THE CANCELLATION PROBLEM IN DIMENSION FOUR

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Report No. 0022 (October 2000)
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Abstract

This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let $k$ be a field of characteristic zero and let $V$ be an algebraic variety over $k$. The Cancellation Problem asks if $V \times k \cong k^n$ implies that $V \cong k^{n-1}$. This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if $A[T] \cong k[X_1, \ldots, X_n]$ implies that $A \cong k[X_1, \ldots, X_{n-1}]$ for an affine $k$-domain $A$. One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on $k[X_1, \ldots, X_n]$ with a slice is isomorphic to $k[X_1, \ldots, X_{n-1}]$. For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert’s Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial local coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing $\mathbb{Q}$, this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing $\mathbb{Q}$ for $n = 3$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for $n = 4$ over a field, including the triangular derivations.

*Partially supported by NSF, grant 9970165
2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let \( A \) be a ring. A derivation on \( A \) is a map \( D: A \to A \) satisfying \( D(a + b) = D(a) + D(b) \) and \( D(ab) = aD(b) + D(a)b \) for all \( a, b \in A \). If \( R \) is a ring and \( A \) is an \( R \)-algebra via \( f: R \to A \), then \( A \) is called an \( R \)-derivation if \( D(f(r)) = 0 \) for all \( r \in R \). A derivation \( D \) is called locally nilpotent if for all \( a \in A \) there is an \( n \in \mathbb{N} \) such that \( D^n(a) = 0 \). The kernel of \( D \) is denoted by \( A^D \). If \( s \in A \) is such that \( D(s) = 1 \), then \( s \) is called a slice of \( D \).

The following proposition (see [Wri81]) is well-known.

**Proposition 2.1.** Let \( A \) be a \( \mathbb{Q} \)-algebra and let \( D \) be a locally nilpotent derivation on \( A \). Assume that \( s \in A \) is a slice of \( D \). Then \( A = A^D[s] \) and \( s \) is algebraically independent over \( A^D \). Furthermore, \( D = d/ds \).

In the applications, \( A \) will invariably be a polynomial ring \( R[X] := R[x_1, \ldots, x_n] \) over a ring \( R \). An \( R \)-derivation \( D \) on such a ring is called triangular if \( D(x_i) \in R[x_{i+1}, \ldots, x_n] \) for all \( i \). Such a derivation is automatically locally nilpotent. An element \( s \in R[X] \) is called a coordinate if there is a polynomial automorphism \( F \) of \( R[X] \) with \( s \) as one of its components. More generally, a sequence \((s_1, \ldots, s_k)\) of elements of \( R[X] \) with \( 1 \leq k \leq n \) is called a partial coordinate system if there are polynomials \( f_{k+1}, \ldots, f_n \in R[X] \) such that \((s_1, \ldots, s_k, f_{k+1}, \ldots, f_n)\) is a polynomial automorphism of \( R[X] \).

Proposition 2.1 implies the following.

**Corollary 2.2.** Let \( R \) be a ring and let \( n \geq 2 \). Let \( D \) be a locally nilpotent \( R \)-derivation on \( R[X] := R[x_1, \ldots, x_n] \) and let \( s \in R[X] \) be a slice of \( D \). Then \( s \) is a coordinate if and only if \( R[X]^D \cong R^{(n-1)} \).

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

**Problem 2.3 (Cancellation Problem).** Let \( k \) be a field of characteristic zero and let \( n \geq 2 \). Let \( D \) be a locally nilpotent \( k \)-derivation on \( k[X] := k[x_1, \ldots, x_n] \) and assume that \( D \) has a slice \( s \in k[X] \). Is then \( k[X]^D \cong k^{(n-1)} \), i.e., is \( s \) a coordinate in \( k[X] \)?

More generally, one can ask the following question.

**Problem 2.4 (Generalized Cancellation Problem).** Let \( k \) be a field of characteristic zero, let \( R \) be an affine \( k \)-domain, and let \( n \geq 2 \). Let \( D \) be a locally nilpotent \( R \)-derivation on \( R[X] := R[x_1, \ldots, x_n] \) and assume that \( D \) has a slice \( s \in R[X] \). Is then \( R[X]^D \cong R^{(n-1)} \), i.e., is \( s \) a coordinate in \( R[X] \)?

