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Abstract

This paper studies polynomials of the form $p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$, mainly on the question when such a polynomial is a coordinate. It is shown that a polynomial of the form $p(X)Y + Q(X, Z, T)$ is a coordinate if and only if $Q(a, Z, T)$ is a coordinate for every zero a of $p(X)$. Connections with the Cancellation Problem are made. As a consequence, the Abhyankar-Sathaye Conjecture has a positive answer for polynomials in four variables of this form. Also, conjectures about possible generalisations of the concept of "coordinate" for elements of general rings are made.

1 Introduction

In his Bourbaki lecture ([11]) Kraft gave a list of various challenging problems in affine geometry. It can be shown that most of these problems are related to the following crucial question: given a polynomial f in an n-variable polynomial ring over $\mathbb{C}$, denoted $\mathbb{C}^{[n]}$, how can one recognise if the zero-set of f is isomorphic to affine $(n-1)$-space over $\mathbb{C}$, or equivalently if $\mathbb{C}^{[n]}/(f)$ is a polynomial ring in $n-1$ variables over $\mathbb{C}$? If $n = 2$ this problem was solved by Abhyankar and Moh in [1]: they showed that $\mathbb{C}^{[2]}/(f) \cong \mathbb{C}^{[1]}$ if and only if f is a coordinate in $\mathbb{C}^{[2]}$ (if R is a commutative ring and $f \in R^{[n]} := R[X_1, \ldots, X_n]$ then f is called a coordinate in $R^{[n]}$ over R if there exists an R-automorphism φ of $R^{[n]}$ such that $\varphi(f) = X_1$). Obviously if f is a coordinate in $\mathbb{C}^{[n]}$ then $\mathbb{C}^{[n]}/(f)$ is isomorphic to $\mathbb{C}^{[n-1]}$. Abhyankar and Sathaye conjectured that the converse is true i.e. if $\mathbb{C}^{[n]}/(f) \cong \mathbb{C}^{[n-1]}$, then f is a coordinate in $\mathbb{C}^{[n]}$. In spite of much research in this direction, the conjecture is still open for all $n \geq 3$. In case $n = 3$ some special results are known: in [17] and [16] Sathaye and Russell show that the Abhyankar-Sathaye Conjecture is correct in case one of the variables appears linearly in f. This result was extended by Bhatwadekar and Dutta in [3], where they replaced $\mathbb{C}$ by a discrete valuation ring. Another special result is obtained by Wright in [18]. In more than three variables it is not known if the Abhyankar-Sathaye Conjecture holds for polynomials in which one variable appears linearly. In 1995 Makar-Limanov showed (see [12]) that for the polynomial $f = X^2Y + X + Z^2 + T^3$, in which Y appears linearly, the quotient ring $\mathbb{C}[X,Y,Z,T]/(f)$ is not isomorphic to $\mathbb{C}^{[3]}$. This result together with similar results for other polynomials formed one of the final ingredients in the complete solution of the linearization conjecture for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ (see [10]). Another example demonstrating the usefulness of studying the question posed in the beginning of this introduction is given in [7]: the authors construct a candidate counterexample to the Cancellation Problem, namely a polynomial in 5 variables over $\mathbb{C}$ in which two variables appear linearly. They conjectured that the corresponding quotient ring is not isomorphic to $\mathbb{C}^{[4]}$. If this is indeed the case then the Cancellation Problem has a negative answer in dimension 5.

In this paper we start a systematic study of polynomials in 4 variables of the form $p(X)Y + q(X,Z,T)$ ($p(X) \neq 0$). We completely characterize when the corresponding quotient ring is isomorphic to $\mathbb{C}^{[3]}$ (see theorem 3.1). One of the characterizations is that $q(\alpha, Z, T)$ is a coordinate in $\mathbb{C}[Z,T]$ for every zero α of $p(X)$. In particular we see that the polynomial $X^2Y + X + Z^2 + T^3$ does not define a hypersurface isomorphic to $\mathbb{C}^3$ (since $Z^2 + T^3$ is clearly no coordinate in $\mathbb{C}^{[2]}$). Another characterization asserts that $p(X)Y + q(X,Z,T)$ is a coordinate in $\mathbb{C}[X,Y,Z,T]$ over $\mathbb{C}[X]$. Also we obtain that the Abhyankar-Sathaye Conjecture is correct for those polynomials. Main ingredients in the proof of theorem 3.1 are a recent result of Kaliman in [9] (see theorem 2.4 below) and a result of Edo-Vénéreau in [5] (see 2.5 below).

Finally, at the end of this paper we discuss some possible definitions of the notion of coordinate to quotients of polynomial rings. We also make several conjectures.

