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THE CANCELLATION PROBLEM
IN DIMENSION FOUR

Harm Derksen, Arno van den Essen, Peter van Rossum

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The Cancellation Problem in Dimension Four

Harm Derksen∗ Arno van den Essen Peter van Rossum

Abstract

This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let \( k \) be a field of characteristic zero and let \( V \) be an algebraic variety over \( k \). The Cancellation Problem asks if \( V \times k \cong k^n \) implies that \( V \cong k^{n-1} \). This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if \( A[T] \cong k[X_1, \ldots, X_n] \) implies that \( A \cong k[X_1, \ldots, X_{n-1}] \) for an affine \( k \)-domain \( A \). One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on \( k[X_1, \ldots, X_n] \) with a slice is isomorphic to \( k[X_1, \ldots, X_{n-1}] \). For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert’s Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial local coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing \( \mathbb{Q} \), this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing \( \mathbb{Q} \) for \( n = 3 \). As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for \( n = 4 \) over a field, including the triangular derivations.

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2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let $A$ be a ring. A derivation on $A$ is a map $D: A \to A$ satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If $R$ is a ring and $A$ is an $R$-algebra via $f: R \to A$, then $A$ is called an $R$-derivation if $D(f(r)) = 0$ for all $r \in R$. A derivation $D$ is called locally nilpotent if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of $D$ is denoted by $A^D$. If $s \in A$ is such that $D(s) = 1$, then $s$ is called a slice of $D$.

The following proposition (see [Wri81]) is well-known.

**Proposition 2.1.** Let $A$ be a $\mathbb{Q}$-algebra and let $D$ be a locally nilpotent derivation on $A$. Assume that $s \in A$ is a slice of $D$. Then $A = A^D[s]$ and $s$ is algebraically independent over $A^D$. Furthermore, $D = d/ds$.

In the applications, $A$ will invariably be a polynomial ring $R[X] := R[X_1, \ldots, X_n]$ over a ring $R$. An $R$-derivation $D$ on such a ring is called triangular if $D(X_i) \in R[X_{i+1}, \ldots, X_n]$ for all $i$. Such a derivation is automatically locally nilpotent. An element $s \in R[X]$ is called a coordinate if there is a polynomial automorphism $F$ of $R[X]$ with $s$ as one of its components. More generally, a sequence $(s_1, \ldots, s_k)$ of elements of $R[X]$ with $1 \leq k \leq n$ is called a partial coordinate system if there are polynomials $f_{k+1}, \ldots, f_n \in R[X]$ such that $(s_1, \ldots, s_k, f_{k+1}, \ldots, f_n)$ is a polynomial automorphism of $R[X]$.

Proposition 2.1 implies the following.

**Corollary 2.2.** Let $R$ be a ring and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and let $s \in R[X]$ be a slice of $D$. Then $s$ is a coordinate if and only if $R[X]^D \cong R^{[n-1]}$.

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

**Problem 2.3 (Cancellation Problem).** Let $k$ be a field of characteristic zero and let $n \geq 2$. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in k[X]$. Is then $k[X]^D \cong k^{[n-1]}$, i.e., is $s$ a coordinate in $k[X]$?

More generally, one can ask the following question.

**Problem 2.4 (Generalized Cancellation Problem).** Let $k$ be a field of characteristic zero, let $R$ be an affine $k$-domain, and let $n \geq 2$. Let $D$ be a locally nilpotent $R$-derivation on $R[X] := R[X_1, \ldots, X_n]$ and assume that $D$ has a slice $s \in R[X]$. Is then $R[X]^D \cong R^{[n-1]}$, i.e., is $s$ a coordinate in $R[X]$?

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

**Theorem 2.5.** Let $k$ be a field of characteristic zero. Let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2]$. Then $k[X]^D \cong k^{[1]}$. 

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Nowadays even stronger results have been obtained by Bhadwadekar and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field \( k \) in the theorem can in fact be replaced by an arbitrary \( \mathbb{Q} \)-algebra \( R \).

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

**Theorem 2.6.** Let \( k \) be a field of characteristic zero and let \( D \) be a locally nilpotent \( k \)-derivation on \( k[X] := k[X_1, X_2, X_3] \). Assume that \( D \) has a slice. Then \( k[X] D \cong k^{[2]} \).

