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Abstract

An approach to approximate a global attractor of a semidynamical system with error estimates in Hausdorff metric is presented. This approach is based on the properties of a function of rate of attraction to an attractor and on some new results for an unstable manifold in a neighborhood of an essential nonhyperbolical point. For some classes of the semidynamical system we construct an unstable manifold in the neighborhood of a fixed isolated point, prove that each trajectory is attracted to the manifold and find the function of attraction.

1 Introduction

The theory of the α, ω-attracting sets for a semidynamical system in a compact space was constructed fifty years ago. Main goal of this theory is to find the minimal closed set attracting each trajectory as the time tends to infinity.

The first results for a semidynamical system corresponding to ordinary differential equation in a noncompact space X were proved in [1] by Dj. Hale. At the same time, O. Ladyzhenskaya [2] constructed the set $\mathcal{M}$ for the Navier-Stokes equations in the 2D case. She proved that $\mathcal{M}$ is defined as a set which is compact, invariant, minimal among the closed sets attracting uniformly any bounded subset $B \subset X$. The set $\mathcal{M}$ was called a minimal global B-attractor. Now the $\mathcal{M}$ is called [4] a global attractor.

Later on, this result was reconstructed for so-called semigroups of genus one [5] whose resolving operators are completely continuous and proved [3] for semigroups of genus two. The Navier–Stokes equations, heat convection equations, equations of

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magnetohydrodynamics for viscous incompressible fluids, quasilinear parabolic systems for which one-valued solvability is proved related to those classes.

The theory of attractors for evolution equations has been studied by A. Babin, M. Vishik, O. Ladyzhenskaya, R. Temam and by their followers.

The compactness property of a global attractor admits to construct an attractor finite approximation ε-net $\mathcal{M}^{\varepsilon}$. There are two basic directions to solve this problem. The first of them [6] is based on attracting property, and the other one [7] is based on a possibility to invert the operator $S(t,h)$ for $h \in \mathcal{M}$. In the current work we deal with the first algorithm and realize numerically the wellknown formula

$$\mathcal{M} = \bigcap_{t \geq 0} [S(t, B_a)]_H,$$

where B_a is a bounded set attracting each trajectory.

The first problem for this approach was pointed out in [5]. There an example was constructed where the attractor for each finite subset of B_a does not equal $\mathcal{M}$. Moreover, when we solve this problem numerically we perturb the original operator $S(t,h)$ to $\tilde{S}(t,h)$ and find the attractor $\tilde{\mathcal{M}}$ of the new semidynamical system.

The problem of closeness of attractors of two semidynamical systems under the closeness (in some sense) of their resolving operators was considered beginning [8, 9] by many authors.

The most powerful test for the attractor $\mathcal{M}_{\lambda}$ being in $O_{\varepsilon}(\mathcal{M}_{\lambda_0})$, with $O_{\varepsilon}(\mathcal{M}_{\lambda_0})$ the ε-neighborhood of $\mathcal{M}_{\lambda_0}$, was proved by Kapitanskij and Kostin, see also [3]. Later on this question was studied in [10]- [18]. For the semilinear parabolic equations this fact was proved in [9], [13]. In [14] this problem was studied for the Navier-Stokes equations in $\Omega \subset R^2$, in [15] it was studied for one modification of the Navier-Stokes equations in $\Omega \subset R^3$ (this modification corresponds to the algebraic turbulence models).

In the general case the closeness is absent. The method of attractor approximation by means of some sets converging to it as the approximate operator tends to the operator of the initial semidynamical system was considered in [6]. In the current work (see [15]) we essentially simplify the structure of approximating sets and give estimates of closeness of an attractor and approximating sets in an explicit form [16] in a Hausdorff metric.

This approach is based on properties of a function of rate of attraction to an attractor. This function has appeared in [3], [17], [5], [18], [19].

We shall construct this function for gradient dynamical systems based on some new results for unstable manifolds in a neighborhood of an essential nonhyperbolical point.

2 An attractor approximation

Let X be a Banach space with norm $\|\cdot\|$, $\mathcal{Q}$ be a nontrivial subgroup of the real numbers R and let $\mathcal{Q}_+ = \mathcal{Q} \cap [0, +\infty[$ be the intersection of $\mathcal{Q}$ and R_+. We shall deal with the abstract semigroup $\{X, \mathcal{Q}_+, S(\cdot)\}$ of a nonlinear operator $S : X \times \mathcal{Q}_+ \to X$. The term semigroup or semidynamical system refers to any family of singlevalued continuous operator S depending on a parameter $t \in \mathcal{Q}_+$ and enjoying the semigroup property:

$$S(t_1, S(t_2, u)) = S(t_1 + t_2, u), \quad \forall t_1, t_2 \in \mathcal{Q}_+, \forall u \in X.$$

A Banach space X is a phase space of a semigroup, $\mathcal{Q}_+$ is a time space and $S(\cdot)$ is an evolution operator. When $\mathcal{Q} = R$ a semigroup is a semigroup with continuous time.