2 Preliminaries

Notations: In this article, $\mathbb{C}^{[n]}$ will denote a ring isomorphic over $\mathbb{C}$ to a polynomial ring in n variables. $LND(\mathbb{C}^{[n]})$ will be the set of all locally nilpotent $\mathbb{C}$-derivations on $\mathbb{C}^{[n]}$, i.e. the set of all $\mathbb{C}$-linear maps $D : \mathbb{C}^{[n]} \longrightarrow \mathbb{C}^{[n]}$ satisfying the Leibnitz rule $D(ab) = D(a)b + aD(b)$ for all $a, b \in \mathbb{C}^{[n]}$ and for all $a \in \mathbb{C}^{[n]}$ there exists an integer $n \in \mathbb{N}$ such that $D^n(a) = 0$. If A is some ring, A^* will be the set of invertible elements.

Definition 2.1. We say $F \in \mathbb{C}^{[n]}$ is a *coordinate in* $\mathbb{C}^{[n]}$ if there exist $F_2, \ldots, F_n \in \mathbb{C}^{[n]}$ such that $\mathbb{C}[F, F_2, \ldots, F_n] = \mathbb{C}^{[n]}$. Similarly, we say that $F \in \mathbb{C}^{[n]}$ is a *stable coordinate (in* $\mathbb{C}^{[n]}$*)* if there exist $m \in \mathbb{N}$ such that F is a coordinate in $\mathbb{C}^{[n+m]}$.

Not every polynomial is a coordinate, as can be seen by several examples. One can deduce the following:

Lemma 2.2. *If $F \in \mathbb{C}^{[n]}$ is a coordinate, then*

(i). F is irreducible, even $F + \alpha$ is irreducible for all $\alpha \in \mathbb{C}$,

(ii). $(\frac{\partial F}{\partial X_1}, \ldots, \frac{\partial F}{\partial X_n}) = (1)$,

(iii). $\mathbb{C}^{[n]}/(F) \cong \mathbb{C}^{[n-1]}$,

(iv). There exists a subring $A \subset \mathbb{C}^{[n]}$ such that F is algebraically independent over A, $A[F] = \mathbb{C}^{[n]}$, and $A \cong \mathbb{C}^{[n-1]}$.

It is an important question to be able to decide whether some polynomial is a coordinate. The question arises whether there exist sufficient properties which imply "coordinate". *(i)* and *(ii)* are by no means sufficient: take $F = XY + ZT + Z + T$, which satisfies both *(i)* and *(ii)* and is no coordinate (by corollary 3.2). Whether *(iii)* is sufficient, is still open for $n \geq 3$:

Abhyankar-Sathaye Conjecture (AS(n)): If $f \in \mathbb{C}^{[n]}$ and $\mathbb{C}^{[n]}/(f) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ then f is a coordinate.

AS(2) was proved by Abhyankar and Moh in [1].

Part *(iv)* of lemma 2.2 gives rise to the following problem:

Cancellation Problem (CP(n)): If $\mathbb{C}^{[n]} = A[T]$ then $A \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$.

This problem had been answered affirmatively for $n = 2$ ([15]) and $n = 3$ ([8]). The following conjecture is a new one. In the rest of the article its significance will become clear.

Commuting Derivations Conjecture (CD(n)) : If $D_1, \ldots, D_{n-1} \in LND(\mathbb{C}^{[n]})$ linearly independent over $\mathbb{C}^{[n]}$ such that $[D_i, D_j] = 0$ for all $1 \leq i, j \leq n-1$ (i.e. they all commute) then

$$\bigcap_{i=1}^{n-1} ker(D_i) = \mathbb{C}[f]$$

where f is a coordinate in $\mathbb{C}^{[n]}$.

The following lemma we will need in the next section.

Lemma 2.3. *Let R be a domain, and $r \in R$ such that rR is a prime ideal. Then r is irreducible in R.*

Proof. Let $I := rR$. Suppose r is reducible, i.e. $r = ab$ for some $a, b \in R$ not invertible. Since $ab \in I$, a prime ideal, we have a or b in I. We may assume $a \in I$, thus $a = rs$ for some $s \in R$, and thus $rsb = ab = r$ and since R is a domain we get $sb = 1$, which means b is invertible, a contradiction. Hence r must be irreducible. □

The following theorem is a special case of the main theorem in [9].

