DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

ALMOST THE FAN THEOREM

Wim Veldman

Report No. 0113 (June 2001)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

Almost the Fan Theorem

Wim Veldman

Dedicated to Arnoud van Rooij on the occasion of his 65th birthday[1]

Maguer, señor Quijote, que sandeces
vos tengan el cerbelo derrumbado,
nunca seréis de alguno reprochado
por home de obras viles y soeces.
Serán vuesas fazañas los joeces,
[7, p. 19]

Maugre the ravings, that are set abroach,
And rumble up and downe thy troubled braine:
Yet none thine acts, Quixote can reproach,
or thy proceedings taxe as vile, or vaine
Thy feats shal be, thy fairest ornament
[8, p. 14]

.

1 Stumbling-blocks

1.1 The *Uniform Continuity Theorem* is a keystone of real analysis, and a jewel:

> *Let f be a function from the real closed segment $[0, 1]$ to the set $\mathbb{R}$ of real numbers, and let m be a natural number. If, for every x in $[0, 1]$, there exists a natural number n such that, for every y in $[0, 1]$, if $|x-y| < \frac{1}{2^n}$, then $|f(x)-f(y)| < \frac{1}{2^{m+1}}$, then there exists a natural number n such that, for all x, y in $[0, 1]$, if $|x-y| < \frac{1}{2^n}$, then $|f(x)-f(y)| < \frac{1}{2^m}$.*
> *Therefore, if f is pointwise continuous on $[0, 1]$, then f is uniformly continuous on $[0, 1]$.*

The usual proof is by contradiction, for instance as follows.
Suppose that, under the circumstances given, there is no suitable n. We build two infinite sequences $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ of points in $[0, 1]$ such that, for every i, $|x_i-y_i| < \frac{1}{2^i}$ and $|f(x_i)-f(y_i)| \geq \frac{1}{2^m}$. We then search for a point x in $[0, 1]$ at which the sequence $x_0, x_1, \ldots$ *accumulates*, that is, for every p there exists $i > p$ such that $|x-x_i| < \frac{1}{2^p}$.

[1]This paper is a slightly revised translation of [20]

We calculate n such that, for every y in $[0,1]$, if $|x-y| < \frac{1}{2^n}$, then $|f(x)-f(y)| < \frac{1}{2^{m+1}}$. Finally we find i such that $i > n$ and $|x-x_i| < \frac{1}{2^{n+1}}$. We conclude: $|x-y_i| < |x-x_i| + |x_i-y_i| < \frac{1}{2^n}$, and therefore: $|f(x_i)-f(y_i)| < |f(x_i-f(x))| + |f(y_i)-f(x)| < \frac{1}{2^m}$. Contradiction. There *must* exist a suitable n.
This fascinating argument leaves us completely in the dark, should we want to actually find and calculate a suitable n.
We are informed only that the assumption that no such n exists leads to a contradiction. We even may put into question the way this modest conclusion is obtained. Sometimes, when given an infinite sequence of points in $[0,1]$, we find ourselves unable to locate a point where the sequence accumulates. Here is an example. Let $d : \mathbb{N} \to \{0,1,\ldots,9\}$ be the decimal expansion of π, that is, $\pi = 3 + \sum_{n=1}^{\infty} d(n) \cdot 10^{-n}$.
We now define a sequence $x_0, x_1, \ldots$ of points in $[0,1]$. For each n, if there is no $i < n$ such that $d(i) = d(i{+}1) = \ldots = d(i{+}9) = 9$, then $z_n := 0$, and if there is one, then $z_n := 1$. Someone who knows how to find an accumulation point of this sequence must be able to decide if his point lies to the right of 0 or to the left of 1. In the former case he has found some i such that $d(i) = d(i{+}1) = \ldots = d(i{+}9) = 9$, in the latter case he is sure that no such i exists. He thus finds the answer to an unsolved problem. If, accidentally, he should have discovered an uninterrupted sequence of 10 nines in the decimal expansion of π, we change the example and make him answer the question if there exists an uninterrupted sequence of 11 nines in the deciaml expansion of π. Clearly, the statement of the Bolzano-Weierstrass Theorem implies that we are in possession of a method for solving every one of a whole class of problems and, obviously, we are not. We have to admit that the Bolzano-Weierstrass Theorem is not true if we read its statement constructively. L.E.J. Brouwer made it clear, by countless such examples, that many famous theorems in a similar way fail to redeem their promises and thereby raised the agonizing question what sense these theorems may be said to make.
We have to fear that also the Uniform Continuity Theorem itself will not survive should Brouwer cast his eye upon it. We first consider another jewel from the treasury of real analysis, the *Theorem of the Greatest Value.*

1.2 *Let f be a function from $[0,1]$ to $\mathbb{R}$ that is continuous at every point in $[0,1]$. There exists a point x in $[0,1]$ at which f assumes its greatest value, that is, for every y in $[0,1]$, $f(y) \leq f(x)$.*

Like the Bolzano-Weierstrass Theorem, this theorem may be proved by the widely applicable method of successive bisection, as follows.
We build an infinite sequence $x_0, x_1, \ldots$ of points in $[0,1]$, step by step. We define $z_0 := 0$, and, for each n, *if* for each y in $[x_n + \frac{1}{2^{n+1}}, x_n + \frac{1}{2^n}]$ there exists z in $[x_n, x_n + \frac{1}{2^{n+1}}]$ such that $f(y) \leq f(z)$, then $x_{n+1} := x_n$, and *if not*, then $x_{n+1} := x_n + \frac{1}{2^{n+1}}$. We first observe that for each n, for each y in $[x_n, x_n + \frac{1}{2^n}]$, there exists z in $[x_{n+1}, x_{n+1} + \frac{1}{2^{n+1}}]$ such that $f(y) \leq f(z)$, and then, by induction, that for every y in $[0,1]$, for each n, there exist z in $[x_n, x_n + \frac{1}{2^n}]$ such that $f(y) \leq f(z)$. The sequence $x_0, x_1, \ldots$ converges, call its limit x. We now show that for every y in $[0,1]$, $f(y) \leq f(x)$. Assume, to this end, that y belongs to $[0,1]$ and $f(y) > f(x)$. Using the

fact that f is continuous in x we find n such that for every z in $[0,1]$, if $|x-z| < \frac{1}{2^n}$, then $f(y) > f(z)$. It follows that for every z in $[x_{n+1}, x_{n+1} + \frac{1}{2^{n+1}}]$, $f(y) > f(z)$, but there exists z in $[x_{n+1}, x_{n+1} + \frac{1}{2^{n+1}}]$ such that $f(y) \leq f(z)$. Contradiction.
We need not be upset that the latter part of this proof is an argument by contradiction. If one wants to show, for certain real numbers a, b, that $a \leq b$, the obvious thing to do is to reduce the assumption that $b < a$ to absurdity. It is alarming, however, that in the first part of the proof a construction is described that cannot be put into practice. How should we decide if it is true that for every y in $[\frac{1}{2}, 1]$ there exists x in $[0, \frac{1}{2}]$ with the property $f(y) \leq f(x)$?
Apart from Don Quixote, who would dare to enact such dream-like constructions in real life?

