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Abstract

In this paper an algorithm is given to decide if a given polynomial in two variables with coefficients in a finitely generated K-algebra is a coordinate and if so, find a mate for this polynomial.

1 Introduction

One of the most fundamental questions in studying problems on affine n-space is the following : given a polynomial F in n variables over a field K, how can one decide if F is a coordinate ? Related problems are the Abhyankar-Sathaye Conjecture, the Cancellation Problem, the Jacobian Conjecture and Linearization Conjectures (see [10],[9] and [7] for more details).
In case $n = 2$ this problem was solved by Chądzyński and Krasiński in [3] and by the second author in [8]; in fact, in the last paper also an algorithm was given to compute a mate of F. However the case $n \geq 3$ remains open.
Recently in [6] a new impuls to the study of coordinates was given. In that paper the authors study polynomials in 2 variables over an almost arbitrary commutative ring, but restrict their attention to coordinates of length at most 2. Nevertheless the results obtained are already rather powerful in the sense that they are used to construct new coordinates in $n(\geq 2)$ variables. These new coordinates are for example used to solve a question posed by A. Choudary and A. Dimca in [5].
In this paper we completely solve the problem of deciding if a given polynomial F in 2 variables over an arbitrary finitely generated K-algebra (where K is a field of characteristic zero) is a coordinate. More precisely, we give an algorithmic solution. Moreover, in case F is a coordinate we also give an algorithm which computes a mate of F, i.e. a $G \in A[X,Y]$ such that $A[F,G] = A[X,Y]$.
The algorithms are based on results concerning locally nilpotent derivations, mostly taken from [2], results of Gröbner basis theory (see [1]) and a particular result of [4].

2 Preliminaries

In the rest of this paper K will be a computable field of characteristic zero and A a finitely generated K-algebra, say $A = K[X_1, \ldots, X_m]/I$ for some ideal I of $K[X_1, \ldots, X_m]$. Let $(f_1, \ldots, f_s)$ be a Gröbner basis of I with respect to some admissable term ordering. Let $F \in A[X,Y]$ and define the derivation D on $A[X,Y]$

by $D := F_Y\partial_X - F_X\partial_Y$, where $F_X = \frac{\partial F}{\partial X}$ and $F_Y = \frac{\partial F}{\partial Y}$. Let's denote its kernel by $A[X,Y]^D$. An element $s \in A[X,Y]$ is called a *slice* of D if $D(s) = 1$. D is called *locally nilpotent* if for every $a \in A[X,Y]$ there exists an $n \in \mathbb{N}^*$ such that $D^n(a) = 0$. If $\mathfrak{a}$ is an ideal of A, $\overline{D}^{\mathfrak{a}}$ will be the induced derivation on $A/\mathfrak{a}[X,Y]$. The nilradical of A will be denoted by η. In case A is a domain, $Q(A)$ is the quotient field of A. First, we will state Theorem 3.5 from [2].

Theorem 2.1. *Any locally nilpotent derivation D on $A[X,Y]$ satisfying $div(D) = 0$ and $1 \in (D(X), D(Y)$ has a slice and satisfies $A[X,Y]^D = A[P]$, where $P \in A[X,Y]$ has the property, that $P_X = -D(Y)$ and $P_Y = D(X)$. This P is unique up to a constant.*

This theorem will be very useful for our algorithm, for it has as its almost immediate consequence the following proposition which gives a nice equivalent description of a coordinate in two variables.

