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ON A PROBLEM OF ADJAMAGBO

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Abstract

In this note we construct an unramified, finite and birational polynomial map from $\mathbb{C}^2$ to an irreducible affine variety V of dimension two which is injective on each line through the origin but which is not injective (hence no isomorphism). More precisely, every point of the singular locus of V has exactly two preimages.

1 Introduction

The Jacobian Conjecture asserts that every unramified polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ (i.e. a map whose Jacobian matrix has maximal rank at each point of $\mathbb{C}^n$) is an isomorphism.
Several attempts have been made to generalise this conjecture by allowing one of the $\mathbb{C}^n$'s to be replaced by an irreducible affine variety of dimension n. One such attempt was made by Bass in [4]. He conjectured that any étale map (i.e. flat and unramified) from a complex irreducible affine and unirational variety of dimension n whose invertible regular functions are all constants to $\mathbb{C}^n$ is an isomorphism. However, based on a result of Kulikov in [8], Adjamagbo showed in [1] that this conjecture is false for any $n \geq 2$ and is true if $n = 1$.
In this paper we consider the following problem, due to Adjamagbo, which arose from an attempt of his to generalise the Jacobian Conjecture.

Adjamagbo's Problem

Let $V \subset \mathbb{C}^m$ $(m \geq 1)$ be an irreducible affine variety of dimension n and $F : \mathbb{C}^n \to V$ an unramified polynomial map. Assume that F satisfies the following conditions
a) F is birational
b) F is finite
c) F is injective on each line through the origin of $\mathbb{C}^n$.
Does it follow that F is an isomorphism ?

Various special cases of this problem have been investigated and seem to indicate that the answer to Adjamagbo's Problem is affirmative: namely if $n = 1$ it is well-known that every unramified injective polynomial map from $\mathbb{C}$ to $\mathbb{C}^m$ is an embedding i.e. $F : \mathbb{C} \to \mathbb{C}^m$ is an isomorphism. Furthermore in case $V = \mathbb{C}^n$ each of the two conditions a) and b) imply that F is an isomorphism: if condition a) is satisfied this is Keller's theorem (see [7] or [3] or [5]). If condition b) is satisfied the result is "classical" (see [3] or [5]). Furthermore if $n = 2$ it was shown by Gwoździewicz in [6] that already injectivity of F on one line is sufficient to imply that $F : \mathbb{C}^2 \to \mathbb{C}^2$ is an isomorphism. So in case $V = \mathbb{C}^2$ each of the three conditions a), b), c) above suffices to imply that F is an isomorphism. However the main result of this paper, Proposition 2.1, shows that there exists an irreducible affine variety V of dimension two in $\mathbb{C}^3$ and an unramified polynomial map $F : \mathbb{C}^2 \to V$ which satisfies all three conditions a), b) and c) and is nevertheless not injective (hence no isomorphism). By extending this example in an obvious way we obtain a negative answer to Adjamagbo's Problem for all $n \geq 2$.

2 The example

Throughout this section we have the following notations: by $f = (f_1, f_2, f_3) : \mathbb{C}^2 \to \mathbb{C}^3$ we denote the polynomial map given by the following polynomials

$$f_1 = X_1 + X_1^2 X_2^2,\ f_2 = X_2 - X_1 \text{ and } f_3 = X_1^2$$

By $V \subset \mathbb{C}^3$ we denote the Zariski closure of $f(\mathbb{C}^2)$ in $\mathbb{C}^3$. So f induces a morphism $F : \mathbb{C}^2 \to V$. Finally S denotes the singular locus of V.

Proposition 2.1 *V is an irreducible affine variety of dimension two such that the invertible elements of its coordinate ring are constants. Furthermore, the map F has the following properties.*
1) F is unramified.
2) F is injective on each line through the origin of $\mathbb{C}^2$.
3) F is finite.
4) F is birational.
5) F is not injective (hence no isomorphism). More precisely, if $y \in S$ then $\#F^{-1}(y) = 2$ and if $y \in V \backslash S$ then $\#F^{-1}(y) = 1$.

