ON A PROBLEM OF ADJAMAGBO

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Abstract

In this note we construct an unramified, finite and birational polynomial map from \( \mathbb{C}^2 \) to an irreducible affine variety \( V \) of dimension two which is injective on each line through the origin but which is not injective (hence no isomorphism). More precisely, every point of the singular locus of \( V \) has exactly two preimages.

1 Introduction

The Jacobian Conjecture asserts that every unramified polynomial map \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \) (i.e. a map whose Jacobian matrix has maximal rank at each point of \( \mathbb{C}^n \)) is an isomorphism.

Several attempts have been made to generalise this conjecture by allowing one of the \( \mathbb{C}^n \)'s to be replaced by an irreducible affine variety of dimension \( n \). One such attempt was made by Bass in [4]. He conjectured that any étale map (i.e. flat and unramified) from a complex irreducible affine and unirational variety of dimension \( n \) whose invertible regular functions are all constants to \( \mathbb{C}^n \) is an isomorphism. However, based on a result of Kulikov in [8], Adjamagbo showed in [1] that this conjecture is false for any \( n \geq 2 \) and is true if \( n = 1 \).

In this paper we consider the following problem, due to Adjamagbo, which arose from an attempt of his to generalise the Jacobian Conjecture.

Adjamagbo’s Problem

Let \( V \subset \mathbb{C}^m \) (\( m \geq 1 \)) be an irreducible affine variety of dimension \( n \) and \( F : \mathbb{C}^n \rightarrow V \) an unramified polynomial map. Assume that \( F \) satisfies the following conditions

a) \( F \) is birational
b) \( F \) is finite
c) \( F \) is injective on each line through the origin of \( \mathbb{C}^n \).

Does it follow that \( F \) is an isomorphism?

Various special cases of this problem have been investigated and seem to indicate that the answer to Adjamagbo’s Problem is affirmative: namely if \( n = 1 \) it is well-known that every unramified injective polynomial map from \( \mathbb{C} \) to \( \mathbb{C}^n \) is an isomorphism (i.e. \( F : \mathbb{C} \rightarrow \mathbb{C}^n \) is an isomorphism. Furthermore in case \( V = \mathbb{C}^n \) each of the two conditions a) and b) imply that \( F \) is an isomorphism: if condition a) is satisfied this is Keller’s theorem (see [7] or [3] or [5]). If condition b) is satisfied the result is

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"classical" (see [3] or [5]). Furthermore if \( n = 2 \) it was shown by Gwoździewicz in [6] that already injectivity of \( F \) on one line is sufficient to imply that \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is an isomorphism. So in case \( V = \mathbb{C}^2 \) each of the three conditions a), b), c) above suffices to imply that \( F \) is an isomorphism. However the main result of this paper, Proposition 2.1, shows that there exists an irreducible affine variety \( V \) of dimension two in \( \mathbb{C}^3 \) and an unramified polynomial map \( F : \mathbb{C}^2 \to V \) which satisfies all three conditions a), b) and c) and is nevertheless not injective (hence no isomorphism). By extending this example in an obvious way we obtain a negative answer to Adjamagbo’s Problem for all \( n \geq 2 \).

2 The example

Throughout this section we have the following notations: by \( f = (f_1, f_2, f_3) : \mathbb{C}^2 \to \mathbb{C}^3 \) we denote the polynomial map given by the following polynomials

\[
\begin{align*}
    f_1 &= X_1 + X_2^2 X_3^2, \\
    f_2 &= X_2 - X_1, \\
    f_3 &= X_1^2
\end{align*}
\]

By \( V \subset \mathbb{C}^3 \) we denote the Zariski closure of \( f(\mathbb{C}^2) \) in \( \mathbb{C}^3 \). So \( f \) induces a morphism \( F : \mathbb{C}^2 \to V \). Finally \( S \) denotes the singular locus of \( V \).

Proposition 2.1 \( V \) is an irreducible affine variety of dimension two such that the invertible elements of its coordinate ring are constants. Furthermore, the map \( F \) has the following properties.

