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Around the Abhyankar-Moh Theorem

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To Professor Shreeram S. Abhyankar on his 70th birthday

Abstract In this paper we first show the impact of the Abhyankar-Moh Theorem on various problems on the affine plane. Then we discuss possible generalizations of the AM-Theorem and describe how these may lead to counterexamples in dimension 5 to several of the problems which were shown to be true in the plane.

1 The Abhyankar-Moh Theorem

Let's start with the following

High School Exercise. Let $f(t), g(t) \in \mathbb{C}[t]\backslash\mathbb{C}$ such that $\mathbb{C}[f(t), g(t)] = \mathbb{C}[t]$. Show that $\deg f | \deg g$ or $\deg g | \deg f$.

This problem first appeared as a lemma in the paper [36] of Segre in 1956 in which he used it to give a "proof" of the 2 dimensional Jacobian Conjecture. However in his paper there are various mistakes, including in the proof of his lemma. In 1970 in [13] Canals and Lluis gave a correction of the proof of Segre's lemma. Also their "proof" contained an error. In 1971 Abhyankar posed Segre's lemma as an advanced problem in the problem section of the American Mathematical Monthly, [1]. Finally in 1975 a proof of the Segre lemma, by Abhyankar and Moh appeared in [4]. This result will from now on be called the Abhyankar-Moh Theorem (abbreviated AM-Theorem). The proof is based on two earlier papers of the authors, [2] and [3]. So the total proof is about 80 pages long. Although it is completely elementary, it is very complicated (the reader should take a look at the papers [2] and [3]!). Fortunately in the meantime various new proofs of the AM-Theorem appeared: Suzuki, [38], 1974, Miyanishi [29], [30], 1978, 1985, Ganong, [21], 1979, Rudolph,[33], 1982, Richman, [32],1986, Kang, [25], 1991, Gurjar-Miyanishi, [22], 1987, A'Campo-Oka, [6], 1995, Nowicki, [31], 1995. A short proof
(8 pages) essentially due to Nowicki is included in my recent book [18], 2000.

In their paper Abhyankar and Moh do not mention any relation with the 2 dimensional Jacobian Conjecture, which was the origin of Segre's lemma. So now Segre's lemma has been proved correctly it is natural to ask.

Question 1. What consequence has the AM-Theorem for the 2 dimensional Jacobian Conjecture?

To answer this question we first discuss a geometric consequence of the AM-Theorem. This will be done in the next section.

2 Embeddings and the 2 dimensional Jacobian Conjecture

A polynomial map

$$\mathbb{C} \ni t \longmapsto f(t) := \big(f_1(t), \ldots, f_n(t)\big) \in \mathbb{C}^n$$

is called an *embedding* of $\mathbb{C}$ in $\mathbb{C}^n$ if via f $\mathbb{C}$ is isomorphic to its image i.e. there exists a polynomial map $F : \mathbb{C}^n \to \mathbb{C}$ such that f and $F_{|_{Imf}}$ are each others inverse.

Example 1. The map $\mathbb{C} \ni t \mapsto (t, t^2) \in \mathbb{C}^2$ is an embedding. Take for F the map $F(x, y) = x$.

In algebraic terms we get: f is an embedding if and only if $\mathbb{C}[f_1(t), \ldots, f_n(t)] = \mathbb{C}[t]$. Furthermore one can also show that this notion of embedding coincides with the one used in differential geometry i.e. f is an embedding if and only if $f'(t) \neq 0$ for all $t \in \mathbb{C}$ and the map $f : \mathbb{C} \to \mathbb{C}^n$ is injective. One of the fundamental questions is

Question 2. How many "essentially different" embeddings of $\mathbb{C}$ in $\mathbb{C}^n$ exist?

By "essentially different" we mean "inequivalent" in the following precise sense.

Definition 1. Two embeddings $f, g : \mathbb{C} \to \mathbb{C}^n$ are called *equivalent* if there exists a polynomial automorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $g = F \circ f$. [F is called a polynomial automorphism of $\mathbb{C}^n$ if there exists a polynomial map $G : \mathbb{C}^n \to \mathbb{C}^n$ such that $F \circ G = 1_{\mathbb{C}^n} = G \circ F$].