Proposition 2.2. *F is a coordinate if and only if D is locally nilpotent on $A[X,Y]$ and $1 \in (F_X, F_Y)$.*

Proof. If F is a coordinate, there exists a $G \in A[X,Y]$ such that $F_XG_Y - F_YG_X \in A[X,Y]^*$, yielding $(F_X, F_Y) = (1)$. Viewing $F_XG_Y - F_YG_X$ modulo η we get $\overline{F}_X\overline{G}_Y - \overline{G}_X\overline{F}_Y \in (A/\eta)^*$, which implies $(\overline{D}^{\eta})^2(\overline{G}) = 0$. Of course $\overline{D}^{\eta}(\overline{F}) = 0$, and because $\overline{F}$ and $\overline{G}$ generate $A/\eta[X,Y]$, it follows that $\overline{D}^{\eta}$ is locally nilpotent on $A/\eta[X,Y]$. So by Lemma 2.1.15 in [7], we conclude, that D is already locally nilpotent on $A[X,Y]$.
Now suppose D is locally nilpotent and $(F_X, F_Y) = (1)$. It is easily computed that $div(D) = 0$, so we can use Theorem 2.1 to conclude, that D has a slice $G \in A[X,Y]$ and $A[X,Y]^D = A[F]$. Using Lemma 2.3 below we conclude, that $A[X,Y] = A[X,Y]^D[G] = A[F,G]$, so F is a coordinate. □

Lemma 2.3. *If R is an A-algebra and D is a locally nilpotent A-derivation on R with a slice $s \in R$, then $R = R^D[s]$, a polynomial ring in s over R^D, and $D = \frac{d}{ds}$ on R.*

For a proof of this lemma we refer to [7]. The statement in Proposition 2.2 above is very useful in the sense that there exists a way to find out whether a given derivation D is locally nilpotent. This is described in Proposition 2.5, which makes use of the following theorem that can be found in [7].

Theorem 2.4. *Let K be a field of characteristic zero. Let $0 \neq D = a\partial_X + b\partial_Y$ be a K-derivation on $K[X,Y]$. Put $d := max\{deg_X(a), deg_Y(a), deg_X(b), deg_Y(b)\}$. Then D is locally nilpotent if and only if $D^{d+2}(X) = D^{d+2}(Y) = 0$.*

Proposition 2.5. *If A is a $\mathbb{Q}$-algebra and D a derivation on $A[X,Y]$, let $d := max\{deg_X(a), deg_Y(a), deg_X(b), deg_Y(b)\}$. Then the following statements are equivalent :*

1. *D is locally nilpotent*

2. $D^{d+2}(X), D^{d+2}(Y) \in \eta A[X,Y]$

Proof. Gathering the coefficients of $D(X)$ and $D(Y)$, we may assume, that A is Noetherian. Let $e \in \mathbb{N}^*$ such that $\eta^e = (0)$.
$1 \Rightarrow 2$: Let $\mathfrak{p} \in Spec(A)$. $\overline{D}$ is locally nilpotent on $A/\mathfrak{p}[X,Y]$ and $A/\mathfrak{p}$ is a domain, so by Theorem 2.4 (take $K = Q(A/\mathfrak{p})$) we certainly have $\overline{D}^{d+2}(X) = \overline{D}^{d+2}(Y) = 0$, i.e. $D^{d+2}(X), D^{d+2}(Y) \in \mathfrak{p}A[X,Y]$. This is true for all prime ideals $\mathfrak{p}$, which means that $D^{d+2}(X), D^{d+2}(Y) \in \eta A[X,Y]$.
$2 \Rightarrow 1$: Clearly $\overline{D}^{\eta}$ is locally nilpotent, so by Lemma 2.1.15 in [7], D is also locally nilpotent. □

Remark 2.6. The statement 'D is a locally nilpotent derivation $\Leftrightarrow D^{d+2}(X) = D^{d+2}(Y) = 0$' is not true in general. For example, take $A = \mathbb{C}[T]/(T^2)$ and $\epsilon = \overline{T}$. Then $F := (X + Y^2, Y) \circ (X, Y + \epsilon X^2) = (X + Y^2 + 2\epsilon X^2 Y, Y + \epsilon X^2)$ is an automorphism, so $D := (F_1)_Y \partial_X - (F_1)_X \partial_Y = (2Y + 2\epsilon X^2)\partial_X - (1 + 4\epsilon XY)\partial_Y$ is locally nilpotent (here $F_1 = X + Y^2 + 2\epsilon X^2 Y$), but $D^4(Y) = -24\epsilon \neq 0$.

