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The Cancellation Problem in Dimension Four

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Abstract

This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let k be a field of characteristic zero and let V be an algebraic variety over k. The Cancellation Problem asks if $V \times k \cong k^n$ implies that $V \cong k^{n-1}$. This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if $A[T] \cong k[X_1, \ldots, X_n]$ implies that $A \cong k[X_1, \ldots, X_{n-1}]$ for an affine k-domain A. One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on $k[X_1, \ldots, X_n]$ with a slice is isomorphic to $k[X_1, \ldots, X_{n-1}]$. For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert's Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial local coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing $\mathbb{Q}$, this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing $\mathbb{Q}$ for $n = 3$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for $n = 4$ over a field, including the triangular derivations.

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2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let A be a ring. A *derivation* on A is a map $D\colon A \to A$ satisfying $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If R is a ring and A is an R-algebra via $f\colon R \to A$, then A is called an R-derivation if $D(f(r)) = 0$ for all $r \in R$. A derivation D is called *locally nilpotent* if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of D is denoted by A^D. If $s \in A$ is such that $D(s) = 1$, then s is called a *slice* of D.

The following proposition (see [Wri81]) is well-known.

Proposition 2.1. *Let A be a $\mathbb{Q}$-algebra and let D be a locally nilpotent derivation on A. Assume that $s \in A$ is a slice of D. Then $A = A^D[s]$ and s is algebraically independent over A^D. Furthermore, $D = d/ds$.* □

In the applications, A will invariably be a polynomial ring $R[X] := R[X_1, \ldots, X_n]$ over a ring R. An R-derivation D on such a ring is called *triangular* if $D(X_i) \in R[X_{i+1}, \ldots, X_n]$ for all i. Such a derivation is automatically locally nilpotent. An element $s \in R[X]$ is called a *coordinate* if there is a polynomial automorphism F of $R[X]$ with s as one of its components. More generally, a sequence $(s_1, \ldots, s_k)$ of elements of $R[X]$ with $1 \le k \le n$ is called a *partial coordinate system* if there are polynomials $f_{k+1}, \ldots, f_n \in R[X]$ such that $(s_1, \ldots, s_k, f_{k+1}, \ldots, f_n)$ is a polynomial automorphism of $R[X]$.

Proposition 2.1 implies the following.

Corollary 2.2. *Let R be a ring and let $n \ge 2$. Let D be a locally nilpotent R-derivation on $R[X] := R[X_1, \ldots, X_n]$ and let $s \in R[X]$ be a slice of D. Then s is a coordinate if and only if $R[X]^D \cong R^{[n-1]}$.* □

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

Problem 2.3 (Cancellation Problem). *Let k be a field of characteristic zero and let $n \ge 2$. Let D be a locally nilpotent k-derivation on $k[X] := k[X_1, \ldots, X_n]$ and assume that D has a slice $s \in k[X]$. Is then $k[X]^D \cong k^{[n-1]}$, i.e., is s a coordinate in $k[X]$?*

More generally, one can ask the following question.

Problem 2.4 (Generalized Cancellation Problem). *Let k be a field of characteristic zero, let R be an affine k-domain, and let $n \ge 2$. Let D be a locally nilpotent R-derivation on $R[X] := R[X_1, \ldots, X_n]$ and assume that D has a slice $s \in R[X]$. Is then $R[X]^D \cong R^{[n-1]}$, i.e., is s a coordinate in $R[X]$?*

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

Theorem 2.5. *Let k be a field of characteristic zero. Let D be a locally nilpotent k-derivation on $k[X] := k[X_1, X_2]$. Then $k[X]^D \cong k^{[1]}$.* □

Nowadays even stronger results have been obtained by Bhadwadekar and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field k in the theorem can in fact be replaced by an arbitrary $\mathbb{Q}$-algebra R.