Let B and $\mathcal{M}$ be bounded subsets of X. We say that B is attracted to $\mathcal{M}$ by the semigroup $S(\cdot)$ if

$$dist(S(t, B), \mathcal{M}) \to 0 \quad as\ t \to \infty.$$

Here

$$dist(A, B) = \sup_{y \in A} \{dist(y, B)\}, \quad dist(y, B) = \inf_{x \in B} \|x - y\|.$$

A set $\mathcal{M}$ is called an attracting set of the semigroup if $\mathcal{M}$ attracts each bounded $B \subset X$. The minimal one among the closed attracting sets is called the global attractor [4] (minimal global B-attractor [5]). The global attractor of a semigroup is defined as the set $\mathcal{M}$ which is compact in X, invariant for $S(\cdot)$, i.e.

$$S(t, \mathcal{M}) = \mathcal{M}, \quad t \geq 0.$$

and which attracts all the bounded sets of X.

Later on we need the following definitions, see [5].

A set B_a is called absorbing if for each bounded $B \subset X$ and for each $\varepsilon > 0$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset B_a, \quad \forall t \geq T.$$

If a semigroup possesses a nonempty bounded attractor $\mathcal{M}$ then for arbitrary $\varepsilon > 0$ the set $O_\varepsilon(\mathcal{M})$ is an absorbing set. Here $O_\varepsilon(\mathcal{M})$ is the ε-neighborhood of $\mathcal{M}$, i.e.

$$O_\varepsilon(\mathcal{M}) = \{u : \exists v \in M, \|u - v\| < \varepsilon\}.$$

A semigroup is called bounded if for each bounded B the set $S(t, B)$ is bounded for any $t > 0$.

A semigroup is called pointwise dissipative if it has a pointwise absorbing set B_0

$$\forall x \in X,\ \exists T(x) : \quad S(t, x) \subset B_a, \ \text{ for any } t \geq T(x).$$

A semigroup is called asymptotically compact if for each bounded B such that $S(t, B)$ is bounded for any $t > 0$ each sequence of the form

$$\{S(t_k, u_k)\}_{k=1}^{\infty}, t_k \uparrow \infty, u_k \in B$$

is precompact.

The following theorem holds, see [5].

Theorem 1 *Let the semigroup $\{X, \mathcal{Q}_+, S(\cdot)\}$ be a continuous bounded pointwise dissipative asymptotically compact semigroup. Then there exists a non-empty attractor $\mathcal{M}$*

$$\mathcal{M} = \bigcap_{t \geq 0} [S(t, B_a)]_X .$$

$\mathcal{M}$ is compact and invariant. If X is connected then $\mathcal{M}$ is also connected.

Consider the problem of approximation of $\mathcal{M}$ with respect to perturbations of the original operator $S(\cdot)$.

Suppose that a semigroup $S_\lambda(\cdot) : X \to X$ is depends on a parameter $\lambda \in \Lambda$. We assume that the following conditions (α) hold:
α_1) Λ is compact with respect to the metric $\|\cdot\|_\Lambda$ and λ_0 is a nonisolated point of Λ.
α_2) For each $\lambda \in \Lambda$ the semigroup $\{X, \mathcal{Q}_+, S_\lambda(\cdot)\}$ possesses a pointwise absorbing set B_λ and non-empty attractor $\mathcal{M}_\lambda$.
α_3) There exists a bounded absorbing set B_a and for each $\lambda \in \Lambda$ a set B_λ belongs to the set B_a.

By definition, each ε-neighborhood $O_\varepsilon(\mathcal{M}_\lambda)$ is an absorbing set. Assume that we know a function $\Theta(\lambda, \varepsilon) = \Theta(\lambda, \varepsilon, B_a)$ such that

$$dist(S_\lambda(t, B_a), \mathrm{M})) \leq \varepsilon, \qquad \text{as } t \geq \Theta(\varepsilon, \lambda).$$

Theorem 2 *Under the assumptions (α)*

(i) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ and a point $T_{\lambda_0} \geq \Theta(\lambda_0, \varepsilon)$ such that

$$\|S_\lambda(T_{\lambda_0}, u) - S_{\lambda_0}(T_{\lambda_0}, u)\| < \varepsilon \quad \forall u \in B_a,\ \forall \lambda \in O_\delta(\lambda_0). \tag{1}$$

Then the attractor $\mathcal{M}_\lambda$ is upper semicontinuous in the point λ_0 and the following estimate holds

$$dist(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon. \tag{2}$$

(ii) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

for arbitrary $\lambda \in O_\delta(\lambda_0)$ there exists a point $T_\lambda = T(\lambda) \geq \Theta(\lambda, \varepsilon)$ satisfies the following estimate

$$\|S_\lambda(T_\lambda, u) - S_{\lambda_0}(T_\lambda, u)\| < \varepsilon \quad \forall u \in B_a. \tag{3}$$

Then the attractor $\mathcal{M}_\lambda$ is lower and upper semicontinuous in the point λ_0 and the following estimate holds

$$\max\{dist(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda), dist(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0})\} \leq 2\varepsilon. \tag{4}$$

As was noticed the most natural set, the attractor $\tilde{\mathcal{M}}$ of the semidynamical system $\left\{X, \mathcal{Q}_+, \tilde{S}(\cdot)\right\}$, allows one to approximate only a part of the attractor $\mathcal{M}$, i.e., $\tilde{\mathcal{M}} \subset O_\varepsilon(\mathcal{M})$. Here, generally speaking, the inclusion is strict. The inverse inclusion, i.e., the continuity, holds only for some classes of the dissipative system.