1.3 The theorem of the Greatest Value promises the impossible. The following example makes this clear. Let us first agree that a *real number* x will be an infinite sequence $x(0), x(1), \ldots$ of rationals satisfying Cauchy's condition, that is, for every m there exists n such that, for every p, $|x(n+p) - x(n)| < \frac{1}{2^m}$. Again consider the decimal expansion $d : \mathbb{N} \to \{0, 1, \ldots, 9\}$ of the real number π. We construct a real number x, as follows. We define $x(0) := 0$ and, for each n, if $x(n) = 0$ and $d(n) = d(n+1) = \ldots = d(n+9) = 9$, then $x(n+1) := (-1)^n \frac{1}{2^n}$; if not, then $x(n+1) := x(n)$. The absolute value of the number x is very small and we cannot make out if x coincides with 0 or lies to the left or to the right of 0. One might say that x is *floating around* 0. Let f be the linear function from $[0,1]$ to $\mathbb{R}$ such that $f(0) = 0$ and $f(1) = x$. Suppose someone knows how to find a point at which this function assumes its highest value. If his point lies to the right of 0, then x itself does not lie to the left of 0 and if his point lies to the left of 1, then x itself does not lie to the right of 0. Obviously, finding such a point is beyond our means.

1.4 Contrary to what must be by now our expectation, Brouwer succeeded in proving the Uniform Continuity Theorem, and even more, in his own *intuitionistic* way, see [3] and [4]. He came to understand that one must know quite a lot in order to be sure that a function from $[0,1]$ to $\mathbb{R}$ may be effectively calculated in every point of its domain, indeed so much that one may use this knowledge as a basis for the conclusion of the Uniform Continuity Theorem. A first indication that this idea could make sense is the observation that a function showing a discontinuity is not really everywhere well defined. Consider for instance the function g from $[0,1]$ to $\mathbb{R}$ such that, for every y, if $y < \frac{1}{2}$, then $g(y) = 0$ and, if $y \geq \frac{1}{2}$, then $g(y) = 1$. We cannot start to approximate the value of this function at a point that is floating around $\frac{1}{2}$.
Brouwer's conclusion is that the following assertions hold true, even if they are understood according to his own strict standards:

1.4.1 *Every function from* $[0,1]$ *to* $\mathbb{R}$ *is continuous at every point of* $[0,1]$.

1.4.2 *Every function from* $[0,1]$ *to* $\mathbb{R}$ *that is continuous at every point of* $[0,1]$, *is uniformly continuous on its domain.*

1.4.3 *Given a function* f *from* $[0,1]$ *to* $\mathbb{R}$, *one may calculate a number* M *such that*

(i) *for every* y *in* $[0,1]$, $f(y) \leq M$, *and*
(ii) *for every* n *there exists* y *in* $[0,1]$ *such that* $f(y) > M - \frac{1}{2^n}$.

Theorem 1.4.3 is a first compensation for the loss of the Theorem of the Greatest Value. It is easily derived from Theorems 1.4.1 and 1.4.2, as follows. Let f be a function from $[0,1]$ to $\mathbb{R}$. Define, for each k, $m_k := \max\limits_{i \leq 2^k} f\left(\frac{i}{2^k}\right)$ and observe that the sequence $m_0, m_1, \ldots$ converges. Call its limit M and remark that M satisfies the requirements.
In Section 1 we shall see that Brouwer has a second compensation to offer.

1.5 In Section 2 we consider Brouwer's arguments for his conclusions 1.4.1 and 1.4.2. They turn out to be valid not only for functions defined on the closed interval $[0,1]$, but for all functions whose domain of definition is, like $[0,1]$, closed and bounded and in addition *catalogued*, that is, it is possible, for every point in $\mathbb{R}$, to calculate the distance from the point to the given domain. The so-called *Fan Theorem* plays a key role. We also give a proof of Brouwer's beautiful but not very well-known result on the existence of *forwardly directed minima.*

In Section 3 we explain that words like "finite" and "uniformly continuous" may be understood intuitionistically in many different ways. Using Brouwer's own means we then obtain some theorems that are like shadows of his results; they may be applied under more general circumstances, but with their weaker conclusions are not so useful. Nonetheless, they are worth our attention as offering alternative intuitionistic readings of non-intuitionistic and therefore not immediately understandable mathematical statements. Brouwer's interpretation of the classical results cannot be said to be canonical.

2 Brouwer's twofold insight

2.1 The set $\mathbb{N}$ of the natural numbers is never complete. It is a well-understood but always unfinished project for producing successively $0, 1, 2, \ldots$.
In the same way, every infinite sequence α of natural numbers is always unfinished, a project under execution, work in progress, $\alpha(0), \alpha(1), \alpha(2), \ldots$
Brouwer, more or less forced by Cantor's diagonal argument, made his fancy go through the many possible forms such a project may take. The project may consist in obediently following the instructions of an algorithm, like the decimal expansion of π: $d(0) = 1, d(1) = 4, d(2) = 1, \ldots$ The algorithm may of course be a more simple one, like: $\underline{0}(0) = 0, \underline{0}(1) = \underline{0}, \underline{0}(2) = 0, \ldots$, that is, always the same value. Brouwer thought: I also may make an agreement with myself, for instance this one: for each n, *if* at the moment I have to produce $\alpha(n)$ I have found a proof that in the decimal expansion of π no uninterrupted sequence of 10 nines occurs, then $\alpha(n) := 1$ and *if not*, then $\alpha(n) := 0$. Not going into the difficulties one may have with such an agreement, we prefer, following Brouwer, to take the further step of not insisting to know which rule or agreement governs the development of the sequence and admitting the possibility that such a rule or agreement perhaps do not exist. The only important

thing is that a next value of the sequence is produced as soon as asked for, but I, who am making the sequence, may choose the value as I want it.
$\mathcal{N}$ is the set of all infinite sequences of natural numbers. We are using the word "set" but should not think, as is sometimes suggested in discussions of the concept of "set", that $\mathcal{N}$ may be imagined to be the result of gathering its elements. $\mathcal{N}$ is better compared with a loom, or a canvas, on which all kinds of beautiful sequences may be executed or embroidered.
$\mathbb{N}^*$ is the set of all finite sequences of natural numbers. Let α belong to $\mathcal{N}$ and s to $\mathbb{N}^*$. If there exists n such that $s = \langle \alpha(0), \ldots, \alpha(n-1) \rangle$ we say: α *starts with* s, or α *goes through* s, or, s *contains* α.

2.2 Brouwer's Continuity Principle:

> *Let R be a subset of $\mathcal{N} \times \mathbb{N}$.*
> *(We will write: "$\alpha R n$" intending: "$\langle \alpha, n \rangle \in R$").*
> *Suppose we are able, given any infinite sequence α, to find a natural number n such that $\alpha R n$.*
> *Then we are able, given any infinite sequence α, to find natural numbers m, n such that for every infinite sequence β, if β starts with $\langle \alpha(0), \ldots, \alpha(m-1) \rangle$, then $\beta R n$.*

For, in whatever way the sequence α is given to us, we are able to find a natural number n that is suitable for α. And every sequence, even a very dull one like the sequence $\underline{0}$, could be the result of a sequence of free choices. And if a sequence α is constructed freely, a number n suitable for α will appear at a moment of time at which only finitely many values of α will have been decided upon.
We need a slightly more general version of this continuity principle. Let X be a subset of $\mathcal{N}$. X is called a *spread* if X is closed, that is every infinite sequence with the property that each of its finite initial parts contains an element of X, itself belongs to X, and moreover we may decide, for every s in $\mathbb{N}^*$, if s contains an element of X or not.

