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AN ALGORITHM CONCERNING ONE DIMENSIONAL RINGS OF CONSTANTS IN POLYNOMIAL RINGS

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An algorithm concerning one dimensional rings of constants in polynomial rings

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Abstract

Let $B$ be a one dimensional $k$-subalgebra of the polynomial ring $k[X] := k[X_1, \ldots, X_n]$, where $k$ is a field of characteristic zero. We describe an algorithm which decides if there exists a $k$-derivation $D$ on $k[X]$ such that $B = k[X]^D$ (= the kernel of the derivation $D$). In case $B$ is a ring of constants the algorithm also gives such a derivation.

1 Introduction

Rings of constants appear in various problems. For example the Cancellation Problem asks if the ring of constants of a locally nilpotent derivation on a polynomial ring having a slice is a polynomial ring, Hilbert's fourteenth problem asks if the ring of constants of a derivation on a polynomial ring over a field $k$ is a finitely generated $k$-algebra and the Jacobian Problem asks if the ring of constants associated to a Jacobian derivation of the form $\frac{\partial}{\partial x_i}$ is a polynomial ring generated by $F_1, \ldots, F_{n-1}$, when $\det JF \in k^*$ (for more details we refer to [6]).

In [8] the second author gives a criterion to decide if a finitely generated $k$-subalgebra of an affine $k$-domain can be realized as the ring of constants of some $k$-derivation. In this paper we discuss an effective counterpart of this result. More precisely we consider one dimensional $k$-subalgebras of a polynomial ring in $n$ variables over a field of characteristic zero and give an algorithm to decide if such rings appear as the ring of constants of a $k$-derivation and in case they do the algorithm gives an explicit derivation whose ring of constants is the given subalgebra. The algorithm is based on the aforementioned result of [8] and an algorithm given in [3] to compute the integral closure of an extension of affine $k$-domains.

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2 Preliminaries

Throughout this paper $k$ denotes a field of characteristic zero and $k[X] := k[X_1, \ldots, X_n]$ is the polynomial ring in $n$ variables over $k$. Starting point of our algorithm is the following result of the second author ([8], Theorem 5.4)

**Theorem 2.1** Let $A$ be a finitely generated $k$-domain and $B$ a $k$-subalgebra of $A$. The following conditions are equivalent
1) There exists a $k$-derivation $D$ of $A$ such that $B = AD$;
2) The ring $B$ is integrally closed in $A$ and $Q(B) \cap A = B$ ($Q(B)$ denotes the quotient field of $B$).

So to get an effective algorithm to decide if $B = AD$ for some $k$-derivation $D$ of $A$, we must first of all be able to decide if $B$ equals its integral closure in $A$, which we denote by $\overline{B}^A$. Therefore we make use of the following result of Brennan and Vasconcelos given in [3].

**Theorem 2.2** Let $A = k[X]/\mathfrak{p}$ be an affine domain over $k$ and let $B$ be a finitely generated $k$-subalgebra of $A$. Write $x_i$ instead of $X_i + \mathfrak{p}$. In [3] an algorithm is given which produces elements $f_1, \ldots, f_s$ in $k[x_1, \ldots, x_n]$ such that $B = k[f_1, \ldots, f_s]$.

According to Theorem 2.1 we must be able to compute $Q(B) \cap A$. In general this intersection need not be a finitely generated $k$-algebra: in case $A = k[X]$ this was exactly the question of Hilbert’s fourteenth problem. Even if we assume that $B$ is integrally closed in $k[X]$ the intersection need not be finitely generated over $k$:
for example the locally nilpotent derivations $D$ defined in [4] and [7] give rise to $k$-subalgebras of $k[X]$ of the form $B = k[X]^D$ which are integrally closed in $k[X]$ but for which $Q(B) \cap k[X]$ is not finitely generated over $k$. Therefore in this paper we will restrict to the situation that $A := k[X]$ and $B$ is a finitely generated $k$-subalgebra of dimension one. This enables us to compute $Q(B) \cap A$. In fact we have

**Proposition 2.3** Let $B$ be a finitely generated $k$-subalgebra of $A := k[X]$ of dimension one. If $B$ is integrally closed in $A$, then
1) $B = k[f]$ for some $f \notin k$.
2) $Q(B) \cap A = B$.
3) The algebraic closure of $Q(B)$ in $Q(A)$ equals $Q(B)$.

**Proof.** The first statement follows from Zaks’ theorem (see [9] or [6], Theorem 1.2.26). So $B$ is a polynomial ring in one variable over $k$, hence a principal ideal domain. Since obviously $A$ is a torsion free $B$-module it follows from [2], Chap.1, §2, no.4, Prop. 3 iii) that $A$ is a flat $B$-module. Furthermore, since $B$ is a UFD and $A^* \cap B = k^* = B^*$ it follows from a result of Bass (see [1] or [6], Proposition D.1.7) that $Q(B) \cap A = B$, which proves 2). Finally to prove 3) let $x = a_1/a_2$ be algebraic over $Q(B)$, where
Then there exists a non-zero element \( b \in B \) such that \( bx \) is integral over \( B \). In particular \( bx \) is integral over \( A \) and hence belongs to \( A \) (since \( A = k[X] \) is integrally closed). Since, as observed, \( bx \) is integral over \( B \) and \( B \) is integrally closed in \( A \) it follows that \( bx \in B \). So \( x \in Q(B) \). □

**Corollary 2.4** Let \( B \) be a finitely generated \( k \)-subalgebra of dimension one of \( A := k[X] \). Then \( B \) is the ring of constants of some \( k \)-derivation \( D \) of \( A \) if and only if \( \overline{B^A} = B \).

**Proof.** Follows directly from Theorem 2.1 and Proposition 2.3 □

The algorithm which we will give in the next section not only decides if \( B \) is a ring of constants of some \( k \)-derivation of \( A \), it also gives an explicit derivation \( D \) on \( A \) such that \( B = A^D \) (in case \( B \) is a ring of constants). In order to find such a \( D \) we need some preliminaries (which can already be found in [8]).

**Lemma 2.5** Let \( D = \partial_1 + X_2 \partial_2 + X_2X_3 \partial_3 + \ldots + X_2 \ldots X_n \partial_n \) on \( k(X) \). Then \( k(X)^D = k \).

A proof of this result, which is due to Derksen, can be found in [5].

We will apply this result as follows: let \( K \subset L \) be fields of characteristic zero and let \( s_1, \ldots, s_m \) be a transcendence basis of \( L \) over \( K \). So \( K(S) := K(s_1, \ldots, s_m) \subset L \) is algebraic and the \( K \)-derivation

\[
D := \frac{\partial}{\partial s_1} + s_2 \frac{\partial}{\partial s_2} + s_2s_3 \frac{\partial}{\partial s_3} + \ldots + s_2 \ldots s_m \frac{\partial}{\partial s_m}
\]

on \( K(S) \) can be extended uniquely to a derivation on \( L \), which we also denote by \( D \).

**Proposition 2.6** If \( K \) is algebraically closed in \( L \), then \( L^D = K \).

**Proof.** Let \( h \in L \) satisfy \( D(h) = 0 \). Since \( h \) is algebraic over \( K(S) \) there exists a minimal \( n \geq 1 \) such that

\[
(*) \quad h^n + a_{n-1}h^{n-1} + \ldots + a_1 h + a_0 = 0, \quad a_i \in K(S).
\]

Applying \( D \) and using that \( D(h) = 0 \) we get that

\[
D(a_{n-1})h^{n-1} + \ldots + D(a_1)h + D(a_0) = 0.
\]

From the minimality of \( n \) it follows that \( D(a_i) = 0 \) for all \( i \) i.e. \( a_i \in K(S)^D = K \) (by Lemma 2.5). So \((*)\) shows that \( h \) is algebraic over \( K \). Since by hypothesis \( K \) is algebraically closed in \( L \) it follows that \( h \in K \). So \( L^D = K \). □
3 The Algorithm

Throughout this section $B$ will be a finitely generated $k$-subalgebra of dimension one of $A = k[X]$ given by $B = k[f_1, \ldots, f_s]$, where $f_i \in k[X] \setminus k$ for all $i$.

Now we will describe an algorithm which decides if $B = AD$ for some $k$-derivation $D$ on $A$ and if it is, gives such a derivation.

Algorithm

Step 1 Compute $B^A$ according Theorem 2.2.

Step 2 Check if all generators of $B^A$ belong to $k[f_1, \ldots, f_s]$: this can be done using the algebra membership algorithm (see for example [6], Proposition C.2.3).

If not, $B$ is not a ring of constants (by Corollary 2.4).

If yes, $B$ is a ring of constants (by Corollary 2.4).

Step 3 Since $f_1 \notin k f_{1X_i} \neq 0$ for some $i$. Let’s assume $i = 1$ (for simplicity) and write $f := f_1$. Put

$$D_0 := \partial_2 + X_3 \partial_3 + X_3 X_4 \partial_4 + \ldots + X_3 \ldots X_n \partial_n$$
on $k(X_1, \ldots, X_n)$

and

$$D := -D_0(f) \partial_1 + fX_1 D_0 \in Der_k k[X].$$

Then $k[X]^D = B$.

Proof of correctness. Extend $D$ to $k(X)$ and denote this extension again by $D$. Observe that $DF = 0$ and that $k(f) \subset Q(B)$ is algebraic (since $\dim B = 1$). So $D = 0$ on $Q(B)$. Furthermore $S := \{X_2, \ldots, X_n\}$ is a transcendence basis of $k(X)$ over $Q(B)$ (since $f_{X_1} \neq 0$). So on $Q(B)(S)$ the derivation $\frac{1}{f}D$ equals the $Q(B)$-derivation which sends $X_2$ to 1, $X_3$ to $X_3$, $X_4$ to $X_3 X_4$, $\ldots$, $X_n$ to $X_3 \ldots X_n$. Since $Q(B)$ is algebraically closed in $k(X)$ (by Proposition 2.3) it follows from Proposition 2.6 that $k(X)^D = Q(B)$. Hence

$$k[X]^D = k(X)^D \cap k[X] = Q(B) \cap k[X] = B$$

(by Proposition 2.3) $\square$

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References


4

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