The following useful lemma is also due to [2]. We will use the technique of this lemma in our algorithm in the next section.

Lemma 2.7. *Let $\overline{D}^{I_i}$ be surjective for the ideals $I_1, \dots, I_r \subseteq A$. Then $\overline{D}^{I_1 \cdots I_r}$ is also surjective.*

Proof. It is enough to show that if $\overline{D}^I, \overline{D}^J$ are surjective $\overline{D}^{IJ}$ is too. Let $a \in A[X,Y]$ be arbitrary. There exists $b \in A[X,Y]$ such that $\overline{D}^I(\overline{b}) = \overline{a}$ hence $D(b) = a + i$ where $i \in IA[X,Y]$. Write $i = \sum_{k=0}^{t} i_k c_k$ where $i_k \in I, c_k \in A[X,Y]$. Then for every c_k there exists some d_k such that $D(d_k) = c_k + j_k$ for some $j_k \in JA[X,Y]$ since $\overline{D}^J$ is surjective. Now $D(b - \sum_{k=0}^{t} i_k d_k) = a - \sum_{k=0}^{t} i_k j_k$. Since $\sum_{k=0}^{t} i_k j_k \in IJA[X,Y]$ we are done. □

One important theorem we will use in the algoritm can be found as Theorem 10 in [4] and it states the following :

Theorem 2.8. *Let $F \in \mathbb{C}[X,Y]$ be monic in X, say $F = X^n + p_{n-1}(Y)X^{n-1} + \cdots + p_1(Y)X + p_0(Y)$ where every $p_i(Y) \in \mathbb{C}[Y]$ and $n \geq 2$. Let S be the Sylvester matrix of F_X and F_Y. If $(a_1(Y), \dots, a_{k+n-1}(Y))$ is the bottom row of the adjoint of S, define $A(X,Y), B(X,Y) \in \mathbb{C}[X,Y]$ as $A(X,Y) = a_1(Y)X^{k-1} + \cdots + a_{k-1}(Y)X + a_k(Y)$ and $B(X,Y) = -(a_{k+1}(Y)X^{n-2} + \cdots + a_{k+n-2}(Y)X + a_{k+n-1}(Y))$. Now suppose $deg_Y F \geq 1$ and let $d = Res_X(F_X, F_Y)$. Then $d = AF_X - BF_Y$ and the following two conditions are equivalent.*

1. *F has a younger mate relative to the X-degree.*

2. *$d \in \mathbb{C}^*$ and $A_X = B_Y$*

If these two conditions are satisfied, then a younger mate G of F relative to the X-degree is given by $G = \frac{1}{d}\left(\int B\, dX + \int A\, dY - \iint A_X\, dY dX\right)$.

Reading the proof of this theorem, it becomes obvious that $\mathbb{C}$ can be replaced by any field of characteristic zero. The polynomial F in the previous theorem must be monic in X and our $F \in A[X,Y]$ need not be, but fortunately we have the following lemma, which can be found as Corollary 3.3.7 in [7].

Lemma 2.9. *Let R be a domain and $F = (F_1, F_2) \in Aut_R R[X,Y]$. Then we have for all i,j : if $deg_{X_j} F_i > 0$, then the leading coefficient of F_i with respect to X_j is a nonzero constant.*

Finally we state the following lemma, which can be found in [7].

Lemma 2.10. *Let $R \subseteq S$ be commutative rings and $F \in R[X]^n$ such that* $\det JF(0) \in R^*$. *If F is invertible over S, then F is invertible over R.*

3 The mate algorithm

First we will give an algorithm to find a mate for a given coordinate in case A is a domain. Then we will use this algorithm in the next one, which decides if a given $F \in A[X,Y]$ is a coordinate (here A is not necessarily a domain), and if so, finds a mate for F. In these algorithms we will use several other ones, which were taken from [1] and [7]. These are stated at the end of this section.

