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ASYMPTOTIC INFERENCE FOR AN UNSTABLE TRIANGULAR SPATIAL AR MODEL

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Abstract

A spatial autoregressive process is investigated, where the autoregressive coefficients are equal, and their sum is one. It is shown that the limiting distribution of the least squares estimator for this coefficient is normal and, in contrast to the doubly geometric process, the rate of convergence is $n^{-5/4}$.

1 Introduction

Consider the AR(1) time series model

$$X_k = \begin{cases} \alpha X_{k-1} + \varepsilon_k, & k \geqslant 1, \\ 0, & k = 0. \end{cases}$$

The least squares estimator $\widehat{\alpha}_n$ of α based on the observations $\{X_k : k = 1, \dots, n\}$ is

$$\widehat{\alpha}_n = \frac{\sum_{k=1}^n X_{k-1}X_k}{\sum_{k=1}^n X_{k-1}^2}.$$

It is well known that in the stable (or, in other words, asymptotically stationary) case when $|\alpha| < 1$, the sequence $(\widehat{\alpha}_n)_{n\geqslant 1}$ is asymptotically normal [Mann and Wald (1943), Anderson (1959)], namely,

$$n^{1/2}(\widehat{\alpha}_n - \alpha) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1-\alpha^2).$$

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In the unstable (or, in other words, unit root) case when $\alpha = 1$, the sequence $(\widehat{\alpha}_n)_{n \geqslant 1}$ is not asymptotically normal but

$$n(\widehat{\alpha}_n - 1) \xrightarrow{\mathcal{D}} \frac{\int_0^1 W(t)\,\mathrm{d}W(t)}{\int_0^1 W^2(t)\,dt},$$

where $\{W(t) : t \in [0,1]\}$ denotes a standard Wiener process [White (1958), Bobkoski (1983), Phillips (1987), Chan and Wei (1987)].

The analysis of spatial models is of interest in many different fields such as geography, geology, biology and agriculture. See Basu and Reinsel (1993) for a discussion on these applications.

The only spatial autoregressive model for which unstability has been studied is the so called doubly geometric spatial autoregressive process

$$X_{k,\ell} = \begin{cases} \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} - \alpha\beta X_{k-1,\ell-1} + \varepsilon_{k,\ell}, & k, \ell \geqslant 1, \\ 0, & k = 0 \ \text{ or } \ \ell = 0, \end{cases}$$

introduced by Martin (1979). It is, in fact, the simplest spatial model, since its nice product structure ensures that it can be considered as some kind of combination of two autoregressive processes on the line, and several properties can be derived by the analogy of one–dimensional autoregressive processes. This model has been used by Jain (1981) in the study of image processing, by Martin (1990), Cullis and Gleeson (1991), Basu and Reinsel (1994) in agricultural trials and by Tjøstheim (1981) in digital filtering.

In the stable (i.e., asymptotically stationary) case when $|\alpha| < 1$ and $|\beta| < 1$, asymptotic normality of several estimators $(\widehat{\alpha}_n, \widehat{\beta}_n)$ of (α, β) based on the observations $\{X_{k,\ell} : k, \ell = 1, \dots, n\}$ has been shown [e.g., Tjøstheim (1978), (1983), Basu and Reinsel (1992, (1993)], namely,

$$n \begin{pmatrix} \widehat{\alpha}_n - \alpha \\ \widehat{\beta}_n - \beta \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_{\alpha,\beta})$$

with some covariance matrix $\Sigma_{\alpha,\beta}$.

In the unstable (i.e., unit root) case when $\alpha = \beta = 1$, in contrast to the AR(1) model, the sequence of Gauss–Newton estimators $(\widehat{\alpha}_n, \widehat{\beta}_n)$ of (α, β) has been shown to be asymptotically normal [Bhattacharyya, Khalil and Richardson (1996), Bhattacharyya Richardson and Franklin (1997)], namely,

$$n^{3/2} \begin{pmatrix} \widehat{\alpha}_n - \alpha \\ \widehat{\beta}_n - \beta \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)$$

with some covariance matrix Σ. In the unstable case $\alpha = 1$, $|\beta| < 1$ the least squares estimator turns out to be asymptotically normal again [Bhattacharyya, Khalil and Richardson (1996)].

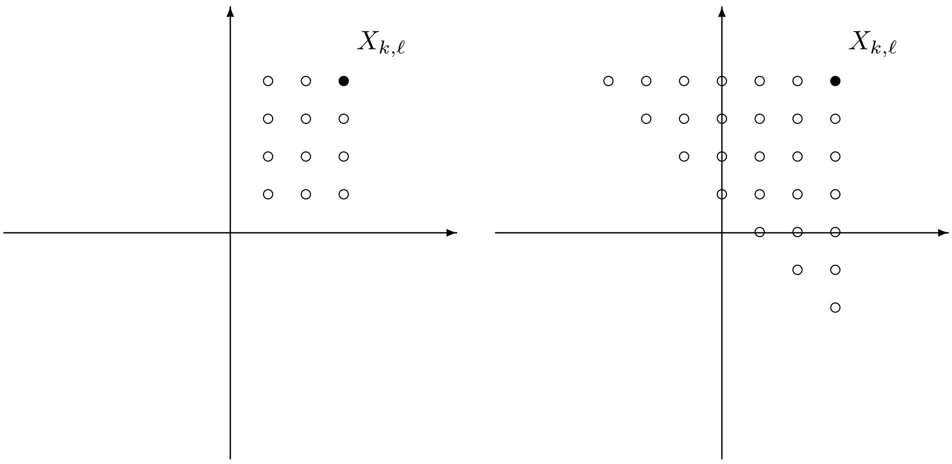

Figure 1: Rectangular and triangular spatial models

In the present paper we study asymptotic properties of the least squares estimator in a spatial model which can be considered as the simplest spatial model, that can not be reduced somehow to autoregressive models on the line (like the doubly geometric model). Note that, because of technical reasons, we consider a triangular model in the sense that $X_{k,\ell}$ can be expressed as a linear combination of $\varepsilon_{i,j}$ from a region of triangular shape (unlike in the above doubly geometric model, where $X_{k,\ell}$ is a linear combination of $\varepsilon_{i,j}$ from a region of rectangular shape; see Figure 1). Eventually, our model is generated on a halfplane (unlike the doubly geometric model, which is generated on the positive quadrant only). We study the stable as well as the unstable case with a unified method which is based on the martingale central limit theorem. Note that the stable case could have been studied in the usual way, i.e., first to prove the result for the stationary solution of the equation (considering it on the whole lattice $\mathbb{Z}^2$), and then to show that the result is the same for the zero start model (examining the difference of the two sequences of estimators). However, we find it interesting to compare the stable and unstable cases by our unified approach. We will find a rather peculiar limiting behaviour of the covariance structure in the unstable case (see Proposition 2.4), and we show that the normalising factor in our unstable model differs from that in the doubly geometric model.

Our zero start triangular spatial autoregressive process $\{X_{k,\ell} : k, \ell \in \mathbb{Z},\ k+\ell \geqslant 0\}$ is defined as

$$X_{k,\ell} = \begin{cases} \alpha(X_{k-1,\ell} + X_{k,\ell-1}) + \varepsilon_{k,\ell}, & \text{for } k+\ell \geqslant 1, \\ 0, & \text{for } k+\ell = 0. \end{cases} \tag{1.1}$$

This model is stable (i.e., asymptotically stationary) in case $|\alpha| < 1/2$ [Whittle (1954), Besag (1972), Basu and Reinsel (1993)], and unstable (i.e., unit root) if $|\alpha| = 1/2$.

For a set $H \subset \{(k,\ell) \in \mathbb{Z}^2 : k+\ell \geqslant 1\}$, the least squares estimator $\widehat{\alpha}_H$ of α based on the observations $\{X_{k,\ell} : (k,\ell) \in H\}$ can be obtained by minimizing the sum of squares

$$\sum_{(k,\ell)\in H} \big(X_{k,\ell} - \alpha(X_{k-1,\ell} + X_{k,\ell-1})\big)^2$$

with respect to α, and it has the form

$$\widehat{\alpha}_H = \frac{\sum_{(k,\ell)\in H}(X_{k-1,\ell} + X_{k,\ell-1})X_{k,\ell}}{\sum_{(k,\ell)\in H}(X_{k-1,\ell} + X_{k,\ell-1})^2}.$$

For $k, \ell \in \mathbb{Z}$, consider the triangle

$$T_{k,\ell} := \{(i,j) \in \mathbb{Z}^2 : i+j \geqslant 1,\ i \leqslant k \ \text{ and } \ j \leqslant \ell\}.$$

Note, that $T_{k,\ell} = \emptyset$ if $k+\ell \leqslant 0$. For simplicity, we shall write $T_n := T_{n,n}$.

1.1 Theorem. *Let $\{\varepsilon_{k,\ell} : k, \ell \in \mathbb{Z},\ k+\ell \geqslant 1\}$ be independent random variables with $\mathsf{E}\,\varepsilon_{k,\ell} = 0$, $\mathsf{Var}\,\varepsilon_{k,\ell} = 1$ and $\sup\{\mathsf{E}\,\varepsilon_{k,\ell}^4 : k, \ell \in \mathbb{Z},\ k+\ell \geqslant 1\} < \infty$. Assume that the model (1.1) is satisfied.*

If $|\alpha| < 1/2$ then

$$(mn)^{1/2}\left(\widehat{\alpha}_{T_{m,n}} - \alpha\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\alpha^2) \qquad \text{as } m, n \to \infty \text{ with } m/n \to \text{constant} > 0,$$

where

$$\sigma_\alpha^2 := \begin{cases} \dfrac{\alpha^2}{2\big((1-4\alpha^2)^{-1/2} - 1\big)}, & \text{for } \alpha \neq 0, \\ 1/4, & \text{for } \alpha = 0. \end{cases}$$

If $|\alpha| = 1/2$ then

$$(mn)^{5/8}\left(\widehat{\alpha}_{T_{m,n}} - \alpha\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \qquad \text{as } m, n \to \infty \text{ with } m/n \to \text{constant} > 0,$$

where

$$\sigma^2 := \frac{15\sqrt{\pi}}{2^{15/2}}.$$

(Note that $\sigma_0^2 = \lim_{\alpha\to 0}\sigma_\alpha^2$, but $\sigma^2 \neq \lim_{\alpha\to 1/2}\sigma_\alpha^2 = 0$.)