In case $n = 2$ the answer to Question 2 is given by

Theorem 1. (Abhyankar, Moh, Suzuki). *All embeddings of $\mathbb{C}$ in $\mathbb{C}^2$ are equivalent. In other words, if $f = \big(f_1(t), f_2(t)\big)$ is an embedding then there exists $F \in Aut_{\mathbb{C}}\mathbb{C}^2$ such that $F\big(f_1(t), f_2(t)\big) = (t, 0)$ i.e. every embedding is equivalent to the standard embedding $t \mapsto (t, 0)$ of $\mathbb{C}$ in $\mathbb{C}^2$.*

In the situation of this theorem we say that f is *rectifiable*. So every embedding of $\mathbb{C}$ in $\mathbb{C}^2$ is rectifiable.

Example 2. Let $f(t) = (t, t^2)$ be as above. Take $F = (X, Y - X^2)$. Then $F \in \mathrm{Aut}_{\mathbb{C}}\mathbb{C}^2$ (its inverse is the map $G = (X, Y + X^2)$) and F rectifies f i.e. $F(t, t^2) = (t, 0)$.

Proof of Theorem 1. Let $f = \big(f_1(t), f_2(t)\big)$ be an embedding of $\mathbb{C}$ in $\mathbb{C}^2$. We use induction on $d(f) := \deg f_1(t) + \deg f_2(t)$ ($\deg 0 := -\infty$). Let $n := \deg f_1(t)$, $m := \deg f_2(t)$ and assume that $1 \leq n \leq m$ (the case $f_1 \in \mathbb{C}$ is easy). Since $\mathbb{C}[f_1(t), f_2(t)] = \mathbb{C}[t]$ it follows from the AM-Theorem that $n|m$, say $m = dn$. So

$$f_1(t) = c_1 t^n + \cdots \text{ and } f_2(t) = c_2 t^{dn} + \cdots, c_1, c_2 \in \mathbb{C}^*.$$

Let $E_1 := (X, Y - (c_2/c_1^d)X^d) \in \mathrm{Aut}_{\mathbb{C}}\mathbb{C}^2$. Then $E_1 \circ f : \mathbb{C} \to \mathbb{C}^2$ is an embedding with $d(E_1 \circ f) < d(f)$. So by the induction hypothesis there exists $F \in \mathrm{Aut}_{\mathbb{C}}\mathbb{C}^2$ with $F \circ (E_1 \circ f) = (t, 0)$, whence $(F \circ E_1)(f) = (t, 0)$. Since $F \circ E_1 \in \mathrm{Aut}\mathbb{C}^2$ we get that f is equivalent to the standard embedding $t \mapsto (t, 0)$. □

Now we are able to answer Question 1.

Theorem 2. (Gwoździewicz [23], 1993). *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map with $\det JF \in \mathbb{C}^*$. If there exists a line $\ell \subset \mathbb{C}^2$ such that $F_{|\ell} : \ell \to \mathbb{C}^2$ is injective, then $F \in Aut\mathbb{C}^2$.*

The proof uses the following result

Lemma 1. *Let $G = (G_1, G_2) \in \mathbb{C}[X, Y]^2$ with $\det JG \in \mathbb{C}^*$.*

i) *If $\deg G_1$ or $\deg G_2$ equal 1, then $G \in Aut\mathbb{C}^2$.*
ii) *If $\deg G_1$ and $\deg G_2 > 1$ then both G_1 and G_2 contain pure X-terms and pure Y-terms of degree ≥ 1.*

The condition i) together with $\det JG \in \mathbb{C}^*$ immediately give that G is an elementary polynomial automorphism. For the proof of ii) we refer to [18], Proposition 10.2.6. Now we are able to give

Proof of Theorem 2. Making a coordinate change we may assume that ℓ equals the X-axis. Now consider

$$f(t) := \big(F_1(t, 0),\ F_2(t, 0)\big) : \mathbb{C} \to \mathbb{C}^2.$$

So by the hypothesis $f : \mathbb{C} \to \mathbb{C}^2$ is injective. Furthermore $f'(t) = (JF)(t, 0)\binom{1}{0} \neq \binom{0}{0}$, since $\det JF \in \mathbb{C}^*$. So $f : \mathbb{C} \to \mathbb{C}^2$ is an embedding. Then by the AM-Theorem there exists $H \in \mathrm{Aut}\mathbb{C}^2$ such that $H\big(f(t)\big) = (t, 0)$ i.e. $H\big(F(t, 0)\big) = (t, 0)$. So if we put $G := H \circ F$ we get

i) $G(t, 0) = (t, 0)$ and
ii) $\det JG \in \mathbb{C}^*$ (by the chain rule).

