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# Generalizations of a lemma of Freudenburg

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**Abstract**

Let $k$ be an algebraically closed field of characteristic zero and $\wp$ a prime ideal in $k[X] := k[X_1, \ldots, X_n]$. Let $g \in k[X]$ and $d \geq 1$. If for all $1 \leq |\alpha| \leq d$ the derivatives $\partial^\alpha g$ belong to $\wp$, then there exists $c \in k$ such that $g - c \in \wp^{(d+1)}$, the $d+1$-th symbolic power of $\wp$. In particular if $\wp$ is a complete intersection it follows that $g - c \in \wp^{d+1}$.

## 1 Introduction

In [4] it was shown by Freudenburg that if $f$ is an irreducible polynomial in $\mathbb{C}[x, y]$ and $g$ any polynomial in $\mathbb{C}[x, y]$ such that both partial derivatives of $g$ are divisible by $f$, then $g - c$ is divisible by $f$ for some $c \in \mathbb{C}$.
In this paper we will consider various generalizations of this result to $n$ variables. More precisely, in section one we show that if $\wp$ is any prime ideal in $k[X]$ then the following holds: if $g$ is an element of $k[X]$ such that all its partial derivatives $\partial_i g$ belong to $\wp$, then for some $c \in k$ also $g - c$ belongs to $\wp$. In section two we show that this result can be improved, namely we obtain that $g - c$ belongs to $\wp^{(2)}$, the second symbolic power of $\wp$. In fact, using a result of Zariski-Nagata we extend this result to higher order partial derivatives and higher order symbolic powers. For the precise formulation we refer to Theorem 3.1.

## 2 A generalization of Freudenburg's lemma to prime ideals

Throughout this paper $k$ will denote an algebraically closed field of characteristic zero and $k[X]$ (resp. $k[[X]]$) denotes the polynomial ring (resp. the power series ring) in $n$ variables over $k$. All rings are commutative and contain 1. By dim $(A)$ we denote the Krull dimension of $A$. The main result of this section is

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**Proposition 2.1** *Let $\wp$ be a prime ideal in $k[X]$ and $g \in k[X]$. If for each $i$ the partial derivative $\partial_i g$ belongs to $\wp$, then there exists $c \in k$ such that $g - c \in \wp$.*

To prove this result we need the following lemma

**Lemma 2.2** *Let $A$ be a finitely generated $k$-domain. Then there exists an injective $k$-algebra homomorphism $\varphi : A \to k[[T_1, \ldots, T_s]]$, where $s = dim(A)$.*

**Proof**. The $k$-algebra $A$ is of the form $k[X_1, \ldots, X_n]/\wp$ for some prime ideal $\wp$ of $k[X_1, \ldots, X_n]$. Let $x$ be a non-singular point of the variety defined by $\wp$ in $k^n$ and let $\mathfrak{m}$ be its corresponding maximal ideal in $A$. Then $A_{\mathfrak{m}}$ is a regular local ring of dimension $s$. Since $A$ contains $\mathbb{Q}$ the complete local ring $\hat{A_{\mathfrak{m}}}$ has equicharacteristic zero, so by Cohen's structure theorem (see [1] or [10], Chap. VIII, §12) we get that $\hat{A_{\mathfrak{m}}} = k[[T_1, \ldots, T_s]]$, a power series ring in $s$ variables over $k$. (Observe that by the Nullstellensatz $k$ is isomorphic to $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$, so indeed $k$ is a field of representatives of $\hat{A_{\mathfrak{m}}}$). Then the result follows from the inclusions $A \subset A_{\mathfrak{m}} \subset \hat{A_{\mathfrak{m}}}$.

**Proof of Proposition 2.1**

Let $A := k[X]/\wp$ and $\varphi : A \to k[[T_1, \ldots, T_s]]$ be the injection of lemma 2.2. Put $f_i := \varphi(\bar{X}_i)$ for each $i$ ($\bar{X}_i := X_i + \wp$). So $\varphi(h + \wp) = h(f_1, \ldots, f_n)$ for all $h \in k[X]$. In particular since $\varphi$ is well-defined it follows that $h(f_1, \ldots, f_n) = 0$ for all $h \in \wp$. Hence $\partial_i g(f_1, \ldots, f_n) = 0$ for all $i$. Now consider $Q(T) := g(f_1(T), \ldots, f_n(T)) \in k[[T_1, \ldots, T_s]]$. Then, using that $\partial_i g(f_1, \ldots, f_n) = 0$ it follows that $\frac{\partial}{\partial T_j} Q(T) = 0$ for all $1 \leq j \leq s$. Consequently $Q(T) = c \in k$. So $(g - c)(f_1, \ldots, f_n) = 0$ i.e. $g - c \in \wp$, as desired.