For the sake of simplicity, we carry out the proof only for $m = n$. The general case can be handled with slight modifications. We can write

$$\widehat{\alpha}_{T_n} - \alpha = \frac{A_n}{B_n}$$

with

$$\begin{aligned} A_n &:= \sum_{(k,\ell)\in T_n} (X_{k-1,\ell} + X_{k,\ell-1})\varepsilon_{k,\ell}, \\ B_n &:= \sum_{(k,\ell)\in T_n} (X_{k-1,\ell} + X_{k,\ell-1})^2, \end{aligned}$$

hence the statement of Theorem 1.1 in case $m = n$ is a consequence of the following two propositions.

1.2 Proposition. *If* $|\alpha| < 1/2$ *then*

$$n^{-2} B_n \stackrel{\mathsf{P}}{\longrightarrow} \frac{1}{\sigma_\alpha^2} \qquad \text{as } n \to \infty.$$

If $|\alpha| = 1/2$ *then*

$$n^{-5/2} B_n \stackrel{\mathsf{P}}{\longrightarrow} \frac{1}{\sigma^2} \qquad \text{as } n \to \infty.$$

1.3 Proposition. *If* $|\alpha| < 1/2$ *then*

$$n^{-1} A_n \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}\left(0, \frac{1}{\sigma_\alpha^2}\right) \qquad \text{as } n \to \infty.$$

If $|\alpha| = 1/2$ *then*

$$n^{-5/4} A_n \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}\left(0, \frac{1}{\sigma^2}\right) \qquad \text{as } n \to \infty.$$

1.4 Corollary. *If* $|\alpha| \leqslant 1/2$ *then*

$$\left(\sum_{(k,\ell)\in T_{m,n}} (X_{k-1,\ell} + X_{k,\ell-1})^2\right)^{1/2} \left(\widehat{\alpha}_{T_{m,n}} - \alpha\right) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}(0,1)$$

as $m, n \to \infty$ *with* $m/n \to$ constant > 0.

The aim of the following discussion is to show that it suffices to prove Propositions 1.2 and 1.3 for $0 \leqslant \alpha \leqslant 1/2$. First we note that the random variable $X_{k,\ell}$ can be expressed as a linear combination of the variables $\{\varepsilon_{i,j} : (i,j) \in T_{k,\ell}\}$, namely,

$$X_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \binom{k+\ell-i-j}{k-i} \alpha^{k+\ell-i-j} \varepsilon_{i,j}, \tag{1.2}$$

for $k, \ell \in \mathbb{Z}$ with $k + \ell \geqslant 0$, where the sum is defined to be 0 if $T_{k,\ell} = \emptyset$, which is the case if $k + \ell = 0$. Now put $\widetilde{\varepsilon}_{k,\ell} := (-1)^{k+\ell} \varepsilon_{k,\ell}$ for $k, \ell \in \mathbb{Z}$ with $k + \ell \geqslant 1$. Then $\{\widetilde{\varepsilon}_{k,\ell} : k, \ell \in \mathbb{Z}, \, k+\ell \geqslant 1\}$ are independent random variables with $\mathsf{E}\, \widetilde{\varepsilon}_{k,\ell} = 0$, $\mathsf{Var}\, \widetilde{\varepsilon}_{k,\ell} = 1$ and $\sup\{\mathsf{E}\, \widetilde{\varepsilon}_{k,\ell}^4 : k, \ell \in \mathbb{Z}, \, k + \ell \geqslant 0\} < \infty$. Consider the zero start triangular spatial AR process $\{\widetilde{X}_{k,\ell} : k, \ell \in \mathbb{Z}, \, k + \ell \geqslant 0\}$ defined by

$$\widetilde{X}_{k,\ell} = \begin{cases} -\alpha(\widetilde{X}_{k-1,\ell} + \widetilde{X}_{k,\ell-1}) + \widetilde{\varepsilon}_{k,\ell}, & \text{for } k + \ell \geqslant 1, \\ 0, & \text{for } k + \ell = 0. \end{cases}$$

Then, by the representation (1.2),

$$\widetilde{X}_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \binom{k+\ell-i-j}{k-i} (-\alpha)^{k+\ell-i-j} \widetilde{\varepsilon}_{i,j} = (-1)^{k+\ell} X_{k,\ell},$$

for $k, \ell \in \mathbb{Z}$ with $k + \ell \geqslant 0$. Hence,

$$\widetilde{A}_n := \sum_{(k,\ell)\in T_n} (\widetilde{X}_{k-1,\ell} + \widetilde{X}_{k,\ell-1}) \widetilde{\varepsilon}_{k,\ell} = -A_n,$$

$$\widetilde{B}_n := \sum_{(k,\ell)\in T_n} (\widetilde{X}_{k-1,\ell} + \widetilde{X}_{k,\ell-1})^2 = B_n.$$

Consequently, in order to prove Propositions 1.2 and 1.3 for $-1/2 \leqslant \alpha < 0$ it suffices to prove them for $0 < \alpha \leqslant 1/2$.

The proof of Propositions 1.2 and 1.3 are provided in Sections 3 and 4, respectively. Section 2 is devoted to the limiting behaviour of the covariance structure of the random field $\{X_{k,\ell} : k, \ell \in \mathbb{Z}, \, k + \ell \geqslant 0\}$.

2 Covariance structure

2.1 Lemma. *For $k_1, \ell_1, k_2, \ell_2 \in \mathbb{Z}$ with $k_1 + \ell_1 \geqslant 0$ and $k_2 + \ell_2 \geqslant 0$, and for all $\alpha \in \mathbb{R}$,*

$$\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) = \sum_{m=1}^{k_1\wedge k_2 + \ell_1 \wedge \ell_2} \binom{k_1+k_2+\ell_1+\ell_2-2m}{k_1+\ell_2-m} \alpha^{k_1+k_2+\ell_1+\ell_2-2m},$$

where an empty sum is defined to be 0, and $x \wedge y := \min\{x; y\}$ for $x, y \in \mathbb{R}$.

Proof. By the representation (1.2),

$$\begin{aligned}&\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) \\ &= \sum_{(i,j)\in T_{k_1,\ell_1}\cap T_{k_2,\ell_2}} \binom{k_1+\ell_1-i-j}{k_1-i}\binom{k_2+\ell_2-i-j}{k_2-i}\alpha^{k_1+\ell_1+k_2+\ell_2-2i-2j}.\end{aligned} \tag{2.1}$$

Obviously, $T_{k_1,\ell_1}\cap T_{k_2,\ell_2} = T_{k_1\wedge k_2,\, \ell_1\wedge \ell_2}$, hence substituting $m := i+j$ we obtain

$$\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) = \sum_{m=1}^{k_1\wedge k_2+\ell_1\wedge\ell_2} \alpha^{k_1+k_2+\ell_1+\ell_2-2m} S_{m,k_1,\ell_1,k_2,\ell_2},$$

where

$$S_{m,k_1,\ell_1,k_2,\ell_2} := \sum_{i=m-\ell_1\wedge\ell_2}^{k_1\wedge k_2} \binom{k_1+\ell_1-m}{k_1-i}\binom{k_2+\ell_2-m}{\ell_2-m+i}.$$

We have

$$S_{m,k_1,\ell_1,k_2,\ell_2} = \binom{k_1+k_2+\ell_1+\ell_2-2m}{k_1+\ell_2-m}$$

applying a simple combinatorial identity. □

In the sequel we make use of the following basic lemma.

2.2 Lemma. *There exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\binom{k}{j} 2^{-k} \leqslant \frac{c_1}{\sqrt{k}}, \tag{2.2}$$

and

$$\left| \binom{k}{j} 2^{-k} - \frac{1}{\sqrt{\pi k/2}} \exp\left\{ -\frac{(j-k/2)^2}{k/2} \right\} \right| \leqslant \frac{c_2}{k}, \tag{2.3}$$

for all integers $k \geqslant 1$ *and* $j \in \{0,1,\dots,k\}$.