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

Theorem 2.6. *Let k be a field of characteristic zero and let D be a locally nilpotent k-derivation on $k[X] := k[X_1, X_2, X_3]$. Assume that D has a slice. Then $k[X]^D \cong k^{[2]}$.* □

This paper now proves the Generalized Cancellation Problem for $n = 3$ in case R is a Dedekind domain over $\mathbb{Q}$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

$$D := a(X_1, X_2, X_3, X_4)\partial_1 + b(X_1, X_2, X_3, X_4)\partial_2 + \\ + c(X_1, X_2, X_3, X_4)\partial_3 + d(X_4)\partial_4$$

for $n = 4$, where ∂_i denotes ∂/∂_{X_i}. In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for $n = 4$. This is especially interesting since [Asa99] (implicitely) and [ER00] (explicitely) give a candidate counterexample to the Cancellation Problem for $n = 5$ which is triangular, namely $D := (2X_4^2 - 3)\partial_1 + (4X_4^3 - 8X_4)\partial_2 + (5X_4^4 - 10)\partial_3 + X_5\partial_4$.

3 Local Coordinates

Let R be a domain, $n \in \mathbb{N}$, and $R[X] := R[X_1, \ldots, X_n]$ the polynomial ring in n variables over R. This section shows that a polynomial in $R[X]$ is a coordinate if and only if it is a coordinate when considered as an element of $R_{\mathfrak{m}}[X]$, for all maximal ideals $\mathfrak{m}$ of R, provided that R is Hermite, and similarly for partial coordinate systems. Recall that R is called *Hermite* if every unimodular row $(r_1, \ldots, r_k)$ can be extended to an invertible square matrix over R.

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

Definition 3.1. Define $\mathrm{Loc}(R) := \{R_r \mid r \in R \setminus \{0\}\}$.

Proposition 3.2 (Quillen Induction). *Let $P \subseteq \mathrm{Loc}(R)$. Write $P(L)$ instead of $L \in P$ for $L \in \mathrm{Loc}(R)$. In that case, L is said to have property P. Assume that*

(a) for all $\mathfrak{m} \in \mathrm{Max}(R)$: there exists an $r \in R \setminus \mathfrak{m}$ such that $P(R_r)$;

(b) for all $r, s, t \in R \setminus \{0\}$: if $rR_t + sR_t = R_t$, $P(R_r)$, and $P(R_s)$, then $P(R_t)$.

Then $P(L)$ for all $L \in \mathrm{Loc}(R)$. In particular $P(R)$.

Proof. Let S be the collection of all $r \in R \setminus \{0\}$ such that $P(R_r)$ together with 0. This is an ideal of R. It is not empty because $0 \in S$, closed under addition because of (b) (for $r, s \in S$ take $t := r + s$), and closed under multiplication with elements of R also because of (b) (for $\tilde{r} \in R$ and $\tilde{s} \in S$, take $r := \tilde{s}$, $s := \tilde{s}$, and $t := \tilde{r}\tilde{s}$).

Suppose that $S \neq R$. Then S is contained in some maximal ideal of R, say $\mathfrak{m}$. By (a) there is an $r \in R \setminus \mathfrak{m}$ such that $P(R_r)$. But then $r \in S \subseteq \mathfrak{m}$, which contradicts $r \notin \mathfrak{m}$. So $S = R$ and therefore $P(L)$ for all $L \in \mathrm{Loc}(R)$. $\square$

Definition 3.3. An element H of $\mathrm{End}_R(R[X])$ is called *nice* if it is of the form $H = (X_1 + \text{h.o.t.}, \ldots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, *i.e.*, terms of degree 2 or greater, and $\mathrm{End}_R(R[X])$ has been identified with $R[X]^n$. A coordinate $h \in R[X]$ is called *nice* if there is a nice $H \in \mathrm{Aut}_R(R[X])$ which has h as its first component. Similarly, a partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is called *nice* if there is a nice $H \in \mathrm{Aut}_R(R[X])$ which has $(h_1, \ldots, h_k)$ as its first k components.

Lemma 3.4. *A partial coordinate system $(h_1, \ldots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + h.o.t., \ldots, X_k + h.o.t.)$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + h.o.t.$.*

Proof. By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with $h_1, \ldots, h_k$ as its first k components. $\square$

Definition 3.5. Let $H \in \mathrm{End}_R(R[X])$ be nice. Then ${}^T H \in \mathrm{End}_{R[T]}(R[T][X])$ is defined by

$${}^T H := T^{-1} H[X_1 := TX_1, \ldots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because H is nice.) If $r \in R$, then ${}^T H[T := r] \in \mathrm{End}_R(R[X])$ is denoted by ${}^r H$.