This paper now proves the Generalized Cancellation Problem for \( n = 3 \) in case \( R \) is a Dedekind domain over \( \mathbb{Q} \). As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

\[
D := a(X_1, X_2, X_3, X_4) \partial_1 + b(X_1, X_2, X_3, X_4) \partial_2 + \\
+ c(X_1, X_2, X_3, X_4) \partial_3 + d(X_4) \partial_4
\]

for \( n = 4 \), where \( \partial_i \) denotes \( \partial / \partial X_i \). In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for \( n = 4 \). This is especially interesting since [Asa99] (implicitly) and [ER00] (explicitly) give a candidate counterexample to the Cancellation Problem for \( n = 5 \) which is triangular, namely

\[
D := (2X_1^2 - 3) \partial_1 + (4X_2^2 - 8X_4) \partial_2 + (5X_4^2 - 10) \partial_3 + X_5 \partial_4.
\]

### 3 Local Coordinates

Let \( R \) be a domain, \( n \in \mathbb{N} \), and \( R[X] := R[X_1, \ldots, X_n] \) the polynomial ring in \( n \) variables over \( R \). This section shows that a polynomial in \( R[X] \) is a coordinate if and only if it is a coordinate when considered as an element of \( R_m[X] \), for all maximal ideals \( m \) of \( R \), provided that \( R \) is Hermite, and similarly for partial coordinate systems.

Recall that \( R \) is called Hermite if every unimodular row \( (r_1, \ldots, r_k) \) can be extended to an invertible square matrix over \( R \).

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

**Definition 3.1.** Define \( \text{Loc}(R) := \{ R_r \mid r \in R \setminus \{0\} \} \).

**Proposition 3.2 (Quillen Induction).** Let \( P \subseteq \text{Loc}(R) \). Write \( P(L) \) instead of \( L \in P \) for \( L \in \text{Loc}(R) \). In that case, \( L \) is said to have property \( P \). Assume that

(a) for all \( m \in \text{Max}(R) \): there exists an \( r \in R \setminus m \) such that \( P(R_r) \);

(b) for all \( r, s, t \in R \setminus \{0\} \): if \( rR_t + sR_t = R_t \), \( P(R_r) \), and \( P(R_s) \), then \( P(R_t) \).

Then \( P(L) \) for all \( L \in \text{Loc}(R) \). In particular \( P(R) \).
Proof. Let $S$ be the collection of all $r \in R \setminus \{0\}$ such that $P(R_r)$ together with $0$. This is an ideal of $R$. It is not empty because $0 \in S$, closed under addition because of (b) (for $r, s \in S$ take $t := r + s$), and closed under multiplication with elements of $R$ also because of (b) (for $r \in R$ and $\tilde{s} \in S$, take $r := \tilde{s}, s := \tilde{s},$ and $t := \tilde{s}\bar{s}$).

Suppose that $S \neq R$. Then $S$ is contained in some maximal ideal of $R$, say $m$. By (a) there is an $r \in R \setminus m$ such that $P(R_r)$. But then $r \in S \subseteq m$, which contradicts $r \notin m$. So $S = R$ and therefore $P(L)$ for all $L \in \text{Loc}(R)$. \hfill \Box

Definition 3.3. An element $H$ of $\text{End}_R(R[X])$ is called nice if it is of the form $H = (X_1 + \text{h.o.t.}, \ldots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, i.e., terms of degree 2 or greater, and $\text{End}_R(R[X])$ has been identified with $R[X]^n$. A coordinate $h \in R[X]$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $h$ as its first component. Similarly, a partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $(h_1, \ldots, h_k)$ as its first $k$ components.

Lemma 3.4. A partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + \text{h.o.t.}$.

Proof. By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with $h_1, \ldots, h_k$ as its first $k$ components. \hfill \Box

Definition 3.5. Let $H \in \text{End}_R(R[X])$ be nice. Then $T^H \in \text{End}_R(R[T][X])$ is defined by

$$T^H := T^{-1}H[X_1 := TX_1, \ldots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because $H$ is nice.) If $r \in R$, then $T^H[T := r] \in \text{End}_R(R[X])$ is denoted by $^rH$.