Proof. The inequality (2.3) is a special case of the expansion in the local central limit theorem [see Chapter VII, Theorem 6 in Petrov (1975)] for Bernoulli random variables. The inequality (2.2) is a consequence of (2.3). □

2.3 Lemma. *Let* $k_1,\ell_1,k_2,\ell_2 \in \mathbb{Z}$ *with* $k_1+\ell_1 \geqslant 0$ *and* $k_2+\ell_2 \geqslant 0$.

If $|\alpha| < 1/2$ *then*

$$|\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant \frac{(2|\alpha|)^{|k_1-k_2|+|\ell_1-\ell_2|}}{1-4\alpha^2}.$$

If $|\alpha| = 1/2$ *then there exists a constant* $C > 0$ *such that*

$$|\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant C\sqrt{k_1+\ell_1+k_2+\ell_2}.$$

Proof. Lemma 2.1 and the inequality $\binom{n}{k} \leqslant 2^n$ for $n = 0, 1, \ldots,$ $0 \leqslant k \leqslant n$, imply

$$|\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant \sum_{m=1}^{k_1\wedge k_2+\ell_1\wedge\ell_2} (2|\alpha|)^{k_1+k_2+\ell_1+\ell_2-2m}.$$

Substituting $j := k_1 \wedge k_2 + \ell_1 \wedge \ell_2 - m$ and using $x + y - 2(x \wedge y) = |x - y|$, $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned}|\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| &\leqslant \sum_{j=0}^{k_1\wedge k_2+\ell_1\wedge\ell_2-1} (2|\alpha|)^{|k_1-k_2|+|\ell_1-\ell_2|+2j} \\ &\leqslant (2|\alpha|)^{|k_1-k_2|+|\ell_1-\ell_2|} \sum_{j=0}^{\infty} (2\alpha)^{2j},\end{aligned}$$

hence we get the statement in case $|\alpha| < 1/2$.

Now let $|\alpha| = 1/2$. If $k_1 + \ell_1 = 0$ or $k_2 + \ell_2 = 0$ then $\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) = 0$. By Lemma 2.1,

$$|\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant \sum_{m=1}^{k_1\wedge k_2+\ell_1\wedge\ell_2} b_m(k_1, \ell_1, k_2, \ell_2),$$

where

$$b_m(k_1, \ell_1, k_2, \ell_2) := \binom{k_1 + k_2 + \ell_1 + \ell_2 - 2m}{k_1 + \ell_2 - m} 2^{-(k_1+k_2+\ell_1+\ell_2-2m)}.$$

Applying the inequality (2.2) of Lemma 2.2, we obtain

$$b_m(k_1, \ell_1, k_2, \ell_2) \leqslant c_1(k_1 + k_2 + \ell_1 + \ell_2 - 2m)^{-1/2}$$

if $k_1 + k_2 + \ell_1 + \ell_2 - 2m \geqslant 1$, which holds if $m \leqslant k_1 \wedge k_2 + \ell_1 \wedge \ell_2 - 1$, since then $k_1 + k_2 + \ell_1 + \ell_2 - 2m \geqslant |k_1 - k_2| + |\ell_1 - \ell_2| + 2 \geqslant 2$. Moreover, $b_m(k_1, \ell_1, k_2, \ell_2) \leqslant 1$ is always satisfied, hence

$$\begin{aligned}|\mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| &\leqslant 1 + \sum_{m=1}^{k_1\wedge k_2+\ell_1\wedge\ell_2-1} \frac{c_1}{\sqrt{k_1 + k_2 + \ell_1 + \ell_2 - 2m}} \\ &\leqslant 1 + c_1 \int_0^{k_1\wedge k_2+\ell_1\wedge\ell_2} \frac{\mathrm{d}x}{\sqrt{k_1 + k_2 + \ell_1 + \ell_2 - 2x}} \\ &= 1 + c_1 \left(\sqrt{k_1 + k_2 + \ell_1 + \ell_2} - \sqrt{|k_1 - k_2| + |\ell_1 - \ell_2|} \right) \\ &\leqslant (1 + c_1)\sqrt{k_1 + k_2 + \ell_1 + \ell_2},\end{aligned}$$

using again the identity $|x - y| + 2(x \wedge y) = x + y$ for $x, y \in \mathbb{R}$. □

For $n \in \mathbb{N}$, let us introduce the piecewise constant random fields

$$Y^{(n)}(s, t) := X_{[ns]+1,[nt]+1}, \quad Z^{(n)}(s, t) := n^{-1/4} X_{[ns]+1,[nt]+1}, \quad s, t \in \mathbb{R}, \quad s + t \geqslant 0.$$

2.4 Proposition. *Let* $s_1, t_1, s_2, t_2 \in \mathbb{R}$ *with* $s_1 + t_1 > 0$, $s_2 + t_2 > 0$.

If $|\alpha| < 1/2$ *then*

$$\lim_{n\to\infty} \mathsf{Cov}(Y^{(n)}(s_1,t_1), Y^{(n)}(s_2,t_2)) = \begin{cases} (1-4\alpha^2)^{-1/2}, & \text{if } s_1 = s_2, \ t_1 = t_2, \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha = 1/2$ *then*

$$\lim_{n\to\infty} \mathsf{Cov}(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)) = K(s_1,t_1,s_2,t_2),$$

where

$$K(s_1,t_1,s_2,t_2) = \begin{cases} \sqrt{\frac{2}{\pi}}\Big(\sqrt{s_1+s_2+t_1+t_2} - \sqrt{|s_1-s_2|+|t_1-t_2|}\Big), & \text{if } s_1 - s_2 = t_1 - t_2, \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha = -1/2$ *then*

$$\lim_{n\to\infty} (-1)^{[ns_1]+[nt_1]+[ns_2]+[nt_2]} \mathsf{Cov}(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)) = K(s_1,t_1,s_2,t_2).$$

2.5 Remark. In case $\alpha = -1/2$ and $s_1 - s_2 \neq t_1 - t_2$ one can easily derive that $\lim_{n\to\infty} \mathsf{Cov}(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)) = 0$, but the sequence $\big(\mathsf{Cov}(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2))\big)_{n\geqslant 1}$ is not convergent for certain s_1, t_1, s_2, t_2 with $s_1 - s_2 = t_1 - t_2$, namely, if the sequence $\big((-1)^{[ns_1]+[nt_1]+[ns_2]+[nt_2]}\big)_{n\geqslant 1}$ is not convergent.

Proof of Proposition 2.4. Let $|\alpha| < 1/2$. By Lemma 2.3,

$$|\mathsf{Cov}(Y^{(n)}(s_1,t_1), Y^{(n)}(s_2,t_2))| \leqslant (1-4\alpha^2)^{-1/2}(2|\alpha|)^{|[ns_1]-[ns_2]|+|[nt_1]-[nt_2]|}.$$

If $s_1 \neq s_2$ then $\mathsf{Cov}(Y^{(n)}(s_1,t_1), Y^{(n)}(s_2,t_2)) \to 0$, since $\big|[ns_1]-[ns_2]\big| \geqslant n|s_1-s_2|/2$ for all sufficiently large n. By symmetry, we also conclude the statement if $t_1 \neq t_2$.

Let $s_1 = s_2$, $t_1 = t_2$ and $s_1 + t_1 > 0$. Then $[ns_1] + [nt_1] \to \infty$ as $n \to \infty$. Hence by Lemma 2.1,

$$\mathsf{Cov}(Y^{(n)}(s_1,t_1), Y^{(n)}(s_2,t_2)) = \sum_{m=0}^{[ns_1]+[nt_1]+1} \binom{2m}{m} \alpha^{2m} \to \frac{1}{\sqrt{1-4\alpha^2}},$$

as $n \to \infty$. We finished the proof in the case $|\alpha| < 1/2$.

Now let $\alpha = 1/2$. By Lemma 2.1,

$$\mathsf{Cov}\big(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)\big) = \frac{1}{\sqrt{n}} \sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]+2} b_{n,m}(s_1,t_1,s_2,t_2), \quad (2.4)$$

where

$$\begin{aligned}
&b_{n,m}(s_1,t_1,s_2,t_2)\\
&:= \binom{[ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m}{[ns_1]+[nt_2]+2-m} 2^{-2([ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m)}.
\end{aligned}$$

Examining the limit as $n \to \infty$, we may omit finitely many terms from the sum in (2.4), since $b_{n,m}(s_1,t_1,s_2,t_2) \leqslant 1$. We consider only the terms in (2.4) with

$$m \in \{2,3,\ldots,[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]\} := H_n(s_1,t_1,s_2,t_2).$$

We want to apply Lemma 2.2 to estimate $b_{n,m}(s_1,t_1,s_2,t_2)$. Let

$$k := [ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m, \qquad j := [ns_1]+[nt_2]+2-m.$$

Then $k \geqslant 1$ for all $m \in H_n(s_1,t_1,s_2,t_2)$ and $j-k/2 = \big([ns_1]-[ns_2]-[nt_1]+[nt_2]\big)/2$, hence

$$|b_{n,m}(s_1,t_1,s_2,t_2) - b^*_{n,m}(s_1,t_1,s_2,t_2)| \leqslant \frac{c_2}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m},$$

where

$$b^*_{n,m}(s_1,t_1,s_2,t_2) := \frac{\exp\left\{-\frac{([ns_1]-[ns_2]-[nt_1]+[nt_2])^2}{2([ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m)}\right\}}{\sqrt{\pi([ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m)/2}}.$$

Obviously, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{m=2}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]} \frac{c_2}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m}\\
&\leqslant \frac{c_2}{\sqrt{n}} \int_2^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]+1} \frac{\mathrm{d}x}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2x}\\
&\leqslant \frac{c_2}{2\sqrt{n}} \log([ns_1]+[ns_2]+[nt_1]+[nt_2]) \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

hence, examining the limit of (2.4) as $n \to \infty$, we may replace $b_{n,m}(s_1,t_1,s_2,t_2)$ by $b^*_{n,m}(s_1,t_1,s_2,t_2)$.