Proof. The ringhomomorphism $f^* : \mathbb{C}[Y] := \mathbb{C}[Y_1, Y_2, Y_3] \to \mathbb{C}[f] := \mathbb{C}[f_1, f_2, f_3]$ defined by $f^*(Y_i) = f_i$ for all i is surjective. Hence, since $\mathbb{C}[f]$ has dimension two the kernel of f^* is generated by one irreducible polynomial in $\mathbb{C}[Y]$. Consequently the Zariski closure of $f(\mathbb{C}^2)$ in $\mathbb{C}^3$, which equals the zero-set of this polynomial in $\mathbb{C}^3$, is an irreducible affine variety of dimension two contained in $\mathbb{C}^3$. The statement concerning the units of the coordinate ring of V is obvious since $\mathbb{C}[f] \subset \mathbb{C}[X_1, X_2]$.
1) To see that F is unramified we need to show that the three 2×2 minors of the Jacobian matrix of F have no common zero in $\mathbb{C}^2$. However from the equalities $\det J(f_1, f_2) = 1 + 2X_1^2 X_2 + 2X_1 X_2^2$ and $\det J(f_1, f_3) = -4X_2 X_1^3$ one readily verifies that already these two minors have no common zero in $\mathbb{C}^3$.
2) To see that F is injective on each line through $0 \in \mathbb{C}^2$ we first consider the lines spanned by a vector of the form $(1, v_2)$, with $v_2 \in \mathbb{C}$ and assume that $F(t, tv_2) = F(s, sv_2)$ for some $t, s \in \mathbb{C}$. Then in particular, looking at the third component of F, we get $t^2 = s^2$, so $t = s$ or $t = -s$. If $t = -s$ then $f_1(t, tv_2) = f_1(s, sv_2)$ implies that $-s + s^4 v_2^4 = s + s^4 v_2^4$, whence $s = 0$ and hence $t = -s = 0$, so $s = t$. Consequently

$t = s$ in any case, so F is injective on the lines spanned by the vectors of the form $(1, v_2)$. Finally, looking at the second component of F, one readily verifies that F is also injective on the Y-axis.
3) To show the last three points of the proposition we compute the reduced Gröbner basis B of the ideal

$$I := (Y_1 - f_1, Y_2 - f_2, Y_3 - f_3) \subset \mathbb{C}[X_1, X_2, Y_1, Y_2, Y_3]$$

with respect to the pure lexicographical ordering with $X_1 > X_2 > Y_1 > Y_2 > Y_3$. The result is that $B = \{b_1, \ldots, b_5\}$, where

$$b_1 = X_1 - X_2 + Y_2$$

$$b_2 = X_2^2 - 2Y_2X_2 + Y_2^2 - Y_3$$

$$b_3 = c_1(Y)X_2 + d_1(Y)$$

$$b_4 = c_2(Y)X_2 + d_2(Y)$$

$$b_5 = -2Y_1Y_3^2 - 4Y_3^2Y_2 + Y_3^4 - 2Y_3^3Y_2^2 - Y_3 + Y_1^2 - 2Y_1Y_3Y_2^2 + Y_3^2Y_2^4$$

$$c_1 = 2Y_1 + Y_2 - 2Y_3^2 \text{ and } d_1 = -Y_3^2Y_2 + Y_2^3Y_3 - 3Y_1Y_2 - Y_2^2 - 2Y_3$$