To approximate the attractor of initial semidynamical system, we consider the following set:

$$B_a^\varepsilon = \left\{ \hat{h}_i \in B_a, i = 1, 2, ..., N_0 \mid \forall h \in B_a \ \exists i_h : \ \|\hat{h}_{i_h} - h\| \leq \varepsilon \right\}.$$

Here B_a is an arbitrary absorbing set.

Let X be a compact metric space where semidynamical systems $\{X, \mathcal{Q}_+, S(\cdot)\}$ and $\left\{X, \mathcal{Q}_+, \tilde{S}(\cdot)\right\}$ are given and these two systems have a common bounded absorbing set B_a and satisfy for $\forall h, \hat{h} \in B_a$ the following inequalities

$$\|S_{\lambda_0}(T, h) - S_{\lambda_0}(T, \hat{h})\| \leq L\|h - \hat{h}\|, \quad \|S_{\lambda_0}(T, h) - S_\lambda(T, h)\| \leq \delta. \tag{5}$$

This estimate means that $S(\cdot)$ is continuous with respect to its second argument and the operators S and $\tilde{S}$ are uniformly close on elements of the absorbing set B_a.

Suppose $T \geq \Theta(\lambda_0, \varepsilon_1, B_a)$, i.e. each trajectory attracts to the ε_1-neighborhood of $\mathcal{M}$:

$$S_{\lambda_0}(t, h) \subset O_{\varepsilon_1}(\mathcal{M}), \quad \forall t \geq T, \forall h \in B_a. \tag{6}$$

The following statement holds.

Theorem 3 *Under the assumptions* (5), (6) *let the set B_a^ε be a finite ε-net in the set B_a. Then the set $B_1 = S_\lambda(T, B_a^\varepsilon)$ satisfies the following injections:*

$$B_1 \subset O_{\varepsilon_1+\delta}(\mathcal{M}), \quad \mathcal{M} \subset O_{\delta+L\varepsilon}(B_1).$$

Note that while the algorithm enables us to approximate the attractor of the initial problem with the required precision by means of a *finite* number of arithmetic operations when we know the function $\Theta(\cdot)$, it is useless now when $\dim(B_a)$ is greater then 10. The reason is that it requires solution for *each* element of B_a^ε on the time point $T \gg 1$. The current algorithm was tested on a parallel computer under a T-system for the Lorenz [4] and for the 1D Chafee-Infante [18] problems. A T-system is a modern programming environment for a parallel computers and clusters which provides dynamic parallelization of programs written in a simple extension of C language. The original sequential C-program for attractor computation achieves efficient parallelization on 32-processor Linux cluster after insertion of a little amount of TC programming language keywords. The nice property of a T-system technology is a transparency of the used C-syntax extension, what makes a possibility to develop and run programs written in TC language without cluster and a T-system installed same way as normal C program. Attractor computation program is a simple but real-life example used as demonstration in a "Super-Computing Initiative — Phoenix" ("SuperKomputernaja Iniciativa — Feniks", SKIF) Russian-Byelorussian supercomputer project. For references see http://cluster.msu.ru and other links.

3 Global attraction to an attractor

Assume that for a bounded B the estimate

$$dist(S^n(B), \mathcal{M}) \leq \Psi(n)$$

holds for each $n \geq 0$. Then we say that $\mathcal{M}$ attracts B via the operator $S(\cdot)$ at the rate Ψ. We shall construct the function Ψ for a gradient semidynamical system possessing a finite number of fixed points. This result is based on the local function of attraction to the attractor in some neighborhoods of fixed points and on a global estimate for a Lyapunov functional.

Let us consider the rigorous results. Later on we deal with a discrete semigroup. Let $\mathcal{T}_+ = \{nt_0, n \in \mathbf{N}_+\}$, $t_0 > 0$, $S(h) = S(t_0, h)$ and $S^k(h) = S(t_0 k, h)$. According to theorem 1 the above result can be adopted to a discrete case.

Assume that the following conditions (β) hold:

β_1) The discrete system $\{S^k(\cdot), k \in N_+\}$ on a bounded subset B_a of a Banach space possesses the compact global attractor $\mathcal{M}$.

β_2) The operator $S(\cdot)$ is a Lipschitz continuous on B_a with a Lipschitz constant L.

β_3) There are a set of neighborhoods $\mathcal{O}_i \subset B_a$, $1 \leq i \leq N$, such that $S(\mathcal{O}_i) \cap \mathcal{O}_j = 0$, as $i \neq j$ and a set of decrease functions $\psi_i(n)$ of rate of attraction to the attractor $\mathcal{M}$, i.e.

$$\mathrm{dist}(S^n(h), \mathcal{M}) \leq \psi_i(n), \quad \text{for } S^k(h) \in \mathcal{O}_i, \text{ as } 0 \leq k \leq n. \tag{7}$$

Define the set $\mathcal{O}_0 = B_a \setminus \bigcup_{i=1}^{N} \mathcal{O}_i$.

