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A VARIATION ON A THEME OF SCHUR

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A variation on a theme of Schur

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Let $f : z \to \sum_{k=0}^{\infty} a_k z^k \in H(\Delta)$. In 1917, Schur [2] gave the following necessary and sufficient condition for f to belong to

$$B = \{f \in H(\Delta) : |f(z)| \leq 1, \forall z \in \Delta\}.$$

For all $m \in \mathbb{N}$ and for all $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ we have

$$\sum_{k=0}^{m} \left| \sum_{l=k}^{m} a_{l-k} \lambda_l \right|^2 \leq \sum_{k=0}^{m} |\lambda_k|^2,$$

i.e. for all $m \in \mathbb{N}$, the linear map associated with the matrix

$$\begin{pmatrix} a_0 & & & & \\ a_1 & a_0 & & \emptyset & \\ a_2 & a_1 & a_0 & & \\ \vdots & & & & \\ a_m & a_{m-1} & a_{m-2} & \ldots & a_0 \end{pmatrix}$$

is norm decreasing on $\mathbb{C}^{m+1}$. [1] contains a simple and elementary proof of this criterion.

In this note we shall present a somewhat similar criterion
Theorem: A necessary and sufficient condition for $f \in H(\Delta)$ to belong to B is that for all $m \in \mathbb{N}$ and for all $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ we have

$$\left| \sum_{l=0}^{m-1} \sum_{k=l+1}^{m} a_{k-l-1} \lambda_l \overline{\lambda}_k \right| \leq \sum_{k=0}^{m} |\lambda_k|^2,$$

i.e. for all $m \in \mathbb{N}$ the linear map $A : \mathbb{C}^{m+1} \to C^{m+1}$, associated with the matrix

$$\begin{pmatrix} 0 & & & & \\ a_0 & 0 & & \emptyset & \\ a_1 & a_1 & 0 & & \\ \vdots & & & & \\ a_{m-1} & a_{m-2} & & \ldots & a_0 & 0 \end{pmatrix}$$

satifies the inequality

$$|\langle A\lambda, \lambda\rangle| \;\leq\; \|\lambda\|^2.$$

We base the proof of this theorem on the following well-known lemma. For the sake of completeness we include a proof of this lemma.

Lemma: Let $p : z \to 1 + 2\sum\limits_{k=1}^{\infty} c_k z^k$ belong to

$$P = \{f \in H(\Delta) : f(0) = 1;\ \text{Re } f(z) > 0, \forall z \in \Delta\}.$$

Let $c_0 = 1$, and let $c_{-k} = \overline{c}_k$ for $k \in \mathbb{N}$. Then we have for all $m \in \mathbb{N}$ and for all $\lambda_0, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$

$$\sum_{k=0}^{m}\sum_{l=0}^{m} c_{k-l}\lambda_l\overline{\lambda}_k \geq 0.$$

Proof: Since $p \in P$ we have for all $r \in (0,1)$ Re $p(re^{it}) > 0$, and hence

$$0 \leq \int_0^{2\pi}\left|\sum_{l=0}^{m}\lambda_l e^{ilt}\right|^2 \text{Re } p(re^{it})dt = \int_0^{2\pi}\left|\sum_{l=0}^{m}\lambda_l e^{ilt}\right|^2 \sum_{n=-\infty}^{\infty} c_n r^{|n|e^{int}}dt =$$
$$\sum_{k=0}^{m}\sum_{l=0}^{m}\sum_{n=-\infty}^{\infty} c_n\lambda_l\overline{\lambda}_k r^{|n|}\int_0^{2\pi} e^{i(l-k+n)t}dt = 2\pi\sum_{k=0}^{m}\sum_{l=0}^{m} c_{k-l}\lambda_l\overline{\lambda}_k r^{|k-l|}.$$

If we let $r \to 1$ we obtain the result.

Proof of the theorem: Note that $f \in B \iff \forall\vartheta \in [0, 2\pi]$ the function $p : z \to 1 + e^{i\vartheta}zf(z) \in P$.

$$p(z) = 1 + e^{i\vartheta}a_0 z + e^{i\vartheta}a_1 z^2 + e^{i\vartheta}a_2 z^3 + \cdots.$$

If we take $c_0 = 1$, $c_n = \frac{1}{2}e^{i\vartheta}a_{n-1}$,$c_{-n} = \frac{1}{2}e^{-i\vartheta}\overline{a}_{n-1}$ $(n \in \mathbb{N}^*)$ we have by the lemma for all $\lambda_0, \lambda_1 \ldots \lambda_m \in \mathbb{C}$

$$\sum_{k=0}^{m}\sum_{l=0}^{m} c_{k-l}\lambda_l\overline{\lambda}_k \geq 0.$$

We split the sum in three parts: $l = k$, $l < k$ and $l > k$.

$$\sum_{k=0}^{m}|\lambda_k|^2 + \frac{1}{2}e^{i\vartheta}\sum_{k=1}^{m}\sum_{l=0}^{k-1} a_{k-l-1}\lambda_l\overline{\lambda}_k + \frac{1}{2}e^{-i\vartheta}\sum_{k=0}^{m-1}\sum_{l=k+1}^{m}\overline{a}_{l-k-1}\lambda_l\overline{\lambda}_k \geq 0.$$