$$c_2 = 1 + 2Y_3Y_2 \text{ and } \quad d_2 = -(Y_1 + Y_3Y_2^2 + Y_2 - Y_3^2).$$

From b_1 and b_2 we see that $X_1 - X_2 + f_2 = 0$ and $X_2^2 - 2f_2X_2 + (f_2^2 - f_3) = 0$. So $\mathbb{C}[X_1, X_2]$ is finite over $\mathbb{C}[f]$ i.e. F is finite.
4) From b_4 we get $X_2 = -d_2(f)/c_2(f) \in \mathbb{C}(f)$ ($c_2(f)$ is non-zero). Hence also $X_1 = X_2 - f_2 \in \mathbb{C}(f)$, whence $\mathbb{C}(X_1, X_2) = \mathbb{C}(f)$ i.e. F is birational.
5) Finally we show that F is not injective. First observe that since $b := b_5$ is the only polynomial in $\mathbb{C}[Y]$ contained in B, it follows from the relation algorithm (see for example Proposition C.2.2 in [5]) that $\ker f^* = (b)$. So S is given by the ideal $(b, b_{Y_1}, b_{Y_2}, b_{Y_3})$, where b_{Y_i} denotes the partial derivative of b with respect to Y_i. Computing the reduced Gröbner basis of this ideal with respect to the pure lexicographical ordering with $Y_1 > Y_2 > Y_3$ we find $\{c_1, c_2\}$! So $S = V(c_1, c_2)$. In particular, looking at c_2 we see that if $y \in S$ then $y_2y_3 \neq 0$. Since $F : \mathbb{C}^2 \to V$ is finite, it is surjective, so $\#F^{-1}(y) \geq 1$ for all $y \in V$. Assume first that $y \in V \backslash S$. So either $c_1(y)$ or $c_2(y)$ is non-zero. Let $x \in F^{-1}(y)$. If $c_1(y) \neq 0$ it follows from b_3 that $x_2 = -d_1(y)/c_1(y)$ and using b_1 we get that $x_1 = -y_2 - d_1(y)/c_1(y)$. So $\#F^{-1}(y) = 1$. A similar argument, using b_4, shows that $\#F^{-1}(y) = 1$ if $c_2(y) \neq 0$. Finally suppose that $y \in S$. So, as observed above $y_3 \neq 0$. Choose $z_1 \in \mathbb{C}$ with $z_1^2 = y_3$.

Claim: $F^{-1}(y) = \{(z_1, z_1 + y_2), (-z_1, -z_1 + y_2)\}$.

The inclusion "$\subset$" follows immediately from $x_2 - x_1 = y_2$ and $x_1^2 = y_3$. To see the converse inclusion it remains to see that $z_1 + z_1^2(z_1 + y_2)^2 = y_1$ and $-z_1 + (-z_1)^2(-z_1 + y_2)^2 = y_1$ or equivalently that $z_1(1 + 2y_3y_2) + y_3y_2^2 + y_3^2 - y_1 = 0$ and $-z_1(1 + 2y_3y_2) + (y_3y_2^2 + y_3^2 - y_1) = 0$. However, $y \in S$ i.e. $c_1(y) = 0$ and $c_2(y) = 0$. So $1 + 2y_3y_2 = c_1(y) = 0$ and $y_3y_2^2 + y_3^2 - y_1 = (-1/2)c_1(y) + (1/2)y_2c_2(y) = 0$, which completes the proof $\square$

Corollary 2.2 *Adjamagbo's Problem has a negative answer for all $n \geq 2$.*

Proof. The case $n = 2$ is 2.1. So let $n \geq 3$ and define $f_i = X_{i-1}$ for all $4 \leq i \leq n+1$ and f_1, f_2, f_3 as in 2.1. Let $f : \mathbb{C}^n \to \mathbb{C}^{n+1}$ be given by $f(x) = (f_1(x), \ldots, f_{n+1}(x))$ for all $x \in \mathbb{C}^n$ and denote by V the Zariski closure of $f(\mathbb{C}^n)$ in $\mathbb{C}^{n+1}$. Then it is left to the reader to verify that V is an irreducible affine variety of dimension n and that the induced map $F : \mathbb{C}^n \to V$ satisfies all five properties of 2.1, thereby supplying a negative answer to Adjamagbo's Problem □

Final remark. The map F constructed above is not étale: namely if F is étale then (by the preservation of normality under étale maps (see [2],(12)i)) V is normal i.e. $\mathbb{C}[f]$ is integrally closed. Since $\mathbb{C}(f) = \mathbb{C}(X)$ and $\mathbb{C}[X]$ is integral over $\mathbb{C}[f]$ this implies that $\mathbb{C}[f] = \mathbb{C}[X]$ i.e. F is an isomorphism, a contradiction.

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