β_4) Each trajectory $\{S^k(h)\}_{k=0}^{\infty}$ possesses in $\mathcal{O}_0$ less than n_0 points, n_0 does not depend on $h \in B_a$, i.e.

$$\mathrm{mes}\Big\{\{S^k(h)\}_{k=0}^{\infty} \cap \mathcal{O}_0\Big\} \leq n_0, \quad \forall h \in B_a.$$

The next theorem gives us the global function of rate of attraction to an attractor.

Theorem 4 *Under the assumptions (β) the following estimate is valid*

$$\mathrm{dist}(S^{2n_0+n}(B_a), \mathcal{M}) \leq L^{2n_0}\psi([\frac{n}{n_0+1}]), \quad \psi(k) = \max_{1 \leq i \leq N} \psi_i(k)$$

for $n \geq 0$.

Proof. Let $\psi_0(n) = L^n$ be a local function of rate of attraction to an attractor as $h \in \mathcal{O}_0$. Then the theorem can be proved in the same way as [18]. The theory of iterated function systems [26] also permits one to construct the function Ψ for $L < 1$.

The conditions $(\beta_{1,2})$ are valid for a semigroup of genus two [3], the condition (β_4) is verified for a gradient dynamical system possessing a Lyapunov functional.

A continuous function $V : B \to R$ is called a Lyapunov functional for $S(\cdot)$, if $V(S(h)) < V(h)$ $\forall h \in B, t > 0$, for $S(h) \neq h$.

For a gradient system we may assume that

$$0 < v \leq V(h) \leq V, \quad \forall\, h \in B_a$$

and rewrite condition β_4) in the following form.

β_4') There is a set of neighborhoods $\mathrm{O}_i \subset \mathcal{O}_i$, $1 \le i \le N$, such that $S(\mathrm{O}_i) \subset \mathcal{O}_i$ and

$$V\big(S(h)\big) \le qV(h), \quad q < 1, \quad \forall h \in \mathrm{O}_0 = B_a \setminus \bigcup_{i=1}^{N} \mathrm{O}_i.$$

The above estimate, together with $n_0 = \left[\log_{1/q} \frac{V}{v}\right] + 1$, gives us the condition (β_4). The condition (β_4') is valid, for example, for the following semidynamical system [21]:

$$\frac{dx}{dt} + \nabla F(x) = 0, \quad F(x) \in C^2(B_a), \ \max_{x \in B_a} F(x) < \inf_{x \in R \setminus B_a} F(x), x \in R^n.$$

Note, that for a gradient system one can prove the continuity property for the attractor.

To construct the local function of attraction to an attractor, which is required in the condition (β_3), let us consider the concept of an unstable manifold [22].

Denote by $\mathcal{W}(S, \mathcal{O})$ an unstable invariant set of an operator $S(\cdot)$ on $\mathcal{O}$

$$\mathcal{W}(S, \mathcal{O}) = \{u_0 \in \mathcal{O} : \exists u_k \in \mathcal{O}, u_k = S(u_{k+1}), \quad k = 0, 1, 2, ..\}.$$

The following theorems hold.

Lemma (O.A. Ladyzhenskaya) *Let a discrete semigroup* $\left\{S^k(\cdot), k \in N_+\right\}$ *in a closed subset* B *of a Banach space possess a compact attractor* $\mathcal{M}$*. Then*

$$\mathcal{M} = \mathcal{W}(S, B).$$

Lemma (O.A. Ladyzhenskaya, I.N. Kostin) *Let a semigroup* $\left\{S^k(\cdot), k \in N_+\right\}$ *in a closed subset* B *of a Banach space possesses a compact attractor* $\mathcal{M}$ *have a Lyapunov functional and the set of fixed points* $Z(S) = \left\{z_i : S(z_i) = z_i\right\}_{i=1}^{N}$ *of* $S(\cdot)$ *be finite. Then*

$$\mathcal{M} = \bigcup_{z \in Z(S)} \bigcup_{k \in \mathbf{N}_+} S^k\left(\mathcal{W}(S, \mathcal{O}_z)\right),$$

the sets $\mathcal{O}_z$ *being arbitrary small neighborhoods of the points* z*.*

4 Neighborhood of a fixed point

The classical results for the hyperbolic sets deal with the structure of the stable and unstable manifolds in a neighborhood $\mathcal{O}$ of a fixed hyperbolic point. One can prove that the unstable manifold $\mathcal{M}^+$ attracts $\mathcal{O}$ via operator $S(\cdot)$ at the same exponential rate ψ. For more details see, for example [22, 23]. However when a spectrum of the linear part of the original operator $S(\cdot)$ has the eigenvalue $\mu : |\mu| = 1$ a rate of attraction is a polynomial and we may use the classical results for the specific semidynamical systems [18] only.

Later on we present a generalization of the hyperbolic theory [18, 24] to the non-hyperbolic case. For some classes of the semidynamical system we shall construct an unstable manifold in a neighborhood of a fixed isolated nonhyperbolic point and estimate the rate of attraction to the manifold. These considerations suggest the concept of a polynomial contraction mapping.