One can easily see that $(\text{det } JH)[X := TX] = \text{det } J^TH$ and that $H$ is invertible if and only if $^rH$ is. Here $JH$ denotes the Jacobian matrix $(\partial H_i/\partial X_j)_{ij}$ of $H$. Even better, if $r \in R \setminus \{0\}$, then $\text{det } J^rH \in R^*$ if and only if $\text{det } JH \in R^*$ and $^rH$ is invertible if and only if $H$ is.

The map $^H$ is called the clearing map because of the following: if $K$ is the quotient field of $R$ and $H \in \text{End}_K(K[X])$ is of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that $^rH \in \text{End}_R(R[X])$. So, the denominators of $H$ are cleared. See Chapter 1 of [Ess00].

Lemma 3.6. Let $r, s \in R \setminus \{0\}$ be such that $rR + sR = R$ and let $H \in \text{Aut}_{R_2}(R_{rs}[X])$ be nice. Then there are nice $H_1 \in \text{Aut}_{R_2}(R_r[X])$ and $H_2 \in \text{Aut}_{R_2}(R_s[X])$ such that $H = H_1H_2$.

Proof. Note that

$$T^H = H_{(1)} + TH_{(2)} + T^2H_{(3)} + \cdots + T^{d-1}H_{(d)}.$$
where each \( H_{(i)} \) is the homogeneous part of degree \( i \) of \( H \) and \( d \) is the degree of \( H \).

Hence
\[
\begin{align*}
1 - TH &= H_{(1)} + (1 - T)H_{(2)} + (1 - T)^2 H_{(3)} + \cdots + (1 - T)^{d-1} H_{(d)} \\
&= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(\text{h.o.t.}) \\
&= H + T(\text{h.o.t.}),
\end{align*}
\]

where, as before, h.o.t. stands for some terms of \( X \)-degree at least two. As a consequence
\[
H^{-1} \circ 1 - TH = H^{-1} \circ (H + T(\text{h.o.t.})) = X + T(\text{h.o.t.}).
\]

Now let \( k \in \mathbb{N} \) be sufficiently large. From \( rR + sR = R \) it follows that \( r^kR + s^kR = R \). Take \( v, w \in R \) with \( r^k v + s^k w = 1 \). If \( k \) is sufficiently large, then \( s^k w H \) and \( s^k w(h^{-1}) \)
are elements of \( \text{End}_{R_t}(R_v[X]) \). They are also each others inverse and hence they are in fact elements of \( \text{Aut}_{R_t}(R_v[X]) \).

Take \( H_1 := s^k w H \) and compute \( H^{-1} H_1 \). This gives
\[
\begin{align*}
H^{-1} H_1 &= H^{-1} \circ T H [T := s^k w] \\
&= H^{-1} \circ 1 - T H [T := r^k v] \\
&= (X + T(\text{h.o.t.})) [T := r^k v]) \\
&= X + r^k v(\text{h.o.t.})
\end{align*}
\]

and similarly
\[
H_1^{-1} H = X + r^k v(\text{h.o.t.}).
\]

For \( k \) sufficiently large, \( H_2 := H_1^{-1} H \) and its inverse apparently are elements of \( \text{Aut}_{R_t}(R_v[X]) \). So now \( H = H_1 H_2 \) with \( H_1 \) and \( H_2 \) are both of the required form.

\( \square \)

**Lemma 3.7.** Let \( r, s \in R \) be such that \( rR + sR = R \). Take \( t \in R_{rs} \) such that \( t \in R_t \cap R_s \). Then \( t \in R \).

**Proof.** Write \( t = v/r^k = w/s^l \) with \( v, w \in R \) and \( k, l \in \mathbb{N} \). Because \( rR + sR = R \), also \( r^k R + s^l R = R \). Write \( r^k x + s^l y = 1 \) for some \( x, y \in R \). Then \( t = (r^k x + s^l y)t = vx + wy \in R \).

