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**ESTIMATION OF THE MEAN OF A
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Estimation of the mean of a Wiener sheet

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Abstract

A standard Wiener sheet shifted by an unknown parameter is observed above a decreasing curve Γ . By the help of a direct discrete approach and under weaker assumptions than in the paper of Arató, N. M. [3], an explicit formula is derived for the maximum likelihood estimator (MLE) of the shift parameter. The MLE is a weighted linear combination of the values at the endpoints of the curve Γ and weighted integrals of the observed process and its normal derivative along the curve Γ .

Keywords. Wiener sheet, Radon–Nikodym derivatives, maximum likelihood estimator, stochastic integral along a curve, weighted integral of the normal derivative of a random process along a curve.

1 Introduction

Let $\{W(s, t) : s, t \geq 0\}$ be a standard Wiener sheet and consider the process $Z(s, t) := W(s, t) + m$, where $m \in \mathbb{R}$ is an unknown parameter. Let $[a, b] \subset (0, \infty)$ and take a continuous, strictly decreasing function $\gamma : [a, b] \rightarrow \mathbb{R}$ with $\gamma(b) > 0$. Consider the curve $\Gamma := \{(s, \gamma(s)) : s \in (a, b)\}$ and the set

$$G := \{(s, t) \in \mathbb{R}^2 : a \leq s \leq b, t \geq \gamma(s) \text{ or } s > b, t \geq \gamma(b)\}.$$

Finally, let $\tilde{G} \subset G$ be a subset containing an ε -strip of Γ , i.e.,

$$\{(s, t) \in \mathbb{R}^2 : s \in [a, a + \varepsilon], t \in [\gamma(s), \gamma(a)] \text{ or } s \in [a + \varepsilon, b], t \in [\gamma(s), \gamma(s) + \varepsilon]\} \subset \tilde{G}$$

with some $\varepsilon > 0$. Arató, N. M. [3] considered a twice continuously differentiable function γ on $[a, b]$ and determined the maximum likelihood estimator (MLE)

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of m based on the observation of $\{Z(s, t) : (s, t) \in \tilde{G}\}$ by the help of partial stochastic differential equations. It turned out that the MLE is a weighted linear combination of the values at the endpoints $(a, \gamma(a))$ and $(b, \gamma(b))$ of the curve Γ and weighted integrals of the observed process and its normal derivative along the curve Γ . The purpose of the present paper is to derive the MLE of m under weaker assumptions on the function γ applying direct discrete approach instead of partial stochastic differential equations. We remark that the results could have been also derived from the general Feldman–Hajek theorem (see, for example, Kuo [7]), but our direct approach seems to be essentially simpler. Moreover, we note that the results derived in Section 2 for finite samples and also the L^2 -integration theory along a curve as derived in the Sections 4 and 5, might be of independent interest.

1.1 Theorem. *Let $[a, b] \subset (0, \infty)$. Suppose that the function $\gamma : [a, b] \rightarrow \mathbb{R}$ is continuous, strictly decreasing with $\gamma(b) > 0$ and twice continuously differentiable on (a, b) . Moreover, assume that the limits $\gamma'(a) := \lim_{s \downarrow a} \gamma'(s) \in [-\infty, 0]$ and $\gamma'(b) := \lim_{s \uparrow b} \gamma'(s) \in [-\infty, 0]$ exist, and*

$$\int_a^b \frac{|\gamma'(s)\gamma''(s)|}{(1 + \gamma'(s)^2)^2} ds < \infty. \quad (1.1)$$

Then the probability measures \mathbb{P}_Z and \mathbb{P}_W , generated on $C(\tilde{G})$ by the sheets Z and W , respectively, are equivalent and the Radon–Nikodym derivative of \mathbb{P}_Z with respect to \mathbb{P}_W equals

$$\frac{d\mathbb{P}_Z}{d\mathbb{P}_W}(Z) = \exp \left\{ -\frac{1}{2}(Am^2 - 2\zeta m) \right\},$$

with

$$A = \frac{1}{b\gamma(b)} + \int_a^b \frac{ds}{s^2\gamma(s)},$$

$$\zeta = c_1 Z(a, \gamma(a)) + c_2 Z(b, \gamma(b)) + \int_{\Gamma} y_1 Z + \int_{\Gamma} y_2 \partial_n Z,$$

where $\partial_n Z$ denotes the normal derivative of Z and

$$c_1 = \begin{cases} \frac{1}{a\gamma(a)(1+\gamma'(a)^2)} & \text{if } \gamma'(a) > -\infty, \\ 0 & \text{if } \gamma'(a) = -\infty, \end{cases} \quad c_2 = \begin{cases} \frac{\gamma'(b)^2}{b\gamma(b)(1+\gamma'(b)^2)} & \text{if } \gamma'(b) > -\infty, \\ \frac{1}{b\gamma(b)} & \text{if } \gamma'(b) = -\infty, \end{cases}$$

$$y_1(s, \gamma(s)) = \frac{\gamma'(s) \left[(\gamma(s)\gamma'(s) - s)(1 + \gamma'(s)^2) - 2s\gamma(s)\gamma''(s) \right]}{s^2\gamma(s)^2 (1 + \gamma'(s)^2)^{5/2}},$$

$$y_2(s, \gamma(s)) = \frac{\gamma'(s)}{s\gamma(s)(1 + \gamma'(s)^2)}.$$

The maximum likelihood estimator of the shift parameter m based on the observations $\{Z(s, t) : (s, t) \in \tilde{G}\}$ has the form $\tilde{m} = \zeta/A$ and it has a normal distribution with mean m and variance $1/A$.

The integrals $\int_{\Gamma} y_1 Z$ and $\int_{\Gamma} y_2 \partial_n Z$ are considered as improper L^2 -Riemann integrals, and will be discussed in the Sections 4 and 5, respectively.

1.2 Remark. In order to calculate Radon–Nikodym derivatives we will apply a discrete approximation, i.e., the sample $\{Z(s, t) : (s, t) \in \tilde{G}\}$ will be approximated by finite samples. For a partition $\mathcal{P} : a = s_1 < s_2 < \dots < s_{N-1} < s_N = b$ of $[a, b]$, let $|\mathcal{P}| := \max_{2 \leq i \leq N} (s_i - s_{i-1})$. Lemma 2.2 implies that the MLE of m based on any finite sample containing the observations

$$\{Z(s_i, \gamma(s_i)) : 1 \leq i \leq N\} \cup \{Z(s_i, \gamma(s_{i-1})) : 2 \leq i \leq N\}$$

and possibly also finitely many observations from

$$\bigcup_{i=1}^N \{Z(s, t) : s \geq s_i, t \geq \gamma(s_i)\}$$

has the form $\tilde{m}_N = \zeta_N/A_N$, where

$$A_N = \sum_{i=1}^N \frac{1}{s_i \gamma(s_i)} - \sum_{i=2}^N \frac{1}{s_i \gamma(s_{i-1})},$$

$$\zeta_N = \sum_{i=1}^N \frac{Z(s_i, \gamma(s_i))}{s_i \gamma(s_i)} - \sum_{i=2}^N \frac{Z(s_i, \gamma(s_{i-1}))}{s_i \gamma(s_{i-1})}.$$

If $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$ is a sequence of partitions with $|\mathcal{P}_N| \rightarrow 0$ then we shall see in the proof of Theorem 1.1 that $\tilde{m}_N \xrightarrow{L^2} \tilde{m}$.

1.3 Example. Let Γ is a part of a circle with centre at the origin, i.e., $\gamma(s) = \sqrt{r^2 - s^2}$ with some $r > 0$, and $[a, b] \subset (0, r)$. Then

$$A = \frac{1}{r^2} \left(\frac{\sqrt{r^2 - a^2}}{a} + \frac{b}{\sqrt{r^2 - b^2}} \right), \quad c_1 = \frac{\sqrt{r^2 - a^2}}{r^2 a}, \quad c_2 = \frac{b}{r^2 \sqrt{r^2 - b^2}},$$

$$y_1(s, \sqrt{r^2 - s^2}) \equiv 0, \quad y_2(s, \sqrt{r^2 - s^2}) \equiv -\frac{1}{r^2},$$

hence

$$\tilde{m} = \frac{\frac{\sqrt{r^2 - a^2}}{a} Z(a, \sqrt{r^2 - a^2}) - \int_{\Gamma} \partial_n Z + \frac{b}{\sqrt{r^2 - b^2}} Z(b, \sqrt{r^2 - b^2})}{\frac{\sqrt{r^2 - a^2}}{a} + \frac{b}{\sqrt{r^2 - b^2}}}.$$

First we derive the Radon–Nikodym derivative of finite samples in Section 2. The proof of Theorem 1.1 is given in Section 3, and it is based on several lemmas given in the Sections 2, 4 and 5.

2 Radon–Nikodym derivatives of finite samples

Let $0 < s_1 < s_2 < \cdots < s_L$ and $0 < t_1 < t_2 < \cdots < t_M$ be real numbers and let $1 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \leq \lambda_{n+1} = L$ and $1 = \mu_1 < \mu_2 < \cdots < \mu_n \leq \mu_{n+1} = M$ be integers. Let

$$\begin{aligned} R &:= \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq L, 1 \leq j \leq M\}, \\ H &:= \bigcup_{k=1}^n \{(i, j) \in \mathbb{N}^2 : \lambda_k \leq i \leq L, \mu_{n-k+1} \leq j \leq M\}, \\ H_+ &:= \{(\lambda_k, \mu_{n-k+1}) : k = 1, \dots, n\}, \\ H_- &:= \{(\lambda_k, \mu_{n-k+2}) : k = 2, \dots, n\}, \\ H_1 &:= \{(i, \mu_k) : k = 1, \dots, n, \lambda_{n-k+1} < i \leq \lambda_{n-k+2}\}, \\ H_2 &:= \{(\lambda_k, j) : k = 1, \dots, n, \mu_{n-k+1} < j \leq \mu_{n-k+2}\}, \\ H_{1,2} &:= \bigcup_{k=1}^n \{(i, j) \in \mathbb{N}^2 : \lambda_k < i \leq L, \mu_{n-k+1} < j \leq M\} = H \setminus (H_+ \cup H_- \cup H_1 \cup H_2). \end{aligned}$$

$$R : \cdot \quad H_+ : \circ \quad H_- : \square \quad H_1 : \triangle \quad H_2 : \nabla \quad H_{1,2} : \diamond$$

Figure 1: An example of the index sets R , H_+ , H_- , H_1 , H_2 and $H_{1,2}$ for $n = 3$.

2.1 Lemma. *The joint density of $\{W(s_i, t_j) : (i, j) \in H\}$ has the form*

$$c \cdot f(x_{i,j} : (i, j) \in H_+ \cup H_-) \cdot g(x_{i,j} : (i, j) \in H), \quad (2.1)$$

where c is a norming constant,

$$\begin{aligned} f(x_{i,j} : (i, j) \in H_+ \cup H_-) &= \exp \left\{ - \sum_{(i,j) \in H_+} \frac{x_{i,j}^2}{2s_i t_j} + \sum_{(i,j) \in H_-} \frac{x_{i,j}^2}{2s_i t_j} \right\}, \\ g(x_{i,j} : (i, j) \in H) &= \exp \left\{ - \sum_{(i,j) \in H_1} \frac{(\Delta_1 x_{i,j})^2}{2(\Delta s_i) t_j} - \sum_{(i,j) \in H_2} \frac{(\Delta_2 x_{i,j})^2}{2s_i (\Delta t_j)} - \sum_{(i,j) \in H_{1,2}} \frac{(\Delta_1 \Delta_2 x_{i,j})^2}{2(\Delta s_i)(\Delta t_j)} \right\}, \end{aligned}$$

where $\Delta_1 x_{i,j} = x_{i,j} - x_{i-1,j}$, $\Delta_2 x_{i,j} = x_{i,j} - x_{i,j-1}$ and $\Delta s_i = s_i - s_{i-1}$.

