A NEW CHARACTERIZATION OF THE UNIT BALL OF $H^2$

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Abstract
We derive a new expression for the norm of $H^2$ functions; we present some well-known results in a different setting.

Introduction
In 1915, Pick [3] proved the following result

**Theorem 1** Let $g$ be an analytic function on the unit disc $\Delta$ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in $\Delta$ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k \overline{z_l}} \lambda_k \overline{\lambda_l} \geq 0.
$$

(1)

Ahlfors [1] page 3, gives an elegant proof of this characterization of the unit ball of $H^\infty$.

In this note we shall present a characterization of the unit ball of $H^2$. Our main tool will be an explicit solution of the “Minimal Interpolation Problem” for $H^2$. [2] page 141. As a byproduct we obtain a new proof of Pick’s theorem.

Description of the main result
Let $z_1, z_2, \ldots, z_n$ be a sequence in $\Delta$, and let $b$ be the Blaschke product generated by this sequence:

$$
b(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - z_j \overline{z}}.
$$

(2)

We shall prove that the following conditions are equivalent for continuous functions $f$ on $\Delta$.

1) $f$ lies in the unit ball of $H^2$.

2) for every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k) \cdot \overline{f(z_l)} - 1}{1 - z_k \overline{z_l}} \leq 1.
$$

(3)
Preliminaries

For mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define

$$\Lambda = \{f \in H^2 : f(z_j) = w_j, j = 1, 2, \ldots, n\}.$$  

$\Lambda$ is not empty; it contains the Lagrange interpolation polynomial

$$\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) \cdot b'(z_k)},$$

where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of $H^p$ spaces it is more natural to work with the Blaschke interpolation function

$$\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \bar{z}_k z}{z - z_k} \frac{w_k}{b'(z_k)(1 - |z_k|^2)},$$

with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with

$$\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) b'(z_k)}.$$  

(4)

$\varphi \in \Lambda$, and $\varphi$ is analytic on some neighbourhood of $\Delta$. $\Lambda$ is a hyperplane in $H^2$.

With $\varphi$ and $b$ defined as in (4) and (2) we have

$$\Lambda = \{\varphi + b \cdot g : g \in H^2\}.$$  

Theorem 2 $\varphi$ is the unique solution of the “Minimal Interpolation Problem”, i.e. for every $f \in \Lambda \setminus \{\varphi\}$ we have $\|f\|_2 > \|\varphi\|_2$.

Proof: It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2$).

From the decomposition $f = \varphi + b \cdot g$ we have

$$\langle f - \varphi, \varphi \rangle = \langle b \cdot g, \varphi \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it}) g(e^{it}) \overline{\varphi(e^{it})} dt$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it}) g(e^{it}) \sum_{k=1}^{n} \frac{w_k}{(e^{-it} - z_k) \cdot b'(z_k)} dt.$$  

Note that $|b(e^{it})|^2 = 1$. Thus

$$\langle f - \varphi, \varphi \rangle = \sum_{k=1}^{n} \frac{w_k}{b'(z_k)} \int_{0}^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it} \bar{z}_k} dt$$

$$= \sum_{k=1}^{n} \frac{w_k}{b'(z_k)} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \bar{z}_k z} dz = 0,$$
because the integrand is analytic on $\Delta$.

It will be convenient to have an explicit expression for $\|\varphi\|_2$.

\[
\|\varphi\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w_l}}{b'(z_k) \overline{b'(z_l)}} \int_0^{2\pi} \frac{dt}{(e^{it} - z_k)(e^{-it} - \overline{z_l})} \\
= \frac{1}{2\pi i} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w_l}}{b'(z_k) \overline{b'(z_l)}} \int \frac{dz}{(z - z_k)(1 - z_l z)} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w_l}}{b'(z_k) \overline{b'(z_l)}}. 
\]

There are of course many other expressions for $\|\varphi\|_2$.

**Theorem 3**

\[
\|\varphi\|_2 = \max \left\{ \left| \sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} \right| : f \in H^2, \|f\|_2 \leq 1 \right\}. 
\]

**Proof:**

\[
\sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} = \frac{1}{2\pi i} \int \frac{f(z) \varphi(z)}{b(z)} dz, 
\]

hence by Schwarz’s inequality we have

\[
\left| \sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})| \cdot |\varphi(e^{it})| dt \leq \|f\|_2 \cdot \|\varphi\|_2 \leq \|\varphi\|_2. 
\]

Equality holds for the function $f : z \rightarrow \frac{1}{\|\varphi\|_2} \sum_{k=1}^{n} \frac{w_k}{(1 - z_k z) \overline{b'(z_k)}}$.