2.3 Brouwer's Continuity Principle, extended version:

> *Let X be a spread and R a subset of $X \times \mathbb{N}$.*
> *Suppose we are able, given any infinite sequence α in X, to find a natural number n such that $\alpha R n$.*
> *Then we are able, given any infinite sequence α in X, to find natural numbers m, n such that for every infinite sequence β in X, if β starts with $\langle \alpha(0), \ldots, \alpha(m-1) \rangle$, then $\beta R n$.*

2.4 We now let ρ be some fixed enumeration of the set $\mathbb{Q}$ of rational numbers.
An element α of $\mathcal{N}$ is called a *real number* if and only if the sequence $\rho(\alpha(0)), \rho(\alpha(1)), \ldots$ satisfies Cauchy's condition.
An element α of $\mathcal{N}$ is called a *canonical real number* if and only if, for each n,

$|\rho(\alpha(n)) - \rho(\alpha(n+1))| < \frac{1}{2^{n+2}}$. Every canonical real number α may be thought of as a sequence $\left(\rho(\alpha(0)) - \frac{1}{2}, \rho(\alpha(0)) + \frac{1}{2}\right), \left(\rho(\alpha(1) - \frac{1}{4}, \rho(\alpha(1) + \frac{1}{4}\right), \ldots$ of open rational intervals, where $\rho(\alpha(0)) - \frac{1}{2} < \rho(\alpha(1)) - \frac{1}{4} < \ldots$ and $\rho(\alpha(0)) + \frac{1}{2} > \rho(\alpha(1)) + \frac{1}{4} > \ldots$ Each one of these intervals "contains" the real number α and may be called an approximation of α.

We let $\mathbb{R}$ be the set of all canonical real numbers. It is very important that $\mathbb{R}$, as a subset of $\mathcal{N}$, is a spread.

For all α, β in $\mathbb{R}$, we define: α *really-coincides with* β, notation $\alpha \equiv_{\mathbb{R}} \beta$ if and only if, for each n, $|\rho(\alpha(n)) - \rho(\beta(n))| < \frac{1}{2^n}$.

Let X, Y be subsets of $\mathbb{R}$. X is a *real subset* of Y if and only if, for every α in X there exists β in Y such that $\alpha \equiv_{\mathbb{R}} \beta$. X *really-coincides* with Y if both X is a real subset of Y and Y is a real subset of X.

Let X be a subset of $\mathbb{R}$ and f a function from X to $\mathbb{R}$. f is called a *real function* from X to $\mathbb{R}$ if and only if, for all α, β in X, if $\alpha \equiv_{\mathbb{R}} \beta$, then $f(\alpha) \equiv_{\mathbb{R}} f(\beta)$.

The operations of addition, subtraction and absolute value are defined as one expects. We only mention, that for all α, β in $\mathbb{R}$, α is called *really-smaller than* β, notation: $\alpha <_{\mathbb{R}} \beta$ if and only if one may calculate n such that $\rho(\alpha(n)) + \frac{1}{2^n} \leq \rho(\beta(n))$. We do not go into the straightforward definition of a natural embedding of $\mathbb{Q}$ into $\mathbb{R}$.

We want to use the fact that for every canonical real number α, for every n, there exists m such that for every canonical real number β, if $|\alpha - \beta| < \frac{1}{2^m}$, then there exists a canonical real number γ going through $\langle \alpha(0), \ldots, \alpha(n-1) \rangle$ such that $\beta \equiv_{\mathbb{R}} \gamma$. It suffices to choose m such that $\frac{1}{2^m}$ is smaller than both $\rho(\alpha(n)) - \rho(\alpha(n-1)) + \frac{1}{2^{n+2}}$ and $\rho(\alpha(n-1)) - \rho(\alpha(n)) + \frac{1}{2^{n+2}}$.

2.5 Continuity Theorem:

> *Let X be a spread and a subset of $\mathbb{R}$*
> *Suppose that for every α in X, for every n, there exists m such that for every β in X, if $|\alpha - \beta| < \frac{1}{2^m}$, then there exists γ in X going through $\langle \alpha(0), \ldots, \alpha(n-1\rangle$ and really-coinciding with β. Then: every real function with domain X is continuous at every point of X. In particular, every real function that is defined everywhere on $\mathbb{R}$, is everywhere continuous.*

Proof: Let α be an element of X and p a natural number. Using the Continuity Principle, we find n in $\mathbb{N}$ such that for every β in X, if β goes through $\langle \alpha(0), \ldots, \alpha(n-1\rangle$, then $(f(\alpha))(p+1) = (f(\beta))(p+1)$ and therefore $|f(\alpha) - f(\beta)| < \frac{1}{2^p}$. We then determine m in $\mathbb{N}$ such that, for every β in X, if $|\alpha - \beta| < \frac{1}{2^m}$, then there exists γ going through $\langle \alpha(0), \ldots, \alpha(n-1) \rangle$ and really-coinciding with β, and therefore $|f(\alpha) - f(\gamma)| < \frac{1}{2^p}$ and $f(\gamma) \equiv_{\mathbb{R}} f(\beta)$, so and also $|f(\alpha) - f(\beta)| < \frac{1}{2^p}$. □

2.6 The Continuity Theorem is due to Brouwer although its above formulation and proof are not precisely his, see [3], [4], [14], [17].

Enunciating the Continuity Principle is only the first step Brouwer takes in analyzing the notion of a real function defined everywhere on $\mathbb{R}$. In order to explain his next step we have to study *stumps*.

Every stump is a decidable subset of the set $\mathbb{N}^*$ of all finite sequences of natural numbers. $\mathbb{N}^*$ contains exactly one element of length 0, the empty sequence notation: $\langle\,\rangle$. For all s, t in $\mathbb{N}^*$, $s * t$ is the element of $\mathbb{N}^*$ that is obtained by putting t behind s. For every s in $\mathbb{N}^*$, for every subset A of $\mathbb{N}^*$ we define: $s * A := \{s * t \mid t \in A\}$.
The set of stumps is given by the following inductive definition:

(i) The set $\{\langle\,\rangle\}$ is a stump.
(ii) Given any sequence $S_0, S_2, \ldots$ of stumps we may form a new stump $S := \{\langle\,\rangle\} \cup \bigcup\limits_{n \in \mathbb{N}} \langle n \rangle * S_n$.
(iii) Every stump is obtained from the basic stump $\{\langle\,\rangle\}$ by the repeated application of construction step (ii).