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

**Theorem 2.5.** Let \( k \) be a field of characteristic zero. Let \( D \) be a locally nilpotent \( k \)-derivation on \( k[X] := k[x_1, x_2] \). Then \( k[X]^D \cong k^{[1]} \).
Nowadays even stronger results have been obtained by Bhadwadekar and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field $k$ in the theorem can in fact be replaced by an arbitrary $\mathbb{Q}$-algebra $R$.

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

**Theorem 2.6.** Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X] := k[X_1, X_2, X_3]$. Assume that $D$ has a slice. Then $k[X]^D \cong k[2]$.

This paper now proves the Generalized Cancellation Problem for $n = 3$ in case $R$ is a Dedekind domain over $\mathbb{Q}$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

$$D := a(X_1, X_2, X_3, X_4)\partial_1 + b(X_1, X_2, X_3, X_4)\partial_2 + c(X_1, X_2, X_3, X_4)\partial_3 + d(X_4)\partial_4$$

for $n = 4$, where $\partial_i$ denotes $\partial/\partial X_i$. In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for $n = 4$. This is especially interesting since [Asa99] (implicitly) and [ER00] (explicitly) give a candidate counterexample to the Cancellation Problem for $n = 5$ which is triangular, namely $D := (2X_1^2 - 3)\partial_1 + (4X_2^2 - 8X_4)\partial_2 + (5X_4^2 - 10)\partial_3 + X_5\partial_4$.

## 3 Local Coordinates

Let $R$ be a domain, $n \in \mathbb{N}$, and $R[X] := R[X_1, \ldots, X_n]$ the polynomial ring in $n$ variables over $R$. This section shows that a polynomial in $R[X]$ is a coordinate if and only if it is a coordinate when considered as an element of $R_m[X]$, for all maximal ideals $m$ of $R$, provided that $R$ is Hermite, and similarly for partial coordinate systems. Recall that $R$ is called Hermite if every unimodular row $(r_1, \ldots, r_k)$ can be extended to an invertible square matrix over $R$.

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

**Definition 3.1.** Define $\text{Loc}(R) := \{R_r : r \in R \setminus \{0\}\}$.

**Proposition 3.2 (Quillen Induction).** Let $P \subseteq \text{Loc}(R)$. Write $P(L)$ instead of $L \in P$ for $L \in \text{Loc}(R)$. In that case, $L$ is said to have property $P$. Assume that

(a) for all $m \in \text{Max}(R)$: there exists an $r \in R \setminus m$ such that $P(R_r)$;

(b) for all $r, s, t \in R \setminus \{0\}$: if $rR_t + sR_t = R_t$, $P(R_r)$, and $P(R_s)$, then $P(R_t)$.

Then $P(L)$ for all $L \in \text{Loc}(R)$. In particular $P(R)$. 

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Proof. Let $S$ be the collection of all $r \in R \setminus \{0\}$ such that $P(R_r)$ together with 0. This is an ideal of $R$. It is not empty because 0 is in $S$, closed under addition because of (b) (for $r, s \in S$ take $t := r + s$), and closed under multiplication with elements of $R$ also because of (b) (for $\tilde{r} \in R$ and $\tilde{s} \in S$, take $r := \tilde{r}, s := \tilde{s}$, and $t := \tilde{r}\tilde{s}$).

Suppose that $S \neq R$. Then $S$ is contained in some maximal ideal of $R$, say $m$. By (a) there is an $r \in R \setminus m$ such that $P(R_r)$. But then $r \in S \subseteq m$, which contradicts $r \not\in m$. So $S = R$ and therefore $P(L)$ for all $L \in \text{Loc}(R)$.

Definition 3.3. An element $H$ of $\text{End}_R[R[X]]$ is called nice if it is of the form $H = (X_1 + \text{h.o.t.}, \ldots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, i.e., terms of degree 2 or greater, and $\text{End}_R[R[X]]$ has been identified with $R[X]^n$. A coordinate $h \in R[X]$ is called nice if there is a nice $H \in \text{Aut}_R[R[X]]$ which has $h$ as its first component. Similarly, a partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is called nice if there is a nice $H \in \text{Aut}_R(R[X])$ which has $(h_1, \ldots, h_k)$ as its first $k$ components.