So let's take a look at the first algorithm, which can be found below. Here I is a prime ideal, so A is a domain. Interchanging X and Y, if necessary, we may assume, that $n := deg_X(F) \geq deg_Y(F)$. If $deg_Y(F) = 0$ then by Proposition 2.2, $1 \in (F_X)$, i.e. $F_X \in A[X,Y]^* = A^*$ (for A is a domain), so $F = \lambda X + \mu$ for some $\lambda \in A^*$ and $\mu \in A[Y]$, which implies that we can take Y as a mate for F. If $n = 1$, F is linear, so 1 is in the ideal of the coefficients of X and Y in F and we can use the algorithm EXTGRÖBNER to determine the coefficients of this linear combination, which give a mate for F.

So from now on, we may assume, that $n \geq 2$. From Lemma 2.9 we get that the coefficient of X^n in F is a nonzero constant. It is easily seen, that in this case Theorem 2.8 can still be applied. Furthermore, F is a tame coordinate over $Q(A)$, so by Corollary 5.1.6 in [7], F has a younger mate relative to the X-degree. The Theorem now tells us, that $d = Res_X(F_X, F_Y) \in Q(A)^* \cap A[X,Y] = A \backslash \{0\}$ and G defined as $G := \int B\, dX + \int A\, dY - \iint A_X\, dY dX$ satisfies $G_Y F_X - G_X F_Y = d$. So $\overline{D}(\overline{G}) = 0$ in $A/(d)[X,Y]$, which implies by Theorem 2.1 that there exists an $H \in A[T]$ such that $\overline{G} = \overline{H}(\overline{F})$. We can use the algorithm MEMBER to find such an H.

Now $G - H(F)$ is divisible by d, so we can use the algorithms EXTGRÖBNER and REDPOL to find $\tilde{G} := \frac{G-H(F)}{d}$. Notice that $Q(A)[F, \frac{G-H(F)}{d}] = Q(A)[F,G] = Q(A)[X,Y]$ and $\det J(F, \frac{G-H(F)}{d}) = 1$, so by Proposition 2.10, $(F, \tilde{G})$ is an invertible polynomial map, which means that $\tilde{G}$ is a mate for F.

Algorithm PRE-MATE(F, I)

Input : A coordinate $F \in A[X,Y]$ with $A := K[X_1,\dots,X_m]/I$, where K is a computable field of characteristic zero and $I = (f_1,\dots,f_s)$ is a prime ideal of $K[X_1,\dots,X_m]$.
Output : A mate $\tilde{G} \in A[X,Y]$ for F.
Begin
  $c := 0$;
  if $deg_X(F) < deg_Y(F)$ **then** $F(X,Y) := F(Y,X)$; $c := 1$ **fi**;
  $n := deg_X(F)$;
  if $deg_Y(F) = 0$ **then** $\tilde{G}(X,Y) := Y$
  else if $n = 1$
    then $(G, \mathcal{G}, \mathcal{F})$:=EXTGRÖBNER($F(0,1)-F(0,0), F(1,0)-F(0,0), f_1,\dots,f_s$);
    $\tilde{G}(X,Y) := -Q_{1,F(0,1)-F(0,0)}X + Q_{1,F(1,0)-F(0,0)}Y$
    else $d := Res_X(F_X, F_Y)$; $S := Sylv(F_X, F_Y)$; $T := adjoint(S)$; $k := deg_X(F_Y)$;
      for i **from** 1 **to** $k+n+1$ **do** $a_i(Y) := T(k+n-1, i)$ **od**;
      $A(X,Y) := a_1(Y)X^{k-1} + \cdots + a_k(Y)$;
      $B(X,Y) := -(a_{k+1}(Y)X^{n-2} + \cdots + a_{k+n-1}(Y))$;
      $G(X,Y) := \int B\,dX + \int A\,dY - \iint A_X\,dY\,dX$;
      H :=MEMBER($G, \{F\}, (f_1,\dots,f_s,d)$); $p_1 := deg_X(G - H(F))$;
      $p_2 := deg_Y(G - H(F))$; $(G_1, \mathcal{G}, \mathcal{F})$:=EXTGRÖBNER($f_1,\dots,f_s,d$);
      for i **from** 1 **to** p_1 **do for** j **from** 1 **to** p_2 **do**
        $a_{i,j} := coeff(G - H(F), X^iY^j)$;
        if $a_{i,j} \neq 0$ **then** $(\mathcal{F}_{i,j}, g_{i,j})$:=REDPOL($a_{i,j}, G_1$) **fi**
      od od;
      $\tilde{G}(X,Y) := \sum_{i,j}(\sum_{g\in G_1} q_g^{i,j} Q_{g,d})X^iY^j$
    fi
  fi; **if** $c = 1$ **then** $\tilde{G}(X,Y) := \tilde{G}(Y,X)$ **fi**;
  return($\tilde{G}(X,Y)$)
end