For every s in $\mathbb{N}^*$ different from the empty sequence we may find t in $\mathbb{N}^*$ and n in $\mathbb{N}$ such that $s = t * \langle n \rangle$; t is called the *immediate predecessor* of s.
For every stump S, for every s in $\mathbb{N}^*$ we define: s is *just outside* S if and only if s is non-empty and the immediate predecessor of s belongs to S but s itself does not.
We also have to consider *bars*. A subset B of $\mathbb{N}^*$ is a *bar* (*in* $\mathcal{N}$) if and only if every infinite sequence α of natural numbers $\alpha(0), \alpha(1), \ldots$ has an initial part $\langle \alpha(0), \ldots, \alpha(n-1) \rangle$ that belongs to B. We say: α *meets* B if and only if some finite initial part of α belongs to B.

2.7 Brouwer's Thesis:

Given any bar B in $\mathcal{N}$, we may build a stump S such that $B \cap S$ is a bar in $\mathcal{N}$.

Brouwer defends this thesis as follows. If we know B to be a bar in $\mathcal{N}$ we must be able to construct a canonical proof of this fact. The *starting-points* of such a canonical proof are statements of the following form:

> s belongs to B, therefore every infinite sequence going through s meets B

There are two kinds of *reasoning steps*:
(i) steps with infinitely many premises:

> Every infinite sequence going through $s * \langle 0 \rangle$ meets B, every infinite sequence going through $s * \langle 1 \rangle$ meets $B, \ldots$
> *therefore*: every infinite sequence through s meets B.

(ii) backward steps, with only one premise, for instance:

> Every infinite sequence going through s meets B
> *therefore*: every infinite sequence through $s * \langle 17 \rangle$ meets B.

The conclusion of the proof is:

> Every infinite sequence going through $\langle\,\rangle$ meets B.
> *Therefore*: B is a bar in $\mathcal{N}$.

Such a canonical proof itself has the structure of a stump. Like every "simple" proof by complete induction, see [13], the proof itself is an infinite mental construction, although, sometimes, such a construction may be called into existence by a finite text. The conclusion of Brouwer's Thesis is obtained by re-using the stump that underlies the canonical proof as a framework for a new reasoning. This new argument consists of statements of the following form:

> There exists a stump S such that $B \cap S$ is a bar in the collection of all infinite sequences α that start with s.

2.8 Let X be a subset of $\mathcal{N}$ and B a subset of $\mathbb{N}^*$. B is called a *bar in* X if and only if every infinite sequence α in X meets the set B.
Let X be a subset of $\mathcal{N}$ and a spread. X is called a *finitary spread* or a *fan* if, for every s in $\mathbb{N}^*$, there exist only finitely many natural numbers n such that $s * \langle n \rangle$ contains an element of X.

2.9 Lemma:

For every stump S, for every fan F,
the collection of all s in $\mathbb{N}^*$ that contain an element of F and are just outside S, is finite.

Proof: We use (transfinite) induction on the set of stumps. The Lemma holds true in case S is the basic stump $\{\langle\,\rangle\}$. Now assume that $S_0, S_1, \ldots$ is a sequence of stumps and that the Lemma holds true for every one of them.
Consider $S := \{\langle\,\rangle\} \cup \bigcup\limits_{n \in \mathbb{N}} \langle n \rangle * S_n$ and let F be a fan. Observe that for every n in $\mathbb{N}$, if $\langle n \rangle$ contains an element of F, then there are only finitely many s in $\mathbb{N}^*$ just outside S_n such that $\langle n \rangle * s$ contains an element of F. As there are but finitely many n in $\mathbb{N}$ such that $\langle n \rangle$ contains an element of F, there are also but finitely many s in $\mathbb{N}^*$ just outside S containing an element of F. $\square$

2.10 Fan Theorem:

Let F be a fan and B an bar in F.

(i) *There exists a finite subset B' of B such that B' is a bar in F*
(ii) *There exists a natural number n such that every α in F has an initial part belonging to B of length at most n.*

Proof: (i) Consider $B^+ := B \cup \{s \mid s \in \mathbb{N}^* \mid s \text{ does not contain an element of } F\}$ and observe that B^+ is a bar in $\mathcal{N}$. Find a stump S such that $B^+ \cap S$ is a bar in $\mathcal{N}$. Let C be the (finite) set of all s in $\mathbb{N}^*$ that are just outside S and contain an element of F. Choose to every s in C an initial part b_s in B and define $B' := \{b_s \mid b \in C\}$.
(ii) easily follows from (i). $\square$

2.11 If, for a moment, the reader puts on his classical opaque spectacles, het will, seeing the Fan Theorem, recornize D. König's Lemma, see [12]. A special case of this Lemma may be formulated as follows:

Let B be a subset of $\mathbb{N}^*$ and suppose that no finite subset B' of B is a bar in $\mathcal{C}$. Then B itself is not a bar in $\mathcal{C}$, that is, we may find an infinite sequence α in $\mathcal{C}$ such that no finite initial part of α belongs to B.

From a constructive point of view, this statement is just as false as the Bolzano-Weierstrass Theorem.

Brouwer's Fan Theorem is a bone of contention, even among constructivists. E. Bishop fullysubscribed to Brouwer's critique and agreed with him that large parts of mathematics are in need of revision, but he could not understand Brouwer's proposals for new asiomx. In [?] and [2] he defends a "*straightforwardly realistic*" constructive mathematics, without "*semi-mystical*" elements. In [10], S.C. Kleene showed that the Fan Theorem is incompatible with an algorithmic conception of the continuum. There exists a Turing-computable subset B of $\mathbb{N}^*$ such that every Turing-computable α in C meets the set B and for every finite subset B' of B there exists a Turing-computable α in C not meeting B'.

Kleene understood very well that Brouwer's conception of the continuum is not the algorithmic one, and the respected Brouwer's view, see [15].

Proof: Let m be a natural number. Let B be the set of all finite sequences $s = \langle s(0), \ldots, s(n-1)\rangle$ in $\mathbb{N}^*$ such that for all α, β in X, if both $|\rho(s(n-1)) - \alpha| < \frac{1}{2^n}$ and $|\rho(s(n-1)) - \beta| < \frac{1}{2^n}$, then $|f(\alpha) - f(\beta)| < \frac{1}{2^m}$. Observe that B is a bar in F. Find a finite subset B' of B that is a bar in F. Now calculate q such that, for every n, for every s in B' of length n, for every p such that $s * \langle p\rangle$ contains an element of F, the number $\frac{1}{2^q}$ is smaller than both $\rho(s(n-1)) - \rho(p) + \frac{1}{2^{n+1}}$ and $\rho(p) - \rho(s(n-1)) + \frac{1}{2^{n+1}}$. Remark that, for all α, β in X, if $|\alpha - \beta| < \frac{1}{2^q}$, then there exist n in $\mathbb{N}$ and $s = \langle s(0), \ldots, s(n-1)\rangle$ in B' such that both $|\rho(s(n-1)) - \alpha| < \frac{1}{2^n}$ and $|\rho(s(n-1)) - \beta| < \frac{1}{2^n}$, therefore $|f(\alpha) - f(\beta)| < \frac{1}{2^m}$. $\square$

2.12 Let X be a (real) subset of $\mathbb{R}$. The *closure* of X, notation $\overline{X}$, is the set of all α in $\mathbb{R}$ with the property that for each n there exists β in X such that $|\alpha - \beta| < \frac{1}{2^n}$. X is a *closed* subset of $\mathbb{R}$ if and only if X really-coincides with $\overline{X}$.

