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# An intuitionistic proof of Kruskal's Theorem

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# An intuitionistic proof of Kruskal's Theorem

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## 0 Introduction

In 1960, J.B. Kruskal published a proof of a conjecture due to A. Vazsonyi. Vazsonyi's conjecture, to be explained in detail in Section 8, says that the collection of all finite trees is well-quasi-ordered by the relation of embeddability, that is, for every infinite sequence $\alpha(0), \alpha(1), \alpha(2), \ldots$ of finite trees there exist $i, j$ such that $i < j$ and $\alpha(i)$ embeds into $\alpha(j)$. Kruskal established an even stronger statement that he called the Tree Theorem. He proved it by a slight extension of an argument developed by G. Higman in 1952.
In 1963, a short proof of Kruskal's Theorem was given by C.St.J.A Nash-Williams, who introduced the elegant and powerful but non-constructive *minimal-bad-sequence* argument.
The purpose of this paper is to show that the arguments given by Higman and Kruskal are essentially constructive and acceptable from an intuitionistic point of view and that the later argument given by Nash-Williams is not.

The paper consists of the following 11 Sections.

1. Dickson's Lemma
2. Almost full relations
3. Brouwer's Thesis
4. Ramsey's Theorem
5. The Finite Sequence Theorem
6. Vazsonyi's Conjecture for binary trees
7. Higman's Theorem
8. Vazsonyi's Conjecture and the Tree Theorem
9. Minimal-Bad-Sequence Arguments
10. The Principle of Open Induction
11. Concluding Remarks

Except for Section 9, we will argue intuitionistically.

# 1 Dickson's Lemma

We start our discussion of Kruskal's Theorem by studying a special case.
John Burgess once asked for a constructive proof of the following statement:

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| For all infinite sequences $\alpha, \beta$ of natural numbers there exist $i, j$ such that $i < j$ and both $\alpha(i) \leq \alpha(j)$ and $\beta(i) \leq \beta(j)$. |

(We are using $i, j, m, n, \ldots$ as variables over the set $\mathbb{N}$ of natural numbers, and $\alpha, \beta, \ldots$ as variables over the set $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$, that is, all infinite sequences of natural numbers).
Let us answer this question and prove an immediate generalization of the above statement.

**Theorem 1.1** *For every $p > 0$, for all infinite sequences $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$ of natural numbers, there exist $i, j$ such that for every $k < p : \alpha_k(i) \leq \alpha_k(j)$.*

**Proof:** First observe that for every $\alpha$ there exists $i \leq \alpha(0)$ such that $\alpha(i) \leq \alpha(i+1)$. This observation proves the case $p = 1$ of the statement of the Theorem. We use induction and assume that $p > 1$ and that we proved the case $p - 1$ of the statement of the Theorem. We handle the case $p$ as follows. Let $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$ be infinite sequences of natural numbers. Define the proposition QED (with a meaning slightly different from the usual one, "quod est demonstrandum", "what is to be proved", rather than "quod erat demonstrandum", "what was to be proved") as follows:

QED := there exists $i, j$ such that for every $k < p$: $\alpha_k(i) \leq \alpha_k(j)$

We *claim*:

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| For every $n$, for every $m$, either QED or there exists $i > m$ such that $\alpha_0(i) \geq n$. |

Observe that the claim holds if $n = 0$. Let $n > 0$ be a natural number and assume that we proved already:

For every $m$, either QED or there exists $i > m$ such that $\alpha_0(i) \geq n - 1$.

Let $m$ be a natural number. Applying the assumption repeatedly, we find a strictly increasing sequence $\gamma(0), \gamma(1), \gamma(2), \ldots$ of natural numbers such that $m < \gamma(0)$ and for every $\ell$: either QED or $\alpha_0(\gamma(\ell)) \geq n - 1$.
Using the induction hypothesis, we calculate $i, j$ such that $i < j$ and for every $k$, if $0 < k < p$, then $\alpha_k(\gamma(i)) \leq \alpha_k(\gamma(j))$.
Now observe: either QED, or $\alpha_0(\gamma(i)) = \alpha_0(\gamma(j)) = n - 1$ and therefore QED, or $\alpha_0(\gamma(i)) \geq n$ or $\alpha_0(\gamma(j)) \geq n$, therefore: either QED or there exists $i > m$ such that $\alpha_0(i) \geq n$. We conclude that our claim is valid. There are two ways to complete the proof.

*First way*:
Using the claim repeatedly, we build a strictly increasing sequence $\gamma$ of natural numbers such that for every $i$: either QED or $\alpha_0\big(\gamma(i+1)\big) \geq \alpha_0\big(\gamma(i)\big)$. We again apply the induction hypothesis and calculate $i, j$ such that $i < j$ and for every $k$, if $1 < k < p$, then $\alpha_k\big(\gamma(i)\big) \leq \alpha_k\big(\gamma(j)\big)$.
Now observe: either QED, or for all $k < p$, $\alpha_k\big(\gamma(i)\big) \leq \alpha_k\big(\gamma(j)\big)$, so also QED, therefore in any case QED.

*Second way*:
Slightly generalizing the result of our claim, we conclude: For every $k < p$, for every $m$, there exists $i > m$ such that either QED or $\alpha_k(i) \geq m$.
We now first build a strictly increasing sequence $\gamma_0$ of natural numbers such that for every $i$: either QED or $\alpha_0(0) \leq \alpha_0\big(\gamma_0(i)\big)$. We then build a strictly increasing subsequence $\gamma_1$ of $\gamma_0$ such that for every $i$: either QED or $\alpha_1(0) \leq \alpha_1\big(\gamma_1(i)\big), \ldots$ and so on.
Finally we build a strictly increasing subsequence $\gamma_{p-1}$ of $\gamma_{p-2}$ such that for every $i$: either QED or $\alpha_{p-1}(0) \leq \alpha_{p-1}\big(\gamma_{p-1}(i)\big)$.
Now observe: either QED or for every $k < p$, $\alpha_k(0) \leq \alpha_k\big(\gamma_{p-1}(1)\big)$, so also QED. $\square$

## 1.2

Theorem 1.1 is known as Dickson's Lemma. (See Dickson 1913). It is often cited in the form: every subset of $\mathbb{N}^p$ contains a finite number of $\leq^p$-minimal elements, where, for all $(a_0, a_1, \ldots, a_{p-1})$ and $(b_0, b_1, \ldots, b_{p-1})$ in $\mathbb{N}^p$, $(a_0, a_1, \ldots, a_{p-1}) \leq^p (b_0, b_1, \ldots, b_{p-1})$ if and only if, for every $k < p$, $a_k \leq b_k$.
Formulated in this way, however, it is not true constructively.
Let us consider why.
Let $A$ be a decidable subset of $\mathbb{N}^p$.
Observe that the set $M(A)$ consisting of the $\leq^p$-minimal elements of $A$ is also a decidable subset of $\mathbb{N}^p$.
The following example shows that $M(A)$ need not be a finite subset of $\mathbb{N}^p$.

Let $d : \mathbb{N} \to \{0, 1, \ldots, 9\}$ be the decimal expansion of $\pi$,
so $\pi = 3 + \sum_{n=1}^{\infty} d(n) \cdot 10^{-n}$.
We define a subset $A$ of $\mathbb{N}^2$ as follows:
$A := \{\langle 1, 1\rangle\} \cup \{\langle 0, n\rangle \mid \text{There exists } i < n \text{ such that for all } k < 99,\ d(i + k) = 9\}$.
Observe that $M(A)$ has at least one and at most two $\leq^2$-minimal elements.
The statement: "$M(A)$ has exactly one element" implies that there is no $i$ such that for all $k < 99$, $d(i + k) = 9$.
The statement: "$M(A)$ has two elements" implies that there exists $i$ such that for all $k < 99$, $d(i + k) = 9$.