Proof. The joint density of $\{W(s_i, t_j) : (i, j) \in R\}$ equals

$$c \exp \left\{ -\frac{x_{1,1}^2}{2s_1 t_1} - \sum_{i=2}^L \frac{(\Delta_1 x_{i,1})^2}{2(\Delta s_i) t_1} - \sum_{j=2}^M \frac{(\Delta_2 x_{1,j})^2}{2s_1 (\Delta t_j)} - \sum_{i=2}^L \sum_{j=2}^M \frac{(\Delta_1 \Delta_2 x_{i,j})^2}{2(\Delta s_i)(\Delta t_j)} \right\},$$

which is, in fact, a special case of the statement for $n = 1$, $1 = \lambda_1 < \lambda_2 = L$, $1 = \mu_1 < \mu_2 = M$, hence for $H = R$ with $H_+ = \{(1, 1)\}$, $H_- = \emptyset$, $H_1 = \{(i, 1) : 2 \leq i \leq L\}$, $H_2 = \{(1, j) : 2 \leq j \leq M\}$, and $H_{1,2} = \{(i, j) : 2 \leq i \leq L, 2 \leq j \leq M\}$. From this we obtain the joint density of $\{W(s_i, t_j) : (i, j) \in H\}$ by integrating with respect to the variables $\{x_{i,j} : (i, j) \in R \setminus H\}$, e.g., in lexicographic order, that is, with respect to the ordering $(i, j) \leq (k, \ell) \iff i \leq k, j \leq \ell$. In each step we obtain a set which can be represented in the same form as H , hence it is sufficient to show that by integrating the formula (2.1) for H with respect to a variable $x_{i,j}$, $(i, j) \in H_+$ with $(i+1, j) \in H$ and $(i, j+1) \in H$ we obtain the formula (2.1) for $\hat{H} := H \setminus \{(i, j)\}$. We have only four terms in the exponent of the formula (2.1) containing the variable $x_{i,j}$, namely

$$S := -\frac{x_{i,j}^2}{2s_i t_j} - \frac{(\Delta_1 x_{i+1,j})^2}{2(\Delta s_{i+1}) t_j} - \frac{(\Delta_2 x_{i,j+1})^2}{2s_i (\Delta t_{j+1})} - \frac{(\Delta_1 \Delta_2 x_{i+1,j+1})^2}{2(\Delta s_{i+1})(\Delta t_{j+1})}.$$

This sum can be written in the following form:

$$\begin{aligned} S = & -\frac{x_{i,j}^2}{2} \left(\frac{1}{s_i t_j} + \frac{1}{(\Delta s_{i+1}) t_j} + \frac{1}{s_i (\Delta t_{j+1})} + \frac{1}{(\Delta s_{i+1})(\Delta t_{j+1})} \right) \\ & + x_{i,j} \left(\frac{x_{i+1,j}}{(\Delta s_{i+1}) t_j} + \frac{x_{i,j+1}}{s_i (\Delta t_{j+1})} + \frac{x_{i+1,j+1} - x_{i+1,j} - x_{i,j+1}}{(\Delta s_{i+1})(\Delta t_{j+1})} \right) \\ & - \frac{x_{i+1,j}^2}{2(\Delta s_{i+1}) t_j} - \frac{x_{i,j+1}^2}{2s_i (\Delta t_{j+1})} - \frac{(x_{i+1,j+1} - x_{i+1,j} - x_{i,j+1})^2}{2(\Delta s_{i+1})(\Delta t_{j+1})}. \end{aligned}$$

The variable $x_{i,j}$ is contained only in the first and second terms. The sum of these terms can be written as follows:

$$S' := -\frac{s_{i+1} t_j x_{i,j}^2}{2s_i t_j (\Delta s_{i+1})(\Delta t_{j+1})} - \frac{x_{i,j} (s_i t_{j+1} x_{i+1,j} + s_{i+1} t_j x_{i,j+1} - s_i t_j x_{i+1,j+1})}{s_i t_j (\Delta s_{i+1})(\Delta t_{j+1})}.$$

Forming a full square in the variable $x_{i,j}$, we obtain

$$\begin{aligned} S' = & -\frac{s_{i+1} t_{j+1}}{2s_i t_j (\Delta s_{i+1})(\Delta t_{j+1})} \left(x_{i,j} - \frac{s_i}{s_{i+1}} x_{i+1,j} - \frac{t_j}{t_{j+1}} x_{i,j+1} - \frac{s_i t_j}{s_{i+1} t_{j+1}} x_{i+1,j+1} \right)^2 \\ & + \frac{(s_i t_{j+1} x_{i+1,j} + s_{i+1} t_j x_{i,j+1} - s_i t_j x_{i+1,j+1})^2}{2s_i t_j (\Delta s_{i+1})(\Delta t_{j+1})}. \end{aligned}$$

The first term of S' disappears after integrating with respect to the variable $x_{i,j}$, while the second term of S' and the remaining last three terms of S can be written as follows:

$$-\frac{x_{i+1,j}^2}{2s_{i+1}t_j} - \frac{x_{i,j+1}^2}{2s_it_{j+1}} + \frac{x_{i+1,j+1}^2}{2s_{i+1}t_{j+1}} - \frac{(\Delta_1 x_{i+1,j+1})^2}{2(\Delta s_{i+1})t_{j+1}} - \frac{(\Delta_2 x_{i+1,j+1})^2}{2s_{i+1}(\Delta t_{j+1})}.$$

We always have

$$\widehat{H}_1 = \left[H_1 \setminus \{(i+1, j)\} \right] \cup \{(i+1, j+1)\}, \quad \widehat{H}_2 = \left[H_2 \setminus \{(i, j+1)\} \right] \cup \{(i+1, j+1)\},$$

$$\widehat{H}_{1,2} = H_{1,2} \setminus \{(i+1, j+1)\}.$$

If $x_{i+1,j} \in H_1$ and $x_{i,j+1} \in H_2$ then

$$\widehat{H}_+ = \left[H_+ \setminus \{(i, j)\} \right] \cup \{(i+1, j), (i, j+1)\}, \quad \widehat{H}_- = H_- \cup \{(i+1, j+1)\}.$$

If $x_{i+1,j} \in H_-$ and $x_{i,j+1} \in H_2$ then

$$\widehat{H}_+ = \left[H_+ \setminus \{(i, j)\} \right] \cup \{(i, j+1)\}, \quad \widehat{H}_- = \left[H_- \setminus \{(i+1, j)\} \right] \cup \{(i+1, j+1)\}.$$

If $x_{i+1,j} \in H_1$ and $x_{i,j+1} \in H_-$ then

$$\widehat{H}_+ = \left[H_+ \setminus \{(i, j)\} \right] \cup \{(i+1, j)\}, \quad \widehat{H}_- = \left[H_- \setminus \{(i, j+1)\} \right] \cup \{(i+1, j+1)\}.$$

If $x_{i+1,j} \in H_-$ and $x_{i,j+1} \in H_-$ then

$$\widehat{H}_+ = H_+ \setminus \{(i, j)\}, \quad \widehat{H}_- = \left[H_- \setminus \{(i+1, j), (i, j+1)\} \right] \cup \{(i+1, j+1)\}.$$

In all cases we obtain the formula (2.1) for the set \widehat{H} , which completes the induction step. \square

2.2 Lemma. *Let \widetilde{H} be a set with $H_+ \cup H_- \subset \widetilde{H} \subset H$. Then the probability measures $\mathbb{P}_Z^{\widetilde{H}}$ and $\mathbb{P}_W^{\widetilde{H}}$, generated by the random variables $\left\{ Z(s_i, t_j) : (i, j) \in \widetilde{H} \right\}$ and $\left\{ W(s_i, t_j) : (i, j) \in \widetilde{H} \right\}$, respectively, are equivalent and the Radon–Nikodym derivative of $\mathbb{P}_Z^{\widetilde{H}}$ with respect to $\mathbb{P}_W^{\widetilde{H}}$ equals*

$$\frac{d\mathbb{P}_Z^{\widetilde{H}}}{d\mathbb{P}_W^{\widetilde{H}}}(x_{i,j} : (i, j) \in \widetilde{H}) = \exp \left\{ -\frac{1}{2}(A_{\widetilde{H}}m^2 - 2y_{\widetilde{H}}m) \right\},$$

where

$$A_{\widetilde{H}} = \sum_{(i,j) \in H_+} \frac{1}{s_it_j} - \sum_{(i,j) \in H_-} \frac{1}{s_it_j}, \quad y_{\widetilde{H}} = \sum_{(i,j) \in H_+} \frac{x_{i,j}}{s_it_j} - \sum_{(i,j) \in H_-} \frac{x_{i,j}}{s_it_j}.$$

The maximum likelihood estimator of the shift parameter m based on the observations $\{Z(s, t) : (s, t) \in \tilde{H}\}$ has the form $\tilde{m}_{\tilde{H}} = \zeta_{\tilde{H}}/A_{\tilde{H}}$ where

$$\zeta_{\tilde{H}} = \sum_{(i,j) \in H_+} \frac{Z(s_i, t_j)}{s_i t_j} - \sum_{(i,j) \in H_-} \frac{Z(s_i, t_j)}{s_i t_j},$$

and $\tilde{m}_{\tilde{H}}$ has a normal distribution with mean m and variance $1/A_{\tilde{H}}$.

Proof. Using Lemma 2.1 we obtain that the joint density of $\{W(s_i, t_j) : (i, j) \in \tilde{H}\}$ is

$$cf(x_{i,j} : (i, j) \in H_+ \cup H_-) \int_{\mathbb{R}^d} g(x_{i,j} : (i, j) \in H) d(x_{i,j} : (i, j) \in H \setminus \tilde{H}),$$

where d denotes the number of points in $H \setminus \tilde{H}$. Clearly, the joint density of $\{Z(s_i, t_j) : (i, j) \in \tilde{H}\}$ is the product of $cf(x_{i,j} - m : (i, j) \in H_+ \cup H_-)$ and

$$\begin{aligned} & \int_{\mathbb{R}^d} g(x_{i,j} - m : (i, j) \in \tilde{H}; x_{i,j} : (i, j) \in H \setminus \tilde{H}) d(x_{i,j} : (i, j) \in H \setminus \tilde{H}) \\ &= \int_{\mathbb{R}^d} g(z_{i,j} : (i, j) \in \tilde{H}; z_{i,j} + m : (i, j) \in H \setminus \tilde{H}) d(z_{i,j} : (i, j) \in H \setminus \tilde{H}). \end{aligned}$$

We have

$$g(z_{i,j} : (i, j) \in \tilde{H}; z_{i,j} + m : (i, j) \in H \setminus \tilde{H}) = g(z_{i,j} : (i, j) \in H),$$

hence

$$\frac{d\mathbb{P}_{\tilde{Z}}^{\tilde{H}}}{d\mathbb{P}_{\tilde{W}}^{\tilde{H}}}(x_{i,j} : (i, j) \in \tilde{H}) = \frac{f(x_{i,j} - m : (i, j) \in H_+ \cup H_-)}{f(x_{i,j} : (i, j) \in H_+ \cup H_-)} = \exp\left\{-\frac{1}{2}(A_{\tilde{H}}m^2 - 2y_{\tilde{H}}m)\right\},$$

which implies the given form of the MLE of m . Clearly, we have $E\tilde{m}_{\tilde{H}} = m$. Finally, introducing the notation

$$\eta_{\tilde{H}} = \sum_{(i,j) \in H_+} \frac{W(s_i, t_j)}{s_i t_j} - \sum_{(i,j) \in H_-} \frac{W(s_i, t_j)}{s_i t_j},$$

we have $E\eta_{\tilde{H}} = 0$ and we can show $D^2(\eta_{\tilde{H}}) = E\eta_{\tilde{H}}^2 = A_{\tilde{H}}$ by induction with respect to n :

$$\begin{aligned} E\eta_{\tilde{H}}^2 &= E\left(\left(\frac{W(s_{\lambda_1}, t_{\mu_n})}{s_{\lambda_1} t_{\mu_n}} - \frac{W(s_{\lambda_2}, t_{\mu_n})}{s_{\lambda_2} t_{\mu_n}} + \dots + \frac{W(s_{\lambda_{n-1}}, t_{\mu_2})}{s_{\lambda_{n-1}} t_{\mu_2}}\right) - \left(\frac{W(s_{\lambda_n}, t_{\mu_2})}{s_{\lambda_n} t_{\mu_2}} - \frac{W(s_{\lambda_n}, t_{\mu_1})}{s_{\lambda_n} t_{\mu_1}}\right)\right)^2 \\ &= \left(\frac{1}{s_{\lambda_1} t_{\mu_n}} - \frac{1}{s_{\lambda_2} t_{\mu_n}} + \dots + \frac{1}{s_{\lambda_{n-1}} t_{\mu_2}}\right) + \left(\frac{1}{s_{\lambda_n} t_{\mu_2}} + \frac{1}{s_{\lambda_n} t_{\mu_1}} - \frac{2}{s_{\lambda_n} t_{\mu_2}}\right) \\ &= \frac{1}{s_{\lambda_1} t_{\mu_n}} - \frac{1}{s_{\lambda_2} t_{\mu_n}} + \dots + \frac{1}{s_{\lambda_{n-1}} t_{\mu_2}} - \frac{1}{s_{\lambda_n} t_{\mu_2}} + \frac{1}{s_{\lambda_n} t_{\mu_1}}, \end{aligned}$$

which implies $D^2(\tilde{m}_{\tilde{H}}) = 1/A_{\tilde{H}}$. \square

We note that the factorisation of the density of $\{Z(s_i, t_j) : (i, j) \in \tilde{H}\}$ shows that $\{Z(s_i, t_j) : (i, j) \in H_+ \cup H_-\}$ is a sufficient statistic for the parameter m .