1. \( F \) is unramified.
2. \( F \) is injective on each line through the origin of \( \mathbb{C}^2 \).
3. \( F \) is finite.
4. \( F \) is birational.
5. \( F \) is not injective (hence no isomorphism). More precisely, if \( y \in S \) then \( \# F^{-1}(y) = 2 \) and if \( y \in \mathbb{C}^2 \setminus S \) then \( \# F^{-1}(y) = 1 \).

Proof. The ring homomorphism \( f^* : \mathbb{C}[Y] := \mathbb{C}[Y_1, Y_2, Y_3] \to \mathbb{C}[f] := \mathbb{C}[f_1, f_2, f_3] \) defined by \( f^*(Y_i) = f_i \) for all \( i \) is surjective. Hence, since \( \mathbb{C}[f] \) has dimension two the kernel of \( f^* \) is generated by one irreducible polynomial in \( \mathbb{C}[Y] \). Consequently the Zariski closure of \( f(\mathbb{C}^2) \) in \( \mathbb{C}^3 \), which equals the zero-set of this polynomial in \( \mathbb{C}^3 \), is an irreducible affine variety of dimension two contained in \( \mathbb{C}^3 \). The statement concerning the units of the coordinate ring of \( V \) is obvious since \( \mathbb{C}[f] \subset \mathbb{C}[X_1, X_3] \).

1) To see that \( F \) is unramified we need to show that the three \( 2 \times 2 \) minors of the Jacobian matrix of \( F \) have no common zero in \( \mathbb{C}^2 \). However from the equalities

\[
\begin{align*}
    \det J(f_1, f_2) &= 1 + 2X_1^2 X_2 + 2X_1 X_2^2, \\
    \det J(f_1, f_3) &= -4X_2 X_3
\end{align*}
\]

one readily verifies that already these two minors have no common zero in \( \mathbb{C}^3 \).

2) To see that \( F \) is injective on each line through \( 0 \in \mathbb{C}^2 \) we first consider the lines spanned by a vector of the form \( (1, v_2) \), with \( v_2 \in \mathbb{C} \) and assume that \( F(t, tv_2) = F(s, sv_2) \) for some \( t,s \in \mathbb{C} \). Then in particular, looking at the third component of \( F \), we get \( t^2 = s^2 \), so \( t = s \) or \( t = -s \). If \( t = -s \) then \( f_1(t, tv_2) = f_1(s, sv_2) \) implies that \( -s + s^4 v_2^2 = s + s^4 v_2^2 \), whence \( s = 0 \) and hence \( t = -s = 0 \), so \( s = t \). Consequently
$t = s$ in any case, so $F$ is injective on the lines spanned by the vectors of the form $(1, v_2)$. Finally, looking at the second component of $F$, one readily verifies that $F$ is also injective on the $Y$-axis.

3) To show the last three points of the proposition we compute the reduced Gröbner basis of the ideal

$$I := (Y_1 - f_1, Y_2 - f_2, Y_3 - f_3) \subseteq \mathbb{C}[X_1, X_2, Y_1, Y_2, Y_3]$$

with respect to the pure lexicographical ordering with $X_1 > X_2 > Y_1 > Y_2 > Y_3$. The result is that $B = \{b_1, \ldots, b_5\}$, where

$$b_1 = X_1 - X_2 + Y_2$$
$$b_2 = X_2^2 - 2Y_2X_2 + Y_2^2 - Y_3$$
$$b_3 = c_1(Y)X_2 + d_1(Y)$$
$$b_4 = c_2(Y)X_2 + d_2(Y)$$
$$b_5 = -2Y_1Y_3^2 - 4Y_3^2Y_2 + Y_3^4 - 2Y_3^3Y_2^2 - Y_3 + Y_1^2 - 2Y_1Y_3Y_2^2 + Y_2^2Y_2^2$$

From $b_1$ and $b_2$ we see that $X_1 - X_2 + f_2 = 0$ and $X_2^2 - 2f_2X_2 + (f_2 - f_3) = 0$. So $\mathbb{C}[X_1, X_2]$ is finite over $\mathbb{C}[f]$ i.e. $F$ is finite.

4) From $b_4$ we get $X_2 = -d_2(f)/c_2(f) \in \mathbb{C}(f)$ ($c_2(f)$ is non-zero). Hence also $X_1 - X_2 - f_2 \in \mathbb{C}(f)$, whence $\mathbb{C}(X_1, X_2) = \mathbb{C}(f)$ i.e. $F$ is birational.

