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AN ALGORITHM CONCERNING ONE DIMENSIONAL RINGS OF CONSTANTS IN POLYNOMIAL RINGS

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An algorithm concerning one dimensional rings of constants in polynomial rings

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Abstract

Let B be a one dimensional k-subalgebra of the polynomial ring $k[X] := k[X_1, \dots, X_n]$, where k is a field of characteristic zero. We describe an algorithm which decides if there exists a k-derivation D on $k[X]$ such that $B = k[X]^D$ (=the kernel of the derivation D). In case B is a ring of constants the algorithm also gives such a derivation.

1 Introduction

Rings of constants appear in various problems. For example the Cancellation Problem asks if the ring of constants of a locally nilpotent derivation on a polynomial ring having a slice is a polynomial ring, Hilbert's fourteenth problem asks if the ring of constants of a derivation on a polynomial ring over a field k is a finitely generated k-algebra and the Jacobian Problem asks if the ring of constants associated to a Jacobian derivation of the form $\frac{\partial}{\partial F_n}$ is a polynomial ring generated by $F_1, \dots, F_{n-1}$, when $\det JF \in k^*$ (for more details we refer to [6]).

In [8] the second author gives a criterion to decide if a finitely generated k-subalgebra of an affine k-domain can be realized as the ring of constants of some k-derivation. In this paper we discuss an effective counterpart of this result. More precisely we consider one dimensional k-subalgebras of a polynomial ring in n variables over a field of characteristic zero and give an algorithm to decide if such rings appear as the ring of constants of a k-derivation and in case they do the algorithm gives an explicite derivation whose ring of constants is the given subalgebra. The algorithm is based on the aforementioned result of [8] and an algorithm given in [3] to compute the integral closure of an extension of affine k-domains.

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2 Preliminaries

Throughout this paper k denotes a field of characteristic zero and $k[X] := k[X_1, \dots, X_n]$ is the polynomial ring in n variables over k. Starting point of our algorithm is the following result of the second author ([8], Theorem 5.4)

Theorem 2.1 *Let A be a finitely generated k-domain and B a k-subalgebra of A. The following conditions are equivalent*
1) There exists a k-derivation D of A such that $B = A^D$;
2) The ring B is integrally closed in A and $Q(B) \cap A = B$ ($Q(B)$ denotes the quotient field of B).

So to get an effective algorithm to decide if $B = A^D$ for some k-derivation D of A, we must first of all be able to decide if B equals its integral closure in A, which we denote by $\overline{B}^A$. Therefore we make use of the following result of Brennan and Vasconcelos given in [3].

Theorem 2.2 *Let $A = k[X]/\wp$ be an affine domain over k and let B be a finitely generated k-subalgebra of A. Write x_i instead of $X_i + \wp$. In [3] an algorithm is given which produces elements $f_1, \dots, f_s$ in $k[x_1, \dots, x_n]$ such that $\overline{B}^A = k[f_1, \dots, f_s]$.*

According to Theorem 2.1 we must be able to compute $Q(B) \cap A$. In general this intersection need not be a finitely generated k-algebra: in case $A = k[X]$ this was exactly the question of Hilbert's fourteenth problem. Even if we assume that B is integrally closed in $k[X]$ the intersection need not be finitely generated over k: for example the locally nilpotent derivations D defined in [4] and [7] give rise to k-subalgebras of $k[X]$ of the form $B = k[X]^D$ which are integrally closed in $k[X]$ but for which $Q(B) \cap k[X]$ is not finitely generated over k. Therefore in this paper we will restrict to the situation that $A := k[X]$ and B is a finitely generated k-subalgebra of dimension one. This enables us to compute $Q(B) \cap A$. In fact we have

Proposition 2.3 *Let B be a finitely generated k-subalgebra of $A := k[X]$ of dimension one. If B is integrally closed in A, then*
1) $B = k[f]$ for some $f \notin k$.
2) $Q(B) \cap A = B$.
3) The algebraic closure of $Q(B)$ in $Q(A)$ equals $Q(B)$.

Proof. The first statement follows from Zaks' theorem (see [9] or [6], Theorem 1.2.26). So B is a polynomial ring in one variable over k, hence a principal ideal domain. Since obviously A is a torsion free B-module it follows from [2], Chap.I, §2, no.4, Prop. 3 iii) that A is a flat B-module. Furthermore, since B is a UFD and $A^* \cap B = k^* = B^*$ it follows from a result of Bass (see [1] or [6], Proposition D.1.7) that $Q(B) \cap A = B$, which proves 2). Finally to prove 3) let $x = a_1/a_2$ be algebraic over $Q(B)$, where