3 Proof of Theorem 1.1

In order to determine Radon–Nikodym derivatives we have developed the following general method based on Section 2.3.2 in Arató [1].

Let X be a separable metric space and let $C(X)$ denote the space of real–valued continuous functions on X with the uniform metric. For a random field $\{\xi_x : x \in X\}$ with continuous trajectories on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the induced mapping $\xi : \Omega \rightarrow C(X)$ is measurable (see, e.g., Billingsley [6, Section 1.3]). Let \mathbb{P}_ξ denote the probability measure, generated by the process ξ on $(C(X), \mathcal{B}(C(X)))$. For a finite set $X' = \{x_1, \dots, x_k\} \subset X$, we denote by $\mathbb{P}_\xi^{X'}$ the probability measure, generated by the random variable $\xi(X') := (\xi_{x_1}, \dots, \xi_{x_k})$ on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$.

3.1 Proposition. *Let $\{\xi_x : x \in X\}$ and $\{\tilde{\xi}_x : x \in X\}$ be random fields with continuous trajectories. Suppose that there exists a measurable function $f : C(X) \rightarrow \mathbb{R}$ such that $\mathbb{E}f(\xi) = 1$ and that for any finite set $X_0 \subset X$, there exists a sequence of finite subsets X_n , $n = 1, 2, \dots$ with $X_0 \subset X_n \subset X$, $n = 1, 2, \dots$, and with*

$$\frac{d\mathbb{P}_\xi^{X_n}}{d\mathbb{P}_\xi^{X_0}}(\xi(X_n)) \xrightarrow{\mathbb{P}} f(\xi) \quad \text{as } n \rightarrow \infty.$$

Then $\mathbb{P}_{\tilde{\xi}}$ is absolutely continuous with respect to \mathbb{P}_ξ and $\frac{d\mathbb{P}_{\tilde{\xi}}}{d\mathbb{P}_\xi} = f$.

(The proof can be found in Baran, Pap and Zuijlen [5].)

We will apply Proposition 3.1 for $X = \tilde{G}$ and for the random fields $\xi = W$ and $\tilde{\xi} = Z$. First we note that the random variable ζ can also be expressed in integrals with respect to the Wiener sheet $\{W(s, t) : s, t \geq 0\}$, namely $\zeta = \eta + mA$, where

$$\eta = c_1 W(a, \gamma(a)) + c_2 W(b, \gamma(b)) + \int_\Gamma y_1 W + \int_\Gamma y_2 \partial_n W.$$

This can be seen from the fact that

$$\zeta = \eta + m \left(c_1 + c_2 + \int_\Gamma y_1 \right),$$

where the integral

$$\int_\Gamma y_1 = \int_a^b y_1(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} ds$$

can be calculated using

$$\frac{d}{ds} \left(\frac{1}{s\gamma(s)(1 + \gamma'(s)^2)} \right) = y_1(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} - \frac{1}{s^2\gamma(s)}.$$

Clearly η has a normal distribution with zero mean and at the end of the proof of the theorem we shall see that its variance equals A . Hence $\tilde{m} = \zeta/A$ has a

normal distribution with mean m and variance $1/A$. Using the moment generating function of η we obtain

$$\mathbb{E} \exp \left\{ -\frac{1}{2}(Am^2 - 2\eta m) \right\} = 1.$$

The subset $\tilde{G} \subset G$ contains an ε -strip of Γ with some $\varepsilon > 0$, hence for a finite subset $\tilde{G}_0 \subset \tilde{G}$, we can put a grid on \tilde{G} , containing the points of \tilde{G}_0 and which is sufficiently fine such that its ‘lower staircase boundary’ is contained in the given ε -strip of Γ . More exactly, there exists $n, L, M \in \mathbb{N}$ with $n \leq L$, $n \leq M$ and real numbers

$$a = s_1 < s_2 < \dots < s_n = b < s_{n+1} < \dots < s_L, \quad \gamma(a) < t_{n+1} < \dots < t_M$$

such that $(\tilde{G}_0 \cup \{(s_i, t_j) : (i, j) \in H_+ \cup H_-\}) \subset (\{(s_i, t_j) : (i, j) \in H\} \cap \tilde{G})$, where $t_i := \gamma(s_{n-i+1})$ for $i = 1, \dots, n$ and

$$H := \bigcup_{k=1}^n \{(i, j) \in \mathbb{N}^2 : k \leq i \leq L, n - k + 1 \leq j \leq M\},$$

$$H_+ := \{(k, n - k + 1) : k = 1, \dots, n\}, \quad H_- := \{(k, n - k + 2) : k = 2, \dots, n\}.$$

Using Lemma 2.2 with $\lambda_k = \mu_k = k$ for $k = 1, \dots, n$, $\lambda_{n+1} = L$, $\mu_{n+1} = M$ and with $\tilde{H} := H_+ \cup H_- \cup \{(i, j) : (s_i, t_j) \in \tilde{G}_0\}$ we conclude that

$$\frac{d\mathbb{P}_{\tilde{Z}}^{\tilde{H}}}{d\mathbb{P}_{\tilde{W}}^{\tilde{H}}}(W(s_i, t_j) : (i, j) \in \tilde{H}) = \exp \left\{ -\frac{1}{2}(A_{\tilde{H}}m^2 - 2\eta_{\tilde{H}}m) \right\},$$

where $\mathbb{P}_{\tilde{Z}}^{\tilde{H}}$ and $\mathbb{P}_{\tilde{W}}^{\tilde{H}}$ denote the probability measures generated by the random variables $\{Z(s_i, t_j) : (i, j) \in \tilde{H}\}$ and $\{W(s_i, t_j) : (i, j) \in \tilde{H}\}$, respectively, and

$$A_{\tilde{H}} = \sum_{i=1}^n \frac{1}{s_i \gamma(s_i)} - \sum_{i=2}^n \frac{1}{s_i \gamma(s_{i-1})}, \quad \eta_{\tilde{H}} = \sum_{i=1}^n \frac{W(s_i, \gamma(s_i))}{s_i \gamma(s_i)} - \sum_{i=2}^n \frac{W(s_i, \gamma(s_{i-1}))}{s_i \gamma(s_{i-1})}.$$

Hence, by Proposition 3.1, it is sufficient to prove that for any sequence of partitions $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_N^{(N)} = b$, $N = 1, 2, \dots$ with $|\mathcal{P}_N| \rightarrow 0$ we have $A_N \rightarrow A$ and $\eta_N \xrightarrow{L^2} \eta$, where

$$A_N = \sum_{i=1}^N \frac{1}{s_i^{(N)} \gamma(s_i^{(N)})} - \sum_{i=2}^N \frac{1}{s_i^{(N)} \gamma(s_{i-1}^{(N)})},$$

$$\eta_N = \sum_{i=1}^N \frac{W(s_i^{(N)}, \gamma(s_i^{(N)}))}{s_i^{(N)} \gamma(s_i^{(N)})} - \sum_{i=2}^N \frac{W(s_i^{(N)}, \gamma(s_{i-1}^{(N)}))}{s_i^{(N)} \gamma(s_{i-1}^{(N)})}.$$

Clearly, we have $A_N \rightarrow A$ since

$$A_N = \frac{1}{s_N^{(N)} \gamma(s_N^{(N)})} + \sum_{i=2}^N \frac{\Delta s_i^{(N)}}{s_{i-1}^{(N)} s_i^{(N)} \gamma(s_{i-1}^{(N)})} \rightarrow \frac{1}{b\gamma(b)} + \int_a^b \frac{ds}{s^2 \gamma(s)} = A.$$

The aim of the following discussion is to prove convergence $\eta_N \xrightarrow{L^2} \eta$ which completes the proof. Note that η_N can be decomposed as

$$\eta_N = \eta_N^{(1)} + \eta_N^{(2)} + \eta_N^{(3)} + \eta_N^{(4)} + \eta_N^{(5)},$$

where

$$\eta_N^{(1)} = \sum_{i=2}^{N-1} \left(-\frac{W(s_{i+1}^{(N)}, \gamma(s_i^{(N)}))}{s_i^{(N)} \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})} + \frac{W(s_i^{(N)}, \gamma(s_i^{(N)}))}{s_i^{(N)} \gamma(s_i^{(N)})} - \frac{W(s_i^{(N)}, \gamma(s_{i-1}^{(N)}))}{s_i^{(N)} \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^2)} \right)$$

$$\eta_N^{(2)} = \sum_{i=2}^{N-1} \left(\frac{1}{s_i^{(N)}} - \frac{1}{s_{i+1}^{(N)}} \right) \frac{W(s_{i+1}^{(N)}, \gamma(s_i^{(N)}))}{\gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})}$$

$$\eta_N^{(3)} = \sum_{i=2}^{N-1} \left(\frac{1}{\gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^2)} - \frac{1}{\gamma(s_{i-1}^{(N)}) (1 + \gamma'(s_{i-1}^{(N)})^2)} \right) \frac{W(s_i^{(N)}, \gamma(s_{i-1}^{(N)}))}{s_i^{(N)}}$$

$$\eta_N^{(4)} = \frac{W(s_1^{(N)}, \gamma(s_1^{(N)}))}{s_1^{(N)} \gamma(s_1^{(N)})} - \frac{W(s_2^{(N)}, \gamma(s_1^{(N)}))}{s_2^{(N)} \gamma(s_1^{(N)}) (1 + \gamma'(s_1^{(N)})^{-2})}$$

$$\eta_N^{(5)} = \frac{W(s_N^{(N)}, \gamma(s_N^{(N)}))}{s_N^{(N)} \gamma(s_N^{(N)})} - \frac{W(s_N^{(N)}, \gamma(s_{N-1}^{(N)}))}{s_N^{(N)} \gamma(s_{N-1}^{(N)}) (1 + \gamma'(s_{N-1}^{(N)})^2)},$$

where $\gamma'(s_1^{(N)})^{-2} = \gamma'(a)^{-2} := 0$ if $\gamma'(a) = -\infty$. (This decomposition can be checked by comparing the coefficients of $W(s_i^{(N)}, \gamma(s_i^{(N)}))$ and $W(s_i^{(N)}, \gamma(s_{i-1}^{(N)}))$.)

The convergence $\eta_N^{(4)} \xrightarrow{L^2} c_1 W(a, \gamma(a))$ can be proved by decomposing the difference:

$$\begin{aligned} c_1 W(a, \gamma(a)) - \eta_N^{(4)} &= \frac{\Delta_1 W(s_2^{(N)}, \gamma(s_1^{(N)}))}{s_2^{(N)} \gamma(s_1^{(N)}) (1 + \gamma'(s_1^{(N)})^{-2})} \\ &\quad - \left(\frac{1}{s_1^{(N)}} - \frac{1}{s_2^{(N)}} \right) \frac{W(s_1^{(N)}, \gamma(s_1^{(N)}))}{\gamma(s_1^{(N)}) (1 + \gamma'(s_1^{(N)})^{-2})} \end{aligned}$$

and taking into account that $\Delta_1 W(s_2^{(N)}, \gamma(s_1^{(N)})) \xrightarrow{L^2} 0$. The convergence $\eta_N^{(5)} \xrightarrow{L^2} c_2 W(b, \gamma(b))$ can be proved in a similar way. The convergence $\eta_N^{(1)} \xrightarrow{L^2} \int_{\Gamma} y_2 \partial_n W$ follows from Theorem 5.2 since

$$\begin{aligned} \eta_N^{(1)} = & - \sum_{i=2}^{N-1} y_2(s_i^{(N)}, \gamma(s_i^{(N)})) \left[\gamma'(s_i^{(N)}) \{W(s_{i+1}^{(N)}, \gamma(s_i^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))\} \right. \\ & \left. + \gamma'(s_i^{(N)})^{-1} \{W(s_i^{(N)}, \gamma(s_{i-1}^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))\} \right], \end{aligned}$$

and the function y_2 satisfies the condition of Theorem 5.2. The convergence $\eta_N^{(2)} + \eta_N^{(3)} \xrightarrow{L^2} \int_{\Gamma} y_1 W$ follows from the Lemmas 4.9 and 4.10.