One can easily see that $(\det JH)[X := TX] = \det J{}^T H$ and that H is invertible if and only if ${}^T H$ is. Here JH denotes the Jacobian matrix $(\partial H_i / \partial X_j)_{ij}$ of H. Even better, if $r \in R \setminus \{0\}$, then $\det J{}^r H \in R^*$ if and only if $\det JH \in R^*$ and ${}^r H$ is invertible if and only if H is.

The map ${}^T H$ is called the *clearing map* because of the following: if K is the quotient field of R and $H \in \mathrm{End}_K(K[X])$ is of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that ${}^r H \in \mathrm{End}_R(R[X])$. So, the denominators of H are cleared. See Chapter 1 of [Ess00].

Lemma 3.6. *Let $r, s \in R \setminus \{0\}$ be such that $rR + sR = R$ and let $H \in \mathrm{Aut}_{R_{rs}}(R_{rs}[X])$ be nice. Then there are nice $H_1 \in \mathrm{Aut}_{R_r}(R_r[X])$ and $H_2 \in \mathrm{Aut}_{R_s}(R_s[X])$ such that $H = H_1 H_2$.*

Proof. Note that

$${}^T H = H_{(1)} + T H_{(2)} + T^2 H_{(3)} + \cdots + T^{d-1} H_{(d)}$$

where each $H_{(i)}$ is the homogeneous part of degree i of H and d is the degree of H. Hence

$$\begin{aligned}
{}^{1-T}H &= H_{(1)} + (1-T)H_{(2)} + (1-T)^2 H_{(3)} + \cdots + (1-T)^{d-1} H_{(d)} \\
&= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(\text{h.o.t.}) \\
&= H + T(\text{h.o.t.}),
\end{aligned}$$

where, as before, h.o.t. stands for some terms of X-degree at least two. As a consequence

$$\begin{aligned}
H^{-1} \circ {}^{1-T}H &= H^{-1} \circ (H + T(\text{h.o.t.})) \\
&= X + T(\text{h.o.t.}).
\end{aligned}$$

Now let $k \in \mathbb{N}$ be sufficiently large. From $rR+sR=R$ it follows that $r^kR+s^kR = R$. Take $v, w \in R$ with $r^kv+s^kw = 1$. If k is sufficiently large, then ${}^{s^kw}H$ and ${}^{s^kw}(H^{-1})$ are elements of $\mathrm{End}_{R_r}(R_r[X])$. They are also each others inverse and hence they are in fact elements of $\mathrm{Aut}_{R_r}(R_r[X])$.

Take $H_1 := {}^{s^kw}H$ and compute $H^{-1}H_1$. This gives

$$\begin{aligned}
H^{-1}H_1 &= H^{-1} \circ {}^{T}H\,[T := s^k w] \\
&= H^{-1} \circ {}^{1-T}H\,[T := r^k v] \\
&= (X + T(\text{h.o.t.}))\,[T := r^k v]) \\
&= X + r^k v(\text{h.o.t.})
\end{aligned}$$

and similarly

$$H_1^{-1}H = X + r^k v(\text{h.o.t.}).$$

For k sufficiently large, $H_2 := H_1^{-1}H$ and its inverse apparently are elements of $\mathrm{Aut}_{R_s}(R_s[X])$. So now $H = H_1H_2$ with H_1 and H_2 are both of the required form. $\square$

Lemma 3.7. *Let $r, s \in R$ be such that $rR + sR = R$. Take $t \in R_{rs}$ such that $t \in R_r \cap R_s$. Then $t \in R$.*

Proof. Write $t = v/r^k = w/s^l$ with $v, w \in R$ and $k, l \in \mathbb{N}$. Because $rR + sR = R$, also $r^kR + s^lR = R$. Write $r^kx + s^ly = 1$ for some $x, y \in R$. Then $t = (r^kx + s^ly)t = vx + wy \in R$. $\square$