Now we distinguish two cases

I) $\deg G_1$ or $\deg G_2 = 1$. Then by lemma 1 i) $G \in \mathrm{Aut}\mathbb{C}^2$. So $F = H^{-1} \circ G \in \mathrm{Aut}\mathbb{C}^2$ and we are done.
II) $\deg G_1$ and $\deg G_2 > 1$. By lemma 1 ii) G_2 has pure X-terms. So $G_2(t, 0) \neq 0$, a contradiction since $G(t, 0) = (t, 0)$ implies that $G_2(t, 0) = 0$.

So only case I) occurs, which completes the proof. □

3 More applications of the AM-Theorem

The previous section already indicates the importance of the AM-Theorem. However to really appreciate the power of the AM-Theorem let me recall a part of the introduction of Kraft's lecture delivered at the Séminaire Bourbaki, June 1995 [26], "Challenging Problems on Affine n-Space".
In this lecture he discusses several of the most important open problems in affine geometry. Here are some of them

The Cancellation Problem (C.P). Does $Y \times \mathbb{C} \simeq \mathbb{C}^n$ imply that $Y \simeq \mathbb{C}^{n-1}$?

The Embedding Problem (E.P). Let $1 \leq k \leq n-1$. Is every closed embedding of $\mathbb{C}^k$ in $\mathbb{C}^n$ equivalent to the standard embedding $\mathbb{C}^k \ni (x_1, \ldots, x_k) \rightarrow (x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{C}^n$?

The Automorphism Problem (A.P). Give an algebraic description of the group of polynomial automorphisms of $\mathbb{C}^n$.

The Linearization Problem (L.P). Is every $F \in \mathrm{Aut}\mathbb{C}^n$ satisfying $F^s = 1_{\mathbb{C}^n}$ for some $s \geq 1$ linearizable i.e. does there exist $\varphi \in \mathrm{Aut}\mathbb{C}^n$ with $\varphi^{-1}F\varphi = L$, a linear map?

The Jacobian Conjecture (J.C). Let $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a polynomial map with $\det JF \in \mathbb{C}^*$. Is F invertible?

The power of the AM-Theorem is clearly demonstrated by the following diagram of implications which hold in case $n = 2$

$$\begin{array}{ccccccc} \text{AM-Theorem} & \Rightarrow & \text{A.P} & \Rightarrow & \text{L.P} & \Rightarrow & \text{C.P} \\ \Downarrow & & & & & & \\ \text{E.P} & & & & & & \end{array}$$

Let us briefly comment on these implications. First the automorphism problem: if we put

$$Aff(2, \mathbb{C}) := \{(F_1, F_2) \in \mathrm{Aut}\mathbb{C}^2 \mid \deg F_1 = \deg F_2 = 1\}$$

and

$$J(2, \mathbb{C}) := \{\big(aX, bY + c(X)\big) \mid a, b \in \mathbb{C}^*,\ c(X) \in \mathbb{C}[X]\}$$

then arguing essentially as in the proof of Theorem 1 it is not difficult to verify that $\mathrm{Aut}\mathbb{C}^2 = \langle Aff(2, \mathbb{C}), J(2, \mathbb{C})\rangle$.
Then considering more closely the products of elements of $Aff(2, \mathbb{C})$ and $J(2, \mathbb{C})$ one can even show that

$$\mathrm{Aut}\mathbb{C}^2 = Aff(2, \mathbb{C}) *_{B_2} J(2, \mathbb{C})$$

where $B_2 := Aff(2, \mathbb{C}) \cap J(2, \mathbb{C})$. This is the so-called Jung-van der Kulk theorem (for more details we refer to [18], Chapter 5).
From the free amalgamated product structure above one easily deduces that every