$a_1, a_2 \in A$ and $a_2 \neq 0$. Then there exists a non-zero element $b \in B$ such that bx is integral over B. In particular bx is integral over A and hence belongs to A (since $A = k[X]$ is integrally closed). Since, as observed, bx is integral over B and B is integrally closed in A it follows that $bx \in B$. So $x \in Q(B)$ □

Corollary 2.4 *Let B be a finitely generated k-subalgebra of dimension one of $A := k[X]$. Then B is the ring of constants of some k-derivation D of A if and only if $\overline{B}^A = B$.*

Proof. Follows directly from Theorem 2.1 and Proposition 2.3 □

The algorithm which we will give in the next section not only decides if B is a ring of constants of some k-derivation of A, it also gives an explicite derivation D on A such that $B = A^D$ (in case B is a ring of constants). In order to find such a D we need some preliminaries (which can already be found in [8]).

Lemma 2.5 *Let $D = \partial_1 + X_2\partial_2 + X_2X_3\partial_3 + \ldots + X_2 \ldots X_n\partial_n$ on $k(X)$. Then $k(X)^D = k$.*

A proof of this result, which is due to Derksen, can be found in [5].

We will apply this result as follows: let $K \subset L$ be fields of characteristic zero and let $s_1, \ldots, s_m$ be a transcendence basis of L over K. So $K(S) := K(s_1, \ldots, s_m) \subset L$ is algebraic and the K-derivation

$$D := \frac{\partial}{\partial s_1} + s_2\frac{\partial}{\partial s_2} + s_2s_3\frac{\partial}{\partial s_3} + \ldots + s_2 \ldots s_m\frac{\partial}{\partial s_m}$$

on $K(S)$ can be extended uniquely to a derivation on L, which we also denote by D.

Proposition 2.6 *If K is algebraically closed in L, then $L^D = K$.*

Proof. Let $h \in L$ satisfy $D(h) = 0$. Since h is algebraic over $K(S)$ there exists a minimal $n \geq 1$ such that

(*) $\quad h^n + a_{n-1}h^{n-1} + \ldots + a_1h + a_0 = 0,\ a_i \in K(S).$

Applying D and using that $D(h) = 0$ we get that

$$D(a_{n-1})h^{n-1} + \ldots + D(a_1)h + D(a_0) = 0.$$

From the minimality of n it follows that $D(a_i) = 0$ for all i i.e. $a_i \in K(S)^D = K$ (by Lemma 2.5). So (*) shows that h is algebraic over K. Since by hypothesis K is algebraically closed in L it follows that $h \in K$. So $L^D = K$ □

3 The Algorithm

Throughout this section B will be a finitely generated k-subalgebra of dimension one of $A = k[X]$ given by $B = k[f_1, \ldots, f_s]$, where $f_i \in k[X] \backslash k$ for all i.

Now we will describe an algorithm which decides if $B = A^D$ for some k-derivation D on A and if it is, gives such a derivation.

Algorithm

Step 1 Compute $\overline{B}^A$ according Theorem 2.2.

Step 2 Check if all generators of $\overline{B}^A$ belong to $k[f_1, \ldots, f_s]$: this can be done using the algebra membership algorithm (see for example [6], Proposition C.2.3).
If not, B is not a ring of constants (by Corollary 2.4).
If yes, B is a ring of constants (by Corollary 2.4).

Step 3 Since $f_1 \notin k$ $f_{1X_i} \neq 0$ for some i. Let's assume $i = 1$ (for simplicity) and write $f := f_1$. Put

$$D_0 := \partial_2 + X_3\partial_3 + X_3X_4\partial_4 + \ldots + X_3 \ldots X_n\partial_n \text{ on } k(X_1, \ldots, X_n)$$

and

$$D := -D_0(f)\partial_1 + f_{X_1} D_0 \in Der_k k[X].$$

Then $k[X]^D = B$.

Proof of correctness. Extend D to $k(X)$ and denote this extension again by D. Observe that $Df = 0$ and that $k(f) \subset Q(B)$ is algebraic (since $\dim B = 1$). So $D = 0$ on $Q(B)$. Furthermore $S := \{X_2, \ldots, X_n\}$ is a transcendence basis of $k(X)$ over $Q(B)$ (since $f_{X_1} \neq 0$). So on $Q(B)(S)$ the derivation $\frac{1}{f_{X_1}} D$ equals the $Q(B)$-derivation which sends X_2 to 1, X_3 to X_3, X_4 to X_3X_4,...,X_n to $X_3 \ldots X_n$. Since $Q(B)$ is algebraically closed in $k(X)$ (by Proposition 2.3) it follows from Proposition 2.6 that $k(X)^D = Q(B)$. Hence

$$k[X]^D = k(X)^D \cap k[X] = Q(B) \cap k[X] = B \text{ (by Proposition 2.3)} \ \square$$

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