An immediate result from Theorem 2 is

**Corollary** For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of $\Delta$ we have

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \cdot \frac{1}{b'(z_k) \overline{b'(z_l)}} \leq 1. 
\]

**Proof:** Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1 \in \Lambda$ and since

\[
\|1\|_2 = 1, 
\]

we have

\[
1 \geq \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \cdot \frac{1}{b'(z_k) \overline{b'(z_l)}}. 
\]

The equality sign certainly occurs if $0 \in \{z_1, z_2, \ldots, z_n\}$:

\[
1 = \varphi(0)^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \cdot \frac{1}{b'(z_k) \overline{b'(z_l)}}. 
\]
If \( 0 \notin \{ z_1, z_2, \ldots, z_n \} \) there is strict inequality:

Because of the uniqueness of \( \varphi \) there can be equality only if

\[
b(z) \sum_{k=1}^{n} \frac{1}{(z-z_k)b'(z_k)} = 1.\]

In this identity for rational functions we let \( z \to \infty \). Since \( z_j \neq 0 \), \( \lim_{z \to \infty} b(z) \) has a finite value. Therefore, the left hand side has limit zero.

The fact that \( \varphi \in \Lambda \) has an interesting reformulation. We start with a lemma.

**Lemma 1** The partial fraction decomposition of \( \varphi \) is

\[
\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \bar{\zeta}_l z)(1 - \bar{\zeta}_l z_k)b'(z_k)b'(z_l)).}
\]

**Proof:** An elegant way to prove this is to compute both sides of the following identity.

For \( z \in \Delta \) we have

\[
\frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) \frac{d\zeta}{1 - \zeta z} = \frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) \frac{d\zeta}{1 - \zeta z}.
\]

The left hand side is equal to

\[
\frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) d\zeta = \varphi(z),
\]

while the right hand side is equal to the complex conjugate of

\[
\frac{1}{2\pi i} \int_{\Gamma} b(\zeta) \sum_{k=1}^{n} \frac{w_k}{(\zeta - z_k)b'(z_k)} \frac{1}{1 - \zeta z} d\zeta,
\]

i.e. to the complex conjugate of

\[
\frac{1}{2\pi i} \int_{\Gamma} b(\zeta) \sum_{k=1}^{n} \frac{w_k}{(1 - \bar{\zeta}_k z)(1 - \bar{\zeta}_k z_k)b'(z_k)b'(z_l)} d\zeta.
\]

Calculation of the residues at the points \( z_1, z_2, \ldots, z_n \) lead to (5).

The condition \( \varphi \in \Lambda \) implies that \( \varphi(z_j) = w_j, j = 1, \ldots, n \) i.e.

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \bar{\zeta}_l z_j)(1 - \bar{\zeta}_l z_k)b'(z_k)b'(z_l)} = w_j.
\]

This is equivalent to the assertion that the matrices

\[
B = (\beta_{lk})
\]

and its conjugate \( \overline{B} = (\overline{\beta}_{lk}) \) where

\[
\beta_{lk} = \frac{1}{(1 - \bar{\zeta}_l z_k)b'(z_k)}
\]

are each others inverse, i.e. \( B \) and \( \overline{B} \) are unitary.
Proof of the main result

Lemma 2 Assume that \( f \) lies in the unit ball of \( H^2 \), and let a sequence of mutually distinct points \( z_1, z_2, \ldots, z_n \) in \( \Delta \) be given. Then (3) holds.

Proof: Define \( w_j = f(z_j) \). \( f \) lies in the hyperplane \( \Lambda \) and the element \( \varphi \) of \( \Lambda \) with minimal norm satisfies
\[
\|\varphi\|_2 \leq \|f\|_2 \leq 1.
\]
Use of the explicit expression for \( \|\varphi\|_2 \) leads to (3).

Lemma 3 Assume that \( f \) is continuous and that \( f \) satisfies (3). We shall show that \( f \in H^2 \) and that \( \|f\|_2 \leq 1 \).