We clearly do not have a proof of either statement, and are unable to show that $M(A)$ is finite.

Theorem 1.1 implies however that, for every decidable subset $A$ of $\mathbb{N}^p$, the set $M(A)$ is *almost-finite* in the following sense:

For every $\gamma : \mathbb{N} \to M(A)$ there exists $i, j$ such that $i < j$ and $\gamma(i) = \gamma(j)$.

The notion of an almost-finite subset of $\mathbb{N}$ is studied in Veldman 1995 and Veldman 1999.

### 1.3

We should perhaps remark that a not unusual classical proof of Theorem 1.1 uses the following fact:

For every $\alpha : \mathbb{N} \to \mathbb{N}$ there exists a strictly increasing function $\gamma : \mathbb{N} \to \mathbb{N}$ such that, for every $i$, $\alpha\big(\gamma(i)\big) \leq \alpha\big(\gamma(i+1)\big)$.

(That is, every sequence of natural numbers contains a monotone subsequence.)
This statement fails constructively, as we may conclude from the following example:

Define $\alpha : \mathbb{N} \to \mathbb{N}$ as follows.
For each $n$, $\alpha(n) \leq 1$, and $\alpha(n) = 1$ if and only if there is no $i < n$ such that for all $k < 99$, $d(i + k) = 9$.
Suppose $\gamma : \mathbb{N} \to \mathbb{N}$ is strictly increasing and for every $i$,
$\alpha\big(\gamma(i)\big) \leq \alpha\big(\gamma(i+1)\big)$. If $\alpha\big(\gamma(0)\big) = 0$, then we have found $i$ such that for every $k < 99$, $d(i + k) = 9$, if $\alpha\big(\gamma(0)\big) = 1$ we are sure that no such $i$ exists.
We are unable to find such $i$ and we cannot find such $\gamma$.

## 2 Almost full relations

### 2.1

Let $A$ be a set and let $R$ be a binary relation on $A$. Let $\alpha : \mathbb{N} \to A$. We say that $\alpha$ *meets* $R$ if and only if there exist $i, j$ such that $i < j$ and $\alpha(i) R \alpha(j)$. We say that $R$ is *almost full* on $A$ if and only if every $\alpha : \mathbb{N} \to A$ meets $R$.
Observe that every relation that is almost full must be reflexive. (For any given $a$ in $A$ one may consider the sequence $\alpha : \mathbb{N} \to A$ such that for every $i$, $\alpha(i) = a$. As $\alpha$ meets $R$ we have $aRa$).
Assume that $R$ is a reflexive relation on the set $A$ and that the set $A$ has a decidable equality, that is, for every $a, b$ in $A$ one may decide $a = b$ or not $a = b$.
The statement "$R$ is almost full" is then equivalent to

> For every one-to-one function $\alpha : \mathbb{N} \to A$
> there exist $i, j$ such that $i < j$ and $\alpha(i) R \alpha(j)$.

We might also have used this as a definition of "$R$ is almost full", thereby allowing some non-reflexive relations to be almost full. In fact, we did so in Veldman and Bezem 1993.

A (reflexive) almost full relation on $A$ that is also transitive is called a *partial quasi-well-ordering* of $A$.
The importance of this notion has been stressed repeatedly see Kruskal 1972.

### 2.2

Let $A, B$ be sets and let $R \subseteq A \times A$ and $T \subseteq B \times B$ be binary relations on $A, B$ respectively.
We define a binary relation on the set $A \times B$, called the ***product*** $R \times T$ of the relations $R$ and $T$, as follows:

> for all $\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle$ in $A \times B$,
> $\langle a_0, b_0 \rangle R \times T \langle a_1, b_1 \rangle$ if and only if both $a_0 R a_1$ and $b_0 T b_1$.

**Theorem 2.3** *Let $R$ be an almost full (reflexive) relation on $\mathbb{N}$. Then $\leq \times R$ is almost full on $\mathbb{N} \times \mathbb{N}$.*

**Proof:** The proof is similar to the proof of Theorem 1.1. Let $\alpha, \beta$ be infinite sequences of natural numbers. Define the proposition QED as follows:

> QED := there exist $i, j$ such that both $\alpha(i) \leq \alpha(j)$ and $\beta(i) R \beta(j)$.

We prove first, by induction, as in the proof of Theorem 1.1:

For every $n$, for every $m$, either QED or there exists $i > m$ such that $\alpha(i) \geq n$.

We then build a strictly increasing sequence $\gamma$ of natural numbers such that

for every $i$, either QED or $\alpha(\gamma(i)) \leq \alpha(\gamma(i+1))$.

Now determine $i, j$ such that $i < j$ and $\beta(\gamma(j)) R \beta(\gamma(j))$.
Observe: either QED or both $\alpha(\gamma(i)) \leq \alpha(\gamma(j))$ and $\beta(\gamma(i)) R \beta(\gamma(j))$, therefore in any case QED. □

## 3 Brouwer's Thesis

Our next goal is to prove that the product of any two almost full relations is almost full. We will do so by transfinite induction. This paragraph is devoted to the intuitionistic treatment of transfinite constructions and proofs, and to Brouwer's Thesis which allows us to use these methods in proving our results.

### 3.1

We first introduce the notion of a stump.
We have taken the word "stump" from Brouwer 1954 but are using it in a sense which is not exactly his. Stumps are decidable subsets of the set $\mathbb{N}^*$ of finite sequences of natural numbers.
$*$ denotes the binary operation on $\mathbb{N}^*$ which consists in the concatenation of finite sequences.
If $s$ belongs to $\mathbb{N}^*$ and $A$ is a subset of $\mathbb{N}^*$ we let $s * A$ be the set of all finite sequences of the form $s * t$, where $t$ belongs to $A$.
The set **Stp** of stumps is given by the following inductive definition:

(i) The empty set, $\emptyset$, is a stump. We sometimes call this set the *basic stump.*

(ii) If $\sigma_0, \sigma_1, \sigma_2, \ldots$ is an infinite sequence of stumps, then the set $\{\langle\ \rangle\} \cup \bigcup_{n \in \mathbb{N}} \langle n \rangle * \sigma_n$ is also a stump.

(iii) Every stump is obtained from the empty stump by repeated applications of the construction step mentioned in (ii).

The stumps $\sigma_0, \sigma_1, \sigma_2, \ldots$ are called the *immediate substumps* of the stump $\sigma := \{\langle\ \rangle\} \cup \bigcup_{n \in \mathbb{N}} \langle n \rangle * \sigma_n$.

We may view a non-empty stump $\sigma$ as an $\omega$-sequence of stumps, that is, as a function from the set $\mathbb{N}$ of natural numbers to the set **Stp** of stumps associating to every natural number $n$ the $n$-th immediate substump of $\sigma$. We therefore sometimes write $\sigma(n)$ for the set $\{s | s \in \mathbb{N}^* | \langle n \rangle * s \in \sigma\}$.

### 3.2

Once we accept the inductive definition of the set of stumps we have to recognize the validity of the following principle of induction.

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| Let $A$ be a subset of the set **Stp** of stumps.<br>Suppose that every stump $\sigma$ belongs to $A$ as soon as every immediate substump of $\sigma$ belongs to $A$.<br>Then every stump belongs to $A$. |

We mention some consequences of the principle of induction on the set of stumps that we want to use in the sequel.

### 3.3

The first one is a principle of double induction.