Thus we obtained $\eta_N \xrightarrow{L^2} \eta$, which implies $E\eta_N^2 \xrightarrow{L^2} E\eta^2$. On the other hand, Lemma 2.2 implies that $E\eta_N^2 = A_N \rightarrow A$, hence we get $E\eta^2 = A$. Consequently, the variance of η equals A , as it was announced at the beginning of the proof of the theorem. \square

4 L^2 -Riemann integral of a process along a curve

First we note the following simple statement about the existence of L^2 -limits (see, e.g., Ash and Gardner [4], Theorem 1.3.2).

4.1 Lemma. *Let $\{Y_n : n \in \mathbb{N}\}$ be a sequence of random variables in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Then there exists a random variable $Y \in L^2(\Omega, \mathcal{A}, \mathbb{P})$ such that $Y_n \xrightarrow{L^2} Y$ if and only if the limit $\lim_{n, m \rightarrow \infty} E(Y_n Y_m)$ exists. Moreover, if $Y_n \xrightarrow{L^2} Y$ then $\lim_{n, m \rightarrow \infty} E(Y_n Y_m) = EY^2$, $\lim_{n \rightarrow \infty} EY_n = EY$, and $\lim_{n \rightarrow \infty} E(XY_n) = E(XY)$ for all $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$.*

Proof. If $Y_n \xrightarrow{L^2} Y$ then $\lim_{n, m \rightarrow \infty} E(Y_n Y_m) = EY^2$ follows from

$$E(Y_n Y_m - Y^2) = E[(Y_n - Y)Y_m] + E[(Y_m - Y)Y] = \langle Y_n - Y, Y_m \rangle_{L^2} + \langle Y_m - Y, Y \rangle_{L^2} \rightarrow 0,$$

since the inner product product $\langle \cdot, \cdot \rangle_{L^2}$ is continuous.

If the limit $\lim_{n, m \rightarrow \infty} E(Y_n Y_m) =: L$ exists then

$$\|Y_n - Y_m\|_{L^2}^2 = E(Y_n - Y_m)^2 = EY_n^2 - 2E(Y_n Y_m) + EY_m^2 \rightarrow L - 2L + L = 0$$

as $n, m \rightarrow \infty$, hence the sequence Y_1, Y_2, \dots is a Cauchy-sequence in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

\square

Instead of $Y_n \xrightarrow{L^2} Y$ we sometimes write $Y = \text{l.i.m.}_{N \rightarrow \infty} Y_N$.

A process $\{Z(s) : s \in [a, b]\}$ is said to be an L^2 -process if $\{Z(s) : s \in [a, b]\} \subset L^2(\Omega, \mathcal{A}, \mathbb{P})$. We will use L^2 -Riemann integrals of L^2 -processes (see, e.g., Ash and Gardner [4]).

4.2 Definition. Let $\{Z(s) : s \in [a, b]\}$ be an L^2 -process. If there exists a random variable I such that for all sequence of partitions $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$ with $|\mathcal{P}_N| \rightarrow 0$ and for all points $\tilde{s}_i^{(N)} \in [s_i^{(N)}, s_{i+1}^{(N)}]$, $i = 1, \dots, N-1$, we have

$$\sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)}) \xrightarrow{L^2} I$$

then we say that Z is L^2 -Riemann integrable on $[a, b]$ and we write

$$I = \int_a^b Z(s) ds.$$

By the help of Lemma 4.1, it is easy to prove the following lemma.

4.3 Lemma. If the Riemann integral $\int_a^b \int_a^b \mathbb{E}[Z(s)Z(t)] ds dt$ exists then Z is L^2 -Riemann integrable on $[a, b]$. Especially, if the function $(s, t) \mapsto \mathbb{E}[Z(s)Z(t)]$ is continuous on $[a, b] \times [a, b]$ then Z is L^2 -Riemann integrable on $[a, b]$. Moreover, if the processes Y and Z are L^2 -Riemann integrable on $[a, b]$ then

$$\mathbb{E} \int_a^b Z(s) ds = \int_a^b \mathbb{E} Z(s) ds, \quad \mathbb{E} \left[Y(s) \int_a^b Z(t) dt \right] = \int_a^b \mathbb{E}[Y(s)Z(t)] dt,$$

$$\mathbb{E} \left[\int_a^b Y(s) ds \int_a^b Z(t) dt \right] = \int_a^b \int_a^b \mathbb{E}[Y(s)Z(t)] ds dt.$$

Proof. In order to show that Z is L^2 -Riemann integrable, it is sufficient to prove that for all sequence of partitions $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$ with $|\mathcal{P}_N| \rightarrow 0$ and for all points $\tilde{s}_i^{(N)} \in [s_i^{(N)}, s_{i+1}^{(N)}]$, $i = 1, \dots, N-1$, the limit $\text{l.i.m.}_{N \rightarrow \infty} \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)})$ exists, since then these limits coincide (because by mixing two such sequences of partitions, the appropriate sequence of sums converges again). We have

$$\mathbb{E} \left[\sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)}) \sum_{j=1}^{M-1} Z(\tilde{s}_j^{(M)})(s_{j+1}^{(M)} - s_j^{(M)}) \right] \rightarrow \int_a^b \int_a^b \mathbb{E}[Z(s)Z(t)] ds dt$$

as $N, M \rightarrow \infty$, hence, by Lemma 4.1, the limit $\text{l.i.m.}_{N \rightarrow \infty} \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)})$ exists.

Again by Lemma 4.1 we obtain

$$\begin{aligned} \mathbb{E} \int_a^b Z(s) ds &= \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)}) \right] = \int_a^b \mathbb{E} Z(t) dt, \\ \mathbb{E} \left[Y(s) \int_a^b Z(t) dt \right] &= \lim_{N \rightarrow \infty} \mathbb{E} \left[Y(s) \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)}) \right] = \int_a^b \mathbb{E}[Y(s)Z(t)] dt, \\ \mathbb{E} \left[\int_a^b Y(s) ds \int_a^b Z(t) dt \right] &= \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^{N-1} Y(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)}) \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)})(s_{i+1}^{(N)} - s_i^{(N)}) \right] \\ &= \int_a^b \int_a^b \mathbb{E}[Y(s)Z(t)] ds dt. \end{aligned}$$

□

Now we define improper L^2 -Riemann integrals.

4.4 Definition. Let $\{Z(s) : s \in (a, b)\}$ be an L^2 -process. Suppose that for all $a < a + \varepsilon < b - \delta < b$, the process Z is L^2 -Riemann integrable on $[a + \varepsilon, b - \delta]$. If $\int_{a+\varepsilon}^{b-\delta} Z(s) ds$ converges in L^2 to some random variable I as $\varepsilon \downarrow 0$ and $\delta \downarrow 0$, then we say that the improper L^2 -Riemann integral $\int_a^b Z(s) ds := I$ exists.

4.5 Lemma. For the existence of the improper L^2 -Riemann integral $\int_a^b Z(s) ds$, one of the following conditions is sufficient:

(i) The limit $\lim_{\varepsilon, \varepsilon', \delta, \delta' \downarrow 0} \int_{a+\varepsilon}^{b-\delta} \int_{a+\varepsilon'}^{b-\delta'} \mathbb{E}[Z(s)Z(t)] ds dt$ exists.

(ii) The improper Riemann integral $\int_a^b \int_a^b \mathbb{E}[Z(s)Z(t)] ds dt$ exists.

(iii) The function $(s, t) \mapsto \mathbb{E}[Z(s)Z(t)]$ is continuous on $(a, b) \times (a, b)$ and

$$\int_a^b \sqrt{\mathbb{E}[Z(s)^2]} ds < \infty.$$

Moreover, if the improper L^2 -Riemann integrals $\int_a^b Y(s) ds$ and $\int_a^b Z(t) dt$ exists then

$$\mathbb{E} \int_a^b Z(s) ds = \int_a^b \mathbb{E} Z(s) ds, \quad \mathbb{E} \left[Y(s) \int_a^b Z(t) dt \right] = \int_a^b \mathbb{E}[Y(s)Z(t)] dt,$$

$$\mathbb{E} \left[\int_a^b Y(s) ds \int_a^b Z(t) dt \right] = \lim_{\varepsilon, \delta \downarrow 0} \int_{a+\varepsilon}^{b-\delta} \int_{a+\varepsilon}^{b-\delta} \mathbb{E}[Y(s)Z(t)] ds dt.$$

If the process Z is L^2 -Riemann integrable on $[a, b]$ then the improper L^2 -Riemann integral $\int_a^b Z(s) ds$ exists and equals the (usual) L^2 -Riemann integral.

Proof. By Lemma 4.1, condition (i) is necessary and sufficient. Condition (ii) obviously implies (i). We will prove that condition (iii) also implies (i). For all $\varepsilon_n \downarrow 0$ and $\delta_n \downarrow 0$, we have

$$\begin{aligned}
& \left| \int_{a+\varepsilon_{n+k}}^{b-\delta_{n+k}} \int_{a+\varepsilon_{m+\ell}}^{b-\delta_{m+\ell}} \mathbb{E}[Z(s)Z(t)] \, ds dt - \int_{a+\varepsilon_n}^{b-\delta_n} \int_{a+\varepsilon_m}^{b-\delta_m} \mathbb{E}[Z(s)Z(t)] \, ds dt \right| \\
& \leq \iint_{[a+\varepsilon_{n+k}, b-\delta_{n+k}] \times [a+\varepsilon_{m+\ell}, b-\delta_{m+\ell}] \setminus [a+\varepsilon_n, b-\delta_n] \times [a+\varepsilon_m, b-\delta_m]} \sqrt{\mathbb{E}[Z(s)^2] \mathbb{E}[Z(t)^2]} \, ds dt \\
& \leq \int_{[a+\varepsilon_{n+k}, b-\delta_{n+k}] \setminus [a+\varepsilon_n, b-\delta_n]} \sqrt{\mathbb{E}[Z(s)^2]} \, ds \int_a^b \sqrt{\mathbb{E}[Z(t)^2]} \, dt \\
& \quad + \int_a^b \sqrt{\mathbb{E}[Z(s)^2]} \, ds \int_{[a+\varepsilon_{m+\ell}, b-\delta_{m+\ell}] \setminus [a+\varepsilon_m, b-\delta_m]} \sqrt{\mathbb{E}[Z(t)^2]} \, dt \\
& \rightarrow 0 \quad \text{as } n, m, k, \ell \rightarrow \infty,
\end{aligned}$$

hence the limit

$$\lim_{n, m \rightarrow \infty} \int_{a+\varepsilon_n}^{b-\delta_n} \int_{a+\varepsilon_m}^{b-\delta_m} \mathbb{E}[Z(s)Z(t)] \, ds dt$$

exists, which implies (i). \square

For further use we prove the following formula of change of variables.

4.6 Lemma. Let $\gamma : [a, b] \rightarrow \mathbb{R}$ be a continuous, strictly increasing or strictly decreasing function, differentiable on (a, b) . Then

$$\int_{\gamma(a)}^{\gamma(b)} Z(t) \, dt = \int_a^b \gamma'(s) Z(\gamma(s)) \, ds$$

if one of the improper L^2 -Riemann integrals exists.

Proof. Suppose that γ is strictly increasing and the improper L^2 -Riemann integral on the right hand side exists. Let $\mathcal{P}_N : \gamma(a) = t_1^{(N)} < t_2^{(N)} < \dots < t_{N-1}^{(N)} < t_N^{(N)} = \gamma(b)$, $N = 1, 2, \dots$, be a sequence of partitions with $|\mathcal{P}_N| \rightarrow 0$, and let $\tilde{t}_i^{(N)} \in [t_i^{(N)}, t_{i+1}^{(N)}]$, $i = 1, \dots, N-1$. Let $s_i^{(N)} := \gamma^{-1}(t_i^{(N)})$, $\tilde{s}_i^{(N)} := \gamma^{-1}(\tilde{t}_i^{(N)})$. Then $\tilde{\mathcal{P}}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$, is a sequence of partitions with $|\tilde{\mathcal{P}}_N| \rightarrow 0$. Moreover, $\tilde{s}_i^{(N)} \in [s_i^{(N)}, s_{i+1}^{(N)}]$, $i = 1, \dots, N-1$, and for all $a < a + \varepsilon < b - \delta < b$, we have

$$\begin{aligned}
\sum_{[t_{i-1}^{(N)}, t_i^{(N)}] \subset [\gamma(a+\varepsilon), \gamma(b-\delta)]} Z(\tilde{t}_i^{(N)})(t_i^{(N)} - t_{i-1}^{(N)}) &= \sum_{[s_{i-1}^{(N)}, s_i^{(N)}] \subset [a+\varepsilon, b-\delta]} Z(\gamma(\tilde{s}_i^{(N)}))(\gamma(s_i^{(N)}) - \gamma(s_{i-1}^{(N)})) \\
&\xrightarrow{L^2} \int_{a+\varepsilon}^{b-\delta} \gamma'(s) Z(\gamma(s)) \, ds \quad \text{as } N \rightarrow \infty \\
&\xrightarrow{L^2} \int_a^b \gamma'(s) Z(\gamma(s)) \, ds \quad \text{as } \varepsilon, \delta \downarrow 0,
\end{aligned}$$

hence the improper L^2 -Riemann integral $\int_{\gamma(a)}^{\gamma(b)} Z(t) dt$ exists and equals the improper L^2 -Riemann integral $\int_a^b \gamma'(s)Z(\gamma(s)) ds$. \square

Next we define L^2 -Riemann integral along a curve $\Gamma := \{(s, \gamma(s)) : s \in (a, b)\}$, where $\gamma : (a, b) \rightarrow \mathbb{R}$ is a differentiable function.

