A NEW CHARACTERIZATION OF THE UNIT BALL OF $H^2$

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Abstract

We derive a new expression for the norm of $H^2$ functions; we present some well-known results in a different setting.

Introduction

In 1915, Pick [3] proved the following result

Theorem 1 Let $g$ be an analytic function on the unit disc $\Delta$ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in $\Delta$ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z_l}} \lambda_k \overline{\lambda_l} \geq 0.$$  

(1)

Ahlfors [1] page 3, gives an elegant proof of this characterization of the unit ball of $H^\infty$.

In this note we shall present a characterization of the unit ball of $H^2$. Our main tool will be an explicit solution of the “Minimal Interpolation Problem” for $H^2$. [2] page 141. As a byproduct we obtain a new proof of Pick’s theorem.

Description of the main result

Let $z_1, z_2, \ldots, z_n$ be a sequence in $\Delta$, and let $b$ be the Blaschke product generated by this sequence:

$$b(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j}z}.$$  

(2)

We shall prove that the following conditions are equivalent for continuous functions $f$ on $\Delta$.

1) $f$ lies in the unit ball of $H^2$.

2) for every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k) \cdot \overline{f(z_l)}}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1.$$  

(3)
Preliminaries

For mutually distinct points \(z_1, z_2, \ldots, z_n\) in \(\Delta\) and for \(w_1, w_2, \ldots, w_n\) in \(\mathbb{C}\) we define
\[
\Lambda = \{ f \in H^2 : f(z_j) = w_j, \ j = 1, 2, \ldots, n \}.
\]
\(\Lambda\) is not empty; it contains the Lagrange interpolation polynomial
\[
\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) \cdot l'(z_k)},
\]
where \(l(z) = \prod_{j=1}^{n} (z - z_j)\).

In the context of \(H^p\) spaces it is more natural to work with the Blaschke interpolation function
\[
\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{z_k} z}{z - z_k} \cdot \frac{w_k}{b'(z_k)(1 - |z_k|^2)},
\]
with \(b(z)\) defined as in (2). Of course \(\beta \in \Lambda\). However, for our purposes we are better off with
\[
\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) b'(z_k)}.
\]
(4)
\(\varphi \in \Lambda\), and \(\varphi\) is analytic on some neighbourhood of \(\Delta\). \(\Lambda\) is a hyperplane in \(H^2\).

With \(\varphi\) and \(b\) defined as in (4) and (2) we have
\[
\Lambda = \{ \varphi + b \cdot g ; g \in H^2 \}.
\]

**Theorem 2** \(\varphi\) is the unique solution of the \textit{“Minimal Interpolation Problem”}, i.e. for every \(f \in \Lambda \setminus \{\varphi\}\) we have \(\|f\|_2 > \|\varphi\|_2\).

**Proof:** It suffices to show that \(\varphi \perp (f - \varphi)\) for every \(f \in \Lambda\) (since under those circumstances \(\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2\)).

From the decomposition \(f = \varphi + b \cdot g\) we have
\[
\langle f - \varphi, \varphi \rangle = \langle b \cdot g, \varphi \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it}) g(e^{it}) \varphi(e^{it}) dt
\]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it}) g(e^{it}) \sum_{k=1}^{n} \frac{w_k}{(e^{-it} - \overline{z_k}) \cdot b'(z_k)} dt.
\]
Note that \(|b(e^{it})|^2 = 1\). Thus
\[
\langle f - \varphi, \varphi \rangle = \sum_{k=1}^{n} \frac{w_k}{2\pi b'(z_k)} \int_{0}^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it} \overline{z_k}} dt
\]
\[
= \sum_{k=1}^{n} \frac{w_k}{b'(z_k)} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \overline{z_k} z} dz = 0,
\]
where \(\Gamma\) is a contour enclosing \(\Delta\). \(\Box\)
because the integrand is analytic on $\Delta$.

It will be convenient to have an explicit expression for $\|\varphi\|_2^2$.

\[
\|\varphi\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l}{b'(z_k) b'(z_l)} \int_0^{2\pi} \frac{dt}{(e^{it} - z_k)(e^{-it} - z_l)}
\]

\[
= \frac{1}{2\pi i} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l}{b'(z_k) b'(z_l)} \int_{\Gamma} \frac{dz}{(z - z_k)(1 - z_l z)} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l}{b'(z_k) b'(z_l)} \frac{1}{z_k - z_l}.
\]

There are of course many other expressions for $\|\varphi\|_2^2$.