Let's look at the second algorithm. Here I denotes an arbitrary proper ideal of $K[X_1,\dots,X_m]$. First we check if F is a coordinate. Proposition 2.2 describes an easy way to test this. To check if D is locally nilpotent, we use Proposition 2.5. We can use the algorithm RADICAL to calculate $G := r(I)$, the radical of I (which corresponds with the nilradical of A), and by the algorithms GRÖBNER and REDPOL we find out if $D^{d+2}(X), D^{d+2}(Y) \in \eta A[X,Y]$. To check if $1 \in (F_X, F_Y)$, use the algorithm GRÖBNER to calculate a Gröbnerbasis of this ideal, and see if it equals $\{1\}$.
Now suppose we know that F is a coordinate. We want to reduce to the case when the ring of coefficients is a domain, so first we calculate a primary decomposition of I using the algorithm PRIMDEC, say $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_l$, where $\mathfrak{q}_1,\dots,\mathfrak{q}_l$ are primary ideals with their respective associated primes $\mathfrak{p}_1,\dots,\mathfrak{p}_l$ (so $r(I) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_l$). For every $k \leq l$ we know that $\overline{F}$ is a coordinate over $A/\mathfrak{p}_k$, so using the previous algorithm we find for every k a mate G_k for $\overline{F}$. Using this information, we construct a mate for $\overline{F}$ over A/η, in the following way. With induction on k, we prove by construction that there is a mate for $\overline{F}$ over $A/(\mathfrak{p}_1 \cdots \mathfrak{p}_k)$ as follows :

we already have G_1 as mate for $\overline{F}$ over $A/\mathfrak{p}_1$. So assume we have a mate H for $\overline{F}$ over $A/(\mathfrak{p}_1 \cdots \mathfrak{p}_k)$. Because $A/\mathfrak{p}_{k+1}[\overline{F}, G_{k+1}] = A/\mathfrak{p}_{k+1}[X,Y]$, we can use the algorithm MEMBER for every monomial X^iY^j in $D(H)-1$ to write $X^iY^j \equiv M_{i,j}(F, G_{k+1})$ $(mod\, \mathfrak{p}_{k+1})$. Let $c_{i,j}$ be the coefficient of each monomial X^iY^j in $D(H)-1$. Now redefine H as $H := H - \sum_{i,j} c_{i,j} \int M_{i,j}(F, G_{k+1})\, dG_{k+1}$. We will prove that this H satisfies $D(H) \equiv 1 (mod\, \mathfrak{p}_1 \cdots \mathfrak{p}_{k+1})$. We know that

$$D(\int M_{i,j}(F, G_{k+1})\, dG_{k+1}) \equiv \frac{d}{dG_{k+1}}(\int M_{i,j}(F, G_{k+1})\, dG_{k+1})$$

$$= M_{i,j}(F, G_{k+1}) \equiv X^iY^j \ (mod\, \mathfrak{p}_{k+1})$$

and by the induction hypothesis $c_{i,j} \in \mathfrak{p}_1 \cdots \mathfrak{p}_k$ (for $D(H) - 1 \in \mathfrak{p}_1 \cdots \mathfrak{p}_k[X,Y]$), so we have