4.7 Definition. Let Z be an L^2 -process given along the curve Γ . Then

$$\int_{\Gamma} Z := \int_a^b Z(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} ds$$

if this improper L^2 -Riemann integral exists.

4.8 Remark. For all $N = 1, 2, \dots$, let $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$ be a partition such that $|\mathcal{P}_N| \rightarrow 0$ as $N \rightarrow \infty$. Let $\tilde{s}_i^{(N)} \in [s_i^{(N)}, s_{i+1}^{(N)}]$, $i = 1, \dots, N-1$. Suppose that $(s, t) \mapsto \mathbb{E}[Z(s, \gamma(s))Z(t, \gamma(t))]$ is continuous on $[a, b] \times [a, b]$ and that γ' can be continuously extended to $[a, b]$. Then the L^2 -Riemann integral $\int_{\Gamma} Z$ exists and

$$\begin{aligned} \int_{\Gamma} Z &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)}, \gamma(\tilde{s}_i^{(N)})) \sqrt{1 + \gamma'(\tilde{s}_i^{(N)})^2} (s_{i+1}^{(N)} - s_i^{(N)}) \\ &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)}, \gamma(\tilde{s}_i^{(N)})) \sqrt{(s_{i+1}^{(N)} - s_i^{(N)})^2 + (\gamma(s_{i+1}^{(N)}) - \gamma(s_i^{(N)}))^2} \\ &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)}, \gamma(\tilde{s}_i^{(N)})) \int_{s_i^{(N)}}^{s_{i+1}^{(N)}} \sqrt{1 + \gamma'(s)^2} ds, \end{aligned}$$

since, for example,

$$\begin{aligned} &\left\| \sum_{i=1}^{N-1} Z(\tilde{s}_i^{(N)}, \gamma(\tilde{s}_i^{(N)})) \left(\sqrt{1 + \gamma'(\tilde{s}_i^{(N)})^2} - \sqrt{1 + \left(\frac{\gamma(s_{i+1}^{(N)}) - \gamma(s_i^{(N)})}{s_{i+1}^{(N)} - s_i^{(N)}} \right)^2} \right) (s_{i+1}^{(N)} - s_i^{(N)}) \right\|_{L^2} \\ &\leq (b-a) \sup_{s \in [a, b]} \sqrt{\mathbb{E}[Z(s, \gamma(s))^2]} \sup_{\substack{a \leq u < v \leq b \\ v-u \leq |\mathcal{P}_N|}} |\sqrt{1 + \gamma'(u)^2} - \sqrt{1 + \gamma'(v)^2}| \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$.

The function y_1 in Theorem 1.1 can be decomposed as $y_1 = y_{1,1} - y_{1,2}$ with

$$y_{1,1}(s, \gamma(s)) := \frac{\gamma'(s)^2}{s^2 \gamma(s) (1 + \gamma'(s)^2)^{3/2}}, \quad y_{1,2}(s, \gamma(s)) := \frac{\gamma'(s) [1 + \gamma'(s)^2 + 2\gamma(s)\gamma''(s)]}{s \gamma(s)^2 (1 + \gamma'(s)^2)^{5/2}}.$$

4.9 Lemma. *With the notation of the proof of Theorem 1.1 we have*

$$\eta_N^{(2)} \xrightarrow{\mathbb{L}^2} \int_{\Gamma} y_{1,1} W.$$

Proof. Since the function

$$\begin{aligned} (s, t) &\mapsto \mathbb{E}[y_{1,1}(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} W(s, \gamma(s)) y_{1,1}(t, \gamma(t)) \sqrt{1 + \gamma'(t)^2} W(t, \gamma(t))] \\ &= \frac{(s \wedge t) \gamma(s \vee t)}{s^2 t^2 \gamma(s) \gamma(t) (1 + \gamma'(s)^{-2}) (1 + \gamma'(t)^{-2})} \end{aligned}$$

can be continuously extended to $[a, b] \times [a, b]$, the L^2 -Riemann integral

$$\int_{\Gamma} y_{1,1} W = \int_a^b \frac{W(s, \gamma(s)) ds}{s^2 \gamma(s) (1 + \gamma'(s)^{-2})}$$

exists and we have

$$\int_a^b \frac{W(s, \gamma(s)) ds}{s^2 \gamma(s) (1 + \gamma'(s)^{-2})} = \text{l.i.m.}_{N \rightarrow \infty} S_N,$$

where

$$S_N := \sum_{i=1}^{N-1} \frac{W(s_i^{(N)}, \gamma(s_i^{(N)})) \Delta s_{i+1}^{(N)}}{(s_i^{(N)})^2 \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})},$$

and $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$, is a sequence of partitions such that $|\mathcal{P}_N| \rightarrow 0$ as $N \rightarrow \infty$. Hence we have to prove $\eta_N^{(2)} - S_N \xrightarrow{\mathbb{L}^2} 0$. We have the decomposition $\eta_N^{(2)} - S_N = S_{N,1} - S_{N,2} - X_N$, where

$$\begin{aligned} S_{N,1} &:= \sum_{i=2}^{N-1} \frac{\Delta_1 W(s_{i+1}^{(N)}, \gamma(s_i^{(N)})) \Delta s_{i+1}^{(N)}}{s_i^{(N)} s_{i+1}^{(N)} \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})}, \\ S_{N,2} &:= \sum_{i=2}^{N-1} \left(\frac{1}{s_i^{(N)}} - \frac{1}{s_{i+1}^{(N)}} \right) \frac{W(s_i^{(N)}, \gamma(s_i^{(N)})) \Delta s_{i+1}^{(N)}}{s_i^{(N)} \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})}, \\ X_N &:= \frac{W(s_1^{(N)}, \gamma(s_1^{(N)})) \Delta s_2^{(N)}}{(s_1^{(N)})^2 \gamma(s_1^{(N)}) (1 + \gamma'(s_1^{(N)})^{-2})}. \end{aligned}$$

Obviously $X_N \xrightarrow{\mathbb{L}^2} 0$. Clearly, $S_{N,1} \xrightarrow{\mathbb{L}^2} 0$, since

$$\|S_{N,1}\|_{\mathbb{L}^2}^2 = \sum_{i=2}^{N-1} \frac{\gamma(s_i^{(N)}) (\Delta s_{i+1}^{(N)})^3}{[s_i^{(N)} s_{i+1}^{(N)} \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})]^2} \leq \frac{b-a}{a^4 \gamma(b)} |\mathcal{P}_N|^2 \rightarrow 0.$$

In a similar way $S_{N,2} \xrightarrow{L^2} 0$ since

$$\begin{aligned} \|S_{N,2}\|_{L^2} &\leq \sum_{i=2}^{N-1} \left| \frac{1}{s_i^{(N)}} - \frac{1}{s_{i+1}^{(N)}} \right| \frac{\|W(s_i^{(N)}, \gamma(s_i^{(N)}))\|_{L^2} \Delta s_{i+1}^{(N)}}{s_i^{(N)} \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})} \\ &= \sum_{i=2}^{N-1} \frac{(\Delta s_{i+1}^{(N)})^2 \sqrt{s_i^{(N)} \gamma(s_i^{(N)})}}{(s_i^{(N)})^2 s_{i+1}^{(N)} \gamma(s_i^{(N)}) (1 + \gamma'(s_i^{(N)})^{-2})} \leq \frac{b-a}{a^{5/2} \gamma(b)^{1/2}} |\mathcal{P}_N| \rightarrow 0, \end{aligned}$$

hence the proof is complete. \square

4.10 Lemma. *With the notation of the proof of Theorem 1.1, we have*

$$\eta_N^{(3)} \xrightarrow{L^2} - \int_{\Gamma} y_{1,2} W.$$

Proof. Condition (1.1) implies

$$\begin{aligned} &\int_a^b \sqrt{\mathbb{E} [y_{1,2}(s, \gamma(s))^2 (1 + \gamma'(s)^2) W(s, \gamma(s))^2]} ds = \\ &\int_a^b \frac{|\gamma'(s) [1 + \gamma'(s)^2 + 2\gamma(s)\gamma''(s)]|}{s^{1/2} \gamma(s)^{3/2} (1 + \gamma'(s)^2)^2} ds < \infty, \end{aligned}$$

hence by Lemma 4.5 the improper L^2 -Riemann integral

$$\int_{\Gamma} y_{1,2} W = \int_a^b \frac{\gamma'(s) [1 + \gamma'(s)^2 + 2\gamma(s)\gamma''(s)] W(s, \gamma(s)) ds}{s \gamma(s)^2 (1 + \gamma'(s)^2)^2}$$

exists and we have

$$\int_{\Gamma} y_{1,2} W = \text{l.i.m.}_{\varepsilon, \delta \downarrow 0} \text{l.i.m.}_{N \rightarrow \infty} S_N(\varepsilon, \delta),$$

where $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$, is a sequence of partitions such that $|\mathcal{P}_N| \rightarrow 0$ as $N \rightarrow \infty$, and

$S_N(\varepsilon, \delta) :=$

$$\sum_{[s_{i-1}^{(N)}, s_i^{(N)}] \subset [a+\varepsilon, b-\delta]} \frac{\gamma'(s_{i-1}^{(N)}) [1 + \gamma'(s_{i-1}^{(N)})^2 + 2\gamma(s_{i-1}^{(N)})\gamma''(s_{i-1}^{(N)})] W(s_{i-1}^{(N)}, \gamma(s_{i-1}^{(N)})) \Delta s_i^{(N)}}{s_{i-1}^{(N)} \gamma(s_{i-1}^{(N)})^2 (1 + \gamma'(s_{i-1}^{(N)})^2)^2}.$$

Hence it is sufficient to prove

$$\lim_{\varepsilon, \delta \downarrow 0} \limsup_{N \rightarrow \infty} \|\eta_N^{(3)} + S_N(\varepsilon, \delta)\|_{L^2} = 0.$$

We have the decomposition $\eta_N^{(3)} + S_N(\varepsilon, \delta) = -S_{N,1}(\varepsilon, \delta) + S_{N,2}(\varepsilon, \delta) + S_{N,3}(\varepsilon, \delta)$, where

$$S_{N,1}(\varepsilon, \delta) :=$$

$$\sum_{[s_{i-1}^{(N)}, s_i^{(N)}] \subset [a+\varepsilon, b-\delta]} \frac{\gamma'(s_{i-1}^{(N)})(1 + \gamma'(s_{i-1}^{(N)})^2 + 2\gamma(s_{i-1}^{(N)})\gamma''(s_{i-1}^{(N)}))\Delta_1 W(s_i^{(N)}, \gamma(s_{i-1}^{(N)}))\Delta s_i^{(N)}}{s_{i-1}^{(N)}\gamma(s_{i-1}^{(N)})^2 \left(1 + \gamma'(s_{i-1}^{(N)})^2\right)^2},$$

$$S_{N,2}(\varepsilon, \delta) := \sum_{[s_{i-1}^{(N)}, s_i^{(N)}] \subset [a+\varepsilon, b-\delta]} \frac{W(s_i^{(N)}, \gamma(s_{i-1}^{(N)}))}{\gamma(s_{i-1}^{(N)}) \left(1 + \gamma'(s_{i-1}^{(N)})^2\right)} \left(\frac{\gamma'(s_{i-1}^{(N)})(1 + \gamma'(s_{i-1}^{(N)})^2 + 2\gamma(s_{i-1}^{(N)})\gamma''(s_{i-1}^{(N)}))\Delta s_i^{(N)}}{s_{i-1}^{(N)}\gamma(s_{i-1}^{(N)}) \left(1 + \gamma'(s_{i-1}^{(N)})^2\right)} - \frac{\gamma(s_i^{(N)}) \left(1 + \gamma'(s_i^{(N)})^2\right) - \gamma(s_{i-1}^{(N)}) \left(1 + \gamma'(s_{i-1}^{(N)})^2\right)}{s_i^{(N)}\gamma(s_i^{(N)}) \left(1 + \gamma'(s_i^{(N)})^2\right)} \right),$$

$$S_{N,3}(\varepsilon, \delta) :=$$

$$\sum_{[s_{i-1}^{(N)}, s_i^{(N)}] \not\subset [a+\varepsilon, b-\delta]} \frac{\left(\gamma(s_{i-1}^{(N)}) \left(1 + \gamma'(s_{i-1}^{(N)})^2\right) - \gamma(s_i^{(N)}) \left(1 + \gamma'(s_i^{(N)})^2\right)\right) W(s_i^{(N)}, \gamma(s_{i-1}^{(N)}))}{s_i^{(N)}\gamma(s_{i-1}^{(N)})\gamma(s_i^{(N)}) \left(1 + \gamma'(s_{i-1}^{(N)})^2\right) \left(1 + \gamma'(s_i^{(N)})^2\right)},$$