Let us call an ordered pair $\langle \sigma_0, \sigma_1 \rangle$ of stumps *more simple* than an ordered pair $\langle \tau_0, \tau_1 \rangle$ of stumps if either $\sigma_0 = \tau_0$ and $\sigma_1$ is an immediate substump of $\tau_1$ or $\sigma_1 = \tau_1$ and $\sigma_0$ is an immediate substump of $\tau_0$.

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| Let $B$ be a subset of the set **Stp**$\times$**Stp** of ordered pairs of stumps.<br>Suppose that every pair $\langle \sigma_0, \sigma_1 \rangle$ belongs to $B$ as soon as every ordered pair of stumps more simple than $\langle \sigma_0, \sigma_1 \rangle$ belongs to $B$.<br>Then every pair of stumps belongs to $B$. |

One may prove this principle by defining

$A := \{\sigma | \sigma \in \mathbf{Stp} |$ for every $\tau$ in **Stp**, $\langle \sigma, \tau \rangle$ belongs to $B\}$

and then using the first principle of induction.

### 3.4

The second one is a principle of induction on finite sequences of stumps.
Let $m$ be a natural number. A finite sequence of length $m$ will be called an $m$-sequence. Let us call an $m$-sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ of stumps *easier than* an $n$-sequence $\langle \tau_0, \tau_1, \ldots, \tau_{n-1} \rangle$ of stumps, if either $m < n$ or $m = n$ and there exists $i < n$ such that $\sigma_i$ is an immediate substump of $\tau_i$ and for all $j$ such that $i < j < n : \sigma_j = \tau_j$.

> Let $C$ be a subset of the set $\mathbf{Stp}^*$ of finite sequences of stumps.
> Suppose that every finite sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$ belongs to $C$ as soon as every finite sequence of stumps easier than $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$ belongs to $C$.
> Then every finite sequence of stumps belongs to $C$.

One proves this principle by complete induction, showing, for each $n$, that every $n$-sequence of stumps belongs to $C$.
Dealing with the case $n$ one defines
$A := \{\sigma | \sigma \in \mathbf{Stp}|$ for every $(n-1)$-sequence $\langle \tau_0, \tau_1, \ldots, \tau_{n-2} \rangle$ of stumps the sequence $\langle \tau_0, \tau_1, \ldots, \tau_{n-2}, \sigma \rangle$ belongs to C$\}$
and uses the first principle of induction on the set of stumps.

### 3.5

The third one is another principle of induction on finite sequences of stumps.
Let us call a finite sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ of stumps *more facile* than a finite sequence $\langle \tau_0, \tau_1, \ldots, \tau_{n-1} \rangle$ of stumps if either $\langle \sigma_0, \sigma_1, \ldots, \sigma_{m-1} \rangle$ is easier than $\langle \tau_0, \tau_1, \ldots, \tau_{n-1} \rangle$ or $m > n$ and $\sigma_{m-1}$ is an immediate substump of $\tau_{n-1}$.

> Let $C$ be a subset of the set $\mathbf{Stp}^*$ of finite sequences of stumps.
> Suppose that every finite sequence $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$ belongs to $C$ as soon as every finite sequence of stumps more facile than $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$ belongs to $C$.
> Then every finite sequences of stumps belongs to $C$.

One may prove this principle as follows.
Define for every stump $\sigma$:

$P(\sigma) :=$ Every finite sequence of stumps $\langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \rangle$ such that $\sigma_{n-1} = \sigma$ belongs to $C$.

Using the first principle of induction one may prove:
for every $\sigma$, $P(\sigma)$.

### 3.6

We find it useful to reformulate the induction principles from Sections 3.4 and 3.5 as follows.
We call a stump $\sigma$ *finitary* if and only if $\sigma \neq \emptyset$ and there exists $i$ such that for every $j > i$, $\sigma(j) = \emptyset$.
Let $\sigma, \tau$ be finitary stumps. We say that $\sigma$ is *easier* than $\tau$ if and only if there exists $i$ such that $\sigma(i)$ is an immediate substump of $\tau(i)$ and for every $j > i$, $\sigma(j) = \tau(j)$.

#### 3.6.1

> *First principle of induction on finitary stumps.*
> Let $C$ be a collection of finitary stumps.
> If every finitary stump $\sigma$ belongs to $C$ as soon as every finitary stump easier than $\sigma$ belongs to $C$, then every finitary stump belongs to $C$.

One may prove this principle in the same way as the principle mentioned in Section 3.4.

For every stump $\sigma$ one may decide if $\sigma = \emptyset$ or not.
For every finitary stump $\sigma$ one may decide if $\sigma = \{\langle\,\rangle\}$ or not, that is, assuming that $\sigma \neq \emptyset$, if for every $i$ $\sigma(i) = \emptyset$, or not. A finitary stump $\sigma$ will be called *nontrivial* if $\sigma \neq \{\langle\,\rangle\}$. Let $\sigma$ be a nontrivial finitary stump.
There exists exactly one natural number $i$ such that $\sigma(i) \neq \emptyset$ and for every $j > i$, $\sigma(j) = \emptyset$. We will call this number the characteristic number of $\sigma$, notation: $i(\sigma)$.
Let $\sigma, \tau$ be finitary stumps. We say that $\sigma$ is *more facile* that $\tau$ if either $\sigma = \{\langle\,\rangle\}$ and $\tau$ is nontrivial or both $\sigma$ and $\tau$ are nontrivial and either $\sigma$ is easier than $\tau$ or $\sigma\big(i(\sigma)\big)$ is an immediate substump of $\tau\big(i(\tau)\big)$.

#### 3.6.2

> *Second principle of induction on finitary stumps.*
> Let $C$ be a collection of finitary stumps.
> If every finitary stump $\sigma$ belongs to $C$ as soon as every finitary stump more facile than $\sigma$ belongs to $C$, then every finitary stump belongs to $C$.

One may prove this principle in the same way as the principle mentioned in Section 3.5.

Let $\sigma$ be a finitary stump.
The set of all natural numbers $i$ such that $\sigma(i) \neq \emptyset$ will be called the domain of $\sigma$, notation: $\mathrm{Dom}(\sigma)$.

### 3.7

We call the following statement *Brouwer's Thesis.*

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| Let $P$ be a subset of the set $\mathbb{N}^*$ of finite sequences of natural numbers.<br>Suppose that for every infinite sequence $\alpha$ of natural numbers there exists $n$ such that $\langle\alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to $P$.<br>Then there exists a stump $\sigma$ such that for every infinite sequence $\alpha$ of natural numbers there exists $n$ such that $\langle\alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to both $\sigma$ and $P$. |

Brouwer came to this Thesis by reflecting on the possible structure of a proof of the statement "Every infinite sequence $\alpha$ has a finite initial part in the set $P$". The argument for his Thesis has been the subject of much debate in the foundations of intuitionistic mathematics. We should warn the reader that our formulation of the Thesis is not literally to be found in Brouwer's writings.

Now let $R$ be an almost full binary relation on $\mathbb{N}$. Brouwer's Thesis implies that there exists a stump $\sigma$ such that for every infinite sequence $\alpha$ there exists $n$ such that $\langle\alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to $\sigma$, and there exists $i, j$ such that $i < j < n$ and $\alpha(i) R \alpha(j)$.
We will say, under these circumstances, that the stump $\sigma$ *secures* the fact that $R$ is almost full.

# 4 Ramsey's Theorem

### 4.1

Let $A$ be a set. An *at-most-binary* relation on $A$ is a subset of $\{\langle\,\rangle\} \cup A^1 \cup A^2$.