Not every closed and bounded subset X of $\mathbb{R}$ really-coincides with a fan. Brouwer has shown that a closed subset X of $\mathbb{R}$ really-coincides with a fan if and only if there exists a sequence $q_0, q_1, \ldots$ of rational numbers *cataloguing* or *demarcating* X, that is, there exists a strictly increasing sequence $n_0, n_1, \ldots$ of natural numbers such that $n_0 = 0$ and, for each k, (i) for each i, if $n_k \leq i < n_{k+1}$, then there exists x in X, such that $|x - q_i| < \frac{1}{2^{2k}}$ and (ii) for every x in X there exists i such that $n_k \leq i < n_{k+1}$ and $|x - q_i| < \frac{1}{2^{2k}}$. (It is not difficult to see that if such sequences $q_0, q_1, \ldots$ and $n_0, n_1, \ldots$ are given, the set X really-coincides with the fan F consisting of all α in $\mathbb{R}$ such that, for each k, there exists i with the property $n_k \leq i \leq n_{k+1}$ and $\rho(\alpha(k)) = q_i$.) A subset X of $\mathbb{R}$ may be demarcated if and only if X is bounded and for each x in $\mathbb{R}$ one may calculate the distance of the point x from the set X, that is, a real number z with the following two properties: (i) for every y in X, $|y - x| \geq z$ and (ii) for each n in $\mathbb{N}$ there exists y in X such that $|y - x| \leq z + \frac{1}{2^n}$. Brouwer calls sets that admit of a demarcation *compactly-catalogued* subsets of $\mathbb{R}$.

An important consequence of the Uniform Continuity Theorem is the fact that every (continuous) function from the closed interval $[0,1]$ to $\mathbb{R}$ is Riemann-integrable. Brouwer also took pride in the following application, the promised second compensation for the loss of the Theorem of the Greatest Value. It may come as a surprise that the second statement of Theorem 2.13 is intuitionistically valid.

2.13 Theorem: (*On the existence of forwardly directed minima*)
Let f be a (uniformly continuous) function from $[0,1]$ to $\mathbb{R}$ such that $f(0) < f(1)$

(i) *For all rational numbers q, r such that $f(0) < q < r < f(1)$ we may construct rational numbers a, b, c such that $0 < a < b < c < 1$ and $q < f(a) < f(b) < f(c) < r$ and, for all y in $[0,1]$, if $b \leq y$, then $f(a) + \frac{1}{5}(r-q) \leq f(y)$, and if $c \leq y$, then $f(b) + \frac{1}{5}(r-q) \leq f(y)$.*

(ii) *For all rational numbers q, r such that $f(0) < q < r < f(1)$ we may construct a real number x such that $q < f(x) < r$ and for all y in $[0,1]$, if $x < y$, then $f(x) < f(y)$.*

Proof: (i) Using the fact that f is uniformly continuous we build a finite sequence of rational rectangles of height $\frac{1}{5}(r-q)$ capturing f, that is, we calculate n in $\mathbb{N}$ and two finite sequence $\langle s_0, \ldots, s_{n-1}, s_n \rangle$ and $\langle t_0, \ldots, t_{n-1} \rangle$ of rational numbers such that $0 = s_0 < \ldots < s_{n-1} < s_n = 1$ and for every x in $[0,1]$ for every $i < n$, if x belongs to $[s_i, s_{i+1}]$, then $f(x)$ belongs to $[t_i, t_i + \frac{1}{5}(r-q)]$.

We calculate $i_0 :=$ the greatest number $i < n$ such that $t_i \leq q$, and $i_1 :=$ the greatest number $i < n$ such that $t_i \leq q + \frac{2}{5}(r-q)$ and $i_2 :=$ the greatest number $i < n$ such that $t_i < q + \frac{4}{5}(r-q)$ and observe $i_0 < i_1 < i_2$. We define $a := s_{i_0+1}$ and $b := s_{i_1+1}$ and $c := s_{i_2+1}$. Then $q < f(a) \leq q + \frac{1}{5}(r-q($ and $q + \frac{2}{5}(r-q) < f(b) \leq q + \frac{3}{5}(r-q)$ and $q + \frac{4}{5}(r-q) \leq f(c) < r$, and for every y in $[0,1]$, if $y \geq b$, then $f(y) > q + \frac{2}{5}(r-q)$ and therefore $f(y) > f(a) + \frac{1}{5}(r-q)$, and if $y \geq c$, then $f(y) \geq q + \frac{4}{5}(r-q)$ and therefore $f(y) \geq f(b) + \frac{1}{5}(r-q)$.
(ii) Repeatedly applying (i), also for other closed intervals than $[0,1]$, we find two infinite sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ of rational numbers, and a sequence $e_0, e_1, \ldots$ of positive rational numbers such that $a_0 = 0$ and $b_0 = 1$ and $e_0, f(1) - f(0)$ and, for

each n, $a_n < a_{n+1} < b_{n+1} < b_n$ and $b_{n+1} - a_{n+1} < \frac{1}{2}(b_n - a_n)$ and $f(a_n) < f(a_{n+1}) < f(b_{n+1}) < f(b_n)$ and for all y in $[a_n, b_n]$, if $y \geq b_{n+1}$ then $f(a_{n+1}) + e_{n+1} \leq f(y)$. In addition, we take care that, for every n, $f(b_{n+1}) < f(a_n) + e_n$.
The sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ now are really-coinciding real numbers. Let x be a canonical real number that really-coincides with them. Observe that for every y in $[0, 1]$, if $x < y$, then $f(x) < f(y)$. For assume $x < y$. Find n in $\mathbb{N}$ such that $b_n < y$. Then $f(x) = \lim_{k \to \infty} f(b_k) < f(b_{n+1}) < f(a_n) + e_n \leq f(y)$. $\square$

Of course one may establish similarly, for functions f from $[0, 1]$ to $\mathbb{R}$ such that $f(0) < f(1)$, the existence of backwardly directed maxima, and for functions f from $[0, 1]$ to $\mathbb{R}$ such that $f(0) > f(1)$, the existence of forwardly directed maxima and backwardly directed minima.

3 Almost

3.1 Let A be a decidable subset of the set $\mathbb{N}$ of natural numbers.
We imagine ourselves that we first decide whether 0 will belong to A or not, then whether 1 will belong to A or not, and so on. So the characteristic function of the set A,

$$\chi_A(0), \chi_A(1), \ldots$$

is called into existence step by step, by an infinite sequence of free decisions. This set A is called *finite* if and only if we may calculate a natural number n such that for every $m > n$, $\alpha(m) = 0$.
The set A will be called *finite-by-delay* if and only if for every n, if n belongs to A, that is, $\chi_A(n) = 1$, then A is finite.
When we are told that a set A is finite-by-delay we do not know immediately the number of elements of A. We have to wait for the first 1 in the sequence $\chi_A(0), \chi_A(1), \ldots$, only then the number of elements of A is revealed.
We mention two examples of a decidable subset A of $\mathbb{N}$ that is finite-by-delay without being finite, $A_0 := \{n \mid n \in \mathbb{N} \mid d(n) = d(n+1) = \ldots = d(n+9) = 9$ and there is no $j < n$ such that $d(j) = d(j+1) = \ldots = d(j+9) = 9\}$ and $A_1 s := \{n \mid n \in \mathbb{N} \mid$ There exists j in A_0 such that $j < n < 2j\}$. Observe that we are sure that A_0 has at most one member but are unable to find an upper bound for the number of elements of A_1.
We now go further and call a decidable subset A of $\mathbb{N}$ *finite-by-double-delay* if and only if for every n, if n belongs to A, then A is finite-by-delay. We mention an example of a decidable subset A of $\mathbb{N}$ that is finite-by-double-delay without being finite-by-delay, $A_2 := \{n \mid n \in \mathbb{N} \mid d(n) = d(n+1) = \ldots = d(n+9) = 9$ and there is at most one $j < n$ such that $d(j) = d(j+1) = \ldots = d(j+9) = 9\}$.
As we perhaps know from experience, delay may be repeated endlessly. For every stump S, for every decidable subset A of $\mathbb{N}$, we define the expression: A *is finite-by-S-fold-delay*, as follows:

(i) A is *finite-by-$\{\langle\,\rangle\}$-fold-delay* $:=$ A is finite.

(ii) for every sequence $S_0, S_1, \ldots$ of stumps we form $S := \{\langle\ \rangle\} \cup \bigcup_{n \in \mathbb{N}} \langle n \rangle * S_n$ and we define: A is *finite-by-S-fold-delay* := for every n, if n belongs to A, then there exists p such that A is finite-by-S_p-fold-delay.

One may show, for all stumps S, T, if S is an immediate substump of T, then every decidable subset of N that is finite-by-S-fold-delay is also finite-by-T-fold-delay. Using Brouwer's Continuity Principle one may show that the converse fails. If S is an immediate substump of T, then not every decidable subset of $\mathbb{N}$ that is finite-by-T-fold-delay will be finite-by-S-fold-delay.
There is one more notion to define. Let A be a decidable subset of $\mathbb{N}$. A is called *almost-finite* if and only if for every strictly increasing sequence $n_0, n_1, \ldots$ of natural numbers one may calculate k such that n_k does not belong to A.
This is a weak and comprehensive notion. Brouwer's Thesis implies, that for every decidable subset A of $\mathbb{N}$, A is almost-finite if and only if there exists a stump S such that A is finite-by-S-fold-delay. These facts, and some related ones, are explained in [15], [16] and [19].
We want to make use of the following observation.

3.2 Lemma: (*On almost-finite sets*)

(i) *For all decidable subsets A, B of $\mathbb{N}$, if both A, B are almost-finite, then also $A \cup B$ is almost-finite.*
(ii) *For every finite sequence $A_0, \ldots, A_{k-1}$ of decidable subsets of $\mathbb{N}$, if each one of $A_0, \ldots, A_{k-1}$ is almost-finite, then $\bigcup_{i<k} A_i$ is almost-finite.*
(iii) *Let $E_0, E_1, \ldots$ be an infinite sequence of decidable subsets of $\mathbb{N}$, such that $\mathbb{N} = \bigcup_{k \in \mathbb{N}} E_k$. Let A be a decidable subset of A such that, for each k, the set $A \cap E_k$ is almost-finite. Also assume that for every strictly increasing sequence $n_0, n_1, \ldots$ of natural numbers there exists k such that $A \cap E_{n_k} = \emptyset$. Then A is almost-finite.*

Proof: (i) Suppose we are given a strictly increasing sequence $n_0, n_1, \ldots$ of natural numbers. We build a subsequence $n_{i_0}, n_{i_1}, \ldots$ of the sequence $n_0, n_1, \ldots$ such that, for every k, $n_{i_k} \notin A$. We then find k such that n_{i_k} and observe $n_{i_k} \notin A \cup B$.
(ii) easily follows from (i).
(iii) Suppose we are given a strictly increasing sequence $n_0, n_1, \ldots$ of natural numbers. We now build two sequences $i_0, i_1, \ldots$ and $k_0, k_1, \ldots$ of natural numbers. We define $i_0 := 0$ and determine k_0 such that $n_0 = n_{i_0}$ belongs to E_{k_0}. We then find i_1 such that $i_0 < i_1$ and either n_{i_1} does not belong to A or n_{i_1} does not belong to E_{k_0}. We then determine k_1 such that $k_1 \neq k_0$ and n_{i_1} does not belong to A or n_{i_1} belongs to E_{k_1}. We then find i_2 such that $i_1 < i_2$ and n_{i_2} does not belong to A or n_{i_2} does not belong to $E_{k_0} \cup E_{k_1}$. We then determine k_2 different from both k_0, k_1 such that n_{i_2} does not belong to A or n_{i_2} belongs to E_{k_2}. In this way we continue. The sequence $i_0, i_1, \ldots$ is strictly increasing and the sequence $k_0, k_1, \ldots$ is one-to-one. For every j, either n_{i_j}

does not belong to A or n_{i_j} belongs to E_{k_j}. We calculate j such that $A \cap E_{k_j} = \emptyset$ and conclude: n_{i_j} does not belong to A. □

3.3 Let X be a spread. X is called an *almost-fan* if, for every s in $\mathbb{N}^*$, the set of all natural numbers n such that $s * \langle n \rangle$ contains an element of X is a decidable and almost-finite subset of $\mathbb{N}$.
Let A be a decidable subset of $\mathbb{N}^*$. A is called *almost-finite* if for every one-to-one sequence $s_0, s_1, \ldots$ of elements of $\mathbb{N}^*$ one may find k such that s_k does not belong to A.

3.4 Lemma: *For every stump S, for every almost-fan F, the set of all s in $\mathbb{N}^*$ that are just outside S and contain an element of F, is decidable and almost-finite.*

Proof: The proof follows the pattern of the proof of Lemma 1.9 and uses Lemma 2.2(iii). We leave the details to the reader. □

3.5 Almost-fan-theorem:
Let F be an almost-fan and let B be a bar in F.
(i) *There exists an almost-finite subset B' of B such that B' is a bar in F.*
(ii) *For every sequence $s_0, s_1, \ldots$ of elements of $\mathbb{N}^*$, if, for every n, s_n contains an element of F and length$(s_n) = n$, then we may find k in $\mathbb{N}$ such that some initial part of s_k belongs to B*

Proof: (i) The proof follows the pattern of the proof of Theorem 2.10(i) and uses Lemma 3.4. We leave the details to the reader.
(ii) Let B' be an almost-finite subset of B that is a bar in F. Let $s_0, s_1, \ldots$ be an infinite sequence of elements of $\mathbb{N}^*$ such that, for each n, s_n contains an element of F and length$(s_n) = n$. For each n, we find b_n in B' such that either b_n is an initial part of s_n or s_n is an initial part of b_n. We determine a strictly increasing sequence $i_0, i_1, \ldots$ of natural numbers such that, for each k, $i_{k+2} >$ length(b_{i_k}). As B' is almost-finite, we may determine k, l such that $k < l$ and $b_{i_k} = b_{i_l}$. But now s_{i_l} has an initial part in B'. □

3.6 The notion of a finite subset of $\mathbb{N}$ is not the only mathematical notion that admits of endless refinement. Here are two more examples, hardly more than suggestions.