**Theorem 3**

\[
\|\varphi\|_2 = \max \left\{ \left| \sum_{k=1}^{n} w_k f(z_k) \right| : f \in H^2, \|f\|_2 \leq 1 \right\}.
\]

**Proof:**

\[
\sum_{k=1}^{n} w_k f(z_k) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z) \varphi(z)}{b(z)} dz,
\]

hence by Schwarz’s inequality we have

\[
\left| \sum_{k=1}^{n} w_k f(z_k) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})||\varphi(e^{it})| dt \leq \|f\|_2 \cdot \|\varphi\|_2 \leq \|\varphi\|_2.
\]

Equality holds for the function $f : z \mapsto \frac{1}{\|\varphi\|_2} \sum_{k=1}^{n} \frac{w_k}{(1 - z_k z) b'(z_k)}$.

An immediate result from Theorem 2 is

**Corollary** For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of $\Delta$ we have

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \cdot \frac{1}{b'(z_k) b'(z_l)} \leq 1.
\]

**Proof:**

Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1 \in \Lambda$ and since

\[
\|1\|_2 = 1,
\]

we have

\[
1 \geq \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \cdot \frac{1}{b'(z_k) b'(z_l)}.
\]

The equality sign certainly occurs if $0 \in \{z_1, z_2, \ldots, z_n\}$:

\[
1 = \varphi(0)^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \cdot \frac{1}{b'(z_k) b'(z_l)}.
\]
If \( 0 \notin \{z_1, z_2, \ldots, z_n\} \) there is strict inequality:
Because of the uniqueness of \( \varphi \) there can be equality only if
\[
b(z) \sum_{k=1}^{n} \frac{1}{(z-z_k)b'(z_k)} = 1.
\]
In this identity for rational functions we let \( z \to \infty \). Since \( z_j \neq 0 \), \( \lim_{z \to \infty} b(z) \) has a finite value. Therefore, the left hand side has limit zero.
The fact that \( \varphi \in \Lambda \) has an interesting reformulation. We start with a lemma.

**Lemma 1** The partial fraction decomposition of \( \varphi \) is
\[
\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - z_l)(1 - z_k) b'(z_k)}.
\]

**Proof:** An elegant way to prove this is to compute both sides of the following identity.
For \( z \in \Delta \) we have
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1 - \zeta z} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1 - \zeta z} \frac{d\zeta}{\zeta},
\]
The left hand side is equal to
\[
\frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) \frac{d\zeta}{\zeta - z} = \varphi(z),
\]
while the right hand side is equal to the complex conjugate of
\[
\frac{1}{2\pi i} \int_{\Gamma} b(\zeta) \sum_{k=1}^{n} \frac{w_k}{(1 - \zeta)(1 - z_k) b'(z_k)} \frac{1}{1 - \zeta z} \frac{d\zeta}{\zeta},
\]
i.e. to the complex conjugate of
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{b(\zeta)} \sum_{k=1}^{n} \frac{w_k}{1 - \zeta z_k b'(z_k)} \frac{d\zeta}{\zeta}.
\]
Calculation of the residues at the points \( z_1, z_2, \ldots, z_n \) lead to (5).
The condition \( \varphi \in \Lambda \) implies that \( \varphi(z_j) = w_j, j = 1, \ldots, n \) i.e.
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - z_l)(1 - z_k) b'(z_k)} = w_j.
\]
This is equivalent to the assertion that the matrices
\[
B = (\beta_{lk})
\]
and its conjugate \( \overline{B} = (\overline{\beta}_{lk}) \) where
\[
\beta_{lk} = \frac{1}{(1 - z_l z_k) b'(z_k)}
\]
are each others inverse, i.e. \( B \) and \( \overline{B} \) are unitary.
Proof of the main result

**Lemma 2** Assume that $f$ lies in the unit ball of $H^2$, and let a sequence of mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ be given. Then (3) holds.

**Proof:** Define $w_j = f(z_j)$. $f$ lies in the hyperplane $\Lambda$ and the element $\varphi$ of $\Lambda$ with minimal norm satisfies

$$\|\varphi\|_2 \leq \|f\|_2 \leq 1.$$ 

Use of the explicit expression for $\|\varphi\|_2$ leads to (3).

**Lemma 3** Assume that $f$ is continuous and that $f$ satisfies (3). We shall show that $f \in H^2$ and that $\|f\|_2 \leq 1$.

**Proof:** We apply (3) for the case $n=1$; an easy computation shows that

$$|f(z)| \leq \frac{1}{\sqrt{1-|z|^2}} \quad \text{(6)}$$

for every choice of $z \in \Delta$.