### 4.2

Let $R$ be an at-most-binary relation on the set $\mathbb{N}$ of natural numbers.
Let $s = \langle s(0), s(1), \dots, s(n-1)\rangle$ be a finite sequence of natural numbers.
We say that $s$ *meets* $R$ if and only if some subsequence of $s$ belongs to $R$, that is, either the empty sequence $\langle\,\rangle$ belongs to $R$ or there exists $i < n$ such that $\langle s(i)\rangle$ belongs to $R$, or there exist $i, j < n$ such that $i < j$ and $\langle s(i), s(j)\rangle$ belongs to $R$. Let $\alpha$ be an infinite sequence of natural numbers. We say that $\alpha$ *meets* $R$ if and only if, for some $n$, the finite sequence $\langle\alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ meets $R$. We say that $R$ is *almost full* if and only if every infinite sequence $\alpha$ of natural numbers meets $R$.

### 4.3

Let $R$ be an at-most-binary relation on $\mathbb{N}$, and let $\sigma$ be a stump. We say that $\sigma$ *secures* that $R$ is almost full if and only if for every $\alpha$ in $\mathcal{N}$ there exists $n$ such that $\langle\alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to $\sigma$ and meets $R$.

Brouwer's Thesis implies: for every at-most-binary relation $R$ on the set $\mathbb{N}$ of natural numbers, if $R$ is almost full, then there exists a stump $\sigma$ that secures that $R$ is almost full.

### 4.4

Let $R, T$ be at-most-binary relations on $\mathbb{N}$.
We let $R \sqcap T$ be the set of all finite sequences $s$ such that either $s$ belongs to $R$ and some initial part of $s$ belongs to $T$ or $s$ belongs to $T$ and some initial part of $s$ belongs to $R$.
We might call $R \sqcap T$ the *open-intersection* of $R$ and $T$, for the following reason. With any at-most-binary relation $R$ on $\mathbb{N}$ we may associate the open subset $R^{\#}$ of Baire space consisting of all infinite sequences $\alpha$ in $\mathcal{N}$ such that either the empty sequence belongs to $R$ or $\langle\alpha(0)\rangle$ belongs to $R$ or $\langle\alpha(0), \alpha(1)\rangle$ belongs to $R$. Observe that for all at-most-binary relations $R, T$ on $\mathbb{N}$, $(R \sqcap T)^{\#} = R^{\#} \cap T^{\#}$.
Observe that the operation $\sqcap$ of open-intersection is idempotent, commutative and associative.

### 4.5

Let $R$ be an at-most-binary relation on $\mathbb{N}$.
For every $n$,we let $R^n$ be the set of all finite sequences $s$ such that $\langle n\rangle * s$ belongs to $R$.
Observe that $R^n$ is also an at-most-binary relation on $\mathbb{N}$, in fact even an at-most-unary relation on $\mathbb{N}$.
Remark finally that for every stump $\sigma$, for every at-most-binary relation $R$ on $\mathbb{N}$:
If $\sigma$ secures that $R$ is almost full then, for every $n$, *either* $\sigma^n = \emptyset$ and $\langle\,\rangle$ belongs to $R$ *or* $\sigma(n)$ secures that the almost-binary relation $R^n \cup R$ is almost full.

**Theorem 4.6** *For every stump $\sigma$, for all at-most-binary relations $R, T$ on $\mathbb{N}$,if $\sigma$ secures that $R$ is almost full, and if also $T$ is almost full, then $R \sqcap T$ is almost full.*

**Proof:** We use induction on the set **Stp** of stumps. We may assume that $\sigma$ is a stump different from $\emptyset$ and that the statement of the Theorem has been verified for every immediate substump $\sigma(n)$ of $\sigma$. We also assume that $R, T$ are almost full at-most-binary relations on $\mathbb{N}$ and that $\sigma$ secures that $R$ is almost full. Observe that, for each $n$, *either* $\sigma(n) = \emptyset$ and $\langle\,\rangle$ belongs to $R$ and $R \sqcap T$ is almost full, *or* $\sigma(n)$ secures that the almost-binary relation $R \cup R^n$ is almost full.
Using this fact repeatedly we conclude first that $(R \cup R^0) \sqcap T$ is almost full, then that $(R \cup R^1) \sqcap (R \cup R^0) \sqcap T$ is almost full, then that $(R \cup R^2) \sqcap (R \cup R^1) \sqcap (R \cup R^0) \sqcap T$ is almost full, and so on. Now let $\alpha$ be an infinite sequence of natural numbers.
We define the proposition QED by:

$$\text{QED} \;:=\; \alpha \text{ meets } R.$$

We construct a sequence $\gamma$ of natural numbers, by induction. We define $\gamma(0) := 0$. Let $n$ be a natural number and suppose we defined already the first $n+1$ values of $\gamma$, say $\gamma(0), \gamma(1), \ldots, \gamma(n)$.
As $\bigsqcap_{k \leq n} (R \cup R^{\alpha(\gamma(k))}) \sqcap T$ is almost full, we determine $i, j$ such that $\gamma(n) < i < j$ and some initial part of $\langle \alpha(i), \alpha(j) \rangle$ belongs to $\bigsqcap_{k \leq n} (R \cup R^{\alpha(\gamma(k))}) \sqcap T$.
We now may distinguish several cases.
If we discover that the empty sequence $\langle\,\rangle$ belongs to either $T$ or $R$, we know $R \sqcap T = R$ or $R \sqcap T = T$ and therefore QED. If we discover that $\langle \alpha(i) \rangle$ belongs to $T$, we may conclude QED if we also find either that $\langle \alpha(i) \rangle$ belongs to $R$ or that $\langle \alpha(i), \alpha(j) \rangle$ belongs to $R$. But if we do not discover that either $\langle\,\rangle$ or $\langle \alpha(i) \rangle$ or $\langle \alpha(i), \alpha(j) \rangle$ belongs to $R$, we find, for each $k \leq n$, either $\langle\,\rangle$ belongs to $R^{\alpha(\gamma(k))}$ or $\alpha(i)$ belongs to $R^{\alpha(\gamma(k))}$, that is, either $\langle \alpha(\gamma(k)) \rangle$ belongs to $R$ or $\langle \alpha(\gamma(k)), \alpha(i) \rangle$ belongs to $R$. So, if we discover that $\langle \alpha(i) \rangle$ belongs to $T$, we may conclude: QED or for every $k \leq n$, either $\langle \alpha(\gamma(k)) \rangle$ belongs to $R$ or $\langle \alpha(\gamma(k)), \alpha(i) \rangle$ belongs to $R$.
On the other hand, if we discover that $\langle \alpha(i), \alpha(j) \rangle$ belongs to $T$, we may conclude QED if we also find that either $\langle\,\rangle$ or $\langle \alpha(i) \rangle$ or $\langle \alpha(i), \alpha(j) \rangle$ belongs to $R$. So also if we discover that $\langle \alpha(i), \alpha(j) \rangle$ belongs to $T$ we may conclude: QED or for every $k \leq n$, either $\langle \alpha(\gamma(k)) \rangle$ belongs to $R$ or $\langle \alpha(\gamma(k)), \alpha(i) \rangle$ belongs to $R$.
We now define: $\gamma(n+1) := i$.