\( \square \)

**Lemma 3.8 (Patching Lemma).** Let \( r, s \in R \) with \( rR + sR = R \). Let \( k \in \{1, \ldots, n\} \) and let \( h_1, \ldots, h_k \in R[X] \) be polynomials of the form \( h_i = X_i + \text{h.o.t.} \). Assume that there is a nice \( F \in \text{Aut}_{R_t}(R_v[X]) \) with first \( k \) components equal to \( h_1, \ldots, h_k \) and that there is a nice \( G \in \text{Aut}_{R_t}(R_v[X]) \) with first \( k \) components equal to \( h_1, \ldots, h_k \). Then there is a nice \( H \in \text{Aut}_{R}(R[X]) \) with first \( k \) components equal to \( h_1, \ldots, h_k \).
Proof. Consider the polynomial map $F^{-1}G \in \text{Aut}_{R_n}(R_n[X])$ and note that it is fact an $R_n[X_1, \ldots, X_k]$-automorphism of $R_n[X] = R_n[X_1, \ldots, X_k][X_{k+1}, \ldots, X_n]$. Now apply Lemma 3.6 to the ring $R[X_1, \ldots, X_k]$ and write $F^{-1}G = H_1H_2$ with $H_1 \in \text{Aut}_{R_n[X_1, \ldots, X_k]}(R_n[X])$ and $H_2 \in \text{Aut}_{R_n[X_1, \ldots, X_k]}(R_n[X])$, where both $H_i$ are of the form $X + \text{h.o.t.}$. Considered as automorphisms over respectively $R_1$ and $R_n$, the first $k$ components of $H_1$ and $H_2$ of course equal $X_1, \ldots, X_k$. Hence $H := FH_1 = GH_2^{-1}$ is a nice polynomial automorphism (over $R_n$, a priori) whose first $k$ components equal $h_1, \ldots, h_k$. It is defined over $R_n$ (because $H = FH_1$ and $F$ and $H_1$ are defined over $R_n$) and it is defined over $R_n$ (because $H = GH_2^{-1}$ and $G$ and $H_2$ are defined over $R_n$). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over $R_n$.

Theorem 3.9. Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that for every maximal ideal $m$ of $R$, $(h_1, \ldots, h_k)$ is a nice partial coordinate system when considered as an element of $R_m[X]^k$. Then $(h_1, \ldots, h_k)$ is a nice partial coordinate system.

Proof. Let $P \subseteq \text{Loc}(R)$ be the collection of all $R_r$, $r \in R \setminus \{0\}$, such that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R_r$. Now check the two conditions for Quillen Induction.

(a) Let $m$ be a maximal ideal of $R$. It is assumed that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R_m$. Using Lemma 3.4, choose $F \in \text{Aut}_{R_m}(R_m[X])$ nice with first $k$ components equal to $h_1, \ldots, h_k$. There are only finitely many elements of $R$ appearing in the denominator of a coefficient of a component of $F$ and its inverse. Denote the product of these denominators by $r$. None of these denominators is an element of $m$ and, because $m$ is prime, $r$ is not an element of $m$ either. Furthermore, obviously, $P(R_r)$.

(b) Let $r, s, t \in R \setminus \{0\}$ be such that $rR_t + sR_t = R_t$ and assume $P(R_r)$ and $P(R_s)$. Then $P(R_t)$ follows by applying the Patching Lemma (Lemma 3.8) to the ring $R_t$.

So, using Quillen Induction (Proposition 3.2), $P(R)$, which means that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R$.

Corollary 3.10. Assume that $R$ is Hermite. Let $k \in \{1, \ldots, n\}$ and $h_1, \ldots, h_k \in R[X]$. Assume that $(h_1, \ldots, h_k)$ is a partial coordinate system when considered as an element of $R_m[X]^k$, for every maximal ideal $m$ of $R$. Then $(h_1, \ldots, h_k)$ is a partial coordinate system.

Proof. First of all note that it is possible to assume that the $h_i$ have no constant part. Write $h_i = r_{i1}X_1 + \cdots + r_{in}X_n + \text{h.o.t.}$ for all $i$, with $r_{ij} \in R$.