Lemma 3.8 (Patching Lemma). *Let $r, s \in R$ with $rR+sR = R$. Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i +$ h.o.t.. Assume that there is a nice $F \in \mathrm{Aut}_{R_r}(R_r[X])$ with first k components equal to $h_1, \ldots, h_k$ and that there is a nice $G \in \mathrm{Aut}_{R_s}(R_s[X])$ with first k components equal to $h_1, \ldots, h_k$. Then there is a nice $H \in \mathrm{Aut}_R(R[X])$ with first k components equal to $h_1, \ldots, h_k$.*

Proof. Consider the polynomial map $F^{-1}G \in \mathrm{Aut}_{R_{rs}}(R_{rs}[X])$ and note that it is fact an $R_{rs}[X_1, \ldots, X_k]$-automorphism of $R_{rs}[X] = R_{rs}[X_1, \ldots, X_k][X_{k+1}, \ldots, X_n]$. Now apply Lemma 3.6 to the ring $R[X_1, \ldots, X_k]$ and write $F^{-1}G = H_1H_2$ with $H_1 \in \mathrm{Aut}_{R_r[X_1,\ldots,X_k]}(R_r[X])$ and $H_2 \in \mathrm{Aut}_{R_s[X_1,\ldots,X_k]}(R_s[X])$, where both H_i are of the form $X + $h.o.t.. Considered as automorphisms over respectively R_r and R_s, the first k components of H_1 and H_2 ofcourse equal $X_1, \ldots, X_k$. Hence $H := FH_1 = GH_2^{-1}$ is a nice polynomial automorphism (over R_{rs}, a priori) whose first k components equal $h_1, \ldots, h_k$. It is defined over R_r (because $H = FH_1$ and F and H_1 are defined over R_r) and it is defined over R_s (because $H = GH_2^{-1}$ and G and H_2 are defined over R_s). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over R. $\square$

Theorem 3.9. *Let $k \in \{1, \ldots, n\}$ and let $h_1, \ldots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + h.o.t.$. Assume that for every maximal ideal $\mathfrak{m}$ of R, $(h_1, \ldots, h_k)$ is a nice partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$. Then $(h_1, \ldots, h_k)$ is a nice partial coordinate system.*

Proof. Let $P \subseteq \mathrm{Loc}(R)$ be the collection of all R_r, $r \in R \setminus \{0\}$, such that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over R_r. Now check the two conditions for Quillen Induction.

(a) Let $\mathfrak{m}$ be a maximal ideal of R. It is assumed that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over $R_{\mathfrak{m}}$. Using Lemma 3.4, choose $F \in \mathrm{Aut}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$ nice with first k components equal to $h_1, \ldots, h_k$. There are only finitely many elements of R appearing in the denominator of a coefficient of a component of F and its inverse. Denote the product of these denominators by r. None of these denominators is an element of $\mathfrak{m}$ and, because $\mathfrak{m}$ is prime, r is not an element of $\mathfrak{m}$ either. Furthermore, obviously, $P(R_r)$.

(b) Let $r, s, t \in R \setminus \{0\}$ be such that $rR_t + sR_t = R_t$ and assume $P(R_r)$ and $P(R_s)$. Then $P(R_t)$ follows by applying the Patching Lemma (Lemma 3.8) to the ring R_t.

So, using Quillen Induction (Proposition 3.2), $P(R)$, which means that $(h_1, \ldots, h_k)$ is a nice partial coordinate system over R. $\square$

Corollary 3.10. *Assume that R is Hermite. Let $k \in \{1, \ldots, n\}$ and $h_1, \ldots, h_k \in R[X]$. Assume that $(h_1, \ldots, h_k)$ is a partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$, for every maximal ideal $\mathfrak{m}$ of R. Then $(h_1, \ldots, h_k)$ is a partial coordinate system.*

Proof. First of all note that it is possible to assume that the h_i have no constant part. Write $h_i = r_{i1}X_1 + \cdots + r_{in}X_n + $h.o.t. for all i, with $r_{ij} \in R$.