Lemma 3.4. A partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + \text{h.o.t.}$.

Proof. By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with $h_1, \ldots, h_k$ as its first $k$ components.

Definition 3.5. Let $H \in \text{End}_R(R[X])$ be nice. Then $\tau H \in \text{End}_R[T](R[T][X])$ is defined by

$$\tau H := T^{-1}H[X_1 := TX_1, \ldots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because $H$ is nice.) If $r \in R$, then $\tau H[T := r] \in \text{End}_R(R[X])$ is denoted by $\tau^r H$.

One can easily see that $(\det JH)[X := TX] = \det J^\tau H$ and that $H$ is invertible if and only if $\tau^r H$ is. Here $JH$ denotes the Jacobian matrix $(\partial H_i / \partial X_j)_{ij}$ of $H$. Even better, if $r \in R \setminus \{0\}$, then $\det J^\tau H \in R^*$ if and only if $\det JH \in R^*$ and $\tau^r H$ is invertible if and only if $H$ is.

The map $\tau^r H$ is called the clearing map because of the following: if $K$ is the quotient field of $R$ and $H \in \text{End}_K(K[X])$ is of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that $\tau^r H \in \text{End}_R(R[X])$. So, the denominators of $H$ are cleared. See Chapter 1 of [Ess00].

Lemma 3.6. Let $r, s \in R \setminus \{0\}$ be such that $rR + sR = R$ and let $H \in \text{Aut}_{R_r}(R_{rs}[X])$ be nice. Then there are nice $H_1 \in \text{Aut}_{R_r}(R_r[X])$ and $H_2 \in \text{Aut}_{R_s}(R_s[X])$ such that $H = H_1 H_2$.

Proof. Note that

$$\tau^r H = H_1 + \tau H_2 + \tau^2 H_3 + \cdots + \tau^{d-1} H_d$$
where each $H_{(i)}$ is the homogeneous part of degree $i$ of $H$ and $d$ is the degree of $H$. Hence

$$1^\cdot T H = H_{(1)} + (1 - T)H_{(2)} + (1 - T)^2H_{(3)} + \cdots + (1 - T)^{d-1}H_{(d)}$$

$$= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(h.o.t.)$$

$$= H + T(h.o.t.),$$

where, as before, $h.o.t.$ stands for some terms of $X$-degree at least two. As a consequence

$$H^{-1} \circ 1^\cdot T H = H^{-1} \circ (H + T(h.o.t.))$$

$$= X + T(h.o.t.).$$

Now let $k \in \mathbb{N}$ be sufficiently large. From $rR + sR = R$ it follows that $r^kR + s^kR = R$. Take $v, w \in R$ with $r^kv + s^kw = 1$. If $k$ is sufficiently large, then $s^w H$ and $s^w(H^{-1})$ are elements of $\text{End}_{R_t}(R_v[X])$. They are also each others inverse and hence they are in fact elements of $\text{Aut}_{R_t}(R_v[X])$.

Take $H_1 := s^wH$ and compute $H^{-1}H_1$. This gives

$$H^{-1}H_1 = H^{-1} \circ T H [T := s^kw]$$

$$= H^{-1} \circ 1^\cdot T H [T := r^kv]$$

$$= (X + T(h.o.t.)[T := r^kv])$$

$$= X + r^kv(h.o.t.)$$

and similarly

$$H_1^{-1}H = X + r^kv(h.o.t.).$$

For $k$ sufficiently large, $H_2 := H_1^{-1}H$ and its inverse apparently are elements of $\text{Aut}_{R_t}(R_v[X])$. So now $H = H_1H_2$ with $H_1$ and $H_2$ are both of the required form. 

**Lemma 3.7.** Let $r, s \in R$ be such that $rR + sR = R$. Take $t \in R_{rs}$ such that $t \in R_r \cap R_s$. Then $t \in R$.