3.6.1 Let x_0, x_1 be a sequence of real numbers.
The sequence $x_0, x_1, \ldots$ is *bounded* if we may calculate M such that, for every n, $|x_n| \leq M$.
The sequence $x_0, x_1, \ldots$ is *perhaps-bounded* if we may calculate M such that, for every n, if $|x_n| > M$, then the sequence $x_0, x_1, \ldots$ is bounded.
The sequence $x_0, x_1, \ldots$ is *perhaps-perhaps-bounded* if we may calculate M such that, for every n, if $|x_n| > M$, then the sequence $x_0, x_1, \ldots$ is perhaps-bounded.
The reader will guess how to define further iterations of "perhaps", even transfinitely many. We also define:

The sequence $x_0, x_1, \ldots$ is *almost-bounded* if, given any strictly increasing sequence of natural numbers, we may find k such that $|x_{n_k}| \leq k$.
This last notion is very weak and comprises the earlier ones.

3.6.2 Let X be a (real) subset of $\mathbb{R}$ and f a real function from X to $\mathbb{R}$. Let e be a positive rational number.
f is e-uniformly-continuous on X, if we may find n such that for all x, y in X, if $|x-y| < \frac{1}{2^n}$, then $|f(x) - f(y)| < e$.
f is *e-perhaps-uniformly-continuous* on X if we may calculate n, such that for all x, y in X, if $|x-y| < \frac{1}{2^n}$ and $|f(x) - f(y)| \geq e$, then f is e-uniformly-continuous on X.
Again, we may iterate:
f is e-perhaps-perhaps-uniformly-continuous on X if we may calculate n such that for all x, y in X, if $|x-y| < \frac{1}{2^n}$ and $|f(x) - f(y)| \geq e$, then f is e-perhaps-uniformly-continuous on X.
The definition of further iterates is left to the reader. We also define:
f is *e-almost-uniformly-continuous* on X if, given any two infinite sequences $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ of elements of X, we may calculate k such that: if $|x_k - y_k| < \frac{1}{2^k}$, then $|f(x) - f(y)| < e$.
This last notion is very weak and comprises the earlier ones.
f is *almost-uniformly-continuous* on X, if, for each positive rational number e, f is e-almost-uniformly-continuous on X.

3.7 Almost-uniform-continuity-theorem:

Let X be a (real) set of real numbers that coincides with a sub-almost-fan F of $\mathbb{R}$, and let f be a (real) function from X to $\mathbb{R}$. If f is continuous in every point of X, then f is almost-uniformly-continuous on X

Proof: Let e be a positive rational number. Let B be the set of all finite sequences $s = \langle s(0), \ldots, s(n-1)\rangle$ in $\mathbb{N}^*$ such that for all α, β in X, if both $|\rho(s(n-1)) - \alpha| < 2^{-n}$ and $|\rho(s(n-1)) - \beta| < 2^{-n}$, then $|f(\alpha) - f(\beta)| < e$. B is a bar in F. We determine an almost finite subset B' of B that is a bar in F. Let $x_0, x_1, \ldots$ and $y_0, y_1, \ldots$ be infinite sequences of elements of F. For every k we let s_k be a finite sequence in B' that contains x_k, we let n_k be the length of s_k, so $s_k = \langle s_k(0), \ldots, s_k(n_k-1)\rangle$, and we let q_k be a natural number such that for every y in F, if $|x_k - y| < \frac{1}{2^{q_k}}$, then $|x_k(n_k-1) - y| < \frac{1}{2^{n_k}}$ and therefore $|f(x_k) - f(y)| < e$. We now consider the sequence $q_0, q_1, \ldots$ We define the proposition QED (quod *est* demonstrandum, that is, the thing that we still have to prove), as follows:

$$\text{QED} := \text{there exists } k \text{ such that } q_k < k.$$

We now build a strictly increasing sequence $m_0, m_1, \ldots$ of natural numbers such that, for every i, $q_{m_i} < q_{m_{i+1}}$ or QED.
As B' is almost finite we may find i, j such that $i < j$ and $s_{m_i} = s_{m_j}$, and therefore $q_{m_i} = q_{m_j}$, and QED, so there exists k such that $q_k < k$, and therefore if $|x_k - y_k| < \frac{1}{2^k}$ then $|x_k - y_k| < \frac{1}{2^{q_k}}$ and $|f(x_k) - f(y_k)| < e$. □

3.8 Theorem:

Let $x_0, x_1, \ldots$ be an almost-bounded set of canonical real numbers. The set $\{x_0, x_1, \ldots\}$ really-coincides with an almost-fan.

Proof: We define a decidable subset C of $\mathbb{N}^*$. The empty sequence $\langle\ \rangle$ belongs to C. For every finite sequence of the form $\langle n\rangle$ we lay down: $\langle n\rangle$ belongs to C if and only if there exists $j \le n$ such that $|\rho(n) - x_j(2)| < \frac{1}{4}$. For every non-empty sequence $t = \langle t(0), \ldots, t(k-1)\rangle$ in $\mathbb{N}^*$, for every n, we prescribe: $t * \langle n\rangle$ belongs to C if and only if t belongs to C and there exists $j \le n$ such that $|\rho(n) - x_j(k+2)| < \frac{1}{2^{k+2}}$ and $|\rho(t(k-1)) - \rho(n)| < \frac{1}{2^{k+1}}$ and there is no $i < n$ such that $t * \langle i\rangle$ belongs to C and $|\rho(i) - \rho(n)| < \frac{1}{2^{k+2}}$.
Let F be the set of all α in $\mathcal{N}$ such that, for every k, $\langle\alpha(0), \ldots, \alpha(k-1\rangle$ belongs to C.
F is an almost-fan and a subspread of $\mathbb{R}$ that really-coincides with $\{x_0, x_1, \ldots\}$. $\square$

3.9 The principle of Open Induction we are about to prove is a remarkable result. It reminds one of Achilles.
Suppose that Achilles has to move from 0 and 1, and that we are given the following information:

(i) For every x in $[0, 1]$, if Achilles reaches x and $x < 1$, then Achilles also reaches some point $y > x$.

(ii) For every x in $[0, 1]$, Achilles reaches x if and only if Achilles reaches every point $y < x$.

Are we able to conclude that Achilles will arrive in 1?
Probably, we should first prove that he reaches $\frac{1}{2}$.
Paradoxically, proving that Achilles will arrive in $\frac{1}{2}$ does not seem in the least easier than proving that he will arrive in 1. Every argument proving that Achilles will reach $\frac{1}{2}$, or any other point, seems to be essentially unconstructive. From a constructive point of view, seem to have no other choice but to wait and see.
Perhaps Achilles reaches $\frac{1}{2}$ in 1 or in 2 steps,..., or in ω steps,..., or in ω^2 steps, or ...
Still, Achilles has some reason to be conficent.

3.10 Theorem: (*Principle of Open Induction*, Th. Coquand.)

Let A be an open subset of $[0, 1]$.
Assume that for every x in $[0, 1]$, if for every $y < x$, $y \in A$, then $x \in A$. Then $[0, 1] = A$.