Proof: We apply (3) for the case \( n = 1 \); an easy computation shows that
\[
|f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}} \tag{6}
\]
for every choice of \( z \in \Delta \).

Let \( 0 < r < \rho < 1 \), and let \( z_1, z_2, z_3, \ldots \) be an enumeration of the rational points of \( \overline{\Delta}_\rho \). For every \( n \) there is a function \( \varphi_n \) with
\[
\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n
\]
and
\[
\|\varphi_n\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k\overline{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
\]
Thus, \( \varphi_n \) lies in the unit ball of \( H^2 \), and so by lemma 2, we have for every sequence \( \zeta_1, \zeta_2, \ldots, \zeta_n \) in \( \Delta \)
\[
\sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\varphi_n(\zeta_k)\overline{\varphi_n(\zeta_l)}}{1 - \zeta_k\overline{\zeta}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
\]
It follows from (6) that
\[
|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1 - |\zeta|^2}},
\]
hence the sequence \( \varphi_1, \varphi_2, \ldots \) is uniformly bounded on \( \overline{\Delta}_\rho \). Therefore, it contains a locally uniformly convergent subsequence \( \varphi_{n_j} \). At the points \( z_1, z_2, \ldots \) the subsequence converges to \( f \). By the continuity of \( f \) and the fact that \( \{z_1, z_2, \ldots\} \) is dense in \( \Delta_\rho \) we see that
\[
\lim_{n_j \to \infty} \varphi_{n_j} = f.
\]
This shows that \( f \) is analytic on \( \Delta_\rho \) for all \( \rho < 1 \). Because of uniform convergence on \( \Gamma_r \) we have
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1.
\]
Thus, \( f \in H^2 \) and \( \|f\|_2 \leq 1 \).

Lemma 2 and lemma 3 together constitute a proof of the main result.

**Corollary** For \( f \in H^2 \) we define

\[
\nu(f) = \sup \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \overline{z_l}} : z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}.
\]

Then \( \nu(f) = \|f\|_2 \).

**Proof:** Assume that \( \nu(f) = 1 \). Then by lemma 3: \( \|f\|_2 \leq 1 \). If \( \|f\|_2 < \lambda^2 < 1 \) for some \( \lambda \), then we have \( \|f\|_2 < 1 \) but \( \nu\left(\frac{f}{\lambda}\right) = 1 \) which is impossible by lemma 2.

In a similar way we can show that \( \nu(f) = 1 \) implies that \( \nu(f) = 1 \). By the homogeneity of \( \nu \) and \( \|\cdot\|_2 \) it follows that for all \( f \in H^2 \): \( \nu(f) = \|f\|_2 \).

**Pick’s theorem**

As an application of our results we shall give a proof of Pick’s theorem.

Let \( g \) belongs to the unit ball of \( H^\infty \), and let \( z_1, z_2, \ldots, z_n \) be a sequence of mutually distinct points in \( \Delta \). Let \( w_1, w_2, \ldots, w_n \) be an arbitrary sequence of complex numbers. We consider the hyperplanes \( \Lambda \) and \( \Lambda_g \) where

\[
\Lambda_g = \{ f \in H^2 : f(z_j) = w_j \cdot g(z_j), j = 1, 2, \ldots, n \}.
\]

Of course, if \( f \in \Delta \), then \( g \cdot f \in \Delta_g \), and by Theorem 2 applied to \( \Lambda_g \) we have

\[
\|g f\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k g(z_k) \cdot w_l g(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}.
\]

Let \( \varphi \) be, as before, the element of \( \Lambda \) with smallest norm. From \( \|g\|_\infty \leq 1 \) we obtain

\[
\|g \varphi\|_2 \leq \|\varphi\|_2.
\]

Combination of these steps leads to

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} = \|\varphi\|_2^2 \geq \|g \varphi\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w_l} g(z_k) \overline{g(z_l)}}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}
\]

i.e. to

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)} \geq 0
\]

and since the sequence \( w_1, w_2, \ldots, w_n \) is arbitrary we have for all choices of \( \lambda_1, \lambda_2, \ldots, \lambda_n \)

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k \overline{z_l}} \cdot \lambda_k \lambda_l \geq 0.
\]

By the choice \( n = 1 \), \( \lambda_1 = 1 \) we see that the converse is trivial.
References