**Remark 2.3** If in Proposition 2.1 we take $\wp$ to be a principal ideal generated by an irreducible polynomial $f$ of $k[X]$ then we obtain that if $g \in k[X]$ is such that $f$ divides all $\partial_i(g)$, then $f$ divides $g - c$ for some $c \in k$. In case $n = 2$ this is Freudenburg's original lemma.

**Remark 2.4** i) The condition "$k$ is algebraically closed" in Proposition 2.1 cannot be dropped: namely take $f = x^2 + 1$ and $g = x^3 + 3x$ in $\mathbb{R}[x]$. Then $f$ is irreducible in $\mathbb{R}[x]$ and divides $g'$. However if $f$ divides $g - c$ for some $c \in \mathbb{R}$, then $x^3 + 3x - c = (x^2 + 1)(x + b)$ for some $b \in \mathbb{R}$. Looking at the coefficient of $x^2$ we see that $b = 0$ and hence, looking at the coefficient of $x$, we get $3 = 1$, a contradiction.

ii) Viewing both polynomials $f$ and $g$ in $\mathbb{C}[x]$, the same argument shows that also the assumption that $f$ is irreducible cannot be dropped i.e. in Proposition 2.1 one cannot replace $\wp$ by a non-prime ideal.

## 3 A further generalization

In the previous section we showed that if for a polynomial $g$ all its derivatives $\partial_i(g)$ belong to a prime ideal $\wp$, then for some $c \in k$ also $g - c \in \wp$. Looking at special cases, for example when $\wp$ is a maximal ideal, one observes that in fact $g - c \in \wp^2$. So one wonders if such a result holds for all prime ideals.
In this section we show that the answer in general is no (see Example 3.3 below) and that the statement is true if we replace $\wp^2$ by $\wp^{(2)}$, its second symbolic power ($\wp^{(n)} := \wp^n R_\wp \cap R$, for all $n \geq 1$). In fact, also allowing higher order derivatives we get the following more general result

**Theorem 3.1** *Let $\wp$ be a prime ideal in $R := k[X_1, \ldots, X_n]$ and $g \in R$. Let $d \geq 1$. If $\partial^\alpha(g) := \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} g \in \wp$ for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $1 \leq |\alpha| \leq d$, then there exists $c \in k$ with $g - c \in \wp^{(d+1)}$.*

**Proof**. Let $\wp^{<d+1>} := \{h \in \wp \,|\, \partial^\alpha h \in \wp \text{ for all } 1 \leq |\alpha| \leq d\}$. Then $\wp^{<d+1>}$ is an ideal in $R$ and in fact $\wp^{<d+1>} = \wp^{(d+1)}$. (See [9], or [3], Theorem 3.14). Now let $c \in k$ be as in Proposition 2.1 and put $h := g - c$. Then $h \in \wp^{<d+1>} = \wp^{(d+1)}$ i.e. $g - c \in \wp^{(d+1)}$ □

**Corollary 3.2** *Notations as in 3.1. If $\wp$ is a complete intersection (i.e. generated by an $R$-sequence), then $g - c \in \wp^{d+1}$. In particular this is the case if $ht\wp = 1$ or $\wp$ is maximal.*

**Proof**. By [5], Proposition 2.1 the hypothesis on $\wp$ implies that $\wp^{(d+1)} = \wp^{d+1}$ □

The question when $\wp^{(d+1)} = \wp^{d+1}$ (and hence the question if $\wp^{<d+1>} = \wp^{d+1}$) is well-studied (see for example [5]). One easily verifies that $\wp^{(d+1)} = \wp^{d+1}$ if and only if $\wp^{d+1}$ is $\wp$-primary (since $\wp^{(d+1)}$ is the primary component of $\wp^{(d+1)}$). So to get a prime ideal $\wp$ such that $\wp^{<2>}$ is not equal to $\wp^2$ we need to have a prime ideal $\wp$ such that $\wp^2$ is not $\wp$-primary. Such an example can be found in [6], page 29, Example 3. Using this prime ideal we give an element $g$ in $\wp$ such that all its derivatives belong to $\wp$ but $g$ does not belong to $\wp^2$. More precisely

