AN ALGORITHM CONCERNING ONE DIMENSIONAL RINGS OF CONSTANTS IN POLYNOMIAL RINGS

Arno van den Essen, Andrzej Nowicki

Report No. 0106 (March 2001)
An algorithm concerning one dimensional rings of constants in polynomial rings

Arno van den Essen        Andrzej Nowicki*

Abstract

Let $B$ be a one dimensional $k$-subalgebra of the polynomial ring $k[X] := k[X_1, \ldots, X_n]$, where $k$ is a field of characteristic zero. We describe an algorithm which decides if there exists a $k$-derivation $D$ on $k[X]$ such that $B = k[X]^D$ (= the kernel of the derivation $D$). In case $B$ is a ring of constants the algorithm also gives such a derivation.

1 Introduction

Rings of constants appear in various problems. For example the Cancellation Problem asks if the ring of constants of a locally nilpotent derivation on a polynomial ring having a slice is a polynomial ring, Hilbert’s fourteenth problem asks if the ring of constants of a derivation on a polynomial ring over a field $k$ is a finitely generated $k$-algebra and the Jacobian Problem asks if the ring of constants associated to a Jacobian derivation of the form $\frac{\partial}{\partial F_n}$ is a polynomial ring generated by $F_1, \ldots, F_{n-1}$, when $\det(JF) \in k^*$ (for more details we refer to [6]).

In [8] the second author gives a criterion to decide if a finitely generated $k$-subalgebra of an affine $k$-domain can be realized as the ring of constants of some $k$-derivation. In this paper we discuss an effective counterpart of this result. More precisely we consider one dimensional $k$-subalgebras of a polynomial ring in $n$ variables over a field of characteristic zero and give an algorithm to decide if such rings appear as the ring of constants of a $k$-derivation and in case they do the algorithm gives an explicit derivation whose ring of constants is the given subalgebra. The algorithm is based on the aforementioned result of [8] and an algorithm given in [3] to compute the integral closure of an extension of affine $k$-domains.

*Supported by KBN Grant 2 PO3A 017 16
2 Preliminaries

Throughout this paper $k$ denotes a field of characteristic zero and $k[X] := k[X_1, \ldots, X_n]$ is the polynomial ring in $n$ variables over $k$. Starting point of our algorithm is the following result of the second author ([8], Theorem 5.4).

**Theorem 2.1** Let $A$ be a finitely generated $k$-domain and $B$ a $k$-subalgebra of $A$. The following conditions are equivalent:
1) There exists a $k$-derivation $D$ of $A$ such that $B = AD$;
2) The ring $B$ is integrally closed in $A$ and $Q(B) \cap A = B$ ($Q(B)$ denotes the quotient field of $B$).

So to get an effective algorithm to decide if $B = AD$ for some $k$-derivation $D$ of $A$, we must first of all be able to decide if $B$ equals its integral closure in $A$, which we denote by $\overline{B}^A$. Therefore we make use of the following result of Brennan and Vasconcelos given in [3].

**Theorem 2.2** Let $A = k[X]/\wp$ be an affine domain over $k$ and let $B$ be a finitely generated $k$-subalgebra of $A$. Write $x_i$ instead of $X_i + \wp$. In [3] an algorithm is given which produces elements $f_1, \ldots, f_s$ in $k[x_1, \ldots, x_n]$ such that $\overline{B}^A = k[f_1, \ldots, f_s]$.

According to Theorem 2.1 we must be able to compute $Q(B) \cap A$. In general this intersection need not be a finitely generated $k$-algebra: in case $A = k[X]$ this was exactly the question of Hilbert’s fourteenth problem. Even if we assume that $B$ is integrally closed in $k[X]$ the intersection need not be finitely generated over $k$: for example the locally nilpotent derivations $D$ defined in [4] and [7] give rise to $k$-subalgebras of $k[X]$ of the form $B = k[X]^D$ which are integrally closed in $k[X]$ but for which $Q(B) \cap k[X]$ is not finitely generated over $k$. Therefore in this paper we will restrict to the situation that $A := k[X]$ and $B$ is a finitely generated $k$-subalgebra of dimension one. This enables us to compute $Q(B) \cap A$. In fact we have

**Proposition 2.3** Let $B$ be a finitely generated $k$-subalgebra of $A := k[X]$ of dimension one. If $B$ is integrally closed in $A$, then
1) $B = k[f]$ for some $f \notin k$.
2) $Q(B) \cap A = B$.
3) The algebraic closure of $Q(B)$ in $Q(A)$ equals $Q(B)$.

**Proof.** The first statement follows from Zaks’ theorem (see [9] or [6], Theorem 1.2.26). So $B$ is a polynomial ring in one variable over $k$, hence a principal ideal domain. Since obviously $A$ is a torsion free $B$-module it follows from [2], Chap.I, §2, no.4, Prop. 3 iii) that $A$ is a flat $B$-module. Furthermore, since $B$ is a UFD and $A^* \cap B = k^* = B^*$ it follows from a result of Bass (see [1] or [6], Proposition D.1.7) that $Q(B) \cap A = B$, which proves 2). Finally to prove 3) let $x = a_1/a_2$ be algebraic over $Q(B)$, where
Let $a_1, a_2 \in A$ and $a_2 \neq 0$. Then there exists a non-zero element $b \in B$ such that $bx$ is integral over $B$. In particular $bx$ is integral over $A$ and hence belongs to $A$ (since $A = k[X]$ is integrally closed). Since, as observed, $bx$ is integral over $B$ and $B$ is integrally closed in $A$ it follows that $bx \in B$. So $x \in Q(B)$ □

**Corollary 2.4** Let $B$ be a finitely generated $k$-subalgebra of dimension one of $A := k[X]$. Then $B$ is the ring of constants of some $k$-derivation $D$ of $A$ if and only if $\overline{B^A} = B$.