Observe that $\gamma$ is a strictly sequence of natural numbers and that for each $i, j$, if $i < j$, then either QED or $\langle \alpha(\gamma(i)) \rangle$ belongs to $R$ or $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to $R$.
We now use the fact that also $T$ is almost full and determine $i, j$ such that $i < j$ and some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to $T$.
Observe: either QED,or some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to $R \sqcap T$, therefore, in any case, QED.
In this way we come to see that every infinite sequence $\alpha$ of natural numbers meets $R \sqcap T$, that is, $R \sqcap T$ is almost full. □

**Corollary 4.7** (*Intuitionistic Ramsey Theorem*)
*For all at-most-binary relations $R, T$ on $\mathbb{N}$, if both $R$ and $T$ are almost full, then $R \sqcap T$ is almost full.*

**Proof:** Use Brouwer's Thesis and apply Theorem 4.3. □

## 4.8

We should perhaps explain why Corollary 4.7 is called the Intuitionistic Ramsey Theorem.
The classical (infinite) Ramsey Theorem, see Ramsey 1928, reads as follows:
For every binary relation $R$ on $\mathbb{N}$ *either* there exists a strictly increasing sequence $\gamma$ of natural numbers such that for all $i, j$, if $i < j$, then $\gamma(i) R \gamma(j)$, (such a sequence

is called *R-homogeneous*) *or* there exists a strictly increasing sequence $\gamma$ of natural numbers such that for all $i, j$, if $i < j$, then not $\gamma(i) R \gamma(j)$ (such a sequence is called $\mathbb{N} \times \mathbb{N} \backslash R$-homogeneous).
As it stands, this statement is constructively false, as the following example shows: define a relation $R$ on $\mathbb{N}$ by:
for all $m, n$, $mRn$ if and only if there exists $i \leq \max(m, n)$ such that for all $k < 99, d(i + k) = 9$.
If there exists an $R$-homogeneous strictly increasing sequence $\gamma$, then there exists $i$ such that for every $k < 99$, $d(i + k) = 9$, if there exists a $\mathbb{N} \times \mathbb{N} \backslash R$-homogeneous strictly increasing sequence $\gamma$, then there exists no such $i$.
We have no proof of either alternative.
Using classical logic, we may reformulate the classical Ramsey Theorem as follows:
For every binary relation $R$ on $\mathbb{N}$, it is impossible that both $R$ and $\mathbb{N} \times \mathbb{N} \backslash R$ are almost full. Observe that we may draw this conclusion, also constructively, from the Intuitionistic Ramsey Theorem.
Conversely, one may prove the Intuitionistic Ramsey Theorem from the classical one, using classical logic, as follows.
Suppose $R, T$ are almost full at-most-binary relations on $\mathbb{N}$. Let $\alpha$ be an infinite sequence of natural numbers. Determine a strictly increasing sequence $\gamma$ of natural numbers such that for all $i, j$ if $i < j$, then some initial part of $\langle \alpha(\gamma(i)), \alpha(\gamma(j)) \rangle$ belongs to $R$. Now determine $i, j$ such that $i < j$ and some initial part of $\langle \alpha \circ \gamma(i), \alpha(\gamma(j)) \rangle$ belongs to $T$. It is clear that $\alpha$ meets $R \sqcap T$.
We may conclude that $R \sqcap T$ is almost full.

## 4.9

We want to give a second proof of the Intuitionistic Ramsey Theorem. This second proof is in two steps. We first prove a result on at-most-unary relations on $\mathbb{N}$. An *at-most-unary relation* on $\mathbb{N}$ is a subset of $\mathbb{N}^0 \cup \mathbb{N}^1$. Observe that an at-most-unary relation on $\mathbb{N}$ is also an at-most-binary relation on $\mathbb{N}$.

**Theorem 4.9.1** *For all stumps $\sigma, \tau$ for all at-most-unary relations $A, B$ on $\mathbb{N}$, if $\sigma$ secures that $A$ is almost full and $\tau$ secures that $B$ is almost full, then $A \sqcap B$ is almost full.*

**Proof:** We use the principle of double induction on the set of stumps that we explained in Section 3.4.
Assume that $\langle \sigma_0, \sigma_1 \rangle$ is an ordered pair of stumps and that the statement of the Theorem has been verified for every pair $\langle \tau_0, \tau_1 \rangle$ of stumps that is more simple than $\langle \sigma_0, \sigma_1 \rangle$, that is, either $\tau_0 = \sigma_0$ and $\tau_1$ is an immediate substump of $\sigma_1$, or $\tau_1 = \sigma_1$ and $\tau_0$ is an immediate substump of $\sigma_0$. Let $A, B$ be at-most-unary relations on $\mathbb{N}$ such that $\sigma_0$ secures that $A$ is almost full and $\sigma_1$ secures that $B$ is almost full. We now prove that $A \sqcap B$ is almost full. Let $\alpha$ be an infinite sequence of natural numbers. Define the proposition QED as follows:

$$\text{QED} := \alpha \text{ meets } A \sqcap B.$$

Consider $\alpha(0)$. Observe that *either* $\sigma_0\big(\alpha(0)\big) = \emptyset$ and $\langle\,\rangle$ belongs to $A$ and $A \sqcap B$ is almost full, *or* $\sigma_0\big(\alpha(0)\big)$ secures that $A^{\alpha(0)} \cup A$ is almost full. Observe also that $\sigma_1$ secures that $B$ is almost full. Consider $\alpha \circ S$, the composition of the sequence $\alpha$ and the successor function $S$. Observe that $\alpha \circ S$ meets $(A^{\alpha(0)} \cup A) \sqcap B$.
Therefore either QED or $\langle\,\rangle$ belongs to $A^{\alpha(0)}$, that is, $\langle\alpha(0)\rangle$ belongs to $A$. Observe that $\sigma_0$ secures that $A$ is almost full and that *either* $\sigma_1\big(\alpha(0)\big) = \emptyset$ and $\langle\,\rangle$ belongs to $B$ and $A \sqcap B$ is almost full, *or* $\sigma_1\big(\alpha(0)\big)$ secures that $B^{\alpha(0)} \cup B$ is almost full. Therefore $\alpha \circ S$ meets $A \sqcap (B^{\alpha(0)} \cup B)$. Therefore either QED or $\langle\,\rangle$ belongs to $B^{\alpha(0)}$, that is, $\langle\alpha(0)\rangle$ belongs to $B$. Combining our conclusions, we find either QED or $\langle\alpha(0)\rangle$ belongs to $A \sqcap B$.
So in any case QED.
We may conclude that every infinite sequence $\alpha$ meets $A \sqcap B$, that is, $A \sqcap B$ is almost full. □

**Theorem 4.9.2** *For all stumps $\sigma_0, \sigma_1$, for all at-most-binary relations $R, T$ on $\mathbb{N}$, if $\sigma_0$ secures that $R$ is almost full and $\sigma_1$ secures that $T$ is almost full, then $R \sqcap T$ is almost full.*

**Proof:** We again use the earlier mentioned principle of double induction on the set of stumps.
Assume that $\sigma_0, \sigma_1$ are stumps and that the statement of the Theorem has been verified for every pair of stumps more simple than $\langle\sigma_0, \sigma_1\rangle$. Let $R, T$ be at-most-binary relations on $\mathbb{N}$ such that $\sigma_0$ secures that $R$ is almost full and $\sigma_1$ secures that $T$ is almost full. We now prove that $R \sqcap T$ is almost full. Let $\alpha$ be an infinite sequence of natural numbers. Define the proposition QED as follows:

$$\text{QED} := \alpha \text{ meets } R \sqcap T.$$

Observe that *either* $\sigma_0\big(\alpha(0)\big) = \emptyset$ and $\langle\,\rangle$ belongs to $R$ and $R \sqcap T$ is almost full *or* $\sigma\big(\alpha(0)\big)$ secures that $R^{\alpha(0)} \cup R$ is almost full. Observe also that $\sigma_1$ secures that $T$ is almost full.
So for every infinite sequence $\beta$ of natural numbers there exist $i, j$ such $i < j$ and some initial part of $\langle\alpha\big(\beta(i)\big), \alpha\big(\beta(j)\big)\rangle$ belongs to $R^{\alpha(0)} \cup R$ and some initial part of $\langle\alpha\big(\beta(i)\big), \alpha\big(\beta(j)\big)\rangle$ belongs to $T$. Spelling out the various possibilities we find: either QED or $\langle\alpha(0)\rangle$ belongs to $R$ or $\langle\alpha(0), \alpha\big(\beta(i)\big)\rangle$ belongs to $R$. Similarly, using the fact that $\sigma_0$ secures that $R$ is almost full and that *either* $\sigma_1\big(\alpha(0)\big) = \emptyset$ *or* $\sigma_1\big(\alpha(0)\big)$ secures that $T^{\alpha(0)} \cup T$ is almost full, we find that for every infinite sequence $\beta$ of natural numbers there exists $i$ such that either QED or $\langle\alpha(0)\rangle$ belongs to $T$ or $\langle\alpha(0), \alpha\big(\beta(i)\big)\rangle$ belongs to $T$.
Using the previous Theorem we find $i$ such that either QED or both an initial part of $\langle\alpha(0), \alpha(i)\rangle$ belongs to $R$ and an initial part of $\langle\alpha(0), \alpha(i)\rangle$ belongs to $T$, therefore again QED.