In case $s_1 - s_2 \neq t_1 - t_2$ we have for all sufficiently large n and for all $m \in H_n(s_1,t_1,s_2,t_2)$ the inequalities

$$\begin{gathered}
\big|[ns_1]-[ns_2]-[nt_1]+[nt_2]\big| \geqslant |s_1-s_2-t_1+t_2|n/2,\\
2 \leqslant [ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m \leqslant (s_1+s_2+t_1+t_2)n,
\end{gathered}$$

consequently,

$$b^*_{n,m}(s_1,t_1,s_2,t_2) \leqslant \pi^{-1/2} \exp\left\{-\frac{(s_1-s_2-t_1+t_2)^2}{8(s_1+s_2+t_1+t_2)} n\right\}.$$

This implies

$$\frac{1}{\sqrt{n}} \sum_{m=2}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]} b^*_{n,m}(s_1,t_1,s_2,t_2) \to 0 \qquad \text{as} \quad n\to\infty,$$

hence we obtain the statement for $s_1-s_2 \ne t_1-t_2$.

In case $s_1-s_2 = t_1-t_2$ we have for all sufficiently large n and for all $m \in H_n(s_1,t_1,s_2,t_2)$ the inequalities

$$\begin{aligned}\big|[ns_1]-[ns_2]-[nt_1]+[nt_2]\big| &\leqslant 2,\\ [ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m &\geqslant \big(|s_1-s_2|+|t_1-t_2|\big)n/2,\end{aligned}$$

consequently,

$$c_n(s_1,t_1,s_2,t_2) b^{**}_{n,m}(s_1,t_1,s_2,t_2) \leqslant b^*_{n,m}(s_1,t_1,s_2,t_2) \leqslant b^{**}_{n,m}(s_1,t_1,s_2,t_2),$$

where

$$b^{**}_{n,m}(s_1,t_1,s_2,t_2) := \frac{1}{\sqrt{\pi([ns_1]+[ns_2]+[nt_1]+[nt_2]+4-2m)/2}},$$

$$c_n(s_1,t_1,s_2,t_2) := \begin{cases} \exp\Big\{-\frac{4}{(|s_1-s_2|+|t_1-t_2|)n}\Big\} & \text{if } s_1-s_2=t_1-t_2 \ne 0,\\ 1 & \text{if } s_1-s_2=t_1-t_2=0.\end{cases}$$

Clearly

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{m=2}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]} b^{**}_{n,m}(s_1,t_1,s_2,t_2)\\ &= \int_0^{s_1\wedge s_2+t_1\wedge t_2} \frac{\mathrm{d}x}{\sqrt{\pi(s_1+s_2+t_1+t_2-2x)/2}} = K(s_1,t_1,s_2,t_2),\end{aligned}$$

hence $\lim_{n\to\infty} c_n(s_1,t_1,s_2,t_2)=1$ implies the statement for $s_1-s_2=t_1-t_2$.

Now let $\alpha=-1/2$. Using the zero start triangular spatial AR process $\{\widetilde{X}_{k,\ell} : k,\ell\in\mathbb{Z},\, k+\ell\geqslant 0\}$ with parameter $-\alpha=1/2$ introduced in Section 1, we have $X_{k,\ell}=(-1)^{k+\ell}\widetilde{X}_{k,\ell}$ for $k,\ell\in\mathbb{Z}$ with $k+\ell\geqslant 0$. Consequently,

$$\mathsf{Cov}\big(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)\big) = (-1)^{[ns_1]+[nt_1]+[ns_2]+[nt_2]} \mathsf{Cov}\big(\widetilde{Z}^{(n)}(s_1,t_1), \widetilde{Z}^{(n)}(s_2,t_2)\big).$$

The proof is complete. $\square$

In order to estimate covariances and moments we make use the following two lemmata.

2.6 Lemma. *Let* $\xi_1, \ldots, \xi_N$ *be independent random variables with* $\mathsf{E}\xi_i = 0$, $\mathsf{E}\xi_i^2 = 1$ *for all* $i = 1, \ldots, N$, *and* $M_4 := \max_{1 \leqslant i \leqslant N} \mathsf{E}\xi_i^4 < \infty$. *Let* $a_1, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{R}$ *with* $a_i b_i \geqslant 0$ *for all* $i = 1, \ldots, N$. *Let*

$$X := \sum_{i=1}^N a_i \xi_i, \qquad Y := \sum_{j=1}^N b_j \xi_j.$$

Then

$$0 \leqslant \mathsf{Cov}(X^2, Y^2) \leqslant 2M_4 \mathsf{Cov}(X, Y)^2, \qquad \mathsf{E}X^2 Y^2 \leqslant 3M_4 \mathsf{E}X^2 \mathsf{E}Y^2.$$

Proof. We have

$$\mathsf{Cov}(X^2, Y^2) = \sum_{i_1, i_2, j_1, j_2 = 1}^N a_{i_1} a_{i_2} b_{j_1} b_{j_2} \mathsf{Cov}(\xi_{i_1}\xi_{i_2}, \xi_{j_1}\xi_{j_2}).$$

It is easy to check that

$$\mathsf{E}\xi_{i_1}\xi_{i_2}\xi_{j_1}\xi_{j_2} = \begin{cases} \mathsf{E}\xi_{i_1}^4, & \text{if } i_1 = i_2 = j_1 = j_2, \\ 1, & \text{if } i_1 = i_2 \neq j_1 = j_2, \text{ or } i_1 = j_1 \neq i_2 = j_2 \text{ or } i_1 = j_2 \neq i_2 = j_1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\mathsf{Cov}(\xi_{i_1}\xi_{i_2}, \xi_{j_1}\xi_{j_2}) = \begin{cases} \mathsf{E}\xi_{i_1}^4 - 1, & \text{if } i_1 = i_2 = j_1 = j_2, \\ 1, & \text{if } i_1 = j_1 \neq i_2 = j_2 \text{ or } i_1 = j_2 \neq i_2 = j_1, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently,

$$\mathsf{Cov}(X^2, Y^2) = \sum_{i=1}^N a_i^2 b_i^2 (\mathsf{E}\xi_i^4 - 1) + 4 \sum_{1 \leqslant i < j \leqslant N} a_i a_j b_i b_j \geqslant 0$$

by the assumption $a_i b_i \geqslant 0$, $i = 1, \ldots, N$, and by the inequality $\mathsf{E}\xi_i^4 \geqslant (\mathsf{E}\xi_i^2)^2 = 1$. Moreover,

$$\mathsf{Cov}(X^2, Y^2) = \sum_{i=1}^N a_i^2 b_i^2 (\mathsf{E}\xi_i^4 - 3) + 2 \left(\sum_{i=1}^N a_i b_i \right)^2 \leqslant \big((M_4 - 3)^+ + 2\big) \left(\sum_{i=1}^N a_i b_i \right)^2,$$

since $\sum_{i=1}^N a_i^2 b_i^2 \leqslant \left(\sum_{i=1}^N a_i b_i \right)^2$ follows from the assumption $a_i b_i \geqslant 0$, $i = 1, \ldots, N$. Furthermore,

$$\mathsf{Cov}(X, Y) = \sum_{i, j=1}^N a_i b_j \mathsf{Cov}(\xi_i, \xi_j) = \sum_{i=1}^N a_i b_i.$$

Clearly $M_4 \geqslant 1$ implies $(M_4-3)^+ + 2 \leqslant 2M_4$, hence we obtain the first inequality.

Further,

$$\begin{aligned}\mathsf{E}X^2Y^2 &= \mathsf{Cov}(X^2, Y^2) + \mathsf{E}X^2\mathsf{E}Y^2 \leqslant 2M_4\mathsf{Cov}(X,Y)^2 + \mathsf{E}X^2\mathsf{E}Y^2 \\ &= 2M_4(\mathsf{E}XY)^2 + \mathsf{E}X^2\mathsf{E}Y^2 \leqslant (2M_4+1)\mathsf{E}X^2\mathsf{E}Y^2 \leqslant 3M_4\mathsf{E}X^2\mathsf{E}Y^2,\end{aligned}$$

hence we obtain the second inequality. □

2.7 Lemma. *Let* $(\xi_N)_{N\geqslant 1}$ *be a sequence of independent random variables with* $\mathsf{E}\xi_N = 0$ *for all* $N \geqslant 1$, *and* $\sup_{N\geqslant 1} \mathsf{E}\xi_N^4 < \infty$. *Let* $\alpha \in \mathbb{R}$ *with* $|\alpha| < 1$. *Let*

$$U_N := \sum_{i=2}^{N} \xi_{i-1}\xi_i, \qquad V_N := \sum_{1\leqslant j<i\leqslant N} \alpha^{i-j}\xi_i\xi_j.$$

Then

$$\mathsf{E}U_N^4 = O(N^2), \qquad \mathsf{E}V_N^4 = O(N^2), \qquad \text{as } N \to \infty.$$

Proof. We have

$$\mathsf{E}U_N^4 = \sum_{i_1,\, i_2,\, i_3,\, i_4=2}^{N} \mathsf{E}\xi_{i_1-1}\xi_{i_1}\xi_{i_2-1}\xi_{i_2}\xi_{i_3-1}\xi_{i_3}\xi_{i_4-1}\xi_{i_4}.$$

Clearly, if $\mathsf{E}\xi_{i_1-1}\xi_{i_1}\xi_{i_2-1}\xi_{i_2}\xi_{i_3-1}\xi_{i_3}\xi_{i_4-1}\xi_{i_4} \neq 0$ then $i_1 = i_2$ and $i_3 = i_4$, or the same relationship holds with a permutation of (i_1, i_2, i_3, i_4). The number of these cases is $O(N^2)$ as $N \to \infty$, and in all of these cases,

$$\begin{aligned}&|\mathsf{E}\xi_{i_1-1}\xi_{i_1}\xi_{i_2-1}\xi_{i_2}\xi_{i_3-1}\xi_{i_3}\xi_{i_4-1}\xi_{i_4}| \\ &\leqslant \sup\left\{\mathsf{E}\xi_{i-1}^2\mathsf{E}\xi_i^2\mathsf{E}\xi_{j-1}^2\mathsf{E}\xi_j^2;\ \mathsf{E}\xi_{i-1}^2\mathsf{E}\xi_i^4\mathsf{E}\xi_{i+1}^2;\ \mathsf{E}\xi_{i-1}^4\mathsf{E}\xi_i^4 : i \geqslant 2,\ j \geqslant i+2\right\} \\ &\leqslant \left(\sup_{N\geqslant 1} \mathsf{E}\xi_N^4\right)^2 < \infty,\end{aligned}$$

by Hölder inequality. Hence we conclude that $\mathsf{E}U_N^4 = O(N^2)$, as $N \to \infty$.