Theorem 2.4. *Let $f \in \mathbb{C}[X,Y,Z]$ such that $\mathbb{C}[X,Y,Z]/(f-\lambda) \cong \mathbb{C}^{[2]}$ for all but finitely many $\lambda \in \mathbb{C}$. Then f is a coordinate.*

Proof. In the main theorem in [9] take $X' = \mathbb{C}^3$, $U := \{\lambda \mid \mathbb{C}[X,Y,Z]/(f-\lambda) \cong \mathbb{C}^{[2]}\}$, $Z := f^{-1}(U)$, $p = f$. Then this theorem states p is a coordinate. □

The following is theorem 7 in [5]. $\eta(R)$ is the nilradical of some ring R.

Theorem 2.5. *Let A be a ring and let $p \in A^*$. Let $a \in A, G, F \in A[X]$ such that F is a coordinate in $A[X]$, $a \mod (pA)$ invertible, and $G(X) \mod (pA) \in \eta((A/pA)[X])$. Then $aF(X) + G(X) + pY$ is a coordinate in $A[X,Y]$.*

3 Coordinates

Theorem 3.1. *Assume AS(n-1), CD(n) and CP(n-1). Let $F := p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$ where $p(X) \neq 0$. Then equivalent are:*

(i). *F is a coordinate in $\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]$*

(ii). *$\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}$*

(iii). *$q(a, Z_1, \ldots, Z_{n-1})$ is a coordinate in $\mathbb{C}[Z_1,\ldots,Z_{n-1}]$ for every zero a of $P(X)$.*

(iv). *F is a coordinate over $\mathbb{C}[X]$ in $\mathbb{C}[X,Y,Z_1,\ldots,Z_{n-1}]$*

Proof. (*of theorem 3.1*)
From 3.5 we have *(iii)*$\Longrightarrow$*(iv)*. *(iv)*$\Longrightarrow$*(i)* and *(i)*$\Longrightarrow$*(ii)* follow since they are weaker statements in general. *(ii)*$\Longrightarrow$*(iii)* follows from 3.7. □

From the fact that AS(2), CP(2) and CD(3) (see 4.6) are true, we can deduce the following corollaries:

Corollary 3.2. *The above equivalences hold for $F = p(X)Y + q(X, Z_1, Z_2)$.*

Corollary 3.3. *AS(4) is true if restricted to polynomials of the form $p(X)Y + q(X,Z,T)$.*

Lemma 3.4. *Let $q(Z_1,\ldots,Z_{n-1}) \in \mathbb{C}[Z_1,\ldots,Z_{n-1}]$. Suppose AS(n-1) and CP(n-1) are true. If $\mathbb{C}[Z_1,\ldots,Z_{n-1},Y]/(q) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ then q is a coordinate in $\mathbb{C}^{[n-1]}$.*

Proof. $\mathbb{C}[Z_1,\ldots,Z_{n-1}]/(q)[Y] \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ so by CP(n-1) we have $\mathbb{C}[Z_1,\ldots,Z_{n-1}]/(q) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}$ and by AS(n-1) we have q is a coordinate in $\mathbb{C}^{[n-1]}$. □

Write

$$p(X) = \Pi_{i=1}^r (X - \alpha_i)^{e_i}$$

for some $e_i \in \mathbb{N}$, and $F := p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$ for some $q \in \mathbb{C}[X, Z_1, \ldots, Z_{n-1}]$.

Theorem 3.5. *Let $q(X, Z_1, \ldots, Z_{n-1})$ be such that $q(\alpha_i, Z_1, \ldots, Z_{n-1})$ is a coordinate in $\mathbb{C}[Z_1, \ldots, Z_{n-1}]$ for every $1 \leq i \leq r$. Then $F := p(X)Y + q(X, Z_1, \ldots, Z_n)$ is a coordinate in $\mathbb{C}[X, Y, Z_1, \ldots, Z_{n-1}]$ over $\mathbb{C}[X]$.*

Proof. Since $\mathbb{C}[X]$ is Hermites, it follows from [4] or [2] that it suffices to prove that F is a coordinate in $\mathbb{C}[X]_{\mathfrak{m}}[Y, Z_1, \ldots, Z_{n-1}]$ over $\mathbb{C}[X]_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset \mathbb{C}[X]$. Let $\mathfrak{m} = (X-\alpha)$ for some $\alpha \in \mathbb{C}$. Notice that if $a(X) \in \mathbb{C}[X]$ we have $a \in \mathbb{C}[X]_{\mathfrak{m}}^*$ if and only if $a(\alpha) \neq 0$. In case $\alpha \neq \alpha_i$ we have $p(\alpha) \neq 0$ and hence F is a coordinate in $\mathbb{C}[X]_{\mathfrak{m}}[Y, Z_1, \ldots, Z_{n-1}]$. Left to prove the case $\alpha = \alpha_1$ ($\alpha = \alpha_i$ has the same proof). Let $q_1(Z_1, \ldots, Z_{n-1}) := q(\alpha, Z_1, \ldots, Z_{n-1})$ (hence a coordinate in $\mathbb{C}^{[n-1]}$), and define