Consider a maximal ideal $\mathfrak{m}$ of R. Then $(h_1, \ldots, h_k)$ is a partial coordinate system over $R_{\mathfrak{m}}$, which means that there are $f_{k+1}, \ldots, f_n \in R_{\mathfrak{m}}[X]$ such that $F := (h_1, \ldots, h_k, f_{k+1}, \ldots, f_n) \in \mathrm{Aut}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$. The f_i can be chosen in such a way that they have no constant part. Then $\det JF \in R_{\mathfrak{m}}[X]^*$ and hence substituting $X_1 := 0, \ldots, X_n := 0$ gives

$$\begin{vmatrix} r_{11} & \ldots & r_{1n} \\ \vdots & & \vdots \\ r_{k1} & \ldots & r_{kn} \\ * & \ldots & * \\ \vdots & & \vdots \\ * & \ldots & * \end{vmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R_{\mathfrak{m}}^*.$$

In particular, the matrix $(r_{ij})_{ij}$ represents a surjective $R_{\mathfrak{m}}$-module homomorphism from $R_{\mathfrak{m}}^n$ to $R_{\mathfrak{m}}^k$.

Because this holds for every maximal ideal of R, it follows that the matrix $(r_{ij})_{ij}$ represents a surjective R-module homomorphism from R^n to R^k. Now R is Hermite, which implies that the matrix $(r_{ij})_{ij}$ can be extended to an invertible square matrix M over R (see [Lam78], Corollary 4.5). Viewing this matrix M as a polynomial automorphism of $R[X]$ and applying its inverse to the polynomials h_i, it follows that one can assume that $(h_1, \ldots, h_k)$ is of the form $(X_1 + \text{h.o.t.}, \ldots, X_k + \text{h.o.t.})$. By Lemma 3.4, $(h_1, \ldots, h_k)$ then is a nice coordinate system in $R_{\mathfrak{m}}[X]$, for every $\mathfrak{m} \in \operatorname{Max}(R)$. Now apply Theorem 3.9. □

The condition that R be Hermite in the previous corollary is necessary. For let R be any non-Hermite ring; say $(a_1, \ldots, a_n)$ is a unimodular row over R that cannot be extended to an invertible square matrix. Then $h := a_1X_1 + \cdots + a_nX_n \in R[X_1, \ldots, X_n]$ is not a coordinate (if it were, the coefficients of the linear part of an automorphism with h as its first component would form an invertible square matrix over R extending $(a_1, \ldots, a_n)$). However, localising in a maximal ideal $\mathfrak{m}$ of R, $(a_1, \ldots, a_n)$ is extendible to an invertible square matrix over $R_{\mathfrak{m}}$ (since $R_{\mathfrak{m}}$ is local) and so h is a coordinate over $R_{\mathfrak{m}}$.

4 Main Result

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two variables over an discrete valuation ring containing $\mathbb{Q}$.

Theorem 4.1. *Let R be a discrete valution ring containing $\mathbb{Q}$. Denote the unique maximal ideal of R by $\mathfrak{m}$, write K for the quotient field $Q(R)$ of R, and write k for the residue field $R/\mathfrak{m}$ of R. Let A be a finitely generated affine R-domain and assume that $K \otimes_R A \cong K^{[2]}$ and that $k \otimes_R A \cong k^{[2]}$. Then $A \cong R^{[2]}$.* □

In order to use this result, a lemma is needed on the behaviour of the kernel of a locally nilpotent derivation with a slice under tensoring.

Lemma 4.2. *Let $s \in R[X] := R[X_1, \ldots, X_n]$ and let A be an R-algebra via the map $\varphi\colon R \to A$. Denote the induced map $R[X] \to A[X]$ by $\varphi_\#$. Then*

$$A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi_\#(s)A[X])$$

In particular, if D is a locally nilpotent R-derivation on $R[X]$ and s is a slice of D, then

$$A \otimes_R R[X]^D \cong A[X]^{\tilde{D}},$$

where $\tilde{D}$ denotes the extension of D to $A[X]$.