We may conclude that every infinite sequence $\alpha$ meets $R \sqcap T$, that is, $R \sqcap T$ is almost full. $\square$

## 4.10

The method of proof of Theorem 4.9.2 is more powerful than the method of proof of Theorem 4.6. We may use a similar double induction to obtain the corresponding result for at-most-ternary relations. It seems impossible to prove this by an argument in the style of the proof of Theorem 4.6. One may go on and prove the result for at-most-$n$-ary relations,where $n$ is a natural number. The strongest result in this direction is the so-called Clopen Ramsey Theorem. We do not go into details, as we will not make use of this Theorem when dealing with the main subject of this paper.

**Corollary 4.10.1**

(*i*) *Let $A, B$ be subsets of $\mathbb{N}$ such that every strictly increasing sequence of natural numbers meets both $A$ and $B$.*
*Then every strictly increasing sequence of natural numbers meets $A \cap B$.*

(*ii*) *Let $R, T$ be binary relations on $\mathbb{N}$ such that every strictly increasing sequence of natural numbers meets both $R$ and $T$. Then every strictly increasing sequence of natural numbers meets $R \cap T$.*

**Proof:**

(i) Define at-most-binary relations $A', B'$ on $\mathbb{N}$ by:
$A' := \{\langle m, n\rangle | m \in A \text{ or } m = n\}$ and
$B' := \{\langle m, n\rangle | m \in B \text{ or } m = n\}$.
Observe that both $A'$ and $B'$ are almost full.
(One proves this as follows. Let $\alpha$ be an infinite sequence of natural numbers. Define an infinite sequence $\alpha'$ of natural numbers, as follows: $\alpha'(0) := \alpha(0)$ and for each $n > 0$, $\alpha'(n) := \alpha(n)$ if there does not exist $i < n$ such that $\alpha'(i) = \alpha(n)$, and $\alpha'(n) := \underset{i<n}{\text{Max}}(\alpha'(i) + 1)$ otherwise. Observe that $\alpha'$ is one-to-one, so it has a strictly increasing subsequence and will meet $A$, say $\alpha'(i)$ belongs to $A$. If $\alpha'(i) = \alpha(i)$, then $\alpha$ meets $A$ and therefore also $A'$, if $\alpha'(i) \neq \alpha(i)$, then $\alpha$ meets $=$ and therefore also $A'$)
From Corollary 4.7, the Intuitionistic Ramsey Theorem, we conclude that $A' \sqcap B'$ is almost full.
So every strictly increasing sequence meets $A' \sqcap B'$, and, as it does not meet $=$, it will meet $A \cap B$.

(ii) We leave the proof of part (ii) to the reader.

$\square$

## 4.11

The statement of Corollary 4.10.1 is called the Intuitionistic Ramsey Theorem in Veldman and Bezem 1993.

**Corollary 4.11.1** *Let $R, T$ be binary relations on $\mathbb{N}$. If both $R, T$ are almost full, then $R \times T$ is almost full.*

**Proof:** Let $J : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be some pairing function, so $J$ is one-to-one and onto and has inverse functions $K, L : \mathbb{N} \to \mathbb{N}$.
For each $n$, $J\big(K(n), L(n)\big) = n$.
Now define binary relations $R'$ and $T''$ on $\mathbb{N}$ by

$$R' := \{\langle m, n\rangle | \langle K(m), K(n)\rangle \text{ belongs to } R\} \text{ and}$$

$$T'' := \{\langle m, n\rangle | \langle L(m), L(n)\rangle \text{ belongs to } T\}.$$

Observe that $R'$ and $T''$ are almost full,therefore $R' \cap T''$ is almost full, and also $R \times T$ is almost full on $\mathbb{N} \times \mathbb{N}$. □

### 4.11.2

Let $A, B$ be sets. The sum or disjoint union of the sets $A, B$ is the set $A \times \{0\} \cup B \times \{1\}$, notation $A \biguplus B$ or $A + B$. Let $R \subseteq A \times A$ and $T \subseteq B \times B$ be binary relations on $A, B$, respectively. We define a binary relation $R + T$, called the sum of the relations $R$ and $T$ as follows: for all $\langle c_0, i_0\rangle$, $\langle c_1, i_1\rangle$ in $A + B$, $\langle c_0, i_0\rangle R + T\langle c_1, i_1\rangle$ if and only if either $i_0 = i_1 = 0$ and $c_0 R c_1$ or $i_0 = i_1 = 1$ and $d_0 T d_1$.

**Corollary 4.11.3** *Let $R, T$ be binary relations on $\mathbb{N}$. If both $R, T$ are almost full on $\mathbb{N}$, then $R + T$ is almost full on $\mathbb{N} \biguplus \mathbb{N}$.*

**Proof:** Observe that $=$ is almost full on $\{0, 1\}$,therefore, by Corollary 4.11.1 $R \times T \times =$ is almost full on $\mathbb{N} \times \mathbb{N} \times \{0, 1\}$. Therefore, for every sequence $\alpha : \mathbb{N} \to \mathbb{N}$ and every sequence $\beta : \mathbb{N} \to \{0, 1\}$ there exist $i, j$ such that $\alpha(i) R \alpha(j)$ and $\alpha(i) T \alpha(j)$ and $\beta(i) = \beta(j)$, that is: $R + T$ is almost full on $\mathbb{N} + \mathbb{N}$. □

# 5 The Finite Sequence Theorem

## 5.1

We consider the set $\mathbb{N}^*$ of all finite sequences of natural numbers. $*$ denotes the binary operation of concatenation of finite sequences. For every nonempty element $s$ of $\mathbb{N}^*$ there exist a natural number $s(0)$ and a finite sequence of natural numbers $\text{Rem}(s)$ ("the remainder of $s$") such that $s = \langle s(0)\rangle * \text{Rem}(s)$. We now define a binary relation $\leq^*$ on $\mathbb{N}^*$ as follows:

| |
|---|
| For all $s, t$ in $\mathbb{N}^*$: |
| $s \leq^* t$ if and only if either $s = \langle\,\rangle$ or both $s$ and $t$ are non-empty and *either* $s \leq^* \text{Rem}(t)$ *or* $s(0) \leq t(0)$ and $\text{Rem}(s) \leq^* \text{Rem}(t)$. |