Clearly $S_{N,1}(\varepsilon, \delta) \xrightarrow{L^2} 0$ for all $a < a + \varepsilon < b - \delta < b$ since

$$\begin{aligned} & \|S_{N,1}(\varepsilon, \delta)\|_{L^2}^2 \\ &= \sum_{[s_{i-1}^{(N)}, s_i^{(N)}] \subset [a+\varepsilon, b-\delta]} \frac{\gamma'(s_{i-1}^{(N)})^2 \left[1 + \gamma'(s_{i-1}^{(N)})^2 + 2\gamma(s_{i-1}^{(N)})\gamma''(s_{i-1}^{(N)})\right]^2 \gamma(s_{i-1}^{(N)}) (\Delta s_i^{(N)})^3}{(s_{i-1}^{(N)})^2 \gamma(s_{i-1}^{(N)})^4 \left(1 + \gamma'(s_{i-1}^{(N)})^2\right)^4} \\ &\leq \frac{b-a}{a^2 \gamma(b)^3} |\mathcal{P}_N|^2 \sup_{s \in [a+\varepsilon, b-\delta]} \gamma'(s)^2 \left[1 + \gamma'(s)^2 + 2\gamma(s)\gamma''(s)\right]^2, \end{aligned}$$

hence $S_{N,1}(\varepsilon, \delta) \xrightarrow{L^2} 0$ as $N \rightarrow \infty$ for all $a < a + \varepsilon < b - \delta < b$.

The mean value theorem implies

$$\begin{aligned} & \gamma(s_i^{(N)}) \left(1 + \gamma'(s_i^{(N)})^2\right) - \gamma(s_{i-1}^{(N)}) \left(1 + \gamma'(s_{i-1}^{(N)})^2\right) \\ &= \gamma'(\tilde{s}_{i-1}^{(N)}) \left[1 + \gamma'(\tilde{s}_{i-1}^{(N)})^2 + 2\gamma(\tilde{s}_{i-1}^{(N)})\gamma''(\tilde{s}_{i-1}^{(N)})\right] \Delta s_i^{(N)} \end{aligned}$$

with some $\tilde{s}_{i-1}^{(N)} \in (s_{i-1}^{(N)}, s_i^{(N)})$, hence

$$\begin{aligned}
& \|S_{N,2}(\varepsilon, \delta)\|_{\mathbb{L}^2} \\
& \leq \frac{b-a}{a^{1/2}\gamma(b)^{3/2}} \sup_{\substack{a+\varepsilon \leq u < v \leq b-\delta \\ v-u \leq |\mathcal{P}_N|}} |\gamma'(u)[1 + \gamma'(u)^2 + 2\gamma(u)\gamma''(u)] \\
& \quad - \gamma'(v)[1 + \gamma'(v)^2 + 2\gamma(v)\gamma''(v)]| \\
& \quad + \frac{(b-a)b^{1/2}}{\gamma(b)^{1/2}} \sup_{s \in [a+\varepsilon, b-\delta]} |\gamma'(s)[1 + \gamma'(s)^2 + 2\gamma(s)\gamma''(s)]| \\
& \quad \times \sup_{\substack{a+\varepsilon \leq u < v \leq b-\delta \\ v-u \leq |\mathcal{P}_N|}} \left| \frac{1}{u\gamma(u)(1 + \gamma'(u)^2)} - \frac{1}{v\gamma(v)(1 + \gamma'(v)^2)} \right|,
\end{aligned}$$

which implies $S_{N,2}(\varepsilon, \delta) \xrightarrow{\mathbb{L}^2} 0$ as $N \rightarrow \infty$ for all $a < a + \varepsilon < b - \delta < b$.

In a similar way

$$\begin{aligned}
& \|S_{N,3}(\varepsilon, \delta)\|_{\mathbb{L}^2} \\
& \leq \frac{1}{a^{1/2}\gamma(b)^{3/2}} \sum_{[s_{i-1}^{(N)}, s_i^{(N)}] \not\subset [a+\varepsilon, b-\delta]} \frac{|\gamma'(\tilde{s}_{i-1}^{(N)})[1 + \gamma'(\tilde{s}_{i-1}^{(N)})^2 + 2\gamma(\tilde{s}_{i-1}^{(N)})\gamma''(\tilde{s}_{i-1}^{(N)})]|}{(1 + \gamma'(s_{i-1}^{(N)})^2)(1 + \gamma'(s_i^{(N)})^2)} \\
& \rightarrow \frac{1}{a^{1/2}\gamma(b)^{3/2}} \int_{[a, a+\varepsilon] \cup [b-\delta, b]} \frac{|\gamma'(s)[1 + \gamma'(s)^2 + 2\gamma(s)\gamma''(s)]|}{(1 + \gamma'(s)^2)^2} ds \quad \text{as } N \rightarrow \infty \\
& \rightarrow 0 \quad \text{as } \varepsilon, \delta \downarrow 0,
\end{aligned}$$

which implies $\lim_{\varepsilon, \delta \downarrow 0} \limsup_{N \rightarrow \infty} \|S_{N,3}(\varepsilon, \delta)\|_{\mathbb{L}^2} \rightarrow 0$. \square

5 Weighted integral of the normal derivative of the Wiener sheet along a curve

We define L^2 -Riemann integral of the product of a function and the normal derivative of an L^2 -process along a curve $\Gamma := \{(s, \gamma(s)) : s \in (a, b)\}$, where $\gamma : (a, b) \rightarrow \mathbb{R}$ is a differentiable function. We take the normal vector $n_\Gamma(s, \gamma(s))$ of the curve Γ at the point $(s, \gamma(s)) \in \Gamma$ defined by

$$n_\Gamma(s, \gamma(s)) := \left(-\frac{\gamma'(s)}{\sqrt{1 + \gamma'(s)^2}}, \frac{1}{\sqrt{1 + \gamma'(s)^2}} \right) = (\psi(s), \sqrt{1 - \psi(s)^2}),$$

where

$$\psi(s) := -\frac{\gamma'(s)}{\sqrt{1 + \gamma'(s)^2}}, \quad s \in (a, b).$$

5.1 Definition. Let Z be an L^2 -process given along the curve Γ . Let $y : \Gamma \rightarrow \mathbb{R}$. Then

$$\int_{\Gamma} y \partial_n Z := \text{l.i.m.}_{h \rightarrow 0} \frac{1}{h} \int_{\Gamma} y(\cdot) (Z(\cdot + h n_{\Gamma}(\cdot)) - Z(\cdot))$$

if the right hand side exists.

Next we give sufficient conditions for the existence of the integral $\int_{\Gamma} y \partial_n W$.

5.2 Theorem. Suppose that $[a, b] \subset (0, \infty)$, the function $\gamma : [a, b] \rightarrow \mathbb{R}$ is continuous, strictly decreasing with $\gamma(b) > 0$, the function γ is twice continuously differentiable on (a, b) , and the limits $\gamma'(a) := \lim_{s \downarrow a} \gamma'(s) \in [-\infty, 0]$ and $\gamma'(b) := \lim_{s \uparrow b} \gamma'(s) \in [-\infty, 0]$ exist. Suppose that the functions $s \mapsto \gamma'(s)y(s, \gamma(s))$ and $s \mapsto \gamma'(s)^{-1}y(s, \gamma(s))$ can be continuously extended to $[a, b]$. Then the integral $\int_{\Gamma} y \partial_n W$ exists and $\int_{\Gamma} y \partial_n W = \text{l.i.m.}_{N \rightarrow \infty} S(N)$, where

$$S(N) := - \sum_{i=2}^{N-1} y(s_i^{(N)}, \gamma(s_i^{(N)})) \left[\gamma'(s_i^{(N)}) \{W(s_{i+1}^{(N)}, \gamma(s_i^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))\} \right. \\ \left. + \gamma'(s_i^{(N)})^{-1} \{W(s_i^{(N)}, \gamma(s_{i-1}^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))\} \right],$$

where $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$, is a sequence of partitions such that $|\mathcal{P}_N| \rightarrow 0$ as $N \rightarrow \infty$.

For the proof of Theorem 5.2 we make use the following lemma.

5.3 Lemma. Let $\mathcal{P}_N : a = s_1^{(N)} < s_2^{(N)} < \dots < s_{N-1}^{(N)} < s_N^{(N)} = b$, $N = 1, 2, \dots$, be a sequence of partitions with $|\mathcal{P}_N| \rightarrow 0$. Let $g : [a, b] \rightarrow \mathbb{R}$ be continuous. Then

$$\lim_{N, M \rightarrow \infty} S_{N, M} = \lim_{h, h' \rightarrow 0} I_{h, h'} = \lim_{\substack{N \rightarrow \infty \\ h \rightarrow 0}} J_{N, h} = \int_a^b \gamma(s) g(s)^2 ds,$$

where

$$S_{N, M} := \sum_{i=2}^{N-1} \sum_{j=2}^{M-1} g(s_i^{(N)}) g(s_j^{(M)}) \mathbb{E} \left[(W(s_{i+1}^{(N)}, \gamma(s_i^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))) \right. \\ \left. (W(s_{j+1}^{(M)}, \gamma(s_j^{(M)})) - W(s_j^{(M)}, \gamma(s_j^{(M)}))) \right],$$

$$I_{h, h'} := \frac{1}{hh'} \int_a^b \int_a^b g(s) g(t) \mathbb{E} \left[(W(s+h, \gamma(s)) - W(s, \gamma(s))) \right. \\ \left. (W(t+h', \gamma(t)) - W(t, \gamma(t))) \right] ds dt,$$

$$J_{N,h} := \frac{1}{h} \sum_{i=2}^{N-1} g(s_i^{(N)}) \int_a^b \mathbb{E} \left[(W(s_{i+1}^{(N)}, \gamma(s_i^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))) \right. \\ \left. (W(t+h, \gamma(t)) - W(t, \gamma(t))) \right] dt.$$

Moreover,

$$\lim_{N,M \rightarrow \infty} \tilde{S}_{N,M} = \int_{\gamma(b)}^{\gamma(a)} \gamma^{-1}(t) g(\gamma^{-1}(t))^2 dt,$$

where

$$\tilde{S}_{N,M} := \sum_{i=2}^{N-1} \sum_{j=2}^{M-1} g(s_i^{(N)}) g(s_j^{(M)}) \mathbb{E} \left[(W(s_i^{(N)}, \gamma(s_{i-1}^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))) \right. \\ \left. (W(s_j^{(M)}, \gamma(s_{j-1}^{(M)})) - W(s_j^{(M)}, \gamma(s_j^{(M)}))) \right].$$

Proof of Lemma 5.3. Obviously,

$$\mathbb{E} \left[(W(s_{i+1}^{(N)}, \gamma(s_i^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))) (W(s_{j+1}^{(M)}, \gamma(s_j^{(M)})) - W(s_j^{(M)}, \gamma(s_j^{(M)}))) \right] \\ = \gamma(s_i^{(N)} \vee s_j^{(M)}) \lambda \left([s_i^{(N)}, s_{i+1}^{(N)}] \cap [s_j^{(M)}, s_{j+1}^{(M)}] \right),$$

where λ denotes the Lebesgue measure on \mathbb{R} . Clearly,

$$\int_a^b \gamma(s) g(s)^2 ds = \lim_{N \rightarrow \infty} \sum_{i=1}^{N-1} \gamma(s_i^{(N)}) g(s_i^{(N)})^2 (s_{i+1}^{(N)} - s_i^{(N)}) \\ = \lim_{N,M \rightarrow \infty} \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} \gamma(s_i^{(N)}) g(s_i^{(N)})^2 \lambda \left([s_i^{(N)}, s_{i+1}^{(N)}] \cap [s_j^{(M)}, s_{j+1}^{(M)}] \right),$$

since for every $N \in \mathbb{N}$

$$s_{i+1}^{(N)} - s_i^{(N)} = \sum_{j=1}^{M-1} \lambda \left([s_i^{(N)}, s_{i+1}^{(N)}] \cap [s_j^{(M)}, s_{j+1}^{(M)}] \right).$$