Moreover, we have

$$\mathsf{E}V_N^4 = \sum\nolimits_H \alpha^{i_1+i_2+i_3+i_4-j_1-j_2-j_3-j_4}\mathsf{E}\xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4},$$

where the summation $\sum_H$ is taken over for the set

$$H := \{(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4) \in \mathbb{N}^8 : 1 \leqslant j_k < i_k \leqslant N,\ k = 1, 2, 3, 4\}.$$

For $(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4) \in H$, $\mathsf{E}\xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4} \neq 0$ only in the following eleven cases:

(1) $j_1 = j_2 = j_3 = j_4 < i_1 = i_2 = i_3 = i_4$;

(2) $j_1 = j_2 = j_3 = j_4 < i_1 = i_2 < i_3 = i_4$, or with a permutation of (i_1, i_2, i_3, i_4);

(3) $j_1 = j_2 = j_3 < j_4 = i_1 = i_2 < i_3 = i_4$, or with a permutation of (i_1, i_2, i_3);

(4) $j_1 = j_2 = j_3 < j_4 = i_1 < i_2 = i_3 = i_4$, or with a permutation of (i_1, i_2, i_3);

(5) $j_1 = j_2 < i_1 = i_2 < j_3 = j_4 < i_3 = i_4$;

(6) $j_1 = j_2 < j_3 = j_4 = i_1 = i_2 < i_3 = i_4$;

(7) $j_1 = j_2 < j_3 = j_4 = i_1 < i_2 = i_3 = i_4$, or with a permutation of (i_1, i_2);

(8) $j_1 = j_2 < j_3 = j_4 < i_1 = i_2 = i_3 = i_4$;

(9) $j_1 = j_2 < j_3 = j_4 < i_1 = i_2 < i_3 = i_4$, or with permutation of (i_1, i_2, i_3, i_4);

(10) $j_1 = j_2 < j_3 = i_1 < j_4 = i_2 < i_3 = i_4$, or with a permutation of (i_1, i_2);

(11) $j_1 = j_2 < j_3 = i_1 < j_4 = i_3 < i_2 = i_4$, or with a permutation of (i_1, i_2);

or with joint permutations of (i_1, i_2, i_3, i_4) and (j_1, j_2, j_3, j_4). Let $\widetilde{H}_k \subset H$, $k = 1, 2, \ldots, 11$ denote the subset belonging to the case (k). Then for all $(i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4) \in \widetilde{H}_k$ and for all $k = 1, 2, \ldots, 11$,

$$
\begin{aligned}
&|\mathsf{E}\xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}| \\
&\leqslant \sup\bigl\{\mathsf{E}\xi_i^2\mathsf{E}\xi_j^2\mathsf{E}\xi_k^2\mathsf{E}\xi_\ell^2;\ \mathsf{E}\xi_i^2\mathsf{E}|\xi_j|^3\mathsf{E}|\xi_k|^3;\ \mathsf{E}\xi_i^2\mathsf{E}\xi_j^2\mathsf{E}\xi_k^4;\ \mathsf{E}\xi_i^4\mathsf{E}\xi_j^4 : i, j, k, \ell \geqslant 1\bigr\} \\
&\leqslant \left(\sup_{N\geqslant 1} \mathsf{E}\xi_N^4\right)^2 < \infty
\end{aligned}
$$

again by Hölder inequality. Further,

$$
\sum\nolimits_{\widetilde{H}_k} \alpha^{i_1+i_2+i_3+i_4-j_1-j_2-j_3-j_4} = O(N^2), \qquad \text{as } N \to \infty
$$

for all $k = 1, 2, \ldots, 11$. For example, consider the case (5). Applying the inequality $\sum_{k=n_1}^{n_2} q^k \leqslant q^{n_2+1}/(q-1)$ for $q > 1$ and for $n_1, n_2 \in \mathbb{Z}$ with $n_1 \leqslant n_2$, we obtain

$$
\begin{aligned}
&\sum\nolimits_{\widetilde{H}_5} \alpha^{i_1+i_2+i_3+i_4-j_1-j_2-j_3-j_4} = \sum_{1\leqslant j_1<i_1<j_3<i_3\leqslant N} \alpha^{2i_1+2i_3-2j_1-2j_3} \\
&\leqslant \frac{1}{\alpha^{-2}-1} \sum_{1\leqslant i_1<j_3<i_3\leqslant N} \alpha^{2i_3-2j_3} \\
&\leqslant \frac{N}{\alpha^{-2}-1} \sum_{1\leqslant j_3<i_3\leqslant N} \alpha^{2i_3-2j_3} \leqslant \frac{N}{(\alpha^{-2}-1)^2} \sum_{i_3=1}^{N} 1 = \frac{N^2}{(\alpha^{-2}-1)^2}.
\end{aligned}
$$

The other cases can be handled similarly. $\square$

3 Proof of Proposition 1.2

In the whole proof of Propositions 1.2 and 1.3 we will use the notation

$$M_4 := \sup\{\mathsf{E}\,\varepsilon_{k,\ell}^4 : k, \ell \in \mathbb{Z},\ k+\ell \geqslant 1\}.$$

The statement of Proposition 1.2 will follow from

$$\begin{aligned}
n^{-2}\mathsf{E}B_n &\to 1/\sigma_\alpha^2, && \text{if } 0 \leqslant \alpha < 1/2, && (3.1)\\
n^{-4}\mathsf{Var}B_n &\to 0, && \text{if } 0 \leqslant \alpha < 1/2, && (3.2)\\
n^{-5/2}\mathsf{E}B_n &\to 1/\sigma^2, && \text{if } \alpha = 1/2, && (3.3)\\
n^{-5}\mathsf{Var}B_n &\to 0, && \text{if } \alpha = 1/2. && (3.4)
\end{aligned}$$

If $\alpha = 0$ then $X_{k,\ell} = \varepsilon_{k,\ell}$ for all $k, \ell \in \mathbb{Z}$ with $k+\ell \geqslant 1$, hence

$$B_n = \sum_{(k,\ell)\in T_n} (\varepsilon_{k-1,\ell} + \varepsilon_{k,\ell-1})^2 = \sum_{k=-n+2}^{n} \sum_{\ell=-k+2}^{n} (\varepsilon_{k-1,\ell} + \varepsilon_{k,\ell-1})^2.$$

Clearly

$$\mathsf{E}B_n = \sum_{k=-n+2}^{n} \sum_{\ell=-k+2}^{n} \mathsf{Var}(\varepsilon_{k-1,\ell} + \varepsilon_{k,\ell-1}) = 2 \sum_{k=-n+2}^{n} \sum_{\ell=-k+2}^{n} 1 = 2n(2n-1),$$

thus $n^{-2}\mathsf{E}B_n \to 4 = 1/\sigma_0^2$, and we obtain (3.1) in case $\alpha = 0$. Moreover,

$$\mathsf{Var}B_n = \sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in T_n} \mathsf{Cov}\left((\varepsilon_{k_1-1,\ell_1} + \varepsilon_{k_1,\ell_1-1})^2, (\varepsilon_{k_2-1,\ell_2} + \varepsilon_{k_2,\ell_2-1})^2\right).$$

By Lemma 2.6,

$$\begin{aligned}
0 &\leqslant \mathsf{Cov}\left((\varepsilon_{k_1-1,\ell_1} + \varepsilon_{k_1,\ell_1-1})^2, (\varepsilon_{k_2-1,\ell_2} + \varepsilon_{k_2,\ell_2-1})^2\right)\\
&\leqslant 2M_4 \mathsf{Cov}\left(\varepsilon_{k_1-1,\ell_1} + \varepsilon_{k_1,\ell_1-1}, \varepsilon_{k_2-1,\ell_2} + \varepsilon_{k_2,\ell_2-1}\right)^2 \leqslant 8M_4,
\end{aligned}$$

so that $\mathsf{Var}B_n \leqslant 8M_4 n(2n+1)$, and consequently, we obtain (3.2) for $\alpha = 0$.