**Proof.** Write $t = v/r^k = w/s^l$ with $v, w \in R$ and $k, l \in \mathbb{N}$. Because $rR + sR = R$, also $r^kR + s^lR = R$. Write $r^kx + s^ly = 1$ for some $x, y \in R$. Then $t = (r^kx + s^ly)t = vx + wy \in R$. 

**Lemma 3.8 (Patching Lemma).** Let $r, s \in R$ with $rR + sR = R$. Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + h.o.t.$. Assume that there is a nice $F \in \text{Aut}_{R_t}(R_t[X])$ with first $k$ components equal to $h_1, \ldots, h_k$ and that there is a nice $G \in \text{Aut}_{R_t}(R_t[X])$ with first $k$ components equal to $h_1, \ldots, h_k$. Then there is a nice $H \in \text{Aut}_{R_t}(R[X])$ with first $k$ components equal to $h_1, \ldots, h_k$. 

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Proof. Consider the polynomial map $F^{-1}G \in \text{Aut}_{R_{rs}}(R_{rs}[X])$ and note that it is fact an $R_{rs}[X_{1}, \ldots, X_{k}]$-automorphism of $R_{rs}[X] = R_{rs}[X_{1}, \ldots, X_{k}][X_{k+1}, \ldots, X_{n}]$. Now apply Lemma 3.6 to the ring $R[X_{1}, \ldots, X_{k}]$ and write $F^{-1}G = H_1H_2$ with $H_1 \in \text{Aut}_{R_{rs}[X_{1}, \ldots, X_{k}]}(R_{rs}[X])$ and $H_2 \in \text{Aut}_{R_{rs}[X_{1}, \ldots, X_{k}]}(R_{rs}[X])$, where both $H_i$ are of the form $X + \text{h.o.t.}$ Considered as automorphisms over respectively $R_r$ and $R_s$, the first $k$ components of $H_1$ and $H_2$ of course equal $X_1, \ldots, X_k$. Hence $H := FH_1 = GH_2^{-1}$ is a nice polynomial automorphism (over $R_{rs}$, a priori) whose first $k$ components equal $h_1, \ldots, h_k$. It is defined over $R_r$ (because $H = FH_1$ and $F$ and $H_1$ are defined over $R_r$) and it is defined over $R_s$ (because $H = GH_2^{-1}$ and $G$ and $H_2$ are defined over $R_s$). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over $R_r$. \hfill $\square$

Theorem 3.9. Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that for every maximal ideal $m$ of $R$, $(h_1, \ldots, h_k)$ is a nice partial coordinate system when considered as an element of $R_m[X]^k$. Then $(h_1, \ldots, h_k)$ is a nice partial coordinate system.

Proof. Let $P \subseteq \text{Loc}(R)$ be the collection of all $R_r$, $r \in R \setminus \{0\}$, such that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R_r$. Now check the two conditions for Quillen Induction.

(a) Let $m$ be a maximal ideal of $R$. It is assumed that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R_m$. Using Lemma 3.4, choose $F \in \text{Aut}_{R_m}(R_m[X])$ nice with first $k$ components equal to $h_1, \ldots, h_k$. There are only finitely many elements of $R$ appearing in the denominator of a coefficient of a component of $F$ and its inverse. Denote the product of these denominators by $r$. None of these denominators is an element of $m$ and, because $m$ is prime, $r$ is not an element of $m$ either. Furthermore, obviously, $P(R_r)$.

(b) Let $r, s, t \in R \setminus \{0\}$ be such that $rR_t + sR_t = R_t$ and assume $P(R_r)$ and $P(R_s)$. Then $P(R_t)$ follows by applying the Patching Lemma (Lemma 3.8) to the ring $R_t$.

So, using Quillen Induction (Proposition 3.2), $P(R)$, which means that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R$. \hfill $\square$

Corollary 3.10. Assume that $R$ is Hermite. Let $k \in \{1, \ldots, n\}$ and $h_1, \ldots, h_k \in R[X]$. Assume that $(h_1, \ldots, h_k)$ is a partial coordinate system when considered as an element of $R_m[X]^k$, for every maximal ideal $m$ of $R$. Then $(h_1, \ldots, h_k)$ is a partial coordinate system.

