangle = \sum_{k=1}^{n} \frac{w_k}{2\pi b'(z_k)} \int_{0}^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it} z_k} dt = \sum_{k=1}^{n} \frac{w_k}{b'(z_k)} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \overline{z}_k z} dz = 0,$$
because the integrand is analytic on $\Delta$.

It will be convenient to have an explicit expression for $\|\varphi\|_2$.

$$
\|\varphi\|_2^2 = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} |\varphi(e^{it})|^2 dt = \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)} \int_{0}^{2\pi} \frac{dt}{(e^{it} - z_k)(e^{-it} - z_l)}
$$

$$
= \frac{1}{2\pi i} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)} \int_{\Gamma} \frac{dz}{(z - z_k)(1 - z_lz)} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)}\frac{1}{1 - z_k \overline{z_l}}.
$$

There are of course many other expressions for $\|\varphi\|_2$.

**Theorem 3**

$$
\|\varphi\|_2 = \max \left\{ \left| \sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} \right| : f \in H^2, \|f\|_2 \leq 1 \right\}.
$$

**Proof:**

$$
\sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)\varphi(z)}{b(z)} dz,
$$

hence by Schwarz’s inequality we have

$$
\left| \sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} \right| \leq \left( \int_{0}^{2\pi} |f(e^{it})| \cdot |\varphi(e^{it})| dt \right) \leq \|f\|_2 \cdot \|\varphi\|_2 \leq \|\varphi\|_2.
$$

Equality holds for the function $f : z \rightarrow \frac{1}{\|\varphi\|_2} \sum_{k=1}^{n} \frac{w_k}{1 - z_k \overline{z_l} b'(z_k)}$.

An immediate result from Theorem 2 is

**Corollary** For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of $\Delta$ we have

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
$$

**Proof:** Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1 \in \Lambda$ and since

$$
\|1\|_2 = 1,
$$

we have

$$
1 \geq \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}.
$$

The equality sign certainly occurs if $0 \in \{z_1, z_2, \ldots, z_n\}$:

$$
1 = \varphi(0)^2 \leq \frac{1}{2\pi} \int_{0}^{2\pi} |\varphi(e^{it})|^2 dt = \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}.
$$

3
If \( 0 \not\in \{ z_1, z_2, \ldots, z_n \} \) there is strict inequality:
Because of the uniqueness of \( \varphi \) there can be equality only if
\[
b(z) \sum_{k=1}^{n} \frac{1}{(z-z_k)b'(z_k)} = 1.
\]
In this identity for rational functions we let \( z \to \infty \). Since \( z_j \neq 0 \), \( \lim_{z \to \infty} b(z) \) has a finite value. Therefore, the left hand side has limit zero.

The fact that \( \varphi \in \Lambda \) has an interesting reformulation. We start with a lemma.

**Lemma 1** The partial fraction decomposition of \( \varphi \) is
\[
\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \overline{z_l}z)(1 - \overline{z_l}z_k)b'(z_k)b'(z_l)}. \tag{5}
\]

**Proof:** An elegant way to prove this is to compute both sides of the following identity.

For \( z \in \Delta \) we have
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1 - \zeta z} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1 - \zeta z} \frac{d\zeta}{\zeta}.
\]

The left hand side is equal to
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} dz = \varphi(z),
\]
while the right hand side is equal to the complex conjugate of
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{b(\zeta)}{\zeta - z} dz = \varphi(z),
\]

i.e. to the complex conjugate of
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{b(\zeta)} \sum_{k=1}^{n} \frac{\overline{w_k}}{(1 - \overline{z_k}\zeta)b'(z_k)} \frac{1}{1 - \overline{\zeta}z} \frac{d\zeta}{\zeta}.
\]

Calculation of the residues at the points \( z_1, z_2, \ldots, z_n \) lead to (5).

The condition \( \varphi \in \Lambda \) implies that \( \varphi(z_j) = w_j, \ j = 1, \ldots, n \) i.e.
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \overline{z_l}z_j)(1 - \overline{z_l}z_k)b'(z_k)b'(z_l)} = w_j.
\]
This is equivalent to the assertion that the matrices
\[
B = (\beta_{lk})
\]
and its conjugate \( \overline{B} = (\overline{\beta}_{lk}) \) where
\[
\beta_{lk} = \frac{1}{(1 - \overline{z_l}z_k)b'(z_k)}
\]
are each others inverse, i.e. \( B \) and \( \overline{B} \) are unitary.
Proof of the main result

**Lemma 2** Assume that $f$ lies in the unit ball of $H^2$, and let a sequence of mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ be given. Then (3) holds.