$$\tilde{p} := \Pi_{i=2}^r (X - \alpha_i)^{e_i} = p(X)(X - \alpha)^{-e_1}.$$

Now

$$F = (X - \alpha)^{e_1} \tilde{p}(X)Y + q_1 + (X - \alpha)h(X, Z_1, \ldots, Z_{n-1})$$

for some h. Notice $\tilde{p} \in \mathbb{C}[X]_{\mathfrak{m}}^*$. But now, using 2.5 we have F is a coordinate in $\mathbb{C}[X]_{\mathfrak{m}}[Y, Z_1, \ldots, Z_{n-1}]$. □

Lemma 3.6. *Let $F = p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$ irreducible. Then there exists $\lambda \in \mathbb{C}$ such that $X - \lambda \mod (F)$ is irreducible in $\mathbb{C}[X, Y, Z_1, \ldots, Z_{n-1}]/(F)$.*

Proof. Take λ such that $p(\lambda) \neq 0$. Then

$$\begin{aligned}&\mathbb{C}[X, Y, Z_1, \ldots, Z_{n-1}]/(F, X - \lambda)\\ &= \mathbb{C}[Y, Z_1, \ldots, Z_{n-1}]/(p(\lambda)Y + q(\lambda, Z_1, \ldots, Z_{n-1})) \cong_{\mathbb{C}} \mathbb{C}^{[n-1]}\end{aligned}$$

which is a domain: hence $(X - \lambda, F)$ is prime, and thus $X - \lambda \mod F$ is irreducible by lemma 2.3. □

Lemma 3.7. *Assume CD(n), CP(n-1) and AS(n-1). Let $F := p(X)Y + q(X, Z_1, \ldots, Z_{n-1})$ and assume $\mathbb{C}^{[n+1]}/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}$. Then $q(a, Z_1, \ldots, Z_{n-1})$ is a coordinate in $\mathbb{C}[Z_1, \ldots, Z_{n-1}]$ for all zeros a of $p(X)$.*

Proof. Let

$$D_i := \frac{\partial q}{\partial Z_i} \frac{\partial}{\partial Y} - p \frac{\partial}{\partial Z_i}$$

for all $1 \le i \le n-1$. Then it's clear that $[D_i, D_j] = 0$, and that the D_i are linearly independent over $\mathbb{C}[X, Y, Z_1, \dots, Z_{n-1}]$. Now we know

$$\mathbb{C}^{[n+1]}/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}.$$

Furthermore $D_i(F) \subset (F)$, so the derivations $\bar{D}_i := D_i \mod (F)$ are well-defined on $\mathbb{C}^{[n+1]}/(F) \cong \mathbb{C}^{[n]}$. Also they are independent over $\mathbb{C}^{[n+1]}/(F)$. Since we assumed CD(n) we have

$$\bigcap_{i=1}^{n-1} ker(\bar{D}_i) = \mathbb{C}[g]$$

for some coordinate g. Since $ker(\bar{D}_i) \supset \mathbb{C}[X]$ we see $\mathbb{C}[g] \supset \mathbb{C}[X]$. By lemma 3.6 we see that $X - a$ is irreducible in $\mathbb{C}^{[n+1]}/(F)$ for some $a \in \mathbb{C}$. Now $X - a = Q(g)$ for some polynomial $Q(T) \in \mathbb{C}[T]$. Decomposing $Q(T)$ into linear factors $T - \lambda_i$ and observing that $g - \lambda_i$ is irreducible in $\mathbb{C}^{[n+1]}/(F)$ (since g is a coordinate in it), it follows that $g - \lambda_i$ divides the irreducible element $X - a$. So $X - a = bg + c$ for some $b \in \mathbb{C}^*, c \in \mathbb{C}$. Thus $\mathbb{C}[g] = \mathbb{C}[X]$, and $X - \alpha$ is a coordinate in $\mathbb{C}^{[n+1]}/(F) \cong_{\mathbb{C}} \mathbb{C}^{[n]}$ for every $\alpha \in \mathbb{C}$. So

$$\mathbb{C}^{[n-1]} \cong_{\mathbb{C}} \mathbb{C}^{[n+1]}/(F, X - \alpha) \text{ for all } \alpha \in \mathbb{C}.$$

In case $p(\alpha) = 0$ we have

$$\mathbb{C}^{[n-1]} \cong_{\mathbb{C}} \mathbb{C}[Y, Z_1, \dots, Z_{n-1}]/(q(\alpha, Z_1, \dots, Z_{n-1})$$

and thus by CP(n-1) and AS(n-1) and lemma 3.4 we have $q(\alpha, Z_1, \dots, Z_{n-1})$ is a coordinate in $\mathbb{C}[Z_1, \dots, Z_{n-1}]$. □

4 Proof of CD(3)

In the following lemma, the derivation δ_i (the restriction of D_i to A^{D_n}) is well-defined: for all $a \in A^{D_n}$ we have $0 = D_i(D_n(a)) = D_n(D_i(a))$, hence $D_i(A^{D_n}) \subseteq A^{D_n}$.