$$c_{i,j} D(\int M_{i,j}(F, G_{k+1})\, dG_{k+1}) \equiv c_{i,j} X^iY^j \ (mod\, \mathfrak{p}_1 \cdots \mathfrak{p}_{k+1})$$

and taking sums over all i and j we get

$$D(\sum_{i,j} c_{i,j} \int M_{i,j}(F, G_{k+1})\, dG_{k+1}) \equiv \sum_{i,j} c_{i,j} X^iY^j = D(H) - 1 \ (mod\, \mathfrak{p}_1 \cdots \mathfrak{p}_{k+1})$$

which implies that our new H satisfies $D(H) \equiv 1\, (mod\, \mathfrak{p}_1 \cdots \mathfrak{p}_{k+1})$. Furthermore, from the locally nilpotency of D the locally nilpotency of $\overline{D}$ follows, so by Proposition 2.2 we get, that H is a mate for $\overline{F}$ over $A/(\mathfrak{p}_1 \cdots \mathfrak{p}_{k+1})$.

So eventually we have an $H \in A[X,Y]$ such that H is a mate for $\overline{F}$ over $A/(\mathfrak{p}_1 \cdots \mathfrak{p}_l)$. But then also H is a mate for $\overline{F}$ over $A/(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_l) = A/\eta$. Furthermore, by calculating for every generator g of the ideal $r(I)$ an $e_g \in \mathbb{N}^*$ such that $g^{e_g} \in I$, we get an $e \in \mathbb{N}^*$ such that $\eta^e = (0)$, namely $e := \sum_{g \in G} e_g$. By using the same method as we've just seen (but now with H in stead of every G_k and η in stead of every $\mathfrak{p}_k$), we find a $\tilde{G} \in A[X,Y]$ such that $\tilde{G}$ is a mate for F over $A/\eta^e = A$.

Algorithm MATE(F, I)

Input : $F \in A[X,Y]$ with $A := K[X_1, \ldots, X_m]/I$, where K is a computable field of characteristic zero and $I = (f_1, \ldots, f_s)$ is a proper ideal of $K[X_1, \ldots, X_m]$.
Output : Either a message that F is no coordinate or a mate $\tilde{G} \in A[X,Y]$ for F.

Begin
 $c := 1$; $G :=$RADICAL$(f_1, \ldots, f_s)$; $d := max(deg_X(F_X), deg_Y(F_X), deg_X(F_Y), deg_Y(F_Y))$;
 $G :=$GRÖBNER(G); $(\mathcal{F}_1, r_1) :=$REDPOL$(D^{d+2}(X), G)$; $(\mathcal{F}_2, r_2) :=$REDPOL$(D^{d+2}(Y), G)$;
 if $r_1 = 0$ **and** $r_2 = 0$ **then** $P :=$GRÖBNER$(F_X, F_Y, f_1, \ldots, f_s)$;
  if $P \neq \{1\}$ **then** $c := 0$ **fi**
  else $c := 0$
 fi;
 if $c = 0$ **then return**("F is no coordinate")
  else $P :=$PRIMDEC$(f_1, \ldots, f_s)$; $H :=$PRE-MATE$(\mathcal{F}, \mathfrak{p}_1)$;
  for k **from** 1 **to** $\#(P) - 1$ **do** $G_k :=$PRE-MATE$(\mathcal{F}, \mathfrak{p}_k)$;
   $p_1 := deg_X(D(H) - 1)$;
   $p_2 := deg_Y(D(H) - 1)$;
   for i **from** 1 **to** p_1 **do for** j **from** 1 **to** p_2 **do**
    $c_{i,j} := coeff(D(H)-1, X^iY^j)$; $M_{i,j} :=$MEMBER$(X^iY^j, \{F, G_{k+1}\}, \mathfrak{p}_{k+1})$
   od od;
   $H := H - \sum_{i,j} c_{i,j} \int M_{i,j}(F, G_{k+1})\, dG_{k+1}$
  od;
  for $g \in G$ **do** $e_g := 1$; $g' := D(g)$; $(\mathcal{F}_3, r_3) :=$REDPOL(g', I);
   if $r_3 \neq 0$ **then** $n := n + 1$; $g' := D(g')$ **fi**
  od;
  $e := \sum_{g \in G} e_g$; $\tilde{G} := H$;
  for k **from** 1 **to** $e - 1$ **do** $p_1 := deg_X(D(\tilde{G}) - 1)$; $p_2 := deg_Y(D(\tilde{G}) - 1)$;
   for i **from** 1 **to** p_1 **do for** j **from** 1 **to** p_2 **do**
    $c_{i,j} := coeff(D(\tilde{G})-1, X^iY^j)$; $M_{i,j} :=$MEMBER$(X^iY^j, \{F, H\}, \eta)$;
   od od;
   $\tilde{G} := \tilde{G} - \sum_{i,j} c_{i,j} \int M_{i,j}(F, H)\, dH$
  od;
  return$(\tilde{G})$
 fi
end.