Let F be a mapping of a metric space U into itself. Then u is called a fixed (stable) point of F if $F(u) = u$. The mapping F is said to be a weak (or polynomial) contraction mapping when there exists $\alpha, p > 0$ such that

$$\rho(F(u_1), F(u_2)) \leq \frac{\rho(u_1, u_2)}{(1 + \alpha\rho^p(u_1, u_2))^{1/p}} \tag{8}$$

for every pair of points $u_1, u_2 \in U$. The great variety of contracting mappings was studied in [25].

Corollary. *Every weak contraction mapping F defined on a complete metric space has a unique stable point*

$$F(u) = u.$$

For each point u^0 the iteration process $u^{n+1} = F(u^n)$ tens to the stable point and the following estimate holds

$$\rho(u, F^n(u^0)) \leq \frac{\rho(u, u^0)}{(1 + n\alpha\rho^p(u, u^0))^{1/p}}.$$

This corollary can be proved in the same way as the classical results for a contraction mapping. Note that for given F we have

$$\rho(F^n(u_1), F^n(u_2)) \leq \frac{\rho(u_1, u_2)}{(1 + n\alpha\rho^p(u_1, u_2))^{1/p}},$$

and the sum $\displaystyle\sum_{n=0}^{\infty} \rho(F^n(u^0), F^{n+1}(u^0))$ is finite and depends only on $\rho(u^0, F(u^0))$.

The weak contraction mapping (8) for the semigroup corresponding to the Chafee-Infante problem was studied in [18]. For some discrete semigroup one can prove that

$$\rho(F(u_1), F(u_2)) \leq \rho(u_1, u_2)(1 - \alpha\rho^{\tilde{p}}(u_1, u_2)),$$

which implies (8) with $p = \max\{1, \tilde{p}\}$.

Let us construct the unstable manifold in a small neighborhood of a fixed point $S(\cdot)$. Let $z = 0$, otherwise replace $S(\cdot) = S(z + \cdot) - z$. We assume that the following conditions (a) hold:

a_0) The operator $S(\cdot) : H \to H$ be the continuous mapping on a Banach space H with respect to the norm $\|\cdot\|$ and $S(0) = 0$; there exist the bounded projections $P_+, P_- : H \to H$, a bounded linear operator $L : H \to H$, the nonlinear mapping $R(h) = S(h) - Lh$ such that

a_1) $P_+ + P_- = I, \quad \|P_+\| = \|P_-\| = 1,$
a_2) $L(P_+H) = P_+H, \quad L(P_-H) \subset P_-H,$
a_3) $\|Lx\| \geq (1 + \delta_+)\|x\|, \quad \forall x \in P_+H, \quad \delta_+ \geq 0,$
a_4) $\|Ly\| \leq (1 - \delta_-)\|y\|, \quad \forall y \in P_-H, \quad \delta_- \geq 0,$
a_5) $\|R(h_1) - R(h_2)\| < \theta\Big(\max\{\|h_1\|, \|h_2\|\}\Big)\|h_1 - h_2\|, \ \forall h_i \in H$

with a continuous positive monotone nondecreasing function

$$\theta(\cdot): \ \theta(0)=0, \ \max_{h\in\mathcal{O}} \theta(\|h\|) < 1/2,$$

where x, y, h_i are arbitrary elements possessing to $\mathcal{O} \subset H$. When $\delta_+, \delta_- > 0$ the point z=0 is a hyperbolic point otherwise is a nonhyperbolic point.

By the (a_5) we have

$$a_{5_1}) \quad \|P_\pm\Big(R(h_1)-R(h_2)\Big)\| \le \theta\big(\max\{\|h_1\|,\|h_2\|\}\big)\Big(\|x_1-x_2\|+\|y_1-y_2)\|\Big),$$

here $x_i = P_+h_i$, $y_i = P_-h_i$.

Replace the operator $S(h) = Lh + R(h)$ for $h = x+y$, $x \in P_+(\mathcal{O})$, $y \in P_-(\mathcal{O})$ in the following way

$$S(h) = \begin{cases} S_+(x+y) = L_+x + R_+(x+y), \text{ here } S_\pm(\cdot) = P_\pm S(\cdot) \\ S_-(x+y) = L_-y + R_-(x+y), \ L_\pm = P_\pm L, \ R_\pm(\cdot) = P_\pm R(\cdot). \end{cases}$$

Let us consider the set $A_\gamma(\mathcal{O})$ of continuous functions $g(x): P_+\mathcal{O} \to P_-\mathcal{O}$, such that

$$g(0)=0, \quad \|g(x_1)-g(x_2)\| \le \gamma\|x_1-x_2\|,$$

and introduce the notation: $S_\pm(x+g(x)) = S_{\pm,g}(x)$, $h = x+g(x)$.