If $0 < \alpha \leqslant 1/2$ then using (1.1), we rewrite B_n in the form

$$B_n = \alpha^{-2} \sum_{(k,\ell)\in T_n} (X_{k,\ell} - \varepsilon_{k,\ell})^2,$$

so that

$$\mathsf{E}B_n = \alpha^{-2} \sum_{(k,\ell)\in T_n} \mathsf{Var}(X_{k,\ell} - \varepsilon_{k,\ell}).$$

By the representation (1.2), $\mathsf{Cov}(X_{k,\ell}, \varepsilon_{k,\ell}) = 1$. Hence $\mathsf{Var}(X_{k,\ell} - \varepsilon_{k,\ell}) = \mathsf{Var} X_{k,\ell} - 1$, and thus

$$\mathsf{E}B_n = \alpha^{-2} \sum_{(k,\ell)\in T_n} \mathsf{Var} X_{k,\ell} - \alpha^{-2} n(2n+1).$$

If $0 < \alpha < 1/2$ then

$$\sum_{(k,\ell)\in T_n} \mathsf{Var} X_{k,\ell} = \sum_{(k,\ell)\in T_n} \mathsf{Var} Y^{(n)}\left(\frac{k-1}{n}, \frac{\ell-1}{n}\right) = n^2 \iint\limits_T \mathsf{Var} Y^{(n)}(s,t)\,\mathrm{d}s\mathrm{d}t,$$

where $T := \{(s,t) \in \mathbb{R}^2 : s+t \geqslant 0,\, s \leqslant 1,\, t \leqslant 1\}$. Hence

$$n^{-2}\mathsf{E}B_n = \alpha^{-2} \iint\limits_T \mathsf{Var} Y^{(n)}(s,t)\,\mathrm{d}s\mathrm{d}t - \frac{2n+1}{n}\alpha^{-2}.$$

By Lemma 2.3, $|\mathsf{Var} Y^{(n)}(s,t)| \leqslant (1-4\alpha^2)^{-1}$, hence the dominated convergence theorem and Proposition 2.4 imply

$$\begin{aligned}\lim_{n\to\infty} n^{-2}\mathsf{E}B_n &= \alpha^{-2} \iint\limits_T \lim_{n\to\infty} \mathsf{Var} Y^{(n)}(s,t)\,\mathrm{d}s\mathrm{d}t - 2\alpha^{-2} \\ &= \alpha^{-2}(1-4\alpha^2)^{-1/2} \iint\limits_T \mathrm{d}s\mathrm{d}t - 2\alpha^{-2} = 2\alpha^{-2}\Big((1-4\alpha^2)^{-1/2} - 1\Big) = \frac{1}{\sigma_\alpha^2}.\end{aligned}$$

Thus we obtain (3.1) for $0 < \alpha < 1/2$. In case $\alpha = 1/2$ we obtain in the same manner

$$n^{-5/2}\mathsf{E}B_n = 4 \iint\limits_T \mathsf{Var} Z^{(n)}(s,t)\,\mathrm{d}s\mathrm{d}t - \frac{4(2n+1)}{n^{3/2}}.$$

By Lemma 2.3, $|\mathsf{Var} Z^{(n)}(s,t)| \leqslant C n^{-1/2}(2[ns] + 2[nt] + 4)^{1/2} \leqslant C(2s+2t+4)^{1/2}$. The function $(s,t) \mapsto C(2s+2t+4)^{1/2}$ is integrable on the triangle T, hence the dominated convergence theorem applies. By Proposition 2.4,

$$\lim_{n\to\infty} n^{-5/2}\mathsf{E}B_n = 4 \iint\limits_T \lim_{n\to\infty} \mathsf{Var} Z^{(n)}(s,t)\,\mathrm{d}s\mathrm{d}t = \frac{8}{\sqrt{\pi}} \iint\limits_T \sqrt{s+t}\,\mathrm{d}s\mathrm{d}t = \frac{1}{\sigma^2},$$

thus we have derived (3.3).

Further, we have for $\alpha > 0$ that

$$\mathsf{Var} B_n = \alpha^{-4} \sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in T_n} \mathsf{Cov}\left((X_{k_1,\ell_1} - \varepsilon_{k_1,\ell_1})^2, (X_{k_2,\ell_2} - \varepsilon_{k_2,\ell_2})^2\right).$$

Applying Lemma 2.6,

$$0 \leqslant \mathsf{Cov}\left((X_{k_1,\ell_1} - \varepsilon_{k_1,\ell_1})^2, (X_{k_2,\ell_2} - \varepsilon_{k_2,\ell_2})^2\right) \leqslant 2M_4 \mathsf{Cov}(X_{k_1,\ell_1} - \varepsilon_{k_1,\ell_1}, X_{k_2,\ell_2} - \varepsilon_{k_2,\ell_2})^2.$$

One can easily show that

$$|\mathsf{Cov}(X_{k_1,\ell_1} - \varepsilon_{k_1,\ell_1}, X_{k_2,\ell_2} - \varepsilon_{k_2,\ell_2})| \leqslant \mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}). \tag{3.5}$$

Indeed,

$$\begin{aligned}&\mathsf{Cov}(X_{k_1,\ell_1} - \varepsilon_{k_1,\ell_1}, X_{k_2,\ell_2} - \varepsilon_{k_2,\ell_2}) \\ &= \mathsf{E}X_{k_1,\ell_1}X_{k_2,\ell_2} - \mathsf{E}\varepsilon_{k_1,\ell_1}X_{k_2,\ell_2} - \mathsf{E}X_{k_2,\ell_2}\varepsilon_{k_1,\ell_1} + \mathsf{E}\varepsilon_{k_1,\ell_1}\varepsilon_{k_2,\ell_2}.\end{aligned}$$

By the representation (1.2), $\mathsf{E}\varepsilon_{k_1,\ell_1}X_{k_2,\ell_2} \geqslant 0$ and $\mathsf{E}X_{k_2,\ell_2}\varepsilon_{k_1,\ell_1} \geqslant 0$. Thus in case $(k_1,\ell_1) \neq (k_2,\ell_2)$, (3.5) follows from $\mathsf{E}\varepsilon_{k_1,\ell_1}\varepsilon_{k_2,\ell_2} = 0$. If $(k_1,\ell_1) = (k_2,\ell_2)$ then $\mathsf{E}\varepsilon_{k_1,\ell_1}X_{k_2,\ell_2} = \mathsf{E}X_{k_2,\ell_2}\varepsilon_{k_1,\ell_1} = \mathsf{E}\varepsilon_{k_1,\ell_1}\varepsilon_{k_2,\ell_2} = 1$ implies (3.5).

If $0 < \alpha < 1/2$ then

$$\begin{aligned}&\sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in T_n} \mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})^2 \\ &= n^4 \iint\limits_T \iint\limits_T \mathsf{Cov}\big(Y^{(n)}(s_1,t_1), Y^{(n)}(s_2,t_2)\big)^2 \, \mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2.\end{aligned}$$

Hence by (3.5),

$$n^{-4}\mathsf{Var}B_n \leqslant 2M_4\alpha^{-4} \iint\limits_T \iint\limits_T \mathsf{Cov}\big(Y^{(n)}(s_1,t_1), Y^{(n)}(s_2,t_2)\big)^2 \, \mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2,$$

which tends to zero by the dominated convergence theorem and by Proposition 2.4. Hence we conclude (3.2) for $0 < \alpha < 1/2$.

If $\alpha = 1/2$ then

$$\begin{aligned}&\sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in T_n} \mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})^2 \\ &= n^5 \iint\limits_T \iint\limits_T \mathsf{Cov}\big(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)\big)^2 \, \mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2.\end{aligned}$$

Again by (3.5),

$$n^{-5}\mathsf{Var}B_n \leqslant 32M_4 \iint\limits_T \iint\limits_T \mathsf{Cov}\big(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)\big) \, \mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2.$$

By Lemma 2.3,

$$\begin{aligned}\mathsf{Cov}\big(Z^{(n)}(s_1,t_1), Z^{(n)}(s_2,t_2)\big)^2 &\leqslant C^2 n^{-1}([ns_1] + [nt_1] + [ns_2] + [nt_2] + 4) \\ &\leqslant C^2(s_1 + t_1 + s_2 + t_2 + 4)\end{aligned}$$

for all $(s_1,t_1),(s_2,t_2) \in T$. The function $(s_1,t_1,s_2,t_2) \mapsto C^2(s_1+t_1+s_2+t_2+4)$ is integrable on T^2. Hence the dominated convergence theorem and Proposition 2.4 imply

$$\lim_{n\to\infty} \iint\limits_T \iint\limits_T \mathsf{Cov}\left(Z^{(n)}(s_1,t_1), Z^{(n)}(s_1,t_1)\right)^2 \, \mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2 = 0,$$

and we conclude (3.4). □

4 Proof of Proposition 1.3

First we show that $(A_n)_{n\geqslant 1}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geqslant 1}$, where $\mathcal{F}_n$ denotes the σ–algebra generated by the random variables $\{\varepsilon_{k,\ell} : (k,\ell) \in T_n\}$.

Let us introduce the notation

$$S_n := T_n \setminus T_{n-1}, \qquad n \geqslant 1,$$

which is a strip between the two triangles T_{n-1} and T_n.

