AN ALGORITHM CONCERNING ONE DIMENSIONAL RINGS OF CONSTANTS IN POLYNOMIAL RINGS

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Abstract

Let $B$ be a one dimensional $k$-subalgebra of the polynomial ring $k[X] := k[X_1, \ldots, X_n]$, where $k$ is a field of characteristic zero. We describe an algorithm which decides if there exists a $k$-derivation $D$ on $k[X]$ such that $B = k[X]^D$ (= the kernel of the derivation $D$). In case $B$ is a ring of constants the algorithm also gives such a derivation.

1 Introduction

Rings of constants appear in various problems. For example the Cancellation Problem asks if the ring of constants of a locally nilpotent derivation on a polynomial ring having a slice is a polynomial ring, Hilbert’s fourteenth problem asks if the ring of constants of a derivation on a polynomial ring over a field $k$ is a finitely generated $k$-algebra and the Jacobian Problem asks if the ring of constants associated to a Jacobian derivation of the form $\frac{\partial}{\partial F_n}$ is a polynomial ring generated by $F_1, \ldots, F_{n-1}$, when $\det JF \in k^*$ (for more details we refer to [6]).

In [8] the second author gives a criterion to decide if a finitely generated $k$-subalgebra of an affine $k$-domain can be realized as the ring of constants of some $k$-derivation. In this paper we discuss an effective counterpart of this result. More precisely we consider one dimensional $k$-subalgebras of a polynomial ring in $n$ variables over a field of characteristic zero and give an algorithm to decide if such rings appear as the ring of constants of a $k$-derivation and in case they do the algorithm gives an explicit derivation whose ring of constants is the given subalgebra. The algorithm is based on the aforementioned result of [8] and an algorithm given in [3] to compute the integral closure of an extension of affine $k$-domains.

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2 Preliminaries

Throughout this paper $k$ denotes a field of characteristic zero and $k[X] := k[X_1, \ldots, X_n]$ is the polynomial ring in $n$ variables over $k$. Starting point of our algorithm is the following result of the second author ([8], Theorem 5.4)

**Theorem 2.1** Let $A$ be a finitely generated $k$-domain and $B$ a $k$-subalgebra of $A$. The following conditions are equivalent

1) There exists a $k$-derivation $D$ of $A$ such that $B = AD$;
2) The ring $B$ is integrally closed in $A$ and $Q(B) \cap A = B$ ($Q(B)$ denotes the quotient field of $B$).

So to get an effective algorithm to decide if $B = AD$ for some $k$-derivation $D$ of $A$, we must first of all be able to decide if $B$ equals its integral closure in $A$, which we denote by $\overline{B}^A$. Therefore we make use of the following result of Brennan and Vasconcelos given in [3].

**Theorem 2.2** Let $A = k[X]/\mathfrak{p}$ be an affine domain over $k$ and let $B$ be a finitely generated $k$-subalgebra of $A$. Write $x_i$ instead of $X_i + \mathfrak{p}$. In [3] an algorithm is given which produces elements $f_1, \ldots, f_s$ in $k[x_1, \ldots, x_n]$ such that $\overline{B}^A = k[f_1, \ldots, f_s]$.

According to Theorem 2.1 we must be able to compute $Q(B) \cap A$. In general this intersection need not be a finitely generated $k$-algebra: in case $A = k[X]$ this was exactly the question of Hilbert’s fourteenth problem. Even if we assume that $B$ is integrally closed in $k[X]$ the intersection need not be finitely generated over $k$: for example the locally nilpotent derivations $D$ defined in [4] and [7] give rise to $k$-subalgebras of $k[X]$ of the form $B = k[X]^D$ which are integrally closed in $k[X]$ but for which $Q(B) \cap k[X]$ is not finitely generated over $k$. Therefore in this paper we will restrict to the situation that $A := k[X]$ and $B$ is a finitely generated $k$-subalgebra of dimension one. This enables us to compute $Q(B) \cap A$. In fact we have

**Proposition 2.3** Let $B$ be a finitely generated $k$-subalgebra of $A := k[X]$ of dimension one. If $B$ is integrally closed in $A$, then

1) $B = k[f]$ for some $f \notin k$.
2) $Q(B) \cap A = B$.
3) The algebraic closure of $Q(B)$ in $Q(A)$ equals $Q(B)$.

**Proof.** The first statement follows from Zaks’ theorem (see [9] or [6], Theorem 1.2.26). So $B$ is a polynomial ring in one variable over $k$, hence a principal ideal domain. Since obviously $A$ is a torsion free $B$-module it follows from [2], Chap.I, §2, no.4, Prop. 3 iii) that $A$ is a flat $B$-module. Furthermore, since $B$ is a UFD and $A^\ast \cap B = k^\ast = B^\ast$ it follows from a result of Bass (see [1] or [6], Proposition D.1.7) that $Q(B) \cap A = B$, which proves 2). Finally to prove 3) let $x = a_1/a_2$ be algebraic over $Q(B)$, where
\(a_1, a_2 \in A\) and \(a_2 \neq 0\). Then there exists a non-zero element \(b \in B\) such that \(bx\) is integral over \(B\). In particular \(bx\) is integral over \(A\) and hence belongs to \(A\) (since \(A = k[X]\) is integrally closed). Since, as observed, \(bx\) is integral over \(B\) and \(B\) is integrally closed in \(A\) it follows that \(bx \in B\). So \(x \in Q(B)\). □

**Corollary 2.4** Let \(B\) be a finitely generated \(k\)-subalgebra of dimension one of \(A := k[X]\). Then \(B\) is the ring of constants of some \(k\)-derivation \(D\) of \(A\) if and only if \(B^A = B\).