Proof. The following diagram is a commutative diagram of R-modules and R-module homomorphism in which the horizontal sequences are exact.

$$\begin{tikzcd}
sR[X] \arrow[r] \arrow[d] & R[X] \arrow[r] \arrow[d] & R[X]/sR[X] \arrow[r] \arrow[d] & 0 \\
A \otimes_R sR[X] \arrow[r] \arrow[d] & A \otimes_R A[X] \arrow[r] \arrow[d] & A \otimes_R R[X]/sR[X] \arrow[r] \arrow[d] & 0 \\
\varphi_{\#}(s)A[X] \arrow[r] & A[X] \arrow[r] & A[X]/(\varphi_{\#}(s)A[X]) \arrow[r] & 0
\end{tikzcd}$$

The map $A \otimes_R sR[X] \to \varphi_{\#}(s)A[X]$ is surjective: take an element $\varphi_{\#}(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum_\alpha c_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}$ with each $c_\alpha \in A$. Then $\varphi_{\#}(s)f$ is the image of $\sum_\alpha c_\alpha \otimes s X_1^{\alpha_1} \dots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \to A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \to A[X]/(\varphi_{\#}(s)A[X])$ is an isomorphism. A priori this is an isomorphism of R-modules. However, since it is an A-module homomorphism, it is even an isomorphism of A-modules.

The second claim follows from the first one using Theorem 2.1. □

Note that this lemma is false if D does not have slice. For instance, let K be some field, $R := K[Y]$, and consider $A := K$ as an R-module by sending elements of K to themselves and Y to 0. Let D be the locally nilpotent derivation $Y\partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension $\tilde{D}$ of D to $A[X]$ is 0 and hence $A[X]^{\tilde{D}} = A[X]$.

Lemma 4.3. *Let R be a discrete valution ring containing $\mathbb{Q}$ and let D be a locally nilpotent R-derivation on $R[X,Y,Z]$ with a slice $s \in R[X,Y,Z]$. Then $R[X,Y,Z]^D \cong R^{[2]}$.*

Proof. Let k be the residue field of R and let K be the quotient field of R. Denote the extension of D to $K \otimes_R R[X,Y,Z] \cong K[X,Y,Z]$ by $\tilde{D}$. By Lemma 4.2 and Theorem 2.6 it follows that

$$K \otimes_R R[X,Y,Z]^D \cong K[X,Y,Z]^{\tilde{D}} \cong K^{[2]}.$$

In exactly the same way it follows that

$$k \otimes_R R[X,Y,Z]^D \cong k^{[2]}.$$

Hence, by Theorem 4.1, $R[X,Y,Z]^D \cong R^{[2]}$. □

Theorem 4.4. *Let R be a Dedekind domain containing $\mathbb{Q}$ and let D be a locally nilpotent R-derivation on $R[X,Y,Z]$ with a slice. Then $R[X,Y,Z]^D \cong R^{[2]}$.*

Proof. Let $s \in R[X,Y,Z]$ be a slice of D. Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass' Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, R is Hermite. By Corollary 3.10 it is enough to show that s is a coordinate in $R_{\mathfrak{m}}[X,Y,Z]$ for every maximal ideal $\mathfrak{m}$ of R.

So let $\mathfrak{m}$ be a maximal ideal of R. Then $R_{\mathfrak{m}}$ is a discrete valuation ring. Because R contains $\mathbb{Q}$, $R_{\mathfrak{m}}$ contains $\mathbb{Q}$ as well. Now Lemma 4.3 implies that $R_{\mathfrak{m}}[X,Y,Z]^D \cong R_{\mathfrak{m}}^{[2]}$. In other words, s is a coordinate in $R_{\mathfrak{m}}[X,Y,Z]$. $\square$

Corollary 4.5. *Let k be a field of characteristic zero and let D be a locally nilpotent k-derivation on $k[X,Y,Z,W]$ of the form*

$$D := a(X,Y,Z,W)\partial_X + b(X,Y,Z,W)\partial_Y + c(X,Y,Z,W)\partial_Z + d(W)\partial_W.$$

Assume that D has a slice. Then $k[X,Y,Z,W]^D \cong k^{[3]}$.

Proof. If $d(W) \neq 0$, then $d(W) \in k^*$, since D is locally nilpotent. So $d^{-1}W$ is a slice of D. This slice is also a coordinate and hence $k[X]^D \cong k^{[3]}$. Otherwise, if $d(W) = 0$, apply Theorem 4.4 with $R = k[W]$. $\square$

References

[AM75] Shreeram S. Abhyankar and Tzuong-tsieng Moh. Embeddings of the line in the plane. *J. Reine Angew. Math.*, 276:148–166, 1975.