Consequently,

$$\left| \int_a^b \gamma(s) g(s)^2 ds - \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} g(s_i^{(N)}) g(s_j^{(M)}) \gamma(s_i^{(N)} \vee s_j^{(M)}) \right. \\ \left. \lambda \left([s_i^{(N)}, s_{i+1}^{(N)}] \cap [s_j^{(M)}, s_{j+1}^{(M)}] \right) \right| \\ \leq \sup_{\substack{a \leq u < v \leq b \\ v-u \leq |\mathcal{P}_N|}} |\gamma(u \vee v) g(v) - \gamma(u) g(u)| \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} |g(s_i^{(N)})| \\ \lambda \left([s_i^{(N)}, s_{i+1}^{(N)}] \cap [s_j^{(M)}, s_{j+1}^{(M)}] \right) \rightarrow 0,$$

since the supremum tends to zero and

$$\begin{aligned} & \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} |g(s_i^{(N)})| \lambda \left([s_i^{(N)}, s_{i+1}^{(N)}] \cap [s_j^{(M)}, s_{j+1}^{(M)}] \right) \\ &= \sum_{i=1}^{N-1} |g(s_i^{(N)})| (s_{i+1}^{(N)} - s_i^{(N)}) \rightarrow \int_a^b |g(s)| ds. \end{aligned}$$

By the same method we get

$$\left| S_{N,M} - \sum_{i=1}^{N-1} \sum_{j=1}^{M-1} g(s_i^{(N)}) g(s_j^{(M)}) \gamma(s_i^{(N)} \vee s_j^{(M)}) \lambda \left([s_i^{(N)}, s_{i+1}^{(N)}] \cap [s_j^{(M)}, s_{j+1}^{(M)}] \right) \right| \rightarrow 0,$$

thus we obtain $\lim_{N,M \rightarrow \infty} S_{N,M} = \int_a^b \gamma(s) g(s)^2 ds$.

In a similar way,

$$\begin{aligned} & \mathbb{E} \left[(W(s+h, \gamma(s)) - W(s, \gamma(s))) (W(t+h', \gamma(t)) - W(t, \gamma(t))) \right] \\ &= \gamma(s \vee t) \lambda([s, s+h] \cap [t, t+h']). \end{aligned}$$

We may suppose that $h' \leq h$ and $a+h' < b-h$. Then we have

$$\int_a^b \lambda([s, s+h] \cap [t, t+h']) dt = \begin{cases} hh' - (a+h'-s)^2/2 & \text{if } s \in [a, a+h'], \\ hh' & \text{if } s \in [a+h', b-h], \\ hh' - (s+h-b)^2/2 & \text{if } s \in [b-h, b-h+h'], \\ hh' - h'^2/2 - h'(s+h-h'-b) & \text{if } s \in [b-h+h', b], \end{cases}$$

hence $\int_a^b \gamma(s) g(s)^2 ds = I_{h,h'}^{(1)} + I_{h,h'}^{(2)}$, where

$$I_{h,h'}^{(1)} = \frac{1}{hh'} \int_a^b \int_a^b \gamma(s) g(s)^2 \lambda([s, s+h] \cap [t, t+h']) ds dt,$$

and $\lim_{h,h' \rightarrow 0} I_{h,h'}^{(2)} = 0$. We have

$$\begin{aligned} & \left| I_{h,h'}^{(1)} - \int_a^b \gamma(s) g(s)^2 ds \right| \\ & \leq \sup_{\substack{a \leq u < v \leq b \\ v-u \leq |\mathcal{P}_N|}} |\gamma(u \vee v) g(v) - \gamma(u) g(u)| \frac{1}{hh'} \int_a^b \int_a^b |g(s)| \lambda([s, s+h] \cap [t, t+h']) ds dt \rightarrow 0, \end{aligned}$$

since

$$\frac{1}{hh'} \int_a^b \int_a^b |g(s)| \lambda([s, s+h] \cap [t, t+h']) ds dt = \int_a^b |g(s)| ds + I_{h,h'}^{(3)},$$

where $\lim_{h, h' \rightarrow 0} I_{h, h'}^{(3)} = 0$.

The statement $\lim_{\substack{N \rightarrow \infty \\ h \rightarrow 0}} J_{N, h} = \int_a^b \gamma(s) g(s)^2 ds$ can be proved similarly.

To prove the last statement, we introduce the notation $t_i^{(N)} := \gamma(s_{N+1-i}^{(N)})$. We have

$$\begin{aligned} \tilde{S}_{N, M} = & \sum_{i=2}^{N-1} \sum_{j=2}^{M-1} g(\gamma^{-1}(t_{N+1-i}^{(N)})) g(\gamma^{-1}(t_{M+1-j}^{(M)})) \gamma^{-1}(t_{N+1-i}^{(N)} \vee t_{M+1-j}^{(M)}) \\ & \lambda \left([t_{N+1-i}^{(N)}, t_{N+2-i}^{(N)}] \cap [s_{M+1-j}^{(M)}, t_{M+2-j}^{(M)}] \right). \end{aligned}$$

Now we can use the first statement of the lemma for the curve $\gamma^{-1} : [\gamma(b), \gamma(a)] \rightarrow \mathbb{R}$ and for the continuous function $g \circ \gamma^{-1}$ on $[\gamma(b), \gamma(a)]$. Clearly, $\tilde{\mathcal{P}}_N : \gamma(b) = t_1^{(N)} < t_2^{(N)} < \dots < t_{N-1}^{(N)} < t_N^{(N)} = \gamma(a)$, $N = 1, 2, \dots$, is a sequence of partitions with $|\tilde{\mathcal{P}}_N| \rightarrow 0$, hence we obtain the last statement. \square

Proof of Theorem 5.2. First we prove the existence of the limit $\text{l.i.m.}_{N \rightarrow \infty} S(N)$. We have $S(N) = S_1(N) + S_2(N)$, where

$$\begin{aligned} S_1(N) &:= - \sum_{i=2}^{N-1} y(s_i^{(N)}, \gamma(s_i^{(N)})) \gamma'(s_i^{(N)}) (W(s_{i+1}^{(N)}, \gamma(s_i^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))), \\ S_2(N) &:= - \sum_{i=2}^{N-1} y(s_i^{(N)}, \gamma(s_i^{(N)})) \gamma'(s_i^{(N)})^{-1} (W(s_i^{(N)}, \gamma(s_{i-1}^{(N)})) - W(s_i^{(N)}, \gamma(s_i^{(N)}))). \end{aligned}$$

By Lemma 4.1, the existence of the limit $\text{l.i.m.}_{N \rightarrow \infty} S_1(N)$ follows from

$$\lim_{N, M \rightarrow \infty} \mathbb{E}[S_1(N) S_1(M)] = \int_a^b \gamma(s) \gamma'(s)^2 y(s, \gamma(s))^2 ds,$$

which is a consequence of Lemma 5.3. Similarly, the limit $\text{l.i.m.}_{N \rightarrow \infty} S_2(N)$ exists, since, again by Lemma 5.3,

$$\lim_{N, M \rightarrow \infty} \mathbb{E}[S_2(N) S_2(M)] = \int_{\gamma(b)}^{\gamma(a)} \gamma^{-1}(t) y(\gamma^{-1}(t), t)^2 \gamma'(\gamma^{-1}(t))^{-2} dt.$$

Hence we obtain the existence of the limit $\text{l.i.m.}_{N \rightarrow \infty} S(N)$.

We have

$$\frac{1}{h} \int_{\Gamma} y(\cdot) (W(\cdot + hn_{\Gamma}(\cdot)) - W(\cdot)) = I_{1,1}(h) + I_{2,1}(h) + I(h),$$

where

$$\begin{aligned}
I_{1,1}(h) &:= \frac{1}{h} \int_a^b y(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} (W(s + h\psi(s), \gamma(s)) - W(s, \gamma(s))) \, ds, \\
I_{2,1}(h) &:= \frac{1}{h} \int_a^b y(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} (W(s, \gamma(s) + h\sqrt{1 - \psi(s)^2}) - W(s, \gamma(s))) \, ds, \\
I(h) &:= \frac{1}{h} \int_a^b y(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} \left[W(s + h\psi(s), \gamma(s) + h\sqrt{1 - \psi(s)^2}) + W(s, \gamma(s)) \right. \\
&\quad \left. - W(s, \gamma(s) + h\sqrt{1 - \psi(s)^2}) - W(s + h\psi(s), \gamma(s)) \right] \, ds,
\end{aligned}$$

hence the statement of the theorem will follow from

$$\lim_{N \rightarrow \infty} S_1(N) = \lim_{h \rightarrow 0} I_{1,1}(h), \quad \lim_{N \rightarrow \infty} S_2(N) = \lim_{h \rightarrow 0} I_{2,1}(h), \quad \lim_{h \rightarrow 0} I(h) = 0.$$

The aim of the following discussion is to prove

$$\lim_{N \rightarrow \infty} S_1(N) \stackrel{\mathbf{A}}{=} \lim_{h \rightarrow 0} I_{1,2}(h) \stackrel{\mathbf{B}}{=} \lim_{h \rightarrow 0} I_{1,1}(h),$$

where

$$I_{1,2}(h) := \frac{1}{h} \int_a^b y(s, \gamma(s)) \psi'(s) \sqrt{1 + \gamma'(s)^2} (W(s + h, \gamma(s)) - W(s, \gamma(s))) \, ds.$$

In order to prove equality **A**, we will show that for a sequence $h_N \downarrow 0$, we have $S_1(N) - I_{1,2}(h_N) \xrightarrow{L^2} 0$, that is,

$$\lim_{N \rightarrow \infty} \mathbf{E}(S_1(N) - I_{1,2}(h_N))^2 = 0.$$

Lemma 5.3 implies

$$\lim_{N \rightarrow \infty} \mathbf{E}S_1(N)^2 = \lim_{N \rightarrow \infty} \mathbf{E}I_{1,2}(h_N)^2 = \lim_{N \rightarrow \infty} \mathbf{E}[S_1(N)I_{1,2}(h_N)] = \int_a^b y(s, \gamma(s))^2 \gamma'(s)^2 \gamma(s) \, ds,$$

which completes the proof of equality **A**.

In order to prove equality **B**, first we note that the function ψ can be continuously extended to $[a, b]$. Consider the functions

$$\begin{aligned}
\psi_N(s) &:= \sum_{i=2}^N \psi(s_i^{(N)}) \mathbf{1}_{[s_{i-1}^{(N)}, s_i^{(N)}]}(s), \quad s \in [a, b]; \\
\psi_N(b) &:= \lim_{s \uparrow b} \psi_N(s), \quad N = 1, 2, \dots
\end{aligned}$$

Then $\|\psi_N - \psi\| \rightarrow 0$, where $\|\cdot\|$ denotes the supremum norm. To obtain equality **B**, it is sufficient to show that

- (i) $\lim_{N \rightarrow \infty} \limsup_{h \rightarrow 0} \mathbb{E}(I_{1,2}(h) - I_{1,3}(h, N))^2 = 0,$
(ii) $\lim_{h \rightarrow 0} \mathbb{E}(I_{1,3}(h, N) - I_{1,4}(h, N))^2 = 0$ for all $N = 1, 2, \dots,$
(iii) $\lim_{N \rightarrow \infty} \limsup_{h \rightarrow 0} \mathbb{E}(I_{1,4}(h, N) - I_{1,1}(h))^2 = 0,$

where

$$I_{1,3}(h, N) := \frac{1}{h} \int_a^b y(s, \gamma(s)) \psi_N(s) \sqrt{1 + \gamma'(s)^2} (W(s+h, \gamma(s)) - W(s, \gamma(s))) ds,$$

$$I_{1,4}(h, N) := \frac{1}{h} \int_a^b y(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} (W(s+h\psi_N(s), \gamma(s)) - W(s, \gamma(s))) ds.$$

We have

$$I_{1,2}(h) - I_{1,3}(h, N) = \frac{1}{h} \int_a^b y(s, \gamma(s)) (\psi(s) - \psi_N(s)) \sqrt{1 + \gamma'(s)^2} (W(s+h, \gamma(s)) - W(s, \gamma(s))) ds.$$

Applying Lemma 5.3 we obtain

$$\begin{aligned} & \mathbb{E}(I_{1,2}(h) - I_{1,3}(h, N))^2 \\ & \leq \frac{\|\psi - \psi_N\|^2}{h^2} \int_a^b \int_a^b |y(s, \gamma(s)) y(t, \gamma(t))| \sqrt{(1 + \gamma'(s)^2)(1 + \gamma'(t)^2)} \\ & \quad \mathbb{E} \left[(W(s+h, \gamma(s)) - W(s, \gamma(s))) (W(t+h, \gamma(t)) - W(t, \gamma(t))) \right] ds dt \\ & \leq c \|\psi - \psi_N\|^2 \end{aligned}$$

with some constant c depending only on the functions y and ψ , thus we conclude (i).