This definition may be thought of as a definition by recursion to length $(s)+$ length $(t)$.
It is useful to think of a finite sequence $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$ as a function with domain $n = \{0, 1, \ldots, n-1\}$. We will write: $\mathrm{Dom}(s) = n$. So the domain of a finite sequence is the same as its length.
One may verify without difficulty:
For all $s, t$ in $\mathbb{N}^*$: $s \leq^* t$ if and only if there exists a strictly increasing function $h$ from $\mathrm{Dom}(s)$ to $\mathrm{Dom}(t)$ such that for all $i$ in $\mathrm{Dom}(s)$ : $s(i) \leq t\big(h(i)\big)$.
For example: $\langle 1, 3, 5\rangle \leq^* \langle 0, 0, 2, 1, 8, 3, 3, 6, 1\rangle$.
We want to prove that $\leq^*$ is almost full on $\mathbb{N}^*$. To this end, we define, for each $n, k$, an element $\langle n\rangle^k$ of $\mathbb{N}^*$ as follows: $\langle n\rangle^0 := \langle\,\rangle$ and for each $k$, $\langle n\rangle^{k+1} := \langle n\rangle * \langle n\rangle^k$. So $\langle n\rangle^k$ is the finite sequence of length $k$ with the constant value $n$. Remark that for all $s_0, t_0, s_1, t_1$ in $\mathbb{N}^*$: If $s_0 \leq^* s_1$ and $t_0 \leq^* t_1$, then $s_0 * t_0 \leq^* s_1 * t_1$.

**Theorem 5.2** *For every infinite sequence $\alpha$ of finite sequences of natural numbers there exist $i, j$ such that $i < j$ and $\alpha(i) \leq^* \alpha(j)$.*

**Proof:** For all natural numbers $n, k$ we define the proposition $P(n, k)$ as follows:

> $P(n, k) :=$ For every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist $i, j$ such that $i < j$ and either $\alpha(i) \leq^* \alpha(j)$ or $\langle n\rangle^{k+1} \leq^* \alpha(i)$.

We want to prove: for all $n, k, P(n, k)$, and do so by double induction.
Observe that $P(0, 0)$ is true.
We now show:

$$\boxed{\text{For all } n, k, \text{ if } P(n,k), \text{ then } P(n,k+1).}$$

So let $n, k$ be natural numbers and assume $P(n, k)$ and let $\alpha : \mathbb{N} \to \mathbb{N}^*$. We construct two functions $\beta_0, \beta_1$ from $\mathbb{N}$ to $\mathbb{N}^*$ and a function $\gamma$ from $\mathbb{N}$ to $\mathbb{N}$ such that for every $i$, $\alpha(i) = \beta_0(i) * \langle\gamma(i)\rangle * \beta_1(i)$ and, if not $\langle n\rangle^{k+1} \leq^* \alpha(i)$, then $\beta_1(i) = \langle\,\rangle$, and, if $\langle n\rangle^{k+1} \leq^* \alpha(i)$, then not $\langle n\rangle^{k+1} \leq^* \beta_0(i)$ but $\langle n\rangle^k \leq^* \beta_0(i)$ and $n \leq \gamma(i)$.
Now observe: for every $i$, not $\langle n\rangle^{k+1} \leq^* \beta_0(i)$, therefore, by the assumption $P(n, k)$, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and $\beta_0\big(\delta(i)\big) \leq^* \beta_0\big(\delta(j)\big)$.
Also, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and $\gamma\big(\delta(i)\big) \leq \gamma\big(\delta(j)\big)$.
Finally, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and either $\beta_1\big(\delta(i)\big) \leq^* \beta_1\big(\delta(j)\big)$ or $\langle n\rangle \leq^* \beta_1\big(\delta(i)\big)$. (This follows from the assumption $P(n, k)$.)
Applying Ramsey's Theorem, we calculate $i, j$ such that $i < j$ and simultaneously $\beta_0(i) \leq^* \beta_0(j)$ and $\gamma(i) \leq \gamma(j)$ and either $\beta_1(i) \leq^* \beta_1(j)$ or $\langle n\rangle \leq^* \beta_1(i)$, therefore either $\alpha(i) \leq^* \alpha(j)$ or $\langle n\rangle \leq^* \beta_1(i)$. Suppose $\langle n\rangle \leq^* \beta_1(i)$, then reconsidering our construction, we find $\langle n\rangle^k \leq^* \beta_0(i)$ and $n \leq \gamma(i)$, therefore $\langle n\rangle^{k+2} \leq^* \alpha(i)$.
We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist $i, j$ such that $i < j$ and either $\alpha(i) \leq^* \alpha(j)$ or $\langle n\rangle^{k+2} \leq^* \alpha(i)$, that is $P(n, k+1)$.
We now want to prove:

$$\boxed{\text{For all } n, \text{ if for all } k,\ P(n,k), \text{ then } P(n+1,0).}$$

So let $n$ be a natural number and assume: for every $k$, $P(n,k)$. We want to prove: $P(n+1,0)$.
For every finite sequence of natural numbers $s$ we define a finite sequence $s'$ of natural numbers of the same length as $s$ such that for every $j < \text{length}(s)$, $s'(j) = \min(s(j),n)$.
Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. Calculate $k = \text{length}(\alpha(0))$. We may assume $k > 0$. Applying $P(n,k-1)$ we find $i,j$ such that $0 < i < j$ and either $\alpha'(i) \leq^* \alpha'(j)$ or $\langle n\rangle^k \leq^* \alpha'(i)$.
Now observe: if $\alpha'(i) \neq \alpha(i)$ or $\alpha'(j) \neq \alpha(j)$, then $\langle n+1\rangle \leq^* \alpha(i)$ or $\langle n+1\rangle \leq^* \alpha(j)$.
On the other hand, if both $\alpha'(i) = \alpha(i)$ and $\alpha'(j) = \alpha(j)$ then either $\alpha(i) \leq^* \alpha(j)$ or $\langle n\rangle^k \leq^* \alpha(i)$.
But if $\langle n\rangle^k \leq^* \alpha(i)$, then also $\alpha'(0) \leq^* \alpha(i)$, as $\alpha'(0) \leq^* \langle n\rangle^k$ and if $\alpha(0)) \neq \alpha'(0)$, then $\langle n+1\rangle \leq^* \alpha(0)$.
We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist $i,j$ such that $i < j$ and either $\alpha(i) \leq^* \alpha(j)$ or $\langle n+1\rangle \leq^* \alpha(i)$, that is, $P(n+1,0)$.

Clearly then, for all $n,k$, $P(n,k)$.
Now let $\alpha : \mathbb{N} \to \mathbb{N}^*$. Calculate $k := \text{length}\big(\alpha(0)\big)$ and $n := \max\{\big(\alpha(0)\big)(j) | j < k\}$. Applying $P(n,k)$ we find $i,j$ such that $0 < i < j$ and either $\langle n\rangle^k \leq^* \alpha(i)$ or $\alpha(i) \leq^* \alpha(j)$, and therefore either $\alpha(0) \leq^* \alpha(i)$ or $\alpha(i) \leq^* \alpha(j)$.
Therefore, every infinite sequence of finite sequences of natural numbers meet $\leq^*$, that is, $\leq^*$ is almost full on $\mathbb{N}^*$. □

## 5.3

We intend to generalize Theorem 5.2.
Let $R$ be an at-most-binary relation on the set $\mathbb{N}$ of natural numbers. We now define a binary relation $R^*$ on the set $\mathbb{N}^*$ of finite sequences of natural numbers, as follows.

| |
|---|
| For all $s,t$ in $\mathbb{N}^*$:<br>$sR^*t$ if and only if either $s = \langle\,\rangle$ or both $s$ and $t$ are non-empty and *either* $sR^*\text{Rem}(t)$ *or* one of the three sequences $\langle\,\rangle$, $\langle s(0)\rangle$ and $\langle s(0),t(0)\rangle$ belongs to $R$ and $\text{Rem}(s)R^*\text{Rem}(t)$. |

This definition may be thought of as a definition by recursion to length $(s)$ + length$(t)$. One may prove without difficulty:
For all $s,t$ in $\mathbb{N}^*$:
$sR^*t$ if and only if there exists a strictly increasing function $h$ from $\text{Dom}(s)$ to $\text{Dom}(t)$ such that for all $i$ in $\text{Dom}(s)$, one of the three sequences $\langle\,\rangle$, $\langle s(i)\rangle$ and $\langle s(i),t\big(h(i)\big)\rangle$ belongs to $R$.
We want to prove:
For every at-most-binary relation $R$ on $\mathbb{N}$, if $R$ is almost full on $\mathbb{N}$, then $R^*$ is almost full on $\mathbb{N}^*$.
We first consider the case that $R$ is decidable, that is, we may decide, for every finite sequences of length at most 2, if $s$ belongs to $R$ or not.