Consider a maximal ideal $m$ of $R$. Then $(h_1, \ldots, h_k)$ is a partial coordinate system over $R_m$, which means that there are $f_{k+1}, \ldots, f_n \in R_m[X]$ such that $F := (h_1, \ldots, h_k, f_{k+1}, \ldots, f_n) \in \text{Aut}_{R_m}(R_m[X])$. The $f_i$ can be chosen in such a way
that they have no constant part. Then det \( JF \in R_m[X]^* \) and hence substituting
\( X_1 := 0, \ldots, X_n := 0 \) gives
\[
\begin{vmatrix}
\tau_{11} & \cdots & \tau_{1n} \\
\vdots & & \vdots \\
\tau_{k1} & \cdots & \tau_{kn} \\
\ast & \cdots & \ast \\
\vdots & & \vdots \\
\ast & \cdots & \ast
\end{vmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R^*_m.
\]

In particular, the matrix \((r_{ij})_{ij}\) represents a surjective \(R_m\)-module homomorphism
from \(R^n_m\) to \(R^k_m\).

Because this holds for every maximal ideal of \(R\), it follows that the matrix \((r_{ij})_{ij}\)
represents a surjective \(R\)-module homomorphism from \(R^n\) to \(R^k\). Now \(R\) is Hermite,
which implies that the matrix \((r_{ij})_{ij}\) can be extended to an invertible square matrix
\(M\) over \(R\) (see [Lam78], Corollary 4.5). Viewing this matrix \(M\) as a polynomial
automorphism of \(R\) and applying its inverse to the polynomials \(h_i\), it follows
that one can assume that \((h_1, \ldots, h_k)\) is of the form \((X_1 + h.o.t., \ldots, X_k + h.o.t.)\).

By Lemma 3.4, \((h_1, \ldots, h_k)\) then is a nice coordinate system in \(R_m[X]\), for every
\(m \in \text{Max}(R)\). Now apply Theorem 3.9.

The condition that \(R\) be Hermite in the previous corollary is necessary. For let \(R\) be
any non-Hermite ring; say \((a_1, \ldots, a_n)\) is a unimodular row over \(R\) that cannot be
extended to an invertible square matrix. Then \(h := a_1X_1 + \cdots + a_nX_n \in R[X_1, \ldots, X_n]\)
is not a coordinate (if it were, the coefficients of the linear part of an automorphism
with \(h\) as its first component would form an invertible square matrix over \(R\) extending
\((a_1, \ldots, a_n)\)). However, localising in a maximal ideal \(m\) of \(R\), \((a_1, \ldots, a_n)\) is extendible
to an invertible square matrix over \(R_m\) (since \(R_m\) is local) and so \(h\) is a coordinate
over \(R_m\).

4 Main Result

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two
variables over an discrete valuation ring containing \(Q\).

**Theorem 4.1.** Let \(R\) be a discrete valuation ring containing \(Q\). Denote the unique
maximal ideal of \(R\) by \(m\), write \(K\) for the quotient field \(Q(R)\) of \(R\), and write \(k\)
for the residue field \(R/m\) of \(R\). Let \(A\) be a finitely generated affine \(R\)-domain and assume
that \(K \otimes R A \cong K[2]\) and that \(k \otimes R A \cong k[2]\). Then \(A \cong R[2]\). \(\square\)

In order to use this result, a lemma is needed on the behaviour of the kernel of a
locally nilpotent derivation with a slice under tensoring.

**Lemma 4.2.** Let \(s \in R[X] := R[X_1, \ldots, X_n]\) and let \(A\) be an \(R\)-algebra via the map
\(\varphi: R \to A\). Denote the induced map \(R[X] \to A[X]\) by \(\varphi_\#\). Then
\[
A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi_\#(s)A[X]).
\]
In particular, if $D$ is a locally nilpotent $R$-derivation on $R[X]$ and $s$ is a slice of $D$, then

$$A \otimes_R R[X]^D \cong A[X]^\hat{D},$$

where $\hat{D}$ denotes the extension of $D$ to $A[X]$.