[Asa99] Teruo Asanuma. Non-linearizable algebraic k^*-actions on affine spaces. *Inven. Math.*, 138:281–306, 1999.

[Bas68] H. Bass. *Algebraic K-Theory.* Benjamin, 1968.

[BCW77] H. Bass, E.H. Connell, and D.L. Wright. Locally polynomial algebras are symmetric algebras. *Inven. Math.*, 38:279–299, 1977.

[BD97] S. Bhadwadekar and A. Dutta. Kernel of locally nilpotent R-derivations on $R[X,Y]$. *Tr. Am. Math. Soc.*, 349:3303–3319, 1997.

[BEM99] Joost Berson, Arno van den Essen, and Stefan Maubach. Derivations having divergence zero on $R[X,Y]$. Report 9918, Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands, April 1999. To appear in Isr. J. Math.

[Dai97] D. Daigle. On some properties of locally nilpotent derivations. *J. Pure Apl. Alg.*, 114:221–230, 1997.

[DF99] D. Daigle and G. Freudenburg. A counterexample to Hilbert's fourteenth problem in dimension five. *J. Alg.*, 221:528–535, 1999.

[ER00] Arno van den Essen and Peter van Rossum. Triangular derivations related to problems on affine n-space. Report 0005, Department of Mathematics, University of Nijmegen, March 2000.

[Ess93] Arno van den Essen. An algorithm to compute the invariant ring of a G_a-action on an affine variety. *J. Symb. Comp.*, 16:551–555, 1993.

[Ess00] Arno van den Essen. *Polynomial Automorphisms and the Jacobian Conjecture*, volume 190 of *Progress in Mathematics*. Birkhäuser-Verlag, Basel-Boston-Berlin, 2000.

[Fre00] G. Freudenburg. A counterexample to Hilbert's fourteenth problem in dimension six. To appear in J. of Transformation Groups, 2000.

[Fuj79] T. Fujita. On Zariski problem. *Proc. Japan Acad. Ser. A. Math. Sci.*, 55:106–110, 1979.

[Kam75] T. Kambayashi. On the absence of nontrivial separable forms of the affine plane. *J. Alg.*, 35:449–456, 1975.

[Kra89] Hanspeter Kraft. Algebraic automorphisms of affine space. In H. Kraft, T. Petrie, and G. Schwarz, editors, *Topological Methods in Algebraic Transformation Groups: Proceedings of the Conference "Topological Methods in Algebraic Transformation Groups" held at Rutgers University, 4–8 April 1988*, Boston-Basel-Berlin, 1989. Birkhäuser Verlag.

[Lam78] T. Lam. *Serre's Conjecture*, volume 635 of *Lecture Notes in Mathematics*. Springer-Verlag, 1978.

[Miy85] M. Miyanishi. Normal affine subalgebras of a polynomial ring. In *Algebraic and Topological Theories*, pages 37–51, Tokyo, 1985.

[MS80] M. Miyanishi and T. Sugie. Affine surfaces containing cylinderlike open sets. *J. Math. Kyoto Univ.*, 20:11–42, 1980.

[Now94] A. Nowicki. *Polynomial Derivations and their Rings of Constants*. Univ. of Toruń, 1994.

[Qui76] Daniel Quillen. Projective modules over polynomial rings. *Inven. Math.*, 36:167–171, 1976.

[Ren68] R. Rentschler. Opérations du groupe additif sur le plan. *C.R. Acad. Sci. Paris*, 267:384–387, 1968.

[Sat83] A. Sathaye. Polynomial ring in two variables over a d.v.r.: A criterion. *Inven. Math.*, 1983.

[Wei00] Charles Weibel. *An Introduction to Algebraic K-Theory.* `http://math.rutgers.edu/w̃eibel/Kbook.html`, 2000.

[Wri81] David Wright. On the Jacobian Conjecture. *Illinois J. Math.*, 25(3):423–440, 1981.

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