$F \in \mathrm{Aut}\mathbb{C}^2$ satisfying $F^s = 1_{\mathbb{C}^2}$, for some $s \geq 1$, is linearizable (a short proof can be found in [26]). This gives the implication "A.P $\Rightarrow$ L.P". Since we already discussed the implication "A.M Theorem $\Rightarrow$ E.P" in section 2, it remains to show the implication "L.P $\Rightarrow$ C.P". In fact this implication holds in any dimension as can be seen from

Proposition 1. *If every $F \in Aut\mathbb{C}^n$ satisfying $F^2 = 1_{\mathbb{C}^n}$ is linearizable, then $Y \times \mathbb{C} \simeq \mathbb{C}^n$ implies that $Y \simeq \mathbb{C}^{n-1}$. In other words "L.P $\Rightarrow$ C.P".*

Proof Suppose $Y \times \mathbb{C} \simeq \mathbb{C}^n$. Identifying $Y \times \mathbb{C}$ with $\mathbb{C}^n$ we define $F : \mathbb{C}^n \to \mathbb{C}^n$ by $F(y,t) = (y, -t)$. Observe that $F^2 = 1_{\mathbb{C}^n}$. So by the hypothesis F is linearizable. In particular Fix $F \simeq \mathbb{C}^d$ for some $d \geq 0$. However one easily verifies that Fix $F = Y \times \{0\}$,so $Y \simeq \mathbb{C}^d$. From $Y \times \mathbb{C} \simeq \mathbb{C}^n$ it then follows that $d = n-1$. □

4 On generalizations of the AM-Theorem

First let us remark that in [4] the AM-Theorem was proved not only for the field $\mathbb{C}$ but for any field k of characteristic zero (in case char$k = p$ there are easy counterexamples: if $f(t) = t^{p^2}$ and $g(t) = t^{p(p+1)} + t$, then $k[f(t), g(t)] = k[t]$ but no one of the degrees of f or g divides the other). Consequently the AM-Theorem also holds for polynomials $f, g \in A[T]$, where A is any domain containing $\mathbb{Q}$. However this is a far as one can go:

Example 3. Let A be a commutative ring containing elements a, b, nonzero such that $ab = 0$. Put

$$F = (X, Y + aX^3) \circ (X + bY^2, Y).$$

So $F \in \mathrm{Aut}_A A[X,Y]$ and using $ab = 0$ one gets that

$$F = (X + bY^2, Y + aX^3).$$

Put $\big(f(t), g(t)\big) := F(t,t)$. Then $A[f(t), g(t)] = A[t]$, however $\deg f(t) = 2$ and $\deg g(t) = 3$.

Therefore in order to get more generalizations of the AM-Theorem we give two equivalent formulations of the AM-Theorem. The first reformulation is the following: in section 2 we already saw that the AM-Theorem implies that every embedding of $\mathbb{C}$ in $\mathbb{C}^2$ is rectifiable i.e. "A.M.Theorem $\Rightarrow$ E.P". In fact one can show that the converse also holds. Therefore the following problem seems a natural generalization of the AM-Theorem.

$$\text{Is every embedding } f : \mathbb{C} \to \mathbb{C}^n \text{ rectifiable if } n \geq 3? \tag{1}$$

Also one can show (see [18], lemma 5.3.13) that in case $n = 2$ E.P (and hence the AM-Theorem) is equivalent to the following statement.

$$\begin{array}{l}\text{Let } f \in \mathbb{C}[X,Y] \text{ such that } \mathbb{C}[X,Y]/(f) \simeq \mathbb{C}^{[1]}, \text{ then} \\ f \text{ is a coordinate i.e. there exists } g \in \mathbb{C}[X,Y] \text{ such that} \\ \mathbb{C}[X,Y] = \mathbb{C}[f,g].\end{array} \tag{2}$$

Before we consider (1) let us first comment on possible generalizations of (2). It was proved by Bhatwadekar and Dutta in [8] that (2) also holds if we replace $\mathbb{C}$ by any commutative noetherian ring A containing $\mathbb{Q}$. Recently, based on results of Bhatwadekar and Dutta in [9] and Berson, van den Essen and Maubach in [12], this result in turn was generalized by van Rossum and the author in [19] to