**Example 3.3** Let $R := k[x, y, z]$ and $\wp$ the prime ideal of the curve $(t^3, t^4, t^5)$ i.e. the set of all $f \in R$ such that $f(t^3, t^4, t^5) = 0$. Then one can verify (or see [6], page 29, Example 3) that $\wp$ is generated by the polynomials $y^2 - xz$, $yz - x^3$ and $z^2 - x^2y$. Now let $g := x^5 + xy^3 - 3x^2yz + z^3$. Then it is easy to see that $g$ and all its partial derivatives (of order 1) belong to $\wp$ (just substitute $x = t^3$, $y = t^4$, $z = t^5$ and check that the result is zero) i.e. $g \in \wp^{<2>}$. However $g \notin \wp^2$: namely all monomials appearing in the generators of $\wp$ have degree $\geq 2$, hence all monomials appearing in the generators of $\wp^2$ have degree $\geq 4$. But $g$ contains a monomial of degree 3, namely $z^3$.

More generally, given $d \geq 1$ and a prime ideal $\wp \subset R := k[X_1, \ldots, X_n]$ we can decide if $\wp^{d+1}$ is equal to $\wp^{<d+1>}$ and if not construct elements $g \in \wp^{<d+1>} \backslash \wp^{d+1}$. To explain this we need some preparations.

Let $I$ be an ideal in $R$ and $d \geq 0$. Define

$$\int I := \{h \in I \,|\, \partial_i h \in I \text{ for all } 1 \leq i \leq n\}$$

and

$$I^{<d+1>} := \{h \in R \,|\, \partial^\alpha h \in I \text{ for all } 0 \leq |\alpha| \leq d\}.$$

Observe that $I^{<1>} = I$ and $I^{<d+1>} = \int I^{<d>}$ for all $d \geq 1$. So in order to compute $I^{<d+1>}$ inductively we only need to give

**An algorithm for computing $\int I$**

Let $f_1, \ldots, f_s$ be generators of $I$. For each $1 \leq i \leq n$ put

$$P_i := \{\sum a_j f_j \mid \sum a_j \partial_i(f_j) \in \wp\}.$$

Then $\int I = P_1 \cap \ldots \cap P_n$. Hence, using standard Gröbner basis techniques it suffices to compute $P_i$. So consider $P_i$ and observe that $\sum a_j \partial_i(f_j) \in \wp$ if and only if there exist $b_1, \ldots, b_s \in R$ such that

$$a_1 \partial_i(f_1) + \ldots + a_s \partial_i(f_s) + b_1 f_1 + \ldots + b_s f_s = 0.$$

By Gröbner basis methods one can compute generators for the module of syzygies between $(\partial_i(f_1), \ldots, \partial_i(f_s), f_1, \ldots, f_s)$. Let

$$(a_1^{(1)}, \ldots, a_s^{(1)}, b_1^{(1)}, \ldots, b_s^{(1)}), \ldots, (a_1^{(N)}, \ldots, a_s^{(N)}, b_1^{(N)}, \ldots, b_s^{(N)})$$

be such generators. Then the elements $a_1^{(1)} f_1 + \ldots + a_s^{(1)} f_s, \ldots, a_1^{(N)} f_1 + \ldots + a_s^{(N)} f_s$ generate the ideal $P_i$.

**Corollary 3.4** *Let $d \geq 1$ and $\wp$ a prime ideal in $R$. Then one can decide if $\wp^{<d+1>} = \wp^{d+1}$ and if there is no equality one can give $g \in \wp^{<d+1>} \backslash \wp^{d+1}$.*

Namely by the algorithm above we can compute inductively $\wp^{<d+1>}$. Then check for each of the generators of $\wp^{<d+1>}$ if they belong to $\wp^{d+1}$ (using the ideal membership algorithm from Gröbner basis theory).

**Remark 3.5** Using the algorithm above we computed $\wp^{<2>}$ for the prime ideal $\wp$ given in Example 3.3. The computation done by the computeralgebra system MAGMA showed that all generators of $\wp^{<2>}$ except one belonged to $\wp^2$. The only exception was the element $g$ described in Example 3.3. It is interesting to remark that $g$ is exactly the same element that appeared in Example 3, page 30 of Northcott: this is not surprising since, as remarked above $g$ is the only obstruction to $\wp^2$ being equal to $\wp^{<2>}$ and hence to $\wp^2$ being equal to $\wp^{(2)}$ or equivalently to $\wp^2$ being $\wp$-primary, the question considered by Northcott !

**Remark 3.6** In [8] and [9] other algorithms are given to compute $\int I$ and symbolic powers.

**Remark 3.7** The notation $\int I$ is taken from the thesis [7] of Pellikaan.

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