A SIMPLE PROOF OF THE MODULAR IDENTITY
FOR THETA FUNCTIONS

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To A.C.M. van Rooij on occasion of his 65th birthday

Abstract. The modular identity arises in the theory of theta functions in one complex variable. It states a relation between theta functions for parameters $\tau$ and $-1/\tau$ situated in the complex upper half plane. A standard proof uses Poisson summation and hence builds on results from Fourier theory. This paper presents an elementary proof using only a uniqueness property and the simple heat equation.

1. The $\theta$ function

Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half plane of all complex numbers with a positive imaginary part. The following series converges locally uniformly in $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ and hence defines a holomorphic function on $\mathbb{C} \times \mathbb{H}$:

$$\theta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{2\pi ikz + \pi k^2 \tau}$$

This function is often called the $\theta_3$ function of Jacobi (some texts use $q = e^{\pi i \tau}$ or replace $2\pi iz$ by $z$). For $z = 0$ it is also called Ramanujan’s theta function. It satisfies the shift relations in $z$

$$(1.1) \quad \theta(z + 1, \tau) = \theta(z, \tau)$$

and

$$(1.2) \quad \theta(z + \tau, \tau) = e^{-2\pi iz - \pi i \tau} \theta(z, \tau)$$

that can easily be verified from its definition. The following heat equation is also apparent from the definition of $\theta$:

$$(1.3) \quad \frac{d^2 \theta}{dz^2} = 4\pi i \frac{d\theta}{d\tau}.$$ 

Let $\Lambda(\tau) = \mathbb{Z} + \mathbb{Z} \tau$ be the lattice spanned by 1 and $\tau$. For fixed parameter $\tau$ the function $\theta$ in $z$ is the only entire function satisfying (1.1) and (1.2) up to complex multiples. This follows from the following theorem:

**Theorem 1.1.** If $f(z)$ is an entire function on $\mathbb{C}$ satisfying the shift relations (1.1) and (1.2) then either $f$ vanishes identically or all its roots equal $(\tau + 1)/2$ modulo the lattice $\Lambda(\tau)$. 

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Suppose \( f \) does not vanish identically. The shift relations for \( f \) imply
\[
\frac{f'(z + 1)}{f(z + 1)} = \frac{f'(z)}{f(z)} \quad \text{and} \quad \frac{f'(z + \tau)}{f(z + \tau)} = \frac{f'(z)}{f(z)} - 2\pi i.
\]
For \( b \in \mathbb{C} \) define a closed fundamental domain \( P \subset \mathbb{C} \) by
\[
P = \{ b + x + y\tau \mid x, y \in [0, 1] \}.
\]
The number of roots of \( f \) on \( P \) and the sum of its roots on \( P \) can be computed by the integrals
\[
\frac{1}{2\pi i} \oint_{\partial P} \frac{f'(z)}{f(z)} \, dz
\]
and
\[
\frac{1}{2\pi i} \oint_{\partial P} z\frac{f'(z)}{f(z)} \, dz
\]
respectively. By varying the number \( b \) we may assume that \( f \) has no roots on \( \partial P \) so both integrals are well defined. Using the shift relations for \( f \) the first integral evaluates to 1, showing that \( f \) has only one root on \( P \). The second integral evaluates to a value equal to \((\tau + 1)/2\) modulo the lattice \( \Lambda(\tau) \). This proves the theorem. \( \square \)

**Corollary 1.2.** If \( f \) is as theorem 1.1, then \( f(z) = c \cdot \theta(z, \tau) \) for some constant \( c \in \mathbb{C} \).

Also by theorem 1.1 we find that \( \theta(0, \tau)f(z) - f(0)\theta(z, \tau) \) must vanish identically as it vanishes at \( z = 0 \) as well as at \((\tau + 1)/2\). \( \square \)

2. The modular identity

We are already in a position to prove the modular identity (2.3) for \( \theta \). A very accessible treatment of this identity using Poisson summation can be found in [1]. For the proof given below, the heat equation suffices. Define an entire function \( \vartheta \) by
\[
\vartheta(z) = e^{\pi i \tau z^2} \theta(\tau z, \tau).
\]
Then \( \vartheta(z + 1) = \vartheta(z) \) and
\[
\vartheta(z - 1/\tau) = e^{-2\pi i z + \pi i / \tau} \vartheta(z).
\]
Hence \( \vartheta(z) = c(\tau) \cdot \theta(z, -1/\tau) \) for some function \( c \) on the upper half plane by corollary 1.2. Substituting \( \tau = i \) and \( z = 0 \) shows that \( c(i) = 1 \). The heat equation for \( \theta \) will produce a simple differential equation for \( c \). Elementary computations show:

\[
\begin{align*}
\frac{d^2 \vartheta}{dz^2}(0) & = 2\pi i \vartheta(0, \tau) + \tau^2 \frac{d^2 \theta}{dz^2}(0, -1/\tau) = c(\tau) \frac{d^2 \theta}{dz^2}(0, -1/\tau) \quad (2.1) \\
\frac{d\vartheta}{d\tau}(0) & = \frac{d\theta}{d\tau}(0, \tau) = c'(\tau) \theta(0, -1/\tau) + c(\tau) \tau^{-2} \frac{d\theta}{d\tau}(0, -1/\tau). \quad (2.2)
\end{align*}
\]

Using the heat equation (1.3) on (2.1) yields
\[
\frac{1}{\tau} \vartheta^{-1}(0, \tau) + \frac{d\vartheta}{d\tau}(0, \tau) = c(\tau) \tau^{-2} \frac{d\vartheta}{d\tau}(0, -1/\tau)
\]
and combining this with (2.2) leads to
\[
\theta(0, \tau) = -2\tau c'(\tau) \theta(0, -1/\tau).
\]
However, substituting \( z = 0 \) in \( \vartheta(z) \) gives
\[
\theta(0, \tau) = \vartheta(0) = c(\tau)\theta(0, -1/\tau)
\]
and as \( \theta \) does not vanish at \( z = 0 \) we find
\[
-2\tau c'(\tau) = c(\tau).
\]
Together with \( c(i) = 1 \) we finally find
\[
c(\tau) = \frac{1}{\sqrt{-\tau}}
\]
and thus the modular identity for the \( \theta \) function:
\[
(2.3) \quad \theta(z, -1/\tau) = \sqrt{-\tau}e^{\pi i z^2}\theta(\tau z, \tau).
\]

References