Proposition 2. *Let A be any commutative $\mathbb{Q}$-algebra. If $A[X,Y]/(f) \simeq A^{[1]}$, then f is a coordinate over A i.e. $A[X,Y] = A[f,g]$ for some $g \in A[X,Y]$.*

Another possibility to generalize (2) is given by the

Abhyankar-Sathaye Conjecture. Let $f \in \mathbb{C}[X_1,\ldots,X_n]$ such that $\mathbb{C}[X]/(f) \simeq \mathbb{C}^{[n-1]}$. Then f is a coordinate i.e. $\mathbb{C}[X] = \mathbb{C}[f, f_2, \ldots, f_n]$ for some $f_i \in \mathbb{C}[X]$.

This conjecture is open for all $n \geq 3$ and not much is known. In case $n = 3$ it was proved by Sathaye in [35] and Russell in [34] that if f is of the form $a(X,Y)Z + b(X,Y)$ then the AS-Conjecture holds (in fact $\mathbb{C}$ may be replaced by any field). This result was used in [10] by Bhatwadekar and Dutta to show that a similar statement holds if $\mathbb{C}$ is replaced by a discrete valuation ring.

Now let us return to (1) i.e. the question if every embedding $f : \mathbb{C} \to \mathbb{C}^n$ is rectifiable i.e. does there exists $F \in \mathrm{Aut}\mathbb{C}^n$ such that $F(f) = (t, 0, \ldots, 0)$, in case $n \geq 3$?

At a conference in Kyoto,1977 (see [5]) Abhyankar made the following conjectures

AC1. For every $n \geq 3$ there exist non-rectifiable embeddings of $\mathbb{C}$ in $\mathbb{C}^n$.

More specifically in case $n = 3$ he conjectured

AC2. For every $d \geq 3$, is the embedding $\gamma_d(t) = (t^{d+2} + t,\, t^{d+1}, t^d)$ not rectifiable.

However, this time Abhyankar was wrong.

Theorem 3. (Craighero [15], 1986/ Jelonek [24],1987). *If $n \geq 4$ then every embedding of $\mathbb{C}$ in $\mathbb{C}^n$ is rectifiable.*

So the only question which remains is

$$\text{Is every embedding of } \mathbb{C} \text{ in } \mathbb{C}^3 \text{ rectifiable?} \tag{3}$$

It turned out that also AC2 is not correct.

Theorem 4. *(*Craighero [14], 1985, [16],1988*). γ_3 and γ_4 are rectifiable.*

As far as I know it is still an open problem if γ_d is rectifiable if $d \geq 5$ (For results in case the coefficient field has characteristic $p > 0$ we refer the reader to [11]).
Since the formula for F which rectifies γ_3 given in [14] is not correct we give a correct formula below:
let $F = (F_1, F_2, F_3)$ be given by

$$F_1 = Z^3Y + 2Z^3 + X - ZX^2$$

$$\begin{aligned}F_2 = & -Y + 5X^4 - 4Z^4 - 6X^5Z + 2X^6Z^2 - 4XZ - 8X^2Z^2 + 24X^3Z^3 - 12X^4Z^4 \\ & -24XZ^5 + 24X^2Z^6 - 16Z^8 - Z^4Y^2 - 4Z^4Y - 2Z^8Y^3 - 12Z^8Y^2 - 24Z^8Y \\ & -4Z^2YX^2 - 2ZXY - 6Z^5Y^2X + 6Z^6Y^2X^2 - 24Z^5YX + 24Z^6YX^2 \\ & +12Z^3YX^3 - 6Z^4YX^4\end{aligned}$$

$$F_3 = Z - F_1^3$$

Then $F \in \mathrm{Aut}\mathbb{C}^3$ and F rectifies γ_3.
In 1992 Shastri,[37] came with a completely different approach. He observed that for all $d \geq 3$ the embedding γ_d defines a trivial knot in $\mathbb{R}^3$. Therefore he suggested to look for embeddings of $\mathbb{C}$ in $\mathbb{C}^3$ which are defined by polynomials with real coefficients which do not define a trivial knot in $\mathbb{R}^3$. The obvious question is: do such embeddings exists? In his paper he answers this question completely. More precisely