We conclude this paper by stating the algorithms used in this section. The first 5 algorithms were taken from [1].

Algorithm REDPOL

Input : A finite subset P of $K[X_1, \ldots, X_m]$ and $f \in K[X_1, \ldots, X_m]$.
Output : A normal form g of f modulo P, and a family $\mathcal{F} = \{q_p\}_{p \in P}$ of polynomials with $f = \sum_{p \in P} q_p p + g$ and $max\{lt(q_p p) \mid p \in P, q_p p \neq 0\} \leq lt(f)$, where for $h \in K[X_1, \ldots, X_m]$, $lt(h)$ is the leading term of h w.r.t. some admissable term ordering.

Algorithm GRÖBNER

Input : A finite subset F of $K[X_1, \ldots, X_m]$.
Output : A finite subset G of $K[X_1, \ldots, X_m]$ with $F \subseteq G$ such that G is a GRÖBNER basis for (F).

Algorithm EXTGRÖBNER

Input : A finite subset F of $K[X_1, \ldots, X_m]$.
Output : A finite subset G of $K[X_1, \ldots, X_m]$ with $F \subseteq G$ such that G is a GRÖBNER basis for (F) and families $\mathcal{G} = \{\{Q_{g,f}\}_{f \in \mathcal{F}}\}_{g \in \mathcal{G}}$ and $\mathcal{F} = \{\{P_{f,g}\}_{g \in \mathcal{G}}\}_{f \in \mathcal{F}}$ such that $g = \sum_{f \in \mathcal{F}} Q_{g,f} f$ and $f = \sum_{g \in \mathcal{G}} P_{f,g} g$ for all $g \in G$ and $f \in F$.

Algorithm RADICAL

Input : A finite subset F of $K[X_1, \ldots, X_m]$.
Output : A finite basis G of $r(F)$.

Algorithm PRIMDEC

Input : A finite subset F of $K[X_1, \ldots, X_m]$.
Output : A set $P = \{(G_1, H_1), \ldots, (G_r, H_r)\}$ of pairs of finite subsets of $K[X_1, \ldots, X_m]$ such that $P = \emptyset$ if $1 \in (F)$, while otherwise

- For all i, $\mathfrak{q}_i := (G_i)$ is primary with associated prime $\mathfrak{p}_i := (H_i)$
- $\mathfrak{q}_i \neq q_j$ and $\mathfrak{p}_i \neq \mathfrak{p}_j$ whenever $i \neq j$
- $(F) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$.

And, finally we used the algorithm below, which was taken from [7].

Algorithm MEMBER

Input : A finitely generated K-algebra $K[x_1, \ldots, x_m]$ and $f_1, \ldots, f_k, g \in K[x_1, \ldots, x_m]$.
output : Either $P \in K[X_1, \ldots, X_m]$ such that $g = P(f_1, \ldots, f_k)$ or a message that $g \notin K[f_1, \ldots, f_k]$.

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