5) Finally we show that $F$ is not injective. First observe that since $x \in F^*$ is the only polynomial in $\mathbb{C}[Y]$ contained in $B$, it follows from the relation algorithm (see for example Proposition C.2.2 in [5]) that $\ker f^* = (b)$. So $S$ is given by the ideal $(b, b_1, b_2, b_3)$, where $b_1$ denotes the partial derivative of $b$ with respect to $Y_1$. Computing the reduced Gröbner basis of this ideal with respect to the pure lexicographical ordering with $Y_1 > Y_2 > Y_3$ we find $\{c_1, c_2\}$ ! So $S = V(c_1, c_2)$. In particular, looking at $c_2$ we see that if $y \in S$ then $y_2y_3 \neq 0$. Since $F : \mathbb{C}^2 \to V$ is finite, it is surjective, so $\#F^{-1}(y) \geq 1$ for all $y \in V$. Assume first that $y \in V \setminus S$. So either $c_1(y)$ or $c_2(y)$ is non-zero. Let $x \in F^{-1}(y)$. If $c_1(y) \neq 0$ it follows from $b_3$ that $x_2 = -d_1(y)/c_1(y)$ and using $b_1$ we get that $x_1 = -y_2 - d_1(y)/c_1(y)$. So $\#F^{-1}(y) = 1$. A similar argument, using $b_4$, shows that $\#F^{-1}(y) = 1$ if $c_2(y) \neq 0$. Finally suppose that $y \in S$. So, as observed above $y_3 \neq 0$. Choose $z_1 \in \mathbb{C}$ with $z_1^2 = y_3$.

Claim: $F^{-1}(y) = \{(z_1, z_1 + y_2), (-z_1, -z_1 + y_2)\}$.

The inclusion “$\subset$” follows immediately from $x_2 - x_1 = y_2$ and $x_1^2 = y_3$. To see the converse inclusion it remains to see that $z_1 + z_1^2(y_1 + y_2) = y_1$ and $-z_1 + (-z_1)^2(y_1 + y_2) = y_1$ or equivalently that $z_1(1 + 2y_3y_2) + y_3^2 + y_3^2 - y_1 = 0$ and $-z_1(1 + 2y_3y_2) + y_3^2 + y_3^2 - y_1 = 0$. However, $y \in S$ i.e. $c_1(y) = 0$ and $c_2(y) = 0$. So $1 + 2y_3y_2 = c_1(y) = 0$ and $y_3^2 + y_3^2 - y_1 = (-1/2)c_1(y) + (1/2)y_2c_2(y) = 0$, which completes the proof $\Box$. 

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Corollary 2.2 Adjmagbo’s Problem has a negative answer for all \( n \geq 2 \).

Proof. The case \( n = 2 \) is 2.1. So let \( n \geq 3 \) and define \( f_i = x_{i-1} \) for all \( 4 \leq i \leq n + 1 \) and \( f_1, f_2, f_3 \) as in 2.1. Let \( f : \mathbb{C}^n \to \mathbb{C}^{n+1} \) be given by \( f(x) = (f_1(x), \ldots, f_{n+1}(x)) \) for all \( x \in \mathbb{C}^n \) and denote by \( V \) the Zariski closure of \( f(\mathbb{C}^n) \) in \( \mathbb{C}^{n+1} \). Then it is left to the reader to verify that \( V \) is an irreducible affine variety of dimension \( n \) and that the induced map \( F : \mathbb{C}^n \to V \) satisfies all five properties of 2.1, thereby supplying a negative answer to Adjmagbo’s Problem.

Final remark. The map \( F \) constructed above is not étale: namely if \( F \) is étale then (by the preservation of normality under étale maps (see [2],(12)i)) \( V \) is normal i.e. \( \mathbb{C}[f] \) is integrally closed. Since \( \mathbb{C}(f) = \mathbb{C}(X) \) and \( \mathbb{C}[X] \) is integral over \( \mathbb{C}[f] \) this implies that \( \mathbb{C}[f] = \mathbb{C}[X] \) i.e. \( F \) is an isomorphism, a contradiction.

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References


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