Lemma 1. *Under the assumptions (a) let $\gamma \le 1$. Then the equation $S_+(x+g(x)) = \chi$ possesses the unique solution $x \in P_+\mathcal{O}$ for any $g(\cdot) \in A_\gamma(\mathcal{O})$ and each $\chi \in P_+\mathcal{O}$.*

Proof. According to (a_4), (a_{5_1}) the operator $L_+^{-1}R_+(g(x)+x)$ is a contraction mapping and the equation $S_+(x+g(x)) \equiv L_+x + R_+(g(x)+x) = \chi$ possesses the unique solution x for any $\chi \in P_+\mathcal{O}$: $x = S_{+,g}^{-1}(\chi)$.

We shall construct the manifold $\mathcal{M}^+ = \{x+\mathbf{g}(x), x \in P_+H\}$ with $\mathbf{g} \in A_\gamma(\mathcal{O})$, prove that $\mathcal{M}^+$ attracts each trajectory and find the rate of attraction.

Theorem 5 . *Let the mapping $S(\cdot)$ in $\mathcal{O} = \{h: |x|,|y| < r\}$ satisfies the above assumptions (a). Suppose in addition that for some γ, corresponding to*

$$\begin{cases} \gamma = 1, & \delta_+ = \delta_- = 0, \\ \dfrac{\gamma(1-\delta_-) + \theta\Big((\gamma+1)r\Big)(\gamma+1)}{1+\delta_+ - [\delta_+]\theta\Big((\gamma+1)r\Big)(\gamma+1)} \le \gamma \le 1, & \delta_+ + \delta_- > 0, \\ \qquad [\delta_+] = \begin{cases} 0, & \delta_+ = 0, \\ 1, & \delta_+ \ne 0 \end{cases} \end{cases}$$

there exist $\alpha > 0, p \ge 1$ such that, for any $x, \ \tilde{x} \in P_+\mathcal{O}$, $y \in P_-\mathcal{O}$ and any $g(\cdot), \tilde{g}(\cdot) \in A_\gamma(\mathcal{O})$ the inequalities (b) hold

$$\begin{aligned} b_1) \quad & \|S_{-,g}(x) - S_{-,\tilde{g}}(x)\| + \gamma\|S_{+,g}(x) - S_{+,\tilde{g}}(x)\| \le \\ & \le \Big(1-\alpha\|g(x)-\tilde{g}(x)\|^p\Big)\|g(x)-\tilde{g}(x)\|, \qquad 1-\alpha|2r|^p > 0, \\ b_2) \quad & \|S_{-,g}(x) - S_{-,g}(\tilde{x})\| \le \|x-\tilde{x}\|, \\ b_3) \quad & \|S_{+,g}(x) - S_{+,g}(\tilde{x})\| \ge \|x-\tilde{x}\|. \end{aligned}$$

Then

1. There exists the manifold $\mathcal{M}^+ = \{x + \mathbf{g}(x), x \in P_+H\}$ assigned by $\mathbf{g} : P_+H \to P_-H$ in $A_\gamma(\mathcal{O})$.

2. The manifold is an invariant set $S(\mathcal{M}^+) = \mathcal{M}^+$, there exists the inverse mapping $S^{-n}(m), n = 1, 2, ...$ for $\forall m \in \mathcal{M}^+$ and $S^{-n}(m) \subset \mathcal{O}$.

3. The following estimate is valid

$$\mathrm{dist}(\mathcal{M}^+, S^n(h)) \leq \frac{|2r|}{(1 + n\alpha|2r|^p)^{1/p}}, \text{ as } S^k(h) \in \mathcal{O}, 0 \leq k \leq n$$

Remark. According to $(a_5), (b_{2,3})$ we have

$$\|S_{-,g}(x) - S_{-,g}(\tilde{x})\| \leq ((1 - \delta_-)\gamma + \theta((\gamma + 1)r)(\gamma + 1))\|x - \tilde{x}\|$$

$$\|S_{+,g}(x) - S_{+,g}(\tilde{x})\| \geq (1 + \delta_+ - [\delta_+]\theta((\gamma + 1)r)(\gamma + 1))\|x - \tilde{x}\|$$

Proof of the theorem. Let us fixed the unstable manifold $\mathcal{M}^+$ in the following way

$$y = g(x), \quad x \in P_+\mathcal{O}.$$

We shall find the mapping $g(x)$ in the class $A_\gamma(\mathcal{O})$ which was described above. The invariance of $\mathcal{M}^+$ gives us

$$g\left(S_+(x + g(x))\right) = S_-\left(x + g(x)\right), \text{ or } g\left(S_{+,g}(x)\right) = S_{-,g}(x).$$

Inverse the operator $S_{+,g}(x) = \chi$ and rewrite the above equality in the following form

$$g(\chi) = S_{-,g}\left(S_{+,g}^{-1}(\chi)\right) \equiv \mathcal{F}(g, \chi).$$