Proof. First of all note that it is possible to assume that the $h_i$ have no constant part. Write $h_i = r_{i1}X_1 + \cdots + r_{in}X_n + \text{h.o.t.}$ for all $i$, with $r_{ij} \in R$.

Consider a maximal ideal $m$ of $R$. Then $(h_1, \ldots, h_k)$ is a partial coordinate system over $R_m$, which means that there are $f_{k+1}, \ldots, f_n \in R_m[X]$ such that $F := (h_1, \ldots, h_k, f_{k+1}, \ldots, f_n) \in \text{Aut}_{R_m}(R_m[X])$. The $f_i$ can be chosen in such a way
that they have no constant part. Then \( \det JF \in R_m[X]^\ast \) and hence substituting \( X_1 := 0, \ldots, X_n := 0 \) gives
\[
\begin{vmatrix}
 r_{11} & \cdots & r_{1n} \\
 \vdots & & \vdots \\
 r_{k1} & \cdots & r_{kn} \\
 * & \cdots & * \\
 \vdots & & \vdots \\
 * & \cdots & *
\end{vmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R_m^\ast.
\]

In particular, the matrix \((r_{ij})_{ij}\) represents a surjective \(R_m\)-module homomorphism from \(R_n^m\) to \(R_k^m\).

Because this holds for every maximal ideal of \(R\), it follows that the matrix \((r_{ij})_{ij}\) represents a surjective \(R\)-module homomorphism from \(R_n^m\) to \(R_k^m\). Now \(R\) is Hermite, which implies that the matrix \((r_{ij})_{ij}\) can be extended to an invertible square matrix \(M\) over \(R\) (see [Lam78], Corollary 4.5). Viewing this matrix \(M\) as a polynomial automorphism of \(R[X]\) and applying its inverse to the polynomials \(h_i\), it follows that one can assume that \((h_1, \ldots, h_k)\) is of the form \((X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})\).

By Lemma 3.4, \((h_1, \ldots, h_k)\) then is a nice coordinate system in \(R_m[X]\), for every \(m \in \text{Max}(R)\). Now apply Theorem 3.9.

The condition that \(R\) be Hermite in the previous corollary is necessary. For let \(R\) be any non-Hermite ring; say \((a_1, \ldots, a_n)\) is a unimodular row over \(R\) that cannot be extended to an invertible square matrix. Then \(h := a_1X_1 + \cdots + a_nX_n \in R[X_1, \ldots, X_n]\) is not a coordinate (if it were, the coefficients of the linear part of an automorphism with \(h\) as its first component would form an invertible square matrix over \(R\) extending \((a_1, \ldots, a_n)\)). However, localising in a maximal ideal \(m\) of \(R\), \((a_1, \ldots, a_n)\) is extendible to an invertible square matrix over \(R_m\) (since \(R_m\) is local) and so \(h\) is a coordinate over \(R_m\).

4 Main Result

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two variables over an discrete valuation ring containing \(\mathbb{Q}\).

**Theorem 4.1.** Let \(R\) be a discrete valuation ring containing \(\mathbb{Q}\). Denote the unique maximal ideal of \(R\) by \(m\), write \(R\) for the quotient field \(Q(R)\) of \(R\), and write \(k\) for the residue field \(R/m\) of \(R\). Let \(A\) be a finitely generated affine \(R\)-domain and assume that \(K \otimes_R A \cong K[2]\) and that \(k \otimes_R A \cong k[2]\). Then \(A \cong R[2]\).

In order to use this result, a lemma is needed on the behaviour of the kernel of a locally nilpotent derivation with a slice under tensoring.

**Lemma 4.2.** Let \(s \in R[X] := R[X_1, \ldots, X_n]\) and let \(A\) be an \(R\)-algebra via the map \(\varphi : R \to A\). Denote the induced map \(R[X] \to A[X]\) by \(\varphi\). Then
\[
A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi(s)A[X]).
\]
In particular, if $D$ is a locally nilpotent $R$-derivation on $R[X]$ and $s$ is a slice of $D$, then

$$A \otimes_R R[X]^D \cong A[X]^\tilde{D},$$

where $\tilde{D}$ denotes the extension of $D$ to $A[X]$.