**Proof.** The following diagram is a commutative diagram of $R$-modules and $R$-module homomorphism in which the horizontal sequences are exact.

\[
\begin{array}{cccccc}
\text{s}R[X] & \rightarrow & R[X] & \rightarrow & R[X]/sR[X] & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A \otimes_R sR[X] & \rightarrow & A \otimes_R A[X] & \rightarrow & A \otimes_R R[X]/sR[X] & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\varphi_#(s)A[X] & \rightarrow & A[X] & \rightarrow & A[X]/(\varphi_#(s)A[X]) & \rightarrow & 0
\end{array}
\]

The map $A \otimes_R sR[X] \rightarrow \varphi_#(s)A[X]$ is surjective: take an element $\varphi_#(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum \alpha \cdot c_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with each $c_\alpha \in A$. Then $\varphi_#(s)f$ is the image of $\sum \alpha \cdot c_\alpha sX_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \rightarrow A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \rightarrow A[X]/(\varphi_#(s)A[X])$ is an isomorphism. A priori this is an isomorphism of $R$-modules. However, since it is an $A$-module homomorphism, it is even an isomorphism of $A$-modules.

The second claim follows from the first one using Theorem 2.1.

Note that this lemma is false if $D$ does not have slice. For instance, let $K$ be some field, $R := K[Y]$, and consider $A := K$ as an $R$-module by sending elements of $K$ to themselves and $Y$ to 0. Let $D$ be the locally nilpotent derivation $Y \partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension $\hat{D}$ of $D$ to $A[X]$ is 0 and hence $A[X]^\hat{D} = A[X]$.

**Lemma 4.3.** Let $R$ be a discrete valuation ring containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice $s \in R[X,Y,Z]$. Then $R[X,Y,Z]^D \cong R^{[2]}$.

**Proof.** Let $k$ be the residue field of $R$ and let $K$ be the quotient field of $R$. Denote the extension of $D$ to $K \otimes_R R[X,Y,Z] \cong K[X,Y,Z]$ by $\hat{D}$. By Lemma 4.2 and Theorem 2.6 it follows that

$$K \otimes_R R[X,Y,Z]^\hat{D} \cong K[X,Y,Z]^\hat{D} \cong K^{[2]}.$$

In exactly the same way it follows that

$$k \otimes_R R[X,Y,Z]^D \cong k^{[2]}.$$

Hence, by Theorem 4.1, $R[X,Y,Z]^D \cong R^{[2]}$. 

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**Theorem 4.4.** Let $R$ be a Dedekind domain containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X, Y, Z]$ with a slice. Then $R[X, Y, Z]^D \cong R[2]$.

**Proof.** Let $s \in R[X, Y, Z]$ be a slice of $D$. Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass’ Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, $R$ is Hermite. By Corollary 3.10 it is enough to show that $s$ is a coordinate in $R_m[X, Y, Z]$ for every maximal ideal $m$ of $R$.

So let $m$ be a maximal ideal of $R$. Then $R_m$ is a discrete valuation ring. Because $R$ contains $\mathbb{Q}$, $R_m$ contains $\mathbb{Q}$ as well. Now Lemma 4.3 implies that $R_m[X, Y, Z]^D \cong R_m[2]$. In other words, $s$ is a coordinate in $R_m[X, Y, Z]$.

**Corollary 4.5.** Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X, Y, Z, W]$ of the form

$$D := a(X, Y, Z, W)\partial_X + b(X, Y, Z, W)\partial_Y + c(X, Y, Z, W)\partial_Z + d(W)\partial_W.$$ 


**Proof.** If $d(W) \neq 0$, then $d(W) \in k^*$, since $D$ is locally nilpotent. So $d^{-1}W$ is a slice of $D$. This slice is also a coordinate and hence $k[X]^D \cong k[3]$. Otherwise, if $d(W) = 0$, apply Theorem 4.4 with $R = k[W]$.

**References**


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Harm Derksen <hderksen@math.lsa.umich.edu>  
Department of Mathematics  
University of Michigan  
525 East University  
Ann Arbor, MI 48109-1109  
U.S.A.

Arno van den Essen <essen@sci.kun.nl>  
Peter van Rossum <petervr@sci.kun.nl>  
Department of Mathematics  
University of Nijmegen  
Toernooiveld 1  
6525 GD Nijmegen  
The Netherlands