This equation defines the mapping $g(\cdot)$. In oder to prove the solvability of this problem we make use of Newton's like method

$$g^{n+1}(\chi) = \mathcal{F}(g^n, \chi). \tag{9}$$

According to remark, for $\delta_+ + \delta_- > 0$, we have

$$\|\mathcal{F}(g, \chi) - \mathcal{F}(g, \tilde{\chi})\| = \|S_{-,g}(S_{+,g}^{-1}(\chi)) - S_{-,g}(S_{+,g}^{-1}(\tilde{\chi}))\| \leq$$

$$\leq ((1 - \delta_-)\gamma + \theta((\gamma + 1)r)(\gamma + 1))\|S_{+,g}^{-1}(\chi) - S_{+,g}^{-1}(\tilde{\chi})\| \leq$$

$$\leq \frac{(1 - \delta_-)\gamma + \theta((\gamma + 1)r)(\gamma + 1)}{1 + \delta_+ - [\delta_+]\theta((\gamma + 1)r)(\gamma + 1)}\|\chi - \tilde{\chi}\| \leq \gamma\|\chi - \tilde{\chi}\|$$

$$\|\mathcal{F}(g, \chi) - \mathcal{F}(g, \tilde{\chi})\| \leq \gamma\|\chi - \tilde{\chi}\|,$$

$$\|\mathcal{F}(g, \chi)\| \leq \gamma\|\chi\|,$$

which implies that the operator $\mathcal{F}$ be a mapping of the set $A_\gamma(\mathcal{O})$ into itself. Similar inequalities one can prove with $(b_{2,3})$ as $\delta_+ = \delta_- = 0$ and $\gamma = 1$.

Now we shall prove, that the operator $\mathcal{F}$ is a weak contraction operator on $A_\gamma(\mathcal{O})$ with respect to the metric $C(P_+\mathcal{O})$: $\max\limits_{\chi\in P_+\mathcal{O}}\|g(\chi)\| = |g|$. Assuming $\delta_+ + \delta_- > 0$ we obtain

$$\|\mathcal{F}(g,\chi) - \mathcal{F}(\tilde{g},\chi)\| = \|S_{-,g}(S^{-1}_{+,g}(\chi)) - S_{-,\tilde{g}}(S^{-1}_{+,\tilde{g}}(\chi))\| \leq$$

$$\leq \|S_{-,g}(S^{-1}_{+,g}(\chi)) - S_{-,\tilde{g}}(S^{-1}_{+,g}(\chi))\| + \|S_{-,\tilde{g}}(S^{-1}_{+,g}(\chi)) - S_{-,\tilde{g}}(S^{-1}_{+,\tilde{g}}(\chi))\| \leq$$

$$\leq \|S_{-,g}(S^{-1}_{+,g}(\chi)) - S_{-,\tilde{g}}(S^{-1}_{+,g}(\chi))\| +$$

$$+((1-\delta_-)\gamma + \theta((\gamma+1)r)(\gamma+1))\|S^{-1}_{+,g}(\chi) - S^{-1}_{+,\tilde{g}}(\chi)\|.$$

By the definition we have $x = S^{-1}_{+,g}(\chi)$, $\tilde{x} = S^{-1}_{+,\tilde{g}}(\chi)$ and $S_{+,g}(x) = \chi = S_{+,\tilde{g}}(\tilde{x})$. Subtract from the both path the term $S_{+,\tilde{g}}(x)$ and with remark we obtain

$$\|S_{+,g}(x) - S_{+,\tilde{g}}(x)\| = \|S_{+,\tilde{g}}(\tilde{x}) - S_{+,\tilde{g}}(x)\| \geq$$

$$\geq (1 + \delta_+ - [\delta_+]\theta((\gamma+1)r)(\gamma+1))\|x - \tilde{x}\|.$$

This implies with (b_1) the following estimate

$$\|\mathcal{F}(g,\chi) - \mathcal{F}(\tilde{g},\chi)\| \leq \|S_{-,g}(x) - S_{-,\tilde{g}}(x)\| + \gamma\|S_{+,g}(x) - S_{+,\tilde{g}}(x)\| \leq$$

$$\leq \|g(x) - \tilde{g}(x)\|\Big(1 - \alpha\|g(x) - \tilde{g}(x)\|^q\Big). \tag{10}$$

This inequality is valid for any χ and we have

$$|\mathcal{F}(g) - \mathcal{F}(\tilde{g})| \leq \|g(x) - \tilde{g}(x)\|\Big(1 - \alpha\|g(x) - \tilde{g}(x)\|^p\Big).$$

The right hand side of the inequality is monotonically increasing in $\|g(x) - \tilde{g}(x)\|$ as $x \in P_+\mathcal{O}$ and hence

$$|\mathcal{F}(g) - \mathcal{F}(\tilde{g})| \leq |g - \tilde{g}|\Big(1 - \alpha|g - \tilde{g}|^p\Big) \leq \frac{|g - \tilde{g}|}{\Big(1 + \alpha|g - \tilde{g}|^p\Big)^{1/p}}.$$

In this way the mapping $\mathcal{F}(g,\chi)$ is a weak contraction mapping. According to lemma this mapping has the unique stable point $\mathbf{g}$, and the iterative process (9) tens to the $\mathbf{g}$ for each initial data, for example, $g^0 \equiv 0$. The manifold $\mathcal{M}^+$ is constructed.