Lemma 4.1. *Let A be a $\mathbb{C}$-domain and $D_1, \dots, D_n$ be commuting locally nilpotent derivations which are linearly independent over A. Let $\delta_i := D_i|_{A^{D_n}}$. Then $\delta_1, \dots, \delta_{n-1}$ are linearly independent over A^{D_n}.*

Proof. Suppose that $\sum a_i \delta_i = 0$ for some $a_i \in A^{D_n}$. Since D_n is nonzero there exists a preslice $p \in A$ for D_n, i.e. an element p which satisfies $d := D_n(p) \neq 0$ and $D_n^2(p) = 0$ (i.e. $d \in A^{D_n}$). Let $s := pd^{-1} \in A[d^{-1}]$. Then $D_n(s) = 1$. Furthermore, by [6] pages 27-28, $A[d^{-1}] = A^{D_n}[d^{-1}][s]$. Let $a := \sum a_i D_i(s) \in A[d^{-1}]$, say $\tilde{a} := d^m a \in A$. So

$$(\sum_{i=1}^{n-1} a_i d^m D_i)(s) = d^m a = \tilde{a} = \tilde{a} D_n(s).$$

Also by our hypothesis

$$\sum_{i=1}^{n-1} a_i d^m D_i - \tilde{a} D_n = 0$$

on A^{D_n}. Since $A \subset A^{D_n}[d^{-1}][s]$ it follows that $\sum a_i d^m D_i = \tilde{a} D_n$. From the linear independence of the D_i over A we deduce that $d^m a_i = 0$ for all i, whence $a_i = 0$ for all i. □

Proposition 4.2. *Let A be a $\mathbb{C}$-domain with $trdeg_{\mathbb{C}} Q(A) = n (\geq 1)$. Let $D_1, \ldots, D_n$ be commuting locally nilpotent $\mathbb{C}$-derivations on A which are linearly independent over A. Then*

(i). There exist s_i in A such that $D_i s_j = \delta_{ij}$ for all i, j and

(ii). $A = \mathbb{C}[s_1, \ldots, s_n]$ a polynomial ring in $s_1, \ldots, s_n$ over $\mathbb{C}$.

Proof. We use induction on n. The case $n = 1$ is well-known (cor. 1.3.33 [6]). So let $n \geq 2$. $trdeg_{\mathbb{C}}(A^{D_n}) = n - 1$ and according to lemma 4.1 the derivations $\delta_i := D_i|_{A^{D_n}}$ $1 \leq i \leq n-1$ satisfy the hypothesis of the proposition. So by induction there exist $s_i \in A^{D_n}$ such that $\delta_i s_j = \delta_{ij}$ and $A^{D_n} = \mathbb{C}[s_1, \ldots, s_{n-1}]$. So the first $n-1$ derivations have a slice in A. Similarly D_n has a slice s_n in $A^{D_1} \subset A$. Then from $A = A^{D_n}[s_n]$ the result follows. □

Lemma 4.3. *Let A be a $\mathbb{C}$-domain with $trdeg_{\mathbb{C}}(Q(A)) = n$. If $D_1, \ldots, D_p$ are commuting locally nilpotent $\mathbb{C}$-derivations which are linearly independent over A , then $trdeg_{\mathbb{C}} Q(A^{D_1} \cap \ldots \cap A^{D_p}) = n - p$.*

Proof. The case $p = 1$ is well-known. Let $B := A^{D_p}$. By lemma 4.1 the derivations $\delta_i := D_i|_B$ for all $1 \leq i \leq p-1$ are linearly independent over B. Hence by induction $trdeg_{\mathbb{C}} Q(B^{\delta_1} \cap \ldots \cap B^{\delta_{p-1}}) = trdeg_{\mathbb{C}} Q(B) - (p-1) = n - 1 - (p-1) = n - p$. Since $B^{\delta_1} \cap \ldots \cap B^{\delta_{p-1}} = A^{D_1} \cap \ldots \cap A^{D_{p-1}} \cap A^{D_p}$ the result follows. □