Proof. The following diagram is a commutative diagram of $R$-modules and $R$-module homomorphism in which the horizontal sequences are exact.

$$
\begin{array}{ccccccc}
  sR[X] & \longrightarrow & R[X] & \longrightarrow & R[X]/sR[X] & \longrightarrow & 0 \\
  \downarrow & & \downarrow & & \downarrow & & \\
  A \otimes_R sR[X] & \longrightarrow & A \otimes_R A[X] & \longrightarrow & A \otimes_R R[X]/sR[X] & \longrightarrow & 0 \\
  \downarrow & & \downarrow & & \downarrow & & \\
  \varphi_\#(s)A[X] & \longrightarrow & A[X] & \longrightarrow & A[X]/(\varphi_\#(s)A[X]) & \longrightarrow & 0
\end{array}
$$

The map $A \otimes_R sR[X] \to \varphi_\#(s)A[X]$ is surjective: take an element $\varphi_\#(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum c_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with each $c_\alpha \in A$. Then $\varphi_\#(s)f$ is the image of $\sum c_\alpha \otimes sX_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \to A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \to A[X]/(\varphi_\#(s)A[X])$ is an isomorphism. A priori this is an isomorphism of $R$-modules. However, since it is an $A$-module homomorphism, it is even an isomorphism of $A$-modules.

The second claim follows from the first one using Theorem 2.1. \qed

Note that this lemma is false if $D$ does not have slice. For instance, let $K$ be some field, $R := K[Y]$, and consider $A := K$ as an $R$-module by sending elements of $K$ to themselves and $Y$ to 0. Let $D$ be the locally nilpotent derivation $Y \partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension $\tilde{D}$ of $D$ to $A[X]$ is 0 and hence $A[X]^\tilde{D} = A[X]$.

Lemma 4.3. Let $R$ be a discrete valuation ring containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice $s \in R[X,Y,Z]$. Then $R[X,Y,Z]^D \cong R[2]$.

Proof. Let $k$ be the residue field of $R$ and let $K$ be the quotient field of $R$. Denote the extension of $D$ to $K \otimes_R R[X,Y,Z] \cong K[X,Y,Z]$ by $\tilde{D}$. By Lemma 4.2 and Theorem 2.6 it follows that

$$K \otimes_R R[X,Y,Z]^D \cong K[X,Y,Z]^\tilde{D} \cong K[2].$$

In exactly the same way it follows that

$$k \otimes_R R[X,Y,Z]^D \cong k[2].$$

Hence, by Theorem 4.1, $R[X,Y,Z]^D \cong R[2]$. \qed
Theorem 4.4. Let $R$ be a Dedekind domain containing $\mathbb{Q}$ and let $D$ be a locally nilpotent $R$-derivation on $R[X,Y,Z]$ with a slice. Then $R[X,Y,Z]^D \cong R[2]$. 

Proof. Let $s \in R[X,Y,Z]$ be a slice of $D$. Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass’ Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, $R$ is Hermite. By Corollary 3.10 it is enough to show that $s$ is a coordinate in $R_m[X,Y,Z]$ for every maximal ideal $m$ of $R$.

So let $m$ be a maximal ideal of $R$. Then $R_m$ is a discrete valuation ring. Because $R$ contains $\mathbb{Q}$, $R_m$ contains $\mathbb{Q}$ as well. Now Lemma 4.3 implies that $R_m[X,Y,Z]^D \cong R_m[2]$. In other words, $s$ is a coordinate in $R_m[X,Y,Z]$.

Corollary 4.5. Let $k$ be a field of characteristic zero and let $D$ be a locally nilpotent $k$-derivation on $k[X,Y,Z,W]$ of the form

$$D := a(X,Y,Z,W)\partial_X + b(X,Y,Z,W)\partial_Y + c(X,Y,Z,W)\partial_Z + d(W)\partial_W.$$ 


Proof. If $d(W) \neq 0$, then $d(W) \in k^*$, since $D$ is locally nilpotent. So $d^{-1}W$ is a slice of $D$. This slice is also a coordinate and hence $k[X]^D \cong k[3]$. Otherwise, if $d(W) = 0$, apply Theorem 4.4 with $R = k[W]$.

References


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