We give a useful decomposition of $A_n - A_{n-1}$, where $A_0 := 0$. If $\alpha \neq 0$ then by the representation (1.2),

$$\begin{aligned} A_n - A_{n-1} &= \alpha^{-1} \sum_{(k,\ell)\in S_n} (X_{k,\ell} - \varepsilon_{k,\ell})\varepsilon_{k,\ell} \\ &= \alpha^{-1} \sum_{(k,\ell)\in S_n} \varepsilon_{k,\ell} \sum_{\substack{(i,j)\in T_{k,\ell} \\ (i,j)\neq(k,\ell)}} \binom{k+\ell-i-j}{k-i} \alpha^{k+\ell-i-j}\varepsilon_{i,j}. \end{aligned}$$

Collecting first the terms containing only $\varepsilon_{i,j}$ with $(i,j) \in S_n$, and then the rest, we obtain the decomposition

$$A_n - A_{n-1} = A_{n,1} + \sum_{(k,\ell)\in S_n} \varepsilon_{k,\ell} A_{n,2,k,\ell}, \tag{4.1}$$

where

$$\begin{aligned} A_{n,1} &:= \alpha^{-1} \sum_{(k,\ell)\in S_n} \varepsilon_{k,\ell} \sum_{\substack{(i,j)\in T_{k,\ell}\setminus T_{n-1} \\ (i,j)\neq(k,\ell)}} \binom{k+\ell-i-j}{k-i} \alpha^{k+\ell-i-j}\varepsilon_{i,j}, \\ A_{n,2,k,\ell} &:= \alpha^{-1} \sum_{(i,j)\in T_{k,\ell}\cap T_{n-1}} \binom{k+\ell-i-j}{k-i} \alpha^{k+\ell-i-j}\varepsilon_{i,j}. \end{aligned}$$

If $\alpha = 0$ then

$$A_n - A_{n-1} = \sum_{k=-n+2}^{n} (\varepsilon_{k-1,n} + \varepsilon_{k,n-1})\varepsilon_{k,n} + \sum_{\ell=-n+2}^{n-1} (\varepsilon_{n-1,\ell} + \varepsilon_{n,\ell-1})\varepsilon_{n,\ell},$$

so that we obtain again the decomposition (4.1) with

$$A_{n,1} := \sum_{k=-n+2}^{n} \varepsilon_{k-1,n}\varepsilon_{k,n} + \sum_{\ell=-n+2}^{n} \varepsilon_{n,\ell-1}\varepsilon_{n,\ell}.,$$

$$A_{n,2,k,\ell} := \begin{cases} \varepsilon_{k,n-1}, & \text{if } -n+2 \leqslant k \leqslant n-1, \ \ell = n, \\ \varepsilon_{n-1,\ell}, & \text{if } k = n, \ -n+2 \leqslant \ell \leqslant n-1, \\ 0, & \text{otherwise.} \end{cases}$$

The term $A_{n,1}$ is a quadratic form of the variables $\{\varepsilon_{i,j} : (i,j) \in S_n\}$, hence $A_{n,1}$ is independent of $\mathcal{F}_{n-1}$. The terms $A_{n,2,k,\ell}$ are linear combinations of the variables $\{\varepsilon_{i,j} : (i,j) \in T_{n-1}\}$, thus they are measurable with respect to $\mathcal{F}_{n-1}$. Hence,

$$\mathsf{E}(A_n - A_{n-1} \mid \mathcal{F}_{n-1}) = \mathsf{E}A_{n,1} + \sum_{(k,\ell)\in S_n} A_{n,2,k,\ell}\mathsf{E}(\varepsilon_{k,\ell} \mid \mathcal{F}_{n-1}) = 0.$$

Consequently, $(A_n)_{n\geqslant 1}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geqslant 1}$.

By the martingale central limit theorem [Jacod and Shiryaev (1987)], in order to prove Proposition 1.3, it suffices to show that the conditional variances of the martingale differences converge in probability and to verify the conditional Lindeberg condition. To be precise, the statement is a consequence of the following two propositions, where $\mathbb{1}_H$ denotes the indicator function of the set H.

4.1 Proposition. *If* $0 \leqslant \alpha < 1/2$ *then*

$$n^{-2} \sum_{m=1}^{n} \mathsf{E}\left((A_m - A_{m-1})^2 \,\middle|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \frac{1}{\sigma_\alpha^2}, \qquad \text{as } n \to \infty.$$

If $\alpha = 1/2$ *then*

$$n^{-5/2} \sum_{m=1}^{n} \mathsf{E}\left((A_m - A_{m-1})^2 \,\middle|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \frac{1}{\sigma^2}, \qquad \text{as } n \to \infty.$$

4.2 Proposition. *If* $0 \leqslant \alpha < 1/2$ *then for all* $\delta > 0$,

$$n^{-2} \sum_{m=1}^{n} \mathsf{E}\left((A_m - A_{m-1})^2 \mathbb{1}_{\{|A_m - A_{m-1}| \geqslant \delta n\}} \,\middle|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0, \qquad \text{as } n \to \infty.$$

If $\alpha = 1/2$ *then for all* $\delta > 0$,

$$n^{-5/2} \sum_{m=1}^{n} \mathsf{E}\left((A_m - A_{m-1})^2 \mathbb{1}_{\{|A_m - A_{m-1}| \geqslant \delta n^{5/4}\}} \,\Big|\, \mathcal{F}_{m-1} \right) \xrightarrow{\mathsf{P}} 0, \qquad \text{as } n \to \infty.$$

Proof of Proposition 4.1. By the decomposition (4.1) and by the measurability of $A_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$ one can derive that

$$\begin{aligned}\mathsf{E}\left((A_m - A_{m-1})^2 \,\big|\, \mathcal{F}_{m-1}\right) &= \mathsf{E}(A_{m,1}^2 \,|\, \mathcal{F}_{m-1}) + 2 \sum_{(k,\ell)\in S_m} A_{m,2,k,\ell} \mathsf{E}(A_{m,1}\varepsilon_{k,\ell} \,|\, \mathcal{F}_{m-1}) \\ &\quad + \sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in S_m} A_{m,2,k_1,\ell_1} A_{m,2,k_2,\ell_2} \mathsf{E}(\varepsilon_{k_1,\ell_1}\varepsilon_{k_2,\ell_2} \,|\, \mathcal{F}_{m-1}).\end{aligned}$$

By the independence of $A_{m,1}$ and $\{\varepsilon_{k,\ell} : (k,\ell) \in S_m\}$ from $\mathcal{F}_{m-1}$, and by $\mathsf{E}(A_{m,1}\varepsilon_{k,\ell}) = 0$, one obtains

$$\mathsf{E}\left((A_m - A_{m-1})^2 \,\big|\, \mathcal{F}_{m-1}\right) = \mathsf{E}A_{m,1}^2 + \sum_{(k,\ell)\in S_m} A_{m,2,k,\ell}^2 =: U_m.$$

The statement will follow from

$$n^{-2} \sum_{m=1}^{n} \mathsf{E}U_m \to 1/\sigma_\alpha^2, \qquad \text{if } 0 \leqslant \alpha < 1/2, \tag{4.2}$$

$$n^{-4} \mathsf{Var}\left(\sum_{m=1}^{n} U_m \right) \to 0, \qquad \text{if } 0 \leqslant \alpha < 1/2, \tag{4.3}$$

$$n^{-5/2} \sum_{m=1}^{n} \mathsf{E}U_m \to 1/\sigma^2, \qquad \text{if } \alpha = 1/2, \tag{4.4}$$

$$n^{-5} \mathsf{Var}\left(\sum_{m=1}^{n} U_m \right) \to 0, \qquad \text{if } \alpha = 1/2. \tag{4.5}$$

First we show that

$$\sum_{m=1}^{n} \mathsf{E}U_m = \mathsf{E}B_n. \tag{4.6}$$

Indeed, $U_m = \mathsf{E}(A_m^2 - 2A_m A_{m-1} + A_{m-1}^2 \,|\, \mathcal{F}_{m-1}) = \mathsf{E}A_m^2 - A_{m-1}^2$, hence $\mathsf{E}U_m = \mathsf{E}A_m^2 - \mathsf{E}A_{m-1}^2$, and consequently, $\sum_{m=1}^n \mathsf{E}U_m = \mathsf{E}A_n^2$. Further, we have

$$\mathsf{E}A_n^2 = \sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in T_n} \mathsf{E}\Big((X_{k_1-1,\ell_1} + X_{k_1,\ell_1-1})(X_{k_2-1,\ell_2} + X_{k_2,\ell_2-1}) \varepsilon_{k_1,\ell_1}\varepsilon_{k_2,\ell_2} \Big).$$

By the representation (1.2), only the terms with $(k_1,\ell_1) = (k_2,\ell_2)$ are nonzero, and the variables $X_{k-1,\ell}$, $X_{k,\ell-1}$ are independent from $\varepsilon_{k,\ell}$, thus

$$\sum_{m=1}^{n} \mathsf{E}U_m = \mathsf{E}A_n^2 = \sum_{(k,\ell)\in T_n} \mathsf{E}(X_{k-1,\ell} + X_{k,\ell-1})^2 \, \mathsf{E}\varepsilon_{k,\ell}^2 = \mathsf{E}B_n,$$

and consequently, we obtained (4.6).

In view of (4.6), the statements (4.2) and (4.4) follow from (3.1) and (3.3), respectively.

Now we prove (4.3) and (4.5). If $\alpha = 0$ then

$$\sum_{m=1}^{n} U_m = \sum_{m=1}^{n} \mathsf{E} A_{m,1}^2 + \sum_{m=2}^{n} \sum_{k=-m+2}^{m-2} \varepsilon_{k,m-1}^2 + \sum_{m=2}^{n} \sum_{\ell=-m+2}^{m-2} \varepsilon_{m-1,\ell}^2 + 2 \sum_{m=2}^{n} \varepsilon_{m-1,m-1}^2$$

is a decomposition in independent terms. Hence

$$\mathsf{Var}\left(\sum_{m=1}^{n} U_m\right) = \sum_{m=2}^{n} \sum_{k=-m+2}^{m-2} 1 + \sum_{m=2}^{n} \sum_{\ell=-m+2}^{m-2} 1 + 4 \sum_{m=2}^{n} 1 = 2n^2,$$

which implies (4.3) in case $\alpha = 0$.