Proposition 4.4. *Let A be an affine $\mathbb{C}$-domain such that $trdeg_{\mathbb{C}} Q(A) = n$ and $A^* = \mathbb{C}^*$. If A is a* UFD *and $D_1, \ldots, D_{n-1}$ are commuting locally nilpotent $\mathbb{C}$-derivations on A which are linearly independent over A, then $\cap A^{D_i} = \mathbb{C}[g]$ for some $g \in A$ which satisfies $g - c$ is irreducible in A for all $c \in \mathbb{C}$.*

Proof. Put $B := \cap A^{D_i}$. By lemma 4.3 we have $trdeg_{\mathbb{C}} B = n - (n-1) = 1$. Also B is a UFD (see [6] cor. 1.3.36) and $B = A \cap Q(B)$. Since $trdeg_{\mathbb{C}} Q(B) = 1$ it follows from special case of Hilbert 14 (using B is normal since it is a UFD) that B is a finitely generated $\mathbb{C}$-algebra. So B is an affine domain of krull dimension one. It is a well-known result that if $B^* = \mathbb{C}^*$, B is a UFD and B is an affine domain of krull dimension one, that $B = \mathbb{C}[g] \cong_{\mathbb{C}} \mathbb{C}^{[1]}$. (See for example [14].) Since $g - c$ is irreducible in $\mathbb{C}[g]$ for all $c \in \mathbb{C}$ and B is factorially closed in A it follows that $g - c$ is also irreducible in A (see [6] exercise 6, 1.3). $\square$

Proposition 4.5. *Let D_1, D_2 be two linearly independent (over $\mathbb{C}[X, Y, Z]$) commuting locally nilpotent $\mathbb{C}$-derivations. Then there exists $g \in \mathbb{C}[X, Y, Z] \backslash \mathbb{C}$ such that*

(i). $\mathbb{C}[X, Y, Z]^{D_1, D_2} = \mathbb{C}[g]$

(ii). $\mathbb{C}[X, Y, Z]_{b(g)} = \mathbb{C}[f, g, p]_{b(g)}$ *for some* $f, p \in \mathbb{C}[X, Y, Z]$ *and* $b(g) \in \mathbb{C}[g] \backslash \{0\}$

(iii). $\mathbb{C}[X, Y, Z]/(g - \lambda) \cong_{\mathbb{C}} \mathbb{C}^{[2]}$ *for all* $\lambda \in \mathbb{C}$ *with* $b(\lambda) \neq 0$.

Proof. (i) $\mathbb{C}[X, Y, Z]^{D_1} = \mathbb{C}[f, g]$ and $\mathbb{C}[X, Y, Z]^{D_2} = \mathbb{C}[p, q]$ by [13]. Since D_1, D_2 commute we have $D_2(\mathbb{C}[f, g]) \subseteq \mathbb{C}[f, g]$. Write $d_2 := D_2|_{\mathbb{C}[f,g]}$. By lemma 4.1 it follows that $d_2 \neq 0$ on $\mathbb{C}[f, g]$. So by Rentschler's theorem we may assume that $d_2 = a(g)\frac{\partial}{\partial f}$ i.e. $D_2(g) = 0$ and $D_2(f) = a(g) \neq 0$. So $\mathbb{C}[X, Y, Z]^{D_1, D_2} = \mathbb{C}[f, g]^{d_2} = \mathbb{C}[g]$ i.e.

$$\mathbb{C}[X, Y, Z]^{D_1, D_2} = \mathbb{C}[g]. \tag{1}$$

Similarly we get $D_1(\mathbb{C}[p, q]) \subset \mathbb{C}[p, q]$ and putting $d_1 := D_1|_{\mathbb{C}[p,q]}$ this gives by Rentschler that we may assume $d_1 = b(q)\frac{\partial}{\partial p}$ for some $b(q) \neq 0$. So

$$\mathbb{C}[X, Y, Z]^{D_1, D_2} = \mathbb{C}[p, q]^{d_1} = \mathbb{C}[q]. \tag{2}$$

From (1) and (2) we deduce that $\mathbb{C}[g] = \mathbb{C}[p]$, whence $g = \lambda q + \mu$ form some $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$. Replacing q by g (and hence $b(q) = b(\lambda^{-1}(g - \mu) = \tilde{b}(g)$ by $\tilde{b}(g)$ we get that we may assume the following

$$\begin{array}{l} \mathbb{C}[X, Y, Z]^{D_1} = \mathbb{C}[f, g], D_1 f = D_1 g = 0, D_1 p = b(g) \neq 0 \\ \mathbb{C}[X, Y, Z]^{D_2} = \mathbb{C}[p, g], D_2 f = a(g) \neq 0, D_2 g = D_2 p = 0. \end{array}$$