**Proof.** Follows directly from Theorem 2.1 and Proposition 2.3. □

The algorithm which we will give in the next section not only decides if \(B\) is a ring of constants of some \(k\)-derivation of \(A\), it also gives an explicit derivation \(D\) on \(A\) such that \(B = A^D\) (in case \(B\) is a ring of constants). In order to find such a \(D\) we need some preliminaries (which can already be found in [8]).

**Lemma 2.5** Let \(D = \partial_1 + X_2 \partial_2 + X_3 \partial_3 + \ldots + X_n \partial_n\) on \(k(X)\). Then \(k(X)^D = k\).

A proof of this result, which is due to Derksen, can be found in [5].

We will apply this result as follows: let \(K \subseteq L\) be fields of characteristic zero and let \(s_1, \ldots, s_n\) be a transcendence basis of \(L\) over \(K\). So \(K(S) := K(s_1, \ldots, s_m) \subseteq L\) is algebraic and the \(K\)-derivation

\[
D := \frac{\partial}{\partial s_1} + s_2 \frac{\partial}{\partial s_2} + s_2 s_3 \frac{\partial}{\partial s_3} + \ldots + s_2 \ldots s_m \frac{\partial}{\partial s_m}
\]

on \(K(S)\) can be extended uniquely to a derivation on \(L\), which we also denote by \(D\).

**Proposition 2.6** If \(K\) is algebraically closed in \(L\), then \(L^D = K\).

**Proof.** Let \(h \in L\) satisfy \(D(h) = 0\). Since \(h\) is algebraic over \(K(S)\) there exists a minimal \(n \geq 1\) such that

\[
(*) \quad h^n + a_{n-1} h^{n-1} + \ldots + a_1 h + a_0 = 0, \quad a_i \in K(S).
\]

Applying \(D\) and using that \(D(h) = 0\) we get that

\[
D(a_{n-1}) h^{n-1} + \ldots + D(a_1) h + D(a_0) = 0.
\]

From the minimality of \(n\) it follows that \(D(a_i) = 0\) for all \(i\) i.e. \(a_i \in K(S)^D = K\) (by Lemma 2.5). So \((*)\) shows that \(h\) is algebraic over \(K\). Since by hypothesis \(K\) is algebraically closed in \(L\) it follows that \(h \in K\). So \(L^D = K\). □
3 The Algorithm

Throughout this section \( B \) will be a finitely generated \( k \)-subalgebra of dimension one of \( A = k[X] \) given by \( B = k[f_1, \ldots, f_s] \), where \( f_i \in k[X]\setminus k \) for all \( i \).

Now we will describe an algorithm which decides if \( B = AD \) for some \( k \)-derivation \( D \) on \( A \) and if it is, gives such a derivation.

**Algorithm**

**Step 1** Compute \( B^A \) according Theorem 2.2.

**Step 2** Check if all generators of \( B^A \) belong to \( k[f_1, \ldots, f_s] \); this can be done using the algebra membership algorithm (see for example [6], Proposition C.2.3).

If not, \( B \) is not a ring of constants (by Corollary 2.4).

If yes, \( B \) is a ring of constants (by Corollary 2.4).

**Step 3** Since \( f_1 \notin k f_1 X_i \neq 0 \) for some \( i \). Let's assume \( i = 1 \) (for simplicity) and write \( f := f_1 \). Put

\[
D_0 := \partial_2 + X_3 \partial_3 + X_3 X_4 \partial_4 + \ldots + X_3 \ldots X_n \partial_n \text{ on } k(X_1, \ldots, X_n)
\]

and

\[
D := -D_0(f) \partial_1 + f_{X_1} D_0 \in \text{Der}_k k[X].
\]

Then \( k[X]^D = B \).

**Proof of correctness.** Extend \( D \) to \( k(X) \) and denote this extension again by \( D \). Observe that \( Df = 0 \) and that \( k(f) \subset Q(B) \) is algebraic (since \( \dim B = 1 \)). So \( D = 0 \) on \( Q(B) \). Furthermore \( S := \{X_2, \ldots, X_n\} \) is a transcendence basis of \( k(X) \) over \( Q(B) \) (since \( f_{X_1} \neq 0 \)). So on \( Q(B)(S) \) the derivation \( \frac{1}{f_{X_1}} D \) equals the \( Q(B) \)-derivation which sends \( X_2 \) to 1, \( X_3 \) to \( X_3 \), \( X_4 \) to \( X_3 X_4 \), \ldots, \( X_n \) to \( X_3 \ldots X_n \). Since \( Q(B) \) is algebraically closed in \( k(X) \) (by Proposition 2.3) it follows from Proposition 2.6 that \( k(X)^D = Q(B) \). Hence

\[
k[X]^D = k(X)^D \cap k[X] = Q(B) \cap k[X] = B \text{ (by Proposition 2.3)} \square
\]

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**References**


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