If $\alpha > 0$ then we consider

$$\mathsf{Var}\left(\sum_{m=1}^{n} U_m\right) = \sum_{m_1,\, m_2=1}^{n} \sum_{(k_1,\ell_1)\in S_{m_1}} \sum_{(k_2,\ell_2)\in S_{m_2}} \mathsf{Cov}(A_{m_1,2,k_1,\ell_1}^2, A_{m_2,2,k_2,\ell_2}^2).$$

By Lemma 2.6, $\mathsf{Cov}(A_{m_1,2,k_1,\ell_1}^2, A_{m_2,2,k_2,\ell_2}^2) \leqslant 2M_4 \mathsf{Cov}(A_{m_1,2,k_1,\ell_1}, A_{m_2,2,k_2,\ell_2})^2$. Moreover,

$$\begin{aligned}
&\mathsf{Cov}(A_{m_1,2,k_1,\ell_1}, A_{m_2,2,k_2,\ell_2}) \\
&= \alpha^{-2} \sum_{(i,j)\in T_{k_1,\ell_1}\cap T_{m_1-1}\cap T_{k_2,\ell_2}\cap T_{m_2-1}} \binom{k_1+\ell_1-i-j}{k_1-i} \binom{k_2+\ell_2-i-j}{k_2-i} \alpha^{k_1+\ell_1+k_2+\ell_2-2i-2j}.
\end{aligned}$$

By formula (2.1), $|\mathsf{Cov}(A_{m_1,2,k_1,\ell_1}, A_{m_2,2,k_2,\ell_2})| \leqslant \alpha^{-2} \mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})$. Furthermore,

$$\sum_{m_1,\, m_2=1}^{n} \sum_{(k_1,\ell_1)\in S_{m_1}} \sum_{(k_2,\ell_2)\in S_{m_2}} \mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) = \sum_{(k_1,\ell_1),\, (k_2,\ell_2)\in T_n} \mathsf{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}),$$

hence one can derive (4.3) in case $\alpha > 0$ and (4.5) as (3.2) and (3.4) have been derived, respectively. $\square$

Proof of Proposition 4.2. We have

$$\begin{aligned}
\mathbb{1}_{\{|A_m - A_{m-1}| \geqslant \delta n\}} &\leqslant \delta^{-2} n^{-2} (A_m - A_{m-1})^2, \\
\mathbb{1}_{\{|A_m - A_{m-1}| \geqslant \delta n^{5/4}\}} &\leqslant \delta^{-2} n^{-5/2} (A_m - A_{m-1})^2,
\end{aligned}$$

hence it suffices to show that

$$n^{-4}\sum_{m=1}^{n}\mathsf{E}\left((A_m-A_{m-1})^4\,\middle|\,\mathcal{F}_{m-1}\right)\stackrel{\mathsf{P}}{\longrightarrow}0\qquad\text{if }\ 0\leqslant\alpha<1/2,\tag{4.7}$$

$$n^{-5}\sum_{m=1}^{n}\mathsf{E}\left((A_m-A_{m-1})^4\,\middle|\,\mathcal{F}_{m-1}\right)\stackrel{\mathsf{P}}{\longrightarrow}0\qquad\text{if }\ \alpha=1/2.\tag{4.8}$$

By the decomposition (4.1) of A_m-A_{m-1} and by the inequality $(x+y)^4\leqslant 2^3(x^4+y^4)$ for $x,y\in\mathbb{R}$,

$$(A_m-A_{m-1})^4\leqslant 2^3A_{m,1}^4+2^3\left(\sum_{(k,\ell)\in S_m}\varepsilon_{k,\ell}A_{m,2,k,\ell}\right)^4.$$

By the independence of $A_{m,1}$ and $\mathcal{F}_{m-1}$, we have $\mathsf{E}(A_{m,1}^4\,|\,\mathcal{F}_{m-1})=\mathsf{E}A_{m,1}^4$. By the measurability of $A_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$, we obtain

$$\begin{aligned}\mathsf{E}\left(\left(\sum_{(k,\ell)\in S_m}\varepsilon_{k,\ell}A_{m,2,k,\ell}\right)^4\,\middle|\,\mathcal{F}_{m-1}\right)&=(M_4-3)\sum_{(k,\ell)\in S_m}A_{m,2,k,\ell}^4+3\left(\sum_{(k,\ell)\in S_m}A_{m,2,k,\ell}^2\right)^2\\&\leqslant\left((M_4-3)^{+}+3\right)\left(\sum_{(k,\ell)\in S_m}A_{m,2,k,\ell}^2\right)^2.\end{aligned}$$

Hence, in order to prove (4.7) and (4.8), it suffices to show that

$$n^{-4}\sum_{m=1}^{n}\mathsf{E}A_{m,1}^4\to0\qquad\text{if }\ 0\leqslant\alpha<1/2,\tag{4.9}$$

$$n^{-4}\sum_{m=1}^{n}\mathsf{E}\left(\sum_{(k,\ell)\in S_m}A_{m,2,k,\ell}^2\right)^2\to0\qquad\text{if }\ 0\leqslant\alpha<1/2.\tag{4.10}$$

$$n^{-5}\sum_{m=1}^{n}\mathsf{E}A_{m,1}^4\to0\qquad\text{if }\ \alpha=1/2,\tag{4.11}$$

$$n^{-5}\sum_{m=1}^{n}\mathsf{E}\left(\sum_{(k,\ell)\in S_m}A_{m,2,k,\ell}^2\right)^2\to0\qquad\text{if }\ \alpha=1/2.\tag{4.12}$$

If $\alpha=0$ then

$$A_{m,1}^4\leqslant 2^3\left(\sum_{k=-m+2}^{m}\varepsilon_{k-1,m}\varepsilon_{k,m}\right)^4+2^3\left(\sum_{\ell=-m+2}^{m}\varepsilon_{m,\ell-1}\varepsilon_{m,\ell}\right)^4.$$

By Lemma 2.7, $\mathsf{E}A_{m,1}^4 = O(m^2)$ as $n \to \infty$, which implies (4.9) for $\alpha = 0$. Moreover, for $m \geqslant 2$,

$$\sum_{(k,\ell)\in S_m} A_{m,2,k,\ell}^2 = \sum_{k=-m+2}^{m-2} \varepsilon_{k,m-1}^2 + \sum_{\ell=-m+2}^{m-2} \varepsilon_{m-1,\ell}^2 + 2\varepsilon_{m-1,m-1}^2,$$

where the decomposition on the right hand side contains independent random variables. Thus

$$\mathsf{E}\left(\sum_{(k,\ell)\in S_m} A_{m,2,k,\ell}^2\right)^2 \leqslant (4m-2)M_4 + (4m-3)(4m-6),$$

and we obtain (4.10) for $\alpha = 0$.

If $\alpha \neq 0$ then separating the terms in $A_{m,1}$ containing $\{\varepsilon_{k,m} : k = -m+1,\dots,m\}$ and $\{\varepsilon_{m,\ell} : \ell = -m+1,\dots,m\}$, we obtain

$$A_{m,1} = \alpha^{-1} \sum_{-m+1\leqslant i<k\leqslant m} \alpha^{k-i}\varepsilon_{k,m}\varepsilon_{i,m} + \alpha^{-1} \sum_{-m+1\leqslant j<\ell\leqslant m} \alpha^{\ell-j}\varepsilon_{m,\ell}\varepsilon_{m,j}.$$

Consequently,

$$A_{m,1}^4 \leqslant 2^3\alpha^{-1}\left(\sum_{-m+1\leqslant i<k\leqslant m} \alpha^{k-i}\varepsilon_{k,m}\varepsilon_{i,m}\right)^4 + 2^3\alpha^{-1}\left(\sum_{-m+1\leqslant j<\ell\leqslant m} \alpha^{\ell-j}\varepsilon_{m,\ell}\varepsilon_{m,j}\right)^4.$$

By Lemma 2.7, $\mathsf{E}A_{m,1}^4 = O(m^2)$ as $n \to \infty$, which implies (4.9) in the case $0 < \alpha < 1/2$, and (4.11).

Furthermore, we have

$$\mathsf{E}\left(\sum_{(k,\ell)\in S_m} A_{m,2,k,\ell}^2\right)^2 = \sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in S_m} \mathsf{E}(A_{m,2,k_1,\ell_1}^2 A_{m,2,k_2,\ell_2}^2).$$

By Lemma 2.6, $\mathsf{E}(A_{m,2,k_1,\ell_1}^2 A_{m,2,k_2,\ell_2}^2) \leqslant 3M_4 \mathsf{E}A_{m,2,k_1,\ell_1}^2 \mathsf{E}A_{m,2,k_2,\ell_2}^2$. By formula (2.1),

$$\mathsf{E}A_{m,2,k,\ell}^2 = \sum_{(i,j)\in T_{k,\ell}\cap T_{m-1}} \binom{k+\ell-i-j}{k-i}^2 \alpha^{2(k+\ell-i-j)} \leqslant \mathsf{Var}X_{k,\ell}.$$

If $0 < \alpha < 1/2$ then by Lemma 2.3, $\mathsf{Var}X_{k,\ell} \leqslant (1-4\alpha^2)^{-1}$, hence

$$\mathsf{E}\left(\sum_{(k,\ell)\in S_m} A_{m,2,k,\ell}^2\right)^2 \leqslant 3M_4(1-4\alpha^2)^{-2} \sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in S_m} 1 = 3M_4(1-4\alpha^2)^{-2}(4m-1)^2,$$

hence we obtain (4.10) in case $0 < \alpha < 1/2$.

If $\alpha = 1/2$ then again by Lemma 2.3, $\mathsf{Var} X_{k,\ell} \leqslant C\sqrt{2(k+\ell)} \leqslant 2C\sqrt{m}$ for $(k,\ell) \in S_m$, thus

$$\mathsf{E}\left(\sum_{(k,\ell)\in S_m} A_{m,2,k,\ell}^2\right)^2 \leqslant 4C^2 m \sum_{(k_1,\ell_1),\,(k_2,\ell_2)\in S_m} 1 = 4C^2 m(4m-1)^2,$$

hence we obtain (4.12). □

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