The existence of the inverse mapping for $S(h)$ when $h \in \mathcal{M}$ is followed from the existence of the inverse mapping for the $S_{+,\mathbf{g}}(x)$. Moreover, this implies with (b_3) the inequalities

$$\|S^{-1}_{+,g}(x)\| \leq \|x\|, \|S^{-n}_{+,g}(x)\| \leq \|x\|,$$

and inclusions $S^{-n}(m) \subset \mathcal{O}$ as $n \geq 0$.

Let us prove that each $h = x+y \in \mathcal{O}$ attracts to $\mathcal{M}^+$. Choose any point $h^1 = S(h)$, there is the point $m^1 \in \mathcal{M}^+$ such that $m^1 = P_+S(h) + g(P_+S(h))$. The invariance of $\mathcal{M}^+$, together with the inequality (b_1), gives us

$$\|h^1 - m^1\| = \|P_-S(h) - g(P_+S(h))\| \leq$$

$$\leq \|P_-S(y+x) - P_-S(g(y)+x)\| + \|g(P_+S(g(y)+x)) - g(P_+S(y+x))\| \leq$$
$$\leq \|P_-S(y+x) - P_-S(g(y)+x)\| + \|P_+S(y+x) - P_+S(g(y)+x)\| \leq$$
$$\leq \frac{\|y-g(x)\|}{(1+\alpha\|y-g(x)\|^p)^{1/p}}.$$

Whence

$$\mathrm{dist}_{\mathrm{H}}(\mathcal{M}^+, S^n(h)) \leq \|h^n - m^n\| \leq$$
$$\leq \frac{\|y-g(x)\|}{(1+n\alpha\|y-g(x)\|^p)^{1/p}},$$

where $m^n = P_+(h^n) + g(P_+(h^n)) \in \mathcal{M}^+$. The theorem is proved.

The above theorem implies.

Theorem 6 *Under the assumptions of Theorem 5 suppose in addition that for the operator $S_{+,g}(x)$ the following estimate is valid*

$$b_4) \quad \|S_{+,g}(x)\| \geq \Big(1+\beta\|x\|^q\Big)\|x\|, \quad \beta > 0, q \geq 1$$

for $\forall x \in \mathcal{O}$. Then for any $m \in \mathcal{M}^+$ we have

$$\begin{aligned}
&\|S_+^n(m)\| \geq \frac{\|x\|}{(1-n\tilde{\beta}\|x\|^q)^{1/q}}, \qquad S^k(m) \in \mathcal{O},\ 0 \leq k \leq n \\
&\|S_+^{-n}(m)\| \leq \frac{|r|}{(1+n\tilde{\beta}|r|^q)^{1/q}}, \qquad \tilde{\beta} = \min\{\beta, |r|^{-p}\},\ n = 1,2,... \\
&\|S^{-n}(m)\| \to 0 \qquad n \to \infty
\end{aligned}$$

Proof. The existence of the inverse mapping for $S(h)$ as $h \in \mathcal{M}$ follows from the assumptions of the theorem. This, together with (b_4), implies

$$\|S_+(x+\mathbf{g}(x))\| \geq \frac{\|x\|}{(1-\tilde{\beta}\|x\|^q)^{1/q}}$$

with some $\tilde{\beta} = \min\{\beta, |r|^{-p}\}$. The above inequality gives us

$$\|S_{+,\mathbf{g}}^{-n}(\chi)\| \leq \frac{\|\chi\|}{(1+n\tilde{\beta}\|\chi\|^q)^{1/q}}, \quad S_{+,\mathbf{g}}(x) = \chi,$$

and $\|S_{+,\mathbf{g}}^{-n}(\chi)\| \to 0$ as $n \to \infty$. Which, together with the continuity of the mapping $\mathbf{g}$, shows that $\|S^{-n}(m)\| \to 0$ as $n \to \infty$. This completes the prove.

It should be noted, that for $\delta_- > 0$ the assumptions (a), (b) and the condition on γ are valid for some $\mathcal{O}_r \subset \mathcal{O}$ and $r > 0$. Moreover, it is easy to prove that

$$\begin{aligned}
b_1') \quad &\|S_{-,g}(x) - S_{-,\tilde{g}}(x)\| + \gamma\|S_{+,g}(x) - S_{+,\tilde{g}}(x)\| \leq \\
&\leq \Big(1 - \delta_- + 2\theta(r(1+\gamma))\Big)\|g(x) - \tilde{g}(x)\|.
\end{aligned}$$

Hence the current result implies the classical theorem [23] for an unstable manifold in a neighborhood of a fixed hyperbolic point:

Theorem 7 *Under the assumptions (a) assume that* $\delta = \delta_+ = \delta_- > 0$ *and for the given operator* $S(\cdot) : H \to H$ *the following estimate holds*

$$\|R(h_1) - R(h_2)\| < \hat{\theta}\|h_1 - h_2\|, \quad h_i \in H, \ \hat{\theta} < \delta/2.$$

Then the inequalities 1,2 of Theorem 5 are valid.

3. *The manifold* $\mathcal{M}^+$ *attracts each point* h *at the exponential rate*

$$\mathrm{dist}(\mathcal{M}^+, S^n(h)) \leq (1 - \delta + 2\hat{\theta})^n 2r.$$

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