Proof: Let $q_0, q_1, \ldots$ and $r_0, r_1, \ldots$ be infinite sequences of rational numbers such that for every x in $[0, 1]$, x belongs to A if and only if, for some n, $q_n < x < r_n$.
Observe that for every x in $[0, 1]$, the closed interval $[0, x]$ really-coincides with a subfan of $\mathbb{R}$, and therefore $[0, x]$ is a real subset of A if and only if there exists n such that $[0, x]$ forms part of the finite union $\bigcup_{i<n} (q_i, r_i)$. Remark that 0 belongs to A. We therefore may assume: $q_0 < 0 < r_0$.
We now define a sequence $x_0, x_1, \ldots$ of points in $[0, 1]$, as follows. For each n, $x_n :=$

the greatest rational number x such that $[0, x)$ forms part of $\bigcup\limits_{i<n} (q_i, r_i)$.
Observe that the sequence $x_0, x_1, \ldots$ is well-defined and that, for each n, $x_n \leq x_{n+1}$ and that the set of all x in $[0, 1]$ such that $[0, x] \subseteq A$ coincides with $\{x_0, x_1, \ldots\}$.
Let F be an almost-fan really-coinciding with $\{x_0, x_1, \ldots\}$.
Let B be the set of all finite sequences $s = \langle s(0), \ldots, s(n-1)\rangle$ in $\mathbb{N}^*$ such that, for some $j < n$, $q_j < \rho(s(n-1)) < r_j$. B is a bar in F. We determine an almost-finite subset B' of B that is a bar in F. We now construct an infinite sequence $z_0, z_1, \ldots$ of elements of F and an infinite sequence of elements of B', as follows: $z_0 := 0$. Let $s_0 = \langle s_0(0), \ldots, s_0(n_0-1)\rangle$ be the shortest initial part of z_0 that belongs to B'. Find j_0 such that $q_{j_0} \langle \rho(s_0(n_0-1)) < r_{j_0}$. Observe that $[0, r_{j_0}]$ forms part of A and let z_1 be an element of F really-coinciding with r_{j_0}. If $z_1 \geq 1$, we define, for every $n > 0$, $z_{n+1} := z_n$ and $s_{n+1} := s_0$. If $z_1 < 1$, the shortest initial part of z_1 that belongs to B'. Find j, such that $q_{j_1} < \rho(s_1(n_1-1)) < r_{j_1}$. Observe that $[0, r_{j_1}]$ forms part of A and let z_2 be an element of F really-coinciding with r_{j_1}. If $z_2 \geq 1$, we define, for every $n > 1$, $z_{n+1} := z_n$ and $s_{n+1} := s_1$. If $z_2 < 1$, we let $s_2 = \langle s_2(0), \ldots, s_2(n_2-1)\rangle$ be the shortest initial part of z_2 that belongs to B'.
We continue in this way.
As B' is almost-finite, there exists n such that $s_{n+1} := s_n$ and therefore $z_{n+1} \geq 1$ and $[0, 1]$ forms part of A. $\square$

References

[1] E.A. Bishop, *Foundations of constructive analysis*, McGraw Hill, New York, 1967.
[2] E.A. Bishop and D.S. Bridges, *Constructive analysis*, Springer, Heidelberg & New York, 1985.
[3] L.E.J. Brouwer, Begründung der Funktionenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten, Erster Teil, Stetigkeit, Messbarkeit, Derivierbarkeit, *Kon. Ned. Ak. Wet. Verhandelingen*, 1ste sectie 13 nr. 2 (1923) [ook in [5], pp. 246–267].
[4] L.E.J. Brouwer, Über Definitionsbereiche von Funktionen, *Math. Annalen* 97 (1927), pp. 60–75 [ook in [5], pp. 390–405].
[5] L.E.J. Brouwer, *Collected Works, Volume I: Philosophy and Foundations of Mathematics*, North Holland, Amsterdam-Oxford, 1975 [edited by A. Heyting].
[6] L.E.J. Brouwer, *Intuitionismus*, Wissenschaftsverlag, Mannheim, 1992 [herausgegeben, eingeleitet und kommentiert von D. van Dalen].
[7] Miguel de Cervantes Saavedra, *El ingenioso hidalgo Don Quijote de la Mancha*, Primera Parte, Espaca-Calpe, S.A., Madrid, 1966.
[8] *The history of Don Quixote of the Mancha*, translated from the Spanish of Miguel de Cervantes by Thomas Shelton annis 1612, 1620, with introductions by James Fitzmaurice-Kelly, Volume I, published by David Nutt, in the Strand, London, 1896.
[9] W.A. Howard and G. Kreisel, Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis, *The Journal of Symbolic Logic* 31 (1966), pp. 325–358.

[10] S.C. Kleene and R.E. Vesley, *The Foundations of Intuitionistic Mathematics, especially in relation to recursive functions*, North Holland, Amsterdam, 1965.

[11] S.C. Kleene, Recursive Functions and intuitionistic mathematics, *Int. Congr. Math.* 1950, Cambridge MA, volume I (1952), pp. 679-685.

[12] D. König, Über eine Schlussweise aus dem Endlichen ins Unendliche, *Acta Univ. Szeged, Sect. Math.* 3(1927), pp. 121-130.

[13] H. Poincaré, Sur la nature du raisonnement mathématique, in: *La Science et l'Hypothèse*, Flammarion, Paris, 1968.

[14] W. Veldman, *On the continuity of functions in intuitionistic real analysis*, Report nr. 8210, Department of Mathematics, University of Nijmegen, 1982.

[15] W. Veldman, Some intuitionistic variations on the notion of a finite set of natural numbers, in: H.C.M. de Swart, L.J.M. Bergmans (ed.), *Perspectives on Negation, Essays in honour of Johan J. de Iongh on his 80th birthday*, Tilburg University Press, Tilburg, 1995, pp. 177–202.

[16] W. Veldman, On sets enclosed between a set and its double complement, in: A. Cantini (ed.), *Logic and Foundations of Mathematics, Proceedings Xth International Congress on Logic, Methodology and Philosophy of Science, Florence, 1995*, Volume III, Kluwer, Dordrecht, 1999, pp. 143–154.

[17] W. Veldman,Understanding and using Brouwer's Continuity Principle, in: U. Berger, H. Osswald, P.A. Schuster (eds.), *Reuniting the Antipodes, constructive and nonstandard views of the continuum, Proceedings of a symposium held in San Servolo / Venice, 1999*, Kluwer, Dordrecht, 2001, pp. 285–302.

[18] W. Veldman, *An intuitionistic proof of Kruskal's Theorem*, Report nr. 0017, Department of Mathematics, University of Nijmegen, 2000.

[19] W. Veldman, *The Borel Hierarchy and the projective hierarchy in intuitionistic mathematics*, Report nr. 0103, Department of Mathematics, University of Nijmegen, 2001.

[20] W. Veldman, Bijna de Waaierstelling, in: W. Schikhof, D. Beckers, O. van Gaans, S. Teerenstra (ed.), *Circumspice, Various papers in and around mathematics in honor of Arnoud van Rooij*, Department of Mathematics, University of Nijmegen, 2001, pp. 301-315.

Subfaculteit Wiskunde
Katholieke Universiteit Nijmegen
Postbus 9010, 6500 GL Nijmegen, the Netherlands
email: veldman@sci.kun.nl