Proposition 3. *(*Shastri, [37], 1992*). Every (open) knot has a representation by real polynomials $\big(f_1(t), f_2(t), f_3(t)\big)$ that define an embedding of $\mathbb{C}$ in $\mathbb{C}^3$.*

For the simplest non-trivial knot, the trefoil, Shastri gave the following parametrization

$$\gamma(t) = (t^3 - 3t,\ t^4 - 4t^2,\ t^5 - 10t).$$

Indeed $\gamma : \mathbb{C} \to \mathbb{C}^3$ is an embedding since $F\big(\gamma(t)\big) = t$, where $F = YZ - X^3 - 5XY + 2Z - 7X$. We will call this embedding the Shastri embedding.

Conjecture (Shastri) $\gamma : \mathbb{C} \to \mathbb{C}^3$ is a non-rectifiable embedding (and hence a counterexample to the Embedding Problem).

As far I know this conjecture is still open. However there are some recent developments. These will be discussed in the next section.

5 The trefoil and possible counterexample to various conjectures on affine n-space

In this section we discuss some recent results obtained by Peter van Rossum and the author in [19]. In particular we describe a relationship between the Shastri embedding and the Cancellation Problem inspired by work of Asanuma in [7].

As already described above, the Cancellation Problem asks if $Y \times \mathbb{C} \simeq_{\mathbb{C}} \mathbb{C}^n$ implies that $Y \simeq_{\mathbb{C}} \mathbb{C}^{n-1}$. In case $n = 2$ the answer is yes (as we saw in §3) and also if $n = 3$ the answer is yes (see [20], [30] and [18]). However the case $n \geq 4$ remains open. To study the Cancellation Problem we reformulate it in terms of locally nilpotent derivations. Recall that a $\mathbb{C}$-derivation D on $\mathbb{C}[X]$ is called *locally nilpotent* if for every $a \in \mathbb{C}[X]$ there exists $m \geq 1$ such that $D^m(a) = 0$. The derivation $\frac{\partial}{\partial X_i}$ is an easy example. Furthermore an element $s \in \mathbb{C}[X]$ is called *slice of* D if $D(s) = 1$. It is not difficult to show that the Cancellation Problem can be reformulated as follows (see [18], §2.2 Exercise 3).

Cancellation Problem. Let D be a locally nilpotent derivation on $\mathbb{C}[X]$ with a slice s. Does it follow that $\mathbb{C}[X]^D (= \operatorname{Ker} D) \simeq_{\mathbb{C}} \mathbb{C}^{[n-1]}$?

Now let $f = \big(f_1(U), \ldots, f_n(U)\big) : \mathbb{C} \to \mathbb{C}^n$ be an embedding. Define on $A := \mathbb{C}[T, X, U] := \mathbb{C}[T, X_1, \ldots, X_n, U]$ the derivation

$$D_f := f_1'(U)\partial_{X_1} + \cdots + f_n'(U)\partial_{X_n} + T\partial_U.$$

Proposition 4. ([19], Theorem 3.1)
i) *$D := D_f$ is locally nilpotent.*
ii) *D has a slice in A.*

Proof. i) Since D is a triangular derivation it is locally nilpotent (see for example Corollary 1.3.17,[18]).
ii) So it remains to show that D has a slice in A. Therefore observe that since f is an embedding there exists $P \in \mathbb{C}[X]$ such that

$$P\big(f_1(U), \ldots, f_n(U)\big) = U. \tag{1}$$

Furthermore $D(f_i(U) - TX_i) = 0$ for all i, which implies that $D\big(P(f_1(U) - TX_1, \ldots, f_n(U) - TX_n)\big) = 0$. Consequently

$$T = D(U) = D\big(U - P(f_1(U) - TX_1, \ldots, f_n(U) - TX_n)\big). \tag{2}$$

From (1) we get

$$U - P(f_1(U) - TX_1, \ldots, f_n(U) - TX_n) = Ts, \text{ for some } s \in A. \tag{3}$$

Then (2) and (3) imply that $T = D(Ts) = TD(s)$, so $D(s) = 1$ as desired. □

Remark 1. So if $f : \mathbb{C} \to \mathbb{C}^n$ is a polynomial map we see that D_f has a slice if f is an embedding. It is shown in [19] that the converse also holds i.e. f is an embedding if and only if D_f has a slice in A.