Moreover,

$$\begin{aligned} & \text{l.i.m.}_{h \rightarrow 0} I_{1,3}(h, N) \\ & = \text{l.i.m.}_{h \rightarrow 0} \frac{1}{h} \sum_{i=2}^N \psi(s_i^{(N)}) \int_{s_{i-1}^{(N)}}^{s_i^{(N)}} y(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} (W(s+h, \gamma(s)) - W(s, \gamma(s))) ds \\ & = \text{l.i.m.}_{h \rightarrow 0} \frac{1}{h} \sum_{i=2}^N \int_{s_{i-1}^{(N)}}^{s_i^{(N)}} y(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} (W(s+h\psi(s_i^{(N)}), \gamma(s)) - W(s, \gamma(s))) ds \\ & = \text{l.i.m.}_{h \rightarrow 0} I_{1,4}(h, N), \end{aligned}$$

for all $N = 1, 2, \dots$, which implies (ii).

Further, we have

$$I_{1,4}(h, N) - I_{1,1}(h) =$$

$$\frac{1}{h} \int_a^b y(s, \gamma(s)) \sqrt{1 + \gamma'(s)^2} (W(s + h\psi_N(s), \gamma(s)) - W(s + h\psi(s), \gamma(s))) ds,$$

hence

$$\mathbb{E}(I_{1,4}(h, N) - I_{1,1}(h))^2 =$$

$$\frac{1}{h^2} \int_a^b \int_a^b y(s, \gamma(s)) y(t, \gamma(t)) \sqrt{(1 + \gamma'(s)^2)(1 + \gamma'(t)^2)} E_N(s, t) ds dt,$$

where

$$E_N(s, t) := \mathbb{E} \left[\left(W(s + h\psi_N(s), \gamma(s)) - W(s + h\psi(s), \gamma(s)) \right) \right. \\ \left. \times \left(W(t + h\psi_N(t), \gamma(t)) - W(t + h\psi(t), \gamma(t)) \right) \right].$$

If $u \in \mathbb{R}$ is between the numbers $s + h\psi_N(s)$ and $s + h\psi(s)$, then

$$u \in \left[s + h\psi_N(s) - h\|\psi_N - \psi\|, s + h\psi_N(s) + h\|\psi_N - \psi\| \right],$$

hence

$$\begin{aligned} & |E_N(s, t)| \\ & \leq \mathbb{E} \left[\left(W(s + h\psi_N(s) + h\|\psi_N - \psi\|, \gamma(s)) - W(s + h\psi_N(s) - h\|\psi_N - \psi\|, \gamma(s)) \right) \right. \\ & \quad \left. \left(W(t + h\psi_N(t) + h\|\psi_N - \psi\|, \gamma(t)) - W(t + h\psi(t) - h\|\psi_N - \psi\|, \gamma(t)) \right) \right] \\ & = \sum_{i,j=1}^N \mathbb{E} \left[\left(W(s + h\psi_N(s_i^{(N)}) + h\|\psi_N - \psi\|, \gamma(s)) - W(s + h\psi_N(s_i^{(N)}) - h\|\psi_N - \psi\|, \gamma(s)) \right) \right. \\ & \quad \left. \left(W(t + h\psi_N(s_j^{(N)}) + h\|\psi_N - \psi\|, \gamma(t)) - W(t + h\psi(s_j^{(N)}) - h\|\psi_N - \psi\|, \gamma(t)) \right) \right] \\ & \quad \mathbf{1}_{[s_{i-1}^{(N)}, s_i^{(N)}] \times [s_{j-1}^{(N)}, s_j^{(N)}]}(s, t). \end{aligned}$$

Consequently,

$$\begin{aligned} & \mathbb{E}(I_{1,4}(h, N) - I_{1,1}(h))^2 \\ & \leq \frac{1}{h^2} \sup_{u \in [a, b]} [y(u, \gamma(u))^2 (1 + \gamma'(u)^2) \gamma(u)] \sum_{i,j=1}^N \int_{s_{i-1}^{(N)}}^{s_i^{(N)}} \int_{s_{j-1}^{(N)}}^{s_j^{(N)}} L_{i,j}^{(N)}(s, t) ds dt, \end{aligned}$$

where

$$L_{i,j}^{(N)}(s, t) := \lambda \left(\left[s + h\psi_N(s_i^{(N)}) - h\|\psi_N - \psi\|, s + h\psi_N(s_i^{(N)}) + h\|\psi_N - \psi\| \right] \right. \\ \left. \cap \left[t + h\psi_N(s_j^{(N)}) - h\|\psi_N - \psi\|, t + h\psi_N(s_j^{(N)}) + h\|\psi_N - \psi\| \right] \right),$$

and λ denotes the Lebesgue measure. It is easy to see that

$$\int_c^d \lambda([u, u+h] \cap [v, v+h]) dv \begin{cases} \leq h^2 & \text{if } u \in [c-h, d+h], \\ = 0 & \text{if } u \notin [c-h, d+h], \end{cases}$$

thus

$$\int_{s_{j-1}^{(N)}}^{s_j^{(N)}} L_{i,j}^{(N)}(s, t) dt \begin{cases} \leq 4h^2\|\psi_N - \psi\|^2 & \text{if } s \in \left[s_{j-1}^{(N)} + h(\psi(s_j^{(N)}) - \psi_N(s_i^{(N)}) - 2\|\psi_N - \psi\|), \right. \\ & \left. s_j^{(N)} + h(\psi(s_j^{(N)}) - \psi_N(s_i^{(N)}) + 2\|\psi_N - \psi\|) \right], \\ = 0 & \text{otherwise.} \end{cases}$$

Hence

$$\frac{1}{h^2} \sum_{i,j=1}^N \int_{s_{i-1}^{(N)}}^{s_i^{(N)}} \int_{s_{j-1}^{(N)}}^{s_j^{(N)}} L_{i,j}^{(N)}(s, t) ds dt \\ \leq 4\|\psi_N - \psi\|^2 \sum_{i,j=1}^N \lambda \left(\left[s_{i-1}^{(N)}, s_i^{(N)} \right] \cap \left[s_{j-1}^{(N)} + h(\psi(s_j^{(N)}) - \psi_N(s_i^{(N)}) - 2\|\psi_N - \psi\|), \right. \right. \\ \left. \left. s_j^{(N)} + h(\psi(s_j^{(N)}) - \psi_N(s_i^{(N)}) + 2\|\psi_N - \psi\|) \right] \right) \\ \leq 4\|\psi_N - \psi\|^2 \sum_{j=1}^N \left(4h\|\psi_N - \psi\| + s_j^{(N)} - s_{j-1}^{(N)} \right) \\ = 4\|\psi_N - \psi\|^2 (4h\|\psi_N - \psi\| + b - a) \rightarrow 4\|\psi_N - \psi\|^2 (b - a) \quad \text{as } h \rightarrow 0,$$

thus

$$\mathbb{E}(I_{1,4}(h, N) - I_{1,1}(h))^2 \leq c\|\psi - \psi_N\|^2,$$

with some constant c depending only on the functions y and ψ , thus we obtain (iii), and the proof of the equality **B** is complete.

Now, let us deal with $S_2(N)$. We have

$$S_2(N) = - \sum_{i=2}^{N-1} y(\gamma^{-1}(t_{N+1-i}^{(N)}), t_{N+1-i}^{(N)}) \gamma'(\gamma^{-1}(t_{N+1-i}^{(N)}))^{-1} \\ \left(W(\gamma^{-1}(t_{N+1-i}^{(N)}), t_{n+2-i}^{(N)}) - W(\gamma^{-1}(t_i^{(N)}), t_{N+1-i}^{(N)}) \right),$$

where $t_i^{(N)} = \gamma(s_{N+1-i}^{(N)})$, and $\tilde{\mathcal{P}}_N : \gamma(a) = t_1^{(N)} < t_2^{(N)} < \dots < t_{N-1}^{(N)} < t_N^{(N)} = \gamma(b)$, $N = 1, 2, \dots$, is a sequence of partitions with $|\tilde{\mathcal{P}}_N| \rightarrow 0$. Applying the same procedure for the curve $\tilde{\gamma}(t) := \gamma^{-1}(t)$ on the interval $[\gamma(b), \gamma(a)]$ as before, we obtain

$$\text{l.i.m.}_{N \rightarrow \infty} S_2(N) = \text{l.i.m.}_{h \rightarrow 0} \tilde{I}_{2,1}(h),$$

where

$$\tilde{I}_{2,1}(h) := \frac{1}{h} \int_{\gamma(b)}^{\gamma(a)} y(\gamma^{-1}(t), t) \sqrt{1 + \tilde{\gamma}'(t)^2} (W(\gamma^{-1}(t), t + h\tilde{\psi}(t)) - W(\gamma^{-1}(t), t)) dt,$$

and

$$\sqrt{1 + \tilde{\gamma}'(t)^2} = \sqrt{1 + \frac{1}{\gamma'(\gamma^{-1}(t))^2}}, \quad \tilde{\psi}(t) = -\frac{\tilde{\gamma}'(t)}{\sqrt{1 + \tilde{\gamma}'(t)^2}} = \frac{1}{\sqrt{1 + \gamma'(\gamma^{-1}(t))^2}}.$$

By Lemma 4.6 we easily obtain $\tilde{I}_{2,1}(h) = I_{2,1}(h)$, hence we conclude $\text{l.i.m.}_{N \rightarrow \infty} S_2(N) = \text{l.i.m.}_{h \rightarrow 0} I_{2,1}(h)$.

Finally we have

$$\mathbb{E}I(h)^2 = \frac{1}{h} \int_a^b \int_a^b y(s, \gamma(s)) y(t, \gamma(t)) \sqrt{(1 + \gamma'(s)^2)(1 + \gamma'(t)^2)} E(s, t) ds dt,$$

where

$$\begin{aligned} E(s, t) := \mathbb{E} \left[\right. & \left. \left\{ W(s + h\psi(s), \gamma(s) + h\sqrt{1 - \psi(s)^2}) - W(s + h\psi(s), \gamma(s)) \right. \right. \\ & \left. \left. - W(s, \gamma(s) + h\sqrt{1 - \psi(s)^2}) + W(s, \gamma(s)) \right\} \right. \\ & \left. \left\{ W(t + h\psi(t), \gamma(t) + h\sqrt{1 - \psi(t)^2}) - W(t + h\psi(t), \gamma(t)) \right. \right. \\ & \left. \left. - W(t, \gamma(t) + h\sqrt{1 - \psi(t)^2}) + W(t, \gamma(t)) \right\} \right]. \end{aligned}$$

Clearly

$$\begin{aligned} E(s, t) &= \lambda \left(\left[s, s + h\psi(s) \right] \times \left[\gamma(s), \gamma(s) + h\sqrt{1 - \psi(s)^2} \right] \right. \\ & \quad \left. \cap \left[t, t + h\psi(t) \right] \times \left[\gamma(t), \gamma(t) + h\sqrt{1 - \psi(t)^2} \right] \right) \\ &\leq \frac{h^2}{4} \mathbf{1}_{\{[s, s+h\psi(s)] \cap [t, t+h\psi(t)] \neq \emptyset\}}(s, t) \\ &\leq \frac{h^2}{4} \mathbf{1}_{\{|s-t| \leq h\}}(s, t) \end{aligned}$$

and

$$\int_a^b \int_a^b \mathbf{1}_{\{|s-t| \leq h\}}(s, t) \, ds dt = 2h(b-a) - h^2,$$

hence

$$EI(h)^2 \leq \frac{2h(b-a) - h^2}{4} \sup_{s \in [a, b]} [y(s, \gamma(s))^2 (1 + \gamma'(s)^2)] \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

thus we obtain that $\lim_{h \rightarrow 0} I(h) = 0$. □

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