We interchange the order of summation in the second sum.

$$\sum_{k=0}^{m}|\lambda_k|^2 + \frac{1}{2}e^{i\vartheta}\sum_{l=0}^{m-1}\sum_{k=l+1}^{m} a_{k-l-1}\lambda_l\overline{\lambda}_k + \frac{1}{2}e^{-i\vartheta}\sum_{k=0}^{m-1}\sum_{l=k+1}^{m}\overline{a}_{l-k-1}\lambda_l\overline{\lambda}_k \geq 0.$$

The third sum is the complex conjugate of the second, so

$$\sum_{k=0}^{m}|\lambda_k|^2 + \text{ Re }\{e^{i\vartheta}\sum_{l=0}^{m-1}\sum_{k=l+1}^{m} a_{k-l-1}\lambda_l\overline{\lambda}_k\} \geq 0.$$

Since this is true for all $\vartheta \in [0, 2\pi]$ we also have

$$\left| \sum_{l=0}^{m-1} \sum_{k=l+1}^{m} a_{k-l-1} \lambda_l \overline{\lambda}_k \right| \leq \sum_{k=0}^{m} |\lambda_k|^2.$$

In order to prove the sufficiency we interchange the order of summation.

$$\left| \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_{k-l-1} \lambda_l \overline{\lambda}_k \right| \leq \sum_{k=0}^{m} |\lambda_k|^2.$$

(Note that the choice $\lambda_j = \lambda_n = 1$, and $\lambda_k = 0$ for all other k shows that $|a_{n-j-1}| \leq 2$; the sequence $a_0, a_1, a_2, \ldots$ is bounded).
Choose $\lambda_k = z^{m-k}$ (with $z \in \Delta$).

$$\left| \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_{k-l-1} z^{m-l} \overline{z}^{m-k} \right| \leq \sum_{k=0}^{m} |z|^{2(m-k)} = \sum_{k=0}^{m} |z|^{2k} \leq \frac{1}{1-|z|^2},$$

i.e.

$$\left| \sum_{k=1}^{m} \sum_{l=0}^{k-1} a_{k-l-1} z^{k-l} |z|^{2(m-k)} \right| \leq \frac{1}{1-|z|^2}.$$

Take as summation index n instead of k where $n = m - k$.

$$\left| \sum_{n=0}^{m-1} \sum_{l=0}^{m-n-1} a_{m-n-l-1} z^{m-n-l} |z|^{2n} \right| \leq \frac{1}{1-|z|^2}.$$

Take as summation index j instead of l where $j = m - n - l$

$$\left| \sum_{n=0}^{m-1} \sum_{j=1}^{m-n} a_{j-1} z^j |z|^{2n} \right| \leq \frac{1}{1-|z|^2},$$

i.e.

$$\left| \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} - \sum_{n=m}^{\infty} \sum_{j=1}^{\infty} - \sum_{n=0}^{m-1} \sum_{j=m}^{\infty} - \sum_{n=1}^{m-1} \sum_{j=m-n+1}^{m} \right| \leq \frac{1}{1-|z|^2}.$$

Since $|a_l| \leq 2$ we have

$$\left| \sum_{n=m}^{\infty} \sum_{j=1}^{\infty} a_{j-1} z^j |z|^{2n} \right| \leq 2 \sum_{n=m}^{\infty} |z|^{2n} \sum_{j=1}^{\infty} |z|^j = 2 \frac{|z|}{1-|z|} \cdot \frac{|z|^{2m}}{1-|z|^2}$$

$$\left| \sum_{n=0}^{m-1} \sum_{j=m}^{\infty} a_{j-1} z^j |z|^{2n} \right| \leq 2 \sum_{n=0}^{m-1} |z|^{2n} \sum_{j=m}^{\infty} |z|^j \leq \frac{2}{1-|z|^2} \cdot \frac{|z|^m}{1-|z|}$$

$$\left|\sum_{n=1}^{m-1}\sum_{j=m-n+1}^{m} a_{j-1}z^j|z|^{2n}\right| \leq 2\sum_{n=1}^{m-1}|z|^{2n}\sum_{j=m-n+1}^{m}|z|^j \leq 2\sum_{n=0}^{m-1}\frac{|z|^{m+n}}{1-|z|} \leq 2\frac{|z|^m}{(1-|z|)^2}.$$

If we let $m \to \infty$ we obtain

$$\left|\sum_{n=0}^{\infty}\sum_{j=1}^{\infty} a_{j-1}z^j|z|^{2n}\right| \leq \frac{1}{1-|z|^2},$$

i.e.

$$\frac{1}{1-|z|^2}|zf(z)| \leq \frac{1}{1-|z|^2},$$

$$|zf(z)| \leq 1,$$

so by Schwarz's lemma $f \in B$.

References

[1] R.A. Kortram. A simple proof for Schur's theorem. To appear in Proc. Amer. Math. Soc.

[2] I. Schur. Über Potenzreihen die im Innern der Einheitskreises beschränkt sind. J. Reine Angew. Math. 147 (1917), 205-232.
(Gesammelte Abhandlungen II 137-164. Springer-Verlag 1973).