AN ALGORITHM CONCERNING ONE DIMENSIONAL RINGS OF CONSTANTS IN POLYNOMIAL RINGS

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An algorithm concerning one dimensional rings of constants in polynomial rings

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Abstract

Let $B$ be a one dimensional $k$-subalgebra of the polynomial ring $k[X] := k[X_1, \ldots, X_n]$, where $k$ is a field of characteristic zero. We describe an algorithm which decides if there exists a $k$-derivation $D$ on $k[X]$ such that $B = k[X]^D$ (=the kernel of the derivation $D$). In case $B$ is a ring of constants the algorithm also gives such a derivation.

1 Introduction

Rings of constants appear in various problems. For example the Cancellation Problem asks if the ring of constants of a locally nilpotent derivation on a polynomial ring having a slice is a polynomial ring, Hilbert’s fourteenth problem asks if the ring of constants of a derivation on a polynomial ring over a field $k$ is a finitely generated $k$-algebra and the Jacobian Problem asks if the ring of constants associated to a Jacobian derivation of the form $\frac{\partial}{\partial F_n}$ is a polynomial ring generated by $F_1, \ldots, F_{n-1}$, when $\det JF \in k^*$ (for more details we refer to [6]).

In [8] the second author gives a criterion to decide if a finitely generated $k$-subalgebra of an affine $k$-domain can be realized as the ring of constants of some $k$-derivation. In this paper we discuss an effective counterpart of this result. More precisely we consider one dimensional $k$-subalgebras of a polynomial ring in $n$ variables over a field of characteristic zero and give an algorithm to decide if such rings appear as the ring of constants of a $k$-derivation and in case they do the algorithm gives an explicit derivation whose ring of constants is the given subalgebra. The algorithm is based on the aforementioned result of [8] and an algorithm given in [3] to compute the integral closure of an extension of affine $k$-domains.

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2 Preliminaries

Throughout this paper \( k \) denotes a field of characteristic zero and \( k[X] := k[X_1, \ldots, X_n] \) is the polynomial ring in \( n \) variables over \( k \). Starting point of our algorithm is the following result of the second author ([8], Theorem 5.4).

**Theorem 2.1** Let \( A \) be a finitely generated \( k \)-domain and \( B \) a \( k \)-subalgebra of \( A \). The following conditions are equivalent

1) There exists a \( k \)-derivation \( D \) of \( A \) such that \( B = A^D \);
2) The ring \( B \) is integrally closed in \( A \) and \( Q(B) \cap A = B \) (\( Q(B) \) denotes the quotient field of \( B \)).

So to get an effective algorithm to decide if \( B = A^D \) for some \( k \)-derivation \( D \) of \( A \), we must first of all be able to decide if \( B \) equals its integral closure in \( A \), which we denote by \( B^A \). Therefore we make use of the following result of Brennan and Vasconcelos given in [3].

**Theorem 2.2** Let \( A = k[X]/\mathfrak{p} \) be an affine domain over \( k \) and let \( B \) be a finitely generated \( k \)-subalgebra of \( A \). Write \( x_i \) instead of \( X_i + \mathfrak{p} \). In [3] an algorithm is given which produces elements \( f_1, \ldots, f_s \) in \( k[x_1, \ldots, x_n] \) such that \( B^A = k[f_1, \ldots, f_s] \).

According to Theorem 2.1 we must be able to compute \( Q(B) \cap A \). In general this intersection need not be a finitely generated \( k \)-algebra: in case \( A = k[X] \) this was exactly the question of Hilbert’s fourteenth problem. Even if we assume that \( B \) is integrally closed in \( k[X] \) the intersection need not be finitely generated over \( k \): for example the locally nilpotent derivations \( D \) defined in [4] and [7] give rise to \( k \)-subalgebras of \( k[X] \) of the form \( B = k[X]^D \) which are integrally closed in \( k[X] \) but for which \( Q(B) \cap k[X] \) is not finitely generated over \( k \). Therefore in this paper we will restrict to the situation that \( A := k[X] \) and \( B \) is a finitely generated \( k \)-subalgebra of dimension one. This enables us to compute \( Q(B) \cap A \). In fact we have

**Proposition 2.3** Let \( B \) be a finitely generated \( k \)-subalgebra of \( A := k[X] \) of dimension one. If \( B \) is integrally closed in \( A \), then

1) \( B = k[f] \) for some \( f \notin k \).
2) \( Q(B) \cap A = B \).
3) The algebraic closure of \( Q(B) \) in \( Q(A) \) equals \( Q(B) \).

**Proof.** The first statement follows from Zaks’ theorem (see [9] or [6], Theorem 1.2.26). So \( B \) is a polynomial ring in one variable over \( k \), hence a principal ideal domain. Since obviously \( A \) is a torsion free \( B \)-module it follows from [2], Chap.I, §2, no.4, Prop. 3 iii) that \( A \) is a flat \( B \)-module. Furthermore, since \( B \) is a UFD and \( A^* \cap B = k^* = B^* \) it follows from a result of Bass (see [1] or [6], Proposition D.1.7) that \( Q(B) \cap A = B \), which proves 2). Finally to prove 3) let \( x = a_1/a_2 \) be algebraic over \( Q(B) \), where
$a_1, a_2 \in A$ and $a_2 \neq 0$. Then there exists a non-zero element $b \in B$ such that $bx$ is integral over $B$. In particular $bx$ is integral over $A$ and hence belongs to $A$ (since $A = k[X]$ is integrally closed). Since, as observed, $bx$ is integral over $B$ and $B$ is integrally closed in $A$ it follows that $bx \in B$. So $x \in Q(B)$.

**Corollary 2.4** Let $B$ be a finitely generated $k$-subalgebra of dimension one of $A := k[X]$. Then $B$ is the ring of constants of some $k$-derivation $D$ of $A$ if and only if $B^A = B$.

**Proof.** Follows directly from Theorem 2.1 and Proposition 2.3.

The algorithm which we will give in the next section not only decides if $B$ is a ring of constants of some $k$-derivation of $A$, it also gives an explicite derivation $D$ on $A$ such that $B = A^D$ (in case $B$ is a ring of constants). In order to find such a $D$ we need some preliminaries (which can already be found in [8]).

**Lemma 2.5** Let $D = \partial_1 + X_2 \partial_2 + X_2X_3\partial_3 + \ldots + X_2\ldots X_n\partial_n$ on $k(X)$. Then $k(X)^D = k$.

A proof of this result, which is due to Derksen, can be found in [5].

We will apply this result as follows: let $K \subset L$ be fields of characteristic zero and let $s_1, \ldots, s_m$ be a transcendence basis of $L$ over $K$. So $K(S) := K(s_1, \ldots, s_m) \subset L$ is algebraic and the $K$-derivation

$$D := \frac{\partial}{\partial s_1} + s_2 \frac{\partial}{\partial s_2} + s_2s_3 \frac{\partial}{\partial s_3} + \ldots + s_2\ldots s_m \frac{\partial}{\partial s_m}$$

on $K(S)$ can be extended uniquely to a derivation on $L$, which we also denote by $D$.

**Proposition 2.6** If $K$ is algebraically closed in $L$, then $L^D = K$.

**Proof.** Let $h \in L$ satisfy $D(h) = 0$. Since $h$ is algebraic over $K(S)$ there exists a minimal $n \geq 1$ such that

$$(*) \quad h^n + a_{n-1}h^{n-1} + \ldots + a_1h + a_0 = 0, \quad a_i \in K(S).$$

Applying $D$ and using that $D(h) = 0$ we get that

$$D(a_{n-1})h^{n-1} + \ldots + D(a_1)h + D(a_0) = 0.$$

From the minimality of $n$ it follows that $D(a_i) = 0$ for all $i$ i.e. $a_i \in K(S)^D = K$ (by Lemma 2.5). So $(*)$ shows that $h$ is algebraic over $K$. Since by hypothesis $K$ is algebraically closed in $L$ it follows that $h \in K$. So $L^D = K$. 

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3 The Algorithm

Throughout this section $B$ will be a finitely generated $k$-subalgebra of dimension one of $A = k[X]$ given by $B = k[f_1, \ldots, f_s]$, where $f_i \in k[X] \setminus k$ for all $i$.

Now we will describe an algorithm which decides if $B = AD$ for some $k$-derivation $D$ on $A$ and if it is, gives such a derivation.

Algorithm

Step 1 Compute $B^A$ according Theorem 2.2.

Step 2 Check if all generators of $B^A$ belong to $k[f_1, \ldots, f_s]$: this can be done using the algebra membership algorithm (see for example [6], Proposition C.2.3).

If not, $B$ is not a ring of constants (by Corollary 2.4).

If yes, $B$ is a ring of constants (by Corollary 2.4).

Step 3 Since $f_1 \notin k f_1 X_i \neq 0$ for some $i$. Let’s assume $i = 1$ (for simplicity) and write $f := f_1$. Put

$$D_0 := \partial_2 + X_3 \partial_3 + X_3 X_4 \partial_4 + \ldots + X_3 \ldots X_n \partial_n \text{ on } k(X_1, \ldots, X_n)$$

and

$$D := -D_0(f) \partial_1 + f X_1 D_0 \in Der_k k[X].$$

Then $k[X]^D = B$.

Proof of correctness. Extend $D$ to $k(X)$ and denote this extension again by $D$. Observe that $Df = 0$ and that $k(f) \subset Q(B)$ is algebraic (since $\text{dim}B = 1$). So $D = 0$ on $Q(B)$. Furthermore $S := \{X_2, \ldots, X_n\}$ is a transcendence basis of $k(X)$ over $Q(B)$ (since $f X_1 \neq 0$). So on $Q(B)(S)$ the derivation $\frac{1}{f X_1} D$ equals the $Q(B)$-derivation which sends $X_2$ to 1, $X_3$ to $X_3$, $X_4$ to $X_3 X_4$, $\ldots$, $X_n$ to $X_3 \ldots X_n$. Since $Q(B)$ is algebraically closed in $k(X)$ (by Proposition 2.3) it follows from Proposition 2.6 that $k(X)^D = Q(B)$. Hence

$$k[X]^D = k(X)^D \cap k[X] = Q(B) \cap k[X] = B \text{ (by Proposition 2.3)} \square$$

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