So due to Proposition 4 the set of derivations D_f where f is an embedding of $\mathbb{C}$ in $\mathbb{C}^n$ gives us a test class for the Cancellation Problem. Hence a crucial question is: does the Cancellation Problem has an affirmative answer for these derivations? A partial answer is given by

Theorem 5. ([19], Theorem 4.1). *If $f : \mathbb{C} \to \mathbb{C}^n$ is rectifiable, then the Cancellation Problem has an affirmative answer for D_f.*

Corollary 1. *If $n \neq 3$ then the Cancellation Problem has an affirmative answer for all derivations D_f where f is an embedding of $\mathbb{C}$ in $\mathbb{C}^n$.*

Proof. One just observes that if $n \neq 3$ every embedding of $\mathbb{C}$ in $\mathbb{C}^n$ is rectifiable due to the AM-Theorem if $n = 2$ and Theorem 3 if $n \geq 4$. □

So the crucial question which remains is

> Does the Cancellation Problem has an affirmative answer for the derivations D_f in case $f : \mathbb{C} \to \mathbb{C}^3$ is an embedding?

Conjecture 1 Let $D := D_f$ on $A = \mathbb{C}[T, X, Y, Z, U]$ where $f(U) = \big(f_1(U), f_2(U), f_3(U)\big) = (U^3 - 3U, U^4 - 4U^2, U^5 - 10U)$ (the Shastri embedding). Then $\mathbb{C}[T, X, Y, Z, U]^D \not\simeq_{\mathbb{C}} \mathbb{C}^{[4]}$ i.e. D gives a counterexample to the Cancellation Problem.

Let us write s for the slice of D (as in this conjecture) constructed in the proof of Proposition 4 i.e.

$$s = \big(U - P(f_1(U) - TX, f_2(U) - TY, f_3(U) - TZ)\big)/T$$

where $P = YZ - X^3 - 5XY + 2Z - 7X$.
Then we know that $A = \mathbb{C}[T, X, Y, Z, U]^D[s]$ and hence that $\mathbb{C}[T, X, Y, Z, U]^D \simeq_{\mathbb{C}} A/(s)$. So Conjecture 1 is equivalent to

Conjecture 2 $\mathbb{C}[T, X, Y, Z, U]/(s) \not\simeq_{\mathbb{C}} \mathbb{C}^{[4]}$ where

$$\begin{aligned} s = {} & -7X + 2Z - T^2X^3 + ZU^4 - 4ZU^2 + YU^5 - YZT + 5YU - 3XU^6 + 13XU^4 + \\ & 3X^2TU^3 - 7XU^2 - 9X^2TU - 5YU^3 + 5XYT. \end{aligned}$$

Unfortunately we do not have a method to decide if a given quotient $\mathbb{C}[X]/(f)$ is $\mathbb{C}$-isomorphic to $\mathbb{C}^{[n-1]}$ if $n \geq 3$!

The missing link

Let us conclude this paper by describing one possible strategy to attack the above conjecture.

In [27],1973 Miyanishi "proved" that if both the Serre Conjecture and the Jacobian Conjecture hold, then the Cancellation Problem has an affirmative answer. Since in the meantime the Serre Conjecture has been proved we get the implication " J.C $\Rightarrow$ C.P". However the proof in [27] is incorrect! Nevertheless it would be of fundamental importance if this "missing link" could be proved. The reason is that we then would have the following diagram of implications

$$\begin{array}{ccc} \text{L.P} \Rightarrow & \text{C.P} \Rightarrow & \text{C.P (for } D_f\text{'s)} \\ & \uparrow \leftarrow & \text{"missing link"} \\ & \text{J.C} & \end{array}$$

The missing link would give the possibility to construct out of a candidate counterexample to the C.P. of the form D_f, a candidate counterexample to the Jacobian Conjecture. The point is that this last candidate counterexample can be tested to be a true counterexample, for example by the invertibility algorithm given in [17]. The diagram of implications shown above would then give a counterexample to the Cancellation Problem of the form D_f (for example where f is the Shastri embedding) and would also give counterexamples to the Linearization Problem, the Embedding Problem and the Jacobian Conjecture!

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