(ii) Also $\mathbb{C}[f, g, p] \cong_{\mathbb{C}} \mathbb{C}^{[3]}$ (for if p depends on $\mathbb{C}[f, g]$ then $D_1 p = 0$, contradiction). Observe that $D_1 p = b(g) \neq 0$ and $D_1^2 p = D_1 b(g) = 0$, so $s := p/b(g) \in \mathbb{C}[X, Y, Z]_{b(g)}$ satisfies $D_1 s = 1$, whence $\mathbb{C}[X, Y, Z]_{b(g)} = \mathbb{C}[f, g]_{b(g)}[s] = \mathbb{C}[f, g, p]_{b(g)}$.
(iii) It remains to show the last statement. Since $g - \lambda$ is irreducible in $\mathbb{C}[f, g]$, for all $\lambda \in \mathbb{C}$ and since $\mathbb{C}[f, g] = \mathbb{C}[X, Y, Z]^{D_1}$ is factorially closed in $\mathbb{C}[X, Y, Z]$, it follows that $g - \lambda$ is irreducible in $\mathbb{C}[X, Y, Z]$ for all $\lambda \in \mathbb{C}$. Now assume $b(\lambda) \neq 0$ i.e. $(g - \lambda)$ does not divide $b(g)$. We will show that $A := \mathbb{C}[X, Y, Z]/(g - \lambda) \cong_{\mathbb{C}} \mathbb{C}^{[2]}$. According to 4.2 it suffices to show that $\bar{D}_1$ and $\bar{D}_2$ are linearly independent derivations over A. Suppose that $a_1, a_2 \in \mathbb{C}[X, Y, Z]$ are such that $\bar{a}_1 \bar{D}_1 + \bar{a}_2 \bar{D}_2 = 0$. ("$\bar{\ }$" means mod $(g - \lambda)$.) Then $(a_1 D_1 + a_2 D_2)(\mathbb{C}[X, Y, Z]) \subset (g - \lambda)\mathbb{C}[X, Y, Z]$. In particular, $a_1(X, Y, Z)b(g) + 0 = a_1 D_1(p) + a_2 D_2(p) \in (g - \lambda)$. Since $g - \lambda$ is irreducible in $\mathbb{C}[X, Y, Z]$ and $g - \lambda \nmid b(g)$ it follows that $(g - \lambda)|a_1$ i.e.

$\bar{a}_1 = 0$. So $\bar{a}_2\bar{D}_2 = 0$ i.e. $a_2D_2(\mathbb{C}[X,T,Z]) \subset (g-\lambda)$. If $(g-\lambda)\not| a_2$, then $g-\lambda | D_2(X), D_2(Y), D_2(Z)$. In this case let $(g-\lambda)^e | D_2(X), D_2(Y), D_2(Z)$, $e \geq 1$ maximal. Then replace D_2 by $\tilde{D}_2 := (g-\lambda)^{-e}D_2$. It then follows that $\bar{D}_1$ and $\bar{\tilde{D}}_2$ are independent over A. Obviously $D_1, \tilde{D}_2$ have the same properties as the pair D_1, D_2 and $\mathbb{C}[X,Y,Z]^{D_1,D_2} = \mathbb{C}[X,Y,Z]^{D_1,\tilde{D}_2}$ which concludes the proof. □

Theorem 4.6. *CD(3) is true, i.e. let D_1, D_2 be two linearly independent (over $\mathbb{C}[X,Y,Z]$) commuting locally nilpotent $\mathbb{C}$-derivations, then $A^{D_1,D_2} = \mathbb{C}[g]$ and g is a coordinate in $\mathbb{C}[X,Y,Z]$.*

Proof. Combining 4.5 and 2.4 gives exactly this result. □

5 An extension of the concept of coordinate

This section deals with a lot of conjectures, and an attempt to generalise the concept of stable coordinate for elements in a quotient ring of a polynomial ring.

Definition 5.1. Let $I = (f_1, \ldots, f_m)$ be an ideal in $\mathbb{C}[X_1, \ldots, X_n] = \mathbb{C}^{[n]}$. Let $r \in \mathbb{C}^{[n]}$. Define $r + (I) \in \mathbb{C}^{[n]}/I$ is a *generalised coordinate in* $\mathbb{C}^{[n]}/I$ if $f_1Y_1 + \ldots + f_mY_m + r \in \mathbb{C}^{[n+m]}$ is a stable coordinate.