**Proof:** Define $w_j = f(z_j)$. $f$ lies in the hyperplane $\Lambda$ and the element $\varphi$ of $\Lambda$ with minimal norm satisfies
\[ \|\varphi\|_2 \leq \|f\|_2 \leq 1. \]

Use of the explicit expression for $\|\varphi\|_2$ leads to (3).

**Lemma 3** Assume that $f$ is continuous and that $f$ satisfies (3). We shall show that $f \in H^2$ and that $\|f\|_2 \leq 1$.

**Proof:** We apply (3) for the case $n = 1$; an easy computation shows that
\[ |f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}} \]
for every choice of $z \in \Delta$.

Let $0 < r < \rho < 1$, and let $z_1, z_2, \ldots$ be an enumeration of the rational points of $\overline{\Delta}_\rho$. For every $n$ there is a function $\varphi_n$ with
\[ \varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n \]
and
\[ \|\varphi_n\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k\overline{z_l}} \frac{1}{b'(z_k)b'(z_l)} \leq 1. \]

Thus, $\varphi_n$ lies in the unit ball of $H^2$, and so by lemma 2, we have for every sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ in $\Delta$
\[ \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{\varphi_n(\zeta_k)\overline{\varphi_n(\zeta_\ell)}}{1 - \zeta_k\overline{\zeta_\ell}} \frac{1}{b'(z_k)b'(z_\ell)} \leq 1. \]

It follows from (6) that
\[ |\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1 - |\zeta|^2}}, \]
hence the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded on $\overline{\Delta}_\rho$. Therefore, it contains a locally uniformly convergent subsequence $\varphi_{n_j}$. At the points $z_1, z_2, \ldots$ the subsequence converges to $f$. By the continuity of $f$ and the fact that $\{z_1, z_2, \ldots\}$ is dense in $\Delta_\rho$ we see that
\[ \lim_{n_j \to \infty} \varphi_{n_j} = f. \]

This shows that $f$ is analytic on $\Delta_\rho$ for all $\rho < 1$. Because of uniform convergence on $\Gamma_r$ we have
\[ \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j \to \infty} \frac{1}{2\pi} \int_{0}^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1. \]
Thus, \( f \in H^2 \) and \( \|f\|_2 \leq 1 \).

Lemma 2 and lemma 3 together constitute a proof of the main result.

**Corollary** For \( f \in H^2 \) we define

\[
\nu(f) = \sup \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} ; z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}.
\]

Then \( \nu(f) = \|f\|_2^2 \).

**Proof:** Assume that \( \nu(f) = 1 \). Then by lemma 3: \( \|f\|_2^2 \leq 1 \). If \( \|f\|_2^2 < \lambda^2 < 1 \) for some \( \lambda \), then we have \( \|\frac{f}{\lambda}\|_2^2 < 1 \) but \( \nu\left(\frac{f}{\lambda}\right) > 1 \) which is impossible by lemma 2.

In a similar way we can show that \( \|f\|_2^2 = 1 \) implies that \( \nu(f) = 1 \). By the homogeneity of \( \nu \) and \( \|\cdot\|_2^2 \) it follows that for all \( f \in H^2 \) \( \nu(f) = \|f\|_2^2 \).

**Pick’s theorem**

As an application of our results we shall give a proof of Pick’s theorem.

Let \( g \) belongs to the unit ball of \( H^\infty \), and let \( z_1, z_2, \ldots, z_n \) be a sequence of mutually distinct points in \( \Delta \). Let \( w_1, w_2, \ldots, w_n \) be an arbitrary sequence of complex numbers.

We consider the hyperplanes \( \Lambda \) and \( \Lambda_g \) where

\[
\Lambda_g = \{ f \in H^2 : f(z_j) = w_j \cdot g(z_j), j = 1, 2, \ldots, n \}.
\]

Of course, if \( f \in \Delta \), then \( g \cdot f \in \Delta_g \), and by Theorem 2 applied to \( \Lambda_g \) we have

\[
\|gf\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k g(z_k) \cdot w_l g(z_l)}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)}.
\]

Let \( \varphi \) be, as before, the element of \( \Lambda \) with smallest norm. From \( \|g\|_\infty \leq 1 \) we obtain

\[
\|gf\|_2^2 \leq \|\varphi\|_2.
\]

Combination of these steps leads to

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} = \|gf\|_2^2 \geq \|\varphi\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l g(z_k) g(z_l)}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)}
\]

i.e. to

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{w_k w_l}{b'(z_k)b'(z_l)} \geq 0
\]

and since the sequence \( w_1, w_2, \ldots, w_n \) is arbitrary we have for all choices of \( \lambda_1, \lambda_2, \ldots, \lambda_n \)

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z}_l} \cdot \lambda_k \lambda_l \geq 0.
\]

By the choice \( n = 1, \lambda_1 = 1 \) we see that the converse is trivial.

6
References