**Theorem 5.4** *For every stump $\sigma$, for every at-most-binary decidable relation $R$ on $\mathbb{N}$,*
*if $\sigma$ secures that $R$ is almost full on $\mathbb{N}$, then $R^*$ is almost full on $\mathbb{N}^*$.*

**Proof:** We use the principle of induction on the set **Stp** of stumps. The statement of the Theorem is obviously true if $\sigma = \emptyset$ as there is no relation $R$ such that the empty stump secures that $R$ is almost full.
Let us assume that $\sigma$ is a non-basic stump and that the statement of the Theorem has been proved for every one of its immediate substumps $\sigma(n)$.
Let $R$ be a decidable at-most-binary relation on $\mathbb{N}$ such that $\sigma$ secures that $R$ is almost full.
Observe that for every $n$, *either* $\sigma(n) = \emptyset$, therefore $\langle\,\rangle$ belongs to $R$, and $R^*$ is almost full, *or* $\sigma(n)$ secures that $R \cup R^n$ is almost full, so we may assume, by the induction hypothesis, that $(R \cup R^n)^*$ is almost full on $\mathbb{N}^*$.
For every finite sequence $s$ of natural numbers we define the proposition $P(s)$ as follows:

$$\begin{array}{ll} P(s) := & \text{For every } \alpha : \mathbb{N} \to \mathbb{N}^* \text{ there exist } i, j \text{ such that } i < j \\ & \text{and either } \alpha(i) R^* \alpha(j) \text{ or } s R^* \alpha(i). \end{array}$$

We want to prove: for every finite sequence $s$ of natural numbers, $P(s)$, and do so by induction on length$(s)$.
Observe that $P(\langle\,\rangle)$ is trivially true.
Observe also that, for every $n$, the proposition $P(\langle n \rangle)$ is equivalent to the statement that $(R \cup R^n)^*$ is almost full on $\mathbb{N}^*$, and therefore true by the induction hypothesis.
Now assume that $s$ is a finite sequence of length at least 2 and that we proved $P\big(\mathrm{Rem}(s)\big)$.
We want to prove $P(s)$. So assume $\alpha : \mathbb{N} \to \mathbb{N}^*$.
We construct two functions $\beta_0, \beta_1$ from $\mathbb{N}$ to $\mathbb{N}^*$ and a function $\gamma$ from $\mathbb{N}$ to $\mathbb{N}$ such that for every $i$, $\alpha(i) = \beta_0(i) * \langle \gamma(i) \rangle * \beta_1(i)$ and not $\langle s(0) \rangle R^* \beta_0(i)$, but if $\langle s(0) \rangle R^* \alpha(i)$, then some initial part of the sequence $\langle s(0), \gamma(i) \rangle$ belongs to $R$. Also, for every $i$, if not $\langle s(0) \rangle R^* \alpha(i)$, then $\beta_1(i) = \langle\,\rangle$.
Observe that for every $i$, not $\langle s(0) \rangle R^* \beta_0(i)$, and by the induction hypothesis, $(R \cup R^{s(0)})^*$ is almost full on $\mathbb{N}^*$,therefore, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such $i < j$ and $\beta_0(i) R^* \beta_0(j)$. Also, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and some initial part of $\langle \gamma\big(\delta(i)\big), \gamma\big(\delta(j)\big) \rangle$ belongs to $R$.
Finally, as we are assuming $P\big(\mathrm{Rem}(s)\big)$, for every $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and either $\beta_1\big(\delta(i)\big) R^* \beta_1\big(\delta(j)\big)$ or $\mathrm{Rem}(s) R^* \beta_1\big(\delta(i)\big)$.
Applying Ramsey's Theorem we calculate $i, j$ such that $i < j$ and simultaneously $\beta_0(i) R^* \beta_0(j)$ and some initial part of $\langle \gamma(i), \gamma(j) \rangle$ belongs to $R$ and either $\beta_1(i) R^* \beta_1(j)$ or $\mathrm{Rem}(s) R^* \beta_1(i)$, therefore either $\alpha(i) R^* \alpha(j)$ or $\mathrm{Rem}(s) R^* \beta_1(i)$.
Suppose $\mathrm{Rem}(s) R^* \beta_1(i)$. Reconsidering our construction we see that $\beta_1(i) \neq \langle\,\rangle$, therefore some initial part of $\langle s(0), \gamma(i) \rangle$ belongs to $R$, so $s R^* \alpha(i)$.
We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist $i, j$ such that $i < j$ and either $\alpha(i) R^* \alpha(j)$ or $s R^* \alpha(i)$, that is: $P(s)$. Clearly then, for every finite sequence $s$ of natural numbers, $P(s)$.

Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. $P\big(\alpha(0)\big)$ implies that $\alpha$ meets $R^*$.
Therefore, every infinite sequence of finite sequences of natural numbers meets $R^*$, that is, $R^*$ is almost full on $\mathbb{N}^*$. $\square$

### 5.5

Theorem 5.2 may easily be derived from Theorem 5.4.
Also the idea at work in the proof of Theorem 5.2 is related to the leading idea in the proof of Theorem 5.4. In the case of Theorem 5.2 think of the stump consisting of all finite sequences of natural numbers that just meet the relation $\leq$. (A finite sequence of natural numbers *just meets* a relation $R$ if it meets the relation $R$ but no proper initial part of it meets the relation $R$.) Observe that the finite sequences in the $n$-th immediate substump of this stump have length at most $n+2$. Proving for every $k$, $P(n,k)$ in the proof of Theorem 5.2 corresponds to showing that this $n$-th immediate substump satisfies Theorem 5.4.

### 5.6

We want to get rid of the assumption that $R$ is a *decidable* at-most-binary relation on $\mathbb{N}$ in Theorem 5.4. Observe that we managed to prove Theorem 4.9.1 and its Corollary, Theorem 4.9.2, the Intuitionistic Ramsey Theorem, without making such an assumption. Our main tool in achieving our goal will be the Fan Theorem. The Fan Theorem probably is the best-known consequence of Brouwer's Thesis.

#### 5.6.1

Let $\delta$ be an infinite sequence of natural numbers.
We let $F_\delta$, the *fan determined by* $\delta$, be the collection of all infinite sequences $\gamma$ of natural numbers such that, for every $n$, $\gamma(i) \leq \delta(i)$. We let $K_\delta$ be the set of all initial parts of members of $F_\delta$, that is, $K_\delta$ is the set of all finite sequences $c$ of natural numbers such that for every $i < \text{length}(c)$, $c(i) \leq \delta(i)$.
We also define a mapping $G_\delta$ of the set $\mathcal{N}$ of all infinite sequences of natural numbers into the set $F_\delta$, as follows:
for every $\alpha$, for every $n$ $\big(G_\delta(\alpha