Let $0 < r < \rho < 1$, and let $z_1, z_2, z_3, \ldots$ be an enumeration of the rational points of $\overline{\Delta}_\rho$. For every $n$ there is a function $\varphi_n$ with

$$\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n$$

and

$$\|\varphi_n\|_2^2 = \sum_{k=1}^n \sum_{l=1}^n \frac{f(z_k)f(z_l)}{1 - z_kz_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$ 

Thus, $\varphi_n$ lies in the unit ball of $H^2$, and so by lemma 2, we have for every sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ in $\Delta$

$$\sum_{k=1}^m \sum_{l=1}^m \frac{\varphi_n(\zeta_k)\varphi_n(\zeta_l)}{1 - \zeta_k\zeta_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$ 

It follows from (6) that

$$|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1-|\zeta|^2}},$$

hence the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded on $\overline{\Delta}_\rho$. Therefore, it contains a locally uniformly convergent subsequence $\varphi_{n_j}$. At the points $z_1, z_2, \ldots$ the subsequence converges to $f$. By the continuity of $f$ and the fact that $\{z_1, z_2, \ldots\}$ is dense in $\Delta_\rho$ we see that

$$\lim_{n_j \to \infty} \varphi_{n_j} = f.$$ 

This shows that $f$ is analytic on $\Delta_\rho$ for all $\rho < 1$. Because of uniform convergence on $\Gamma_r$ we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1.$$
Thus, $f \in H^2$ and $\|f\|_2 \leq 1$.
Lemma 2 and lemma 3 together constitute a proof of the main result.

**Corollary** For $f \in H^2$ we define

$$
\nu(f) = \sup \left\{ \sum_{k=1}^n \sum_{l=1}^n \frac{f(z_k)g(z_l)}{1 - z_k \bar{z}_l} \frac{1}{b'(z_k)b'(z_l)} : z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}.
$$

Then $\nu(f) = \|f\|_2^2$.

**Proof:** Assume that $\nu(f) = 1$. Then by lemma 3: $\|f\|_2^2 \leq 1$. If $\|f\|_2^2 < \lambda^2 < 1$ for some $\lambda$, then we have $\|f\|_2 < 1$ but $\nu \left( \frac{f}{\lambda} \right) > 1$ which is impossible by lemma 2.

In a similar way we can show that $\|f\|_2^2 = 1$ implies that $\nu(f) = 1$. By the homogeneity of $\nu$ and $\|\cdot\|_2^2$ it follows that for all $f \in H^2$: $\nu(f) = \|f\|_2^2$.

**Pick’s theorem**

As an application of our results we shall give a proof of Pick’s theorem.

Let $g$ belongs to the unit ball of $H^\infty$, and let $z_1, z_2, \ldots, z_n$ be a sequence of mutually distinct points in $\Delta$. Let $w_1, w_2, \ldots, w_n$ be an arbitrary sequence of complex numbers. We consider the hyperplanes $\Lambda$ and $\Lambda_g$ where

$$
\Lambda_g = \{ f \in H^2 : f(z_j) = w_j \cdot g(z_j), j = 1, 2, \ldots, n \}.
$$

Of course, if $f \in \Delta$, then $g \cdot f \in \Lambda_g$, and by Theorem 2 applied to $\Lambda_g$ we have

$$
\|g f\|_2^2 \geq \sum_{k=1}^n \sum_{l=1}^n \frac{w_k g(z_k) \cdot w_l g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)}.
$$

Let $\varphi$ be, as before, the element of $\Lambda$ with smallest norm. From $\|g\|_\infty \leq 1$ we obtain

$$
\| g \varphi \|_2 \leq \| \varphi \|_2.
$$

Combination of these steps leads to

$$
\sum_{k=1}^n \sum_{l=1}^n \frac{w_k w_l}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} = \| \varphi \|_2^2 \geq \| g \varphi \|_2^2 \geq \sum_{k=1}^n \sum_{l=1}^n \frac{w_k \overline{w_l} g(z_k)g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)}
$$

i.e.

$$
\sum_{k=1}^n \sum_{l=1}^n \frac{1 - g(z_k)g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)} \geq 0
$$

and since the sequence $w_1, w_2, \ldots, w_n$ is arbitrary we have for all choices of $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$
\sum_{k=1}^n \sum_{l=1}^n \frac{1 - g(z_k)g(z_l)}{1 - z_k \bar{z}_l} \cdot \lambda_k \lambda_l \geq 0.
$$

By the choice $n = 1$, $\lambda_1 = 1$ we see that the converse is trivial.
References