The definition does not depend on the generators of I as can be seen from

Lemma 5.2. *Let $I = (f_1, \ldots, f_m) = (g_1, \ldots, g_l)$ be an ideal in $\mathbb{C}[X_1, \ldots, X_n] = \mathbb{C}^{[n]}$. Let $r \in \mathbb{C}^{[n]}$. Then $f_1Y_1 + \ldots + f_mY_m + r \in \mathbb{C}^{[n+m]}$ can be mapped to $g_1Z_1 + \ldots + g_lZ_l + r$ by an automorphism of $\mathbb{C}[X,Y,Z] = \mathbb{C}^{[n+m+l]}$.*

Proof. Let $F := f_1Y_1 + \ldots + f_mY_m + r$ and $G := g_1Z_1 + \ldots + g_lZ_l + r$. We will show that there is an automorphism of $\mathbb{C}[X,Y,Z]$ sending F to G. Since $(g_1, \ldots, g_l) = (f_1, \ldots, f_m)$ in $\mathbb{C}[X]$ we have $g_i = a_{i1}f_1 + \ldots + a_{im}f_m$ for some $a_{ij} \in \mathbb{C}[X]$. Let $L_j := a_{1j}Z_1 + \ldots + a_{lj}Z_l$ for $1 \leq j \leq m$.Notice that

$$G = f_1L_1 + \ldots + f_mL_m + r.$$

Now let φ be the elementary automorphism sending Y_j to $Y_j + L_j$ for each j and leaving other variables invariant. Then

$$\begin{aligned} \varphi(F) &= f_1\varphi(Y_1) + \ldots + f_m\varphi(Y_m) + r \\ &= f_1(Y_1 + L_1) + \ldots + f_m(Y_m + L_m) + r \\ &= F + f_1L_1 + \ldots + f_mL_m \\ &= F + G - r \end{aligned}$$

In the same way we can make an automorphism τ sending G to $G + F - r$, so F can be mapped to G by $\tau^{-1}\varphi$. □

Conjecture 5.3. "Generalised coordinate" is an extension of the concept of "stable" coordinate. In other words, if I is an ideal in $\mathbb{C}^{[n+m]}$ and if $r \in \mathbb{C}^{[n+m]}/I$ is a generalised coordinate, and $\mathbb{C}^{[n+m]}/I \cong_{\mathbb{C}} \mathbb{C}^{[n]}$ then r is a stable coordinate.

A question related to this conjecture, induced by corollary 3.3, is the following:

Question: When is $p(X,Z)Y + q(X,Z,T)$ a coordinate?

The answer may have something to do with the zero set $p(X,Z) = 0$ but it's not quite clear exactly how. Anyway, examining polynomials of the form $P(X_1,\ldots,X_n)Y + Q(X_1,\ldots,X_n)$ might be a good idea in combination with the next question:

Question: Is there an algorithm which decides of (lots of) $F \in \mathbb{C}[X_1,\ldots,X_n]$ if there exists a ringautomorphism φ such that $\varphi(F)$ is linear in X_n ?

Another possible different approach of extending the concept of (stable) coordinate to a more general ring is looking for (stable) slices in such a ring:

Definition 5.4.

(i). Let R be a finitely generated $\mathbb{C}$-algebra. Say $s \in R$ is a *slice in R* if there exists a locally nilpotent $\mathbb{C}$-derivation on R such that $D(s) = 1$.

(ii). Let R be a finitely generated $\mathbb{C}$-algebra. Say $s \in R$ is a *stable slice in R* if there exists some $n \in \mathbb{N}$ and a locally nilpotent $\mathbb{C}$-derivation on $R[T_1,\ldots,T_n]$ such that $D(s) = 1$.

"Slice" and "stable slice" are extensions of the concept of coordinate, since every coordinate over a polynomial ring induces a locally nilpotent derivation having the coordinate as slice. Compare also lemma 2.2 part 4. So we can ask the same question for "stable slice" as we did for "generalised coordinate" (conjecture 5.3):

Conjecture 5.5. "Stable slice" is an extension of the concept of "stable coordinate". In other words: let $(f_1,\ldots,f_m) = I \subset \mathbb{C}^{[n]}$ be an ideal. Let $s \in \mathbb{C}^{[n]}$. Then s is a stable slice in $\mathbb{C}^{[n]}/I$ if and only if $s + f_1T_1 + \ldots + f_mT_m$ is a stable coordinate in $\mathbb{C}^{[n+m]}$.

Independently of the conjectures 5.3 and 5.5 one can make the following (two) conjecture(s):

Conjecture 5.6. Let $s \in \mathbb{C}[X_1,\ldots,X_n]$. Then

(i). s is a stable slice $\Longrightarrow$ s is a generalised coordinate.

(ii). s is a generalised coordinate $\Longrightarrow$ s is a stable slice.

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