**Proof.** Follows directly from Theorem 2.1 and Proposition 2.3 □

The algorithm which we will give in the next section not only decides if $B$ is a ring of constants of some $k$-derivation of $A$, it also gives an explicite derivation $D$ on $A$ such that $B = A^D$ (in case $B$ is a ring of constants). In order to find such a $D$ we need some preliminaries (which can already be found in [8]).

**Lemma 2.5** Let $D = \partial_1 + X_2\partial_2 + X_2X_3\partial_3 + \ldots + X_2\ldots X_n\partial_n$ on $k(X)$. Then $k(X)^D = k$.

A proof of this result, which is due to Derksen, can be found in [5].

We will apply this result as follows: let $K \subseteq L$ be fields of characteristic zero and let $s_1, \ldots, s_m$ be a transcendence basis of $L$ over $K$. So $K(S) := K(s_1, \ldots, s_m) \subseteq L$ is algebraic and the $K$-derivation

$$D := \frac{\partial}{\partial s_1} + s_2 \frac{\partial}{\partial s_2} + s_2s_3 \frac{\partial}{\partial s_3} + \ldots + s_2\ldots s_m \frac{\partial}{\partial s_m}$$

on $K(S)$ can be extended uniquely to a derivation on $L$, which we also denote by $D$.

**Proposition 2.6** If $K$ is algebraically closed in $L$, then $L^D = K$.

**Proof.** Let $h \in L$ satisfy $D(h) = 0$. Since $h$ is algebraic over $K(S)$ there exists a minimal $n \geq 1$ such that

$$(*) \quad h^n + a_{n-1}h^{n-1} + \ldots + a_1h + a_0 = 0, \quad a_i \in K(S).$$

Applying $D$ and using that $D(h) = 0$ we get that

$$D(a_{n-1})h^{n-1} + \ldots + D(a_1)h + D(a_0) = 0.$$ 

From the minimality of $n$ it follows that $D(a_i) = 0$ for all $i$ i.e. $a_i \in K(S)^D = K$ (by Lemma 2.5). So $(*)$ shows that $h$ is algebraic over $K$. Since by hypothesis $K$ is algebraically closed in $L$ it follows that $h \in K$. So $L^D = K$ □
3 The Algorithm

Throughout this section B will be a finitely generated k-subalgebra of dimension one of A = k[X] given by B = k[f_1, \ldots, f_s], where f_i \in k[X] \setminus k for all i.

Now we will describe an algorithm which decides if B = A^D for some k-derivation D on A and if it is, gives such a derivation.

Algorithm

Step 1 Compute \( B^A \) according Theorem 2.2.

Step 2 Check if all generators of \( B^A \) belong to \( k[f_1, \ldots, f_s] \): this can be done using the algebra membership algorithm (see for example [6], Proposition C.2.3).

If not, B is not a ring of constants (by Corollary 2.4).
If yes, B is a ring of constants (by Corollary 2.4).

Step 3 Since \( f_1 \notin k \) \( f_1 X_i \neq 0 \) for some i. Let’s assume \( i = 1 \) (for simplicity) and write \( f := f_1 \).

Put

\[
D_0 := \partial_2 + X_3 \partial_3 + X_3 X_4 \partial_4 + \ldots + X_3 \ldots X_n \partial_n \text{ on } k(X_1, \ldots, X_n)
\]

and

\[
D := -D_0(f) \partial_1 + f \partial_1 D_0 \in \text{Der}_k k[X].
\]

Then \( k[X]^D = B \).

Proof of correctness. Extend \( D \) to \( k(X) \) and denote this extension again by \( D \).

Observe that \( Df = 0 \) and that \( k(f) \subset Q(B) \) is algebraic (since \( \dim B = 1 \)). So \( D = 0 \) on \( Q(B) \). Furthermore \( S := \{ X_2, \ldots, X_n \} \) is a transcendence basis of \( k(X) \) over \( Q(B) \) (since \( f X_1 \neq 0 \)). So on \( Q(B)(S) \) the derivation \( \frac{1}{f X_1} D \) equals the \( Q(B) \)-derivation which sends \( X_2 \) to 1, \( X_3 \) to \( X_3 \), \( X_4 \) to \( X_3 X_4 \), \ldots, \( X_n \) to \( X_3 \ldots X_n \). Since \( Q(B) \) is algebraically closed in \( k(X) \) (by Proposition 2.3) it follows from Proposition 2.6 that \( k(X)^D = Q(B) \). Hence

\[
k[X]^D = k(X)^D \cap k[X] = Q(B) \cap k[X] = B \text{ (by Proposition 2.3)} \square
\]

Acknowledgement

The second author wants to thank the University of Toruń for its great hospitality during his stay in February 2001, when this work was initiated.

References


Authors addresses:
Arno van den Essen, Dep. of Math., Univ. of Nijmegen, The Netherlands. Email: essen@sci.kun.nl
Andrzej Nowicki, Fac. of Math. and Informatics, N. Copernicus Univ., 87-100, Toruń, Poland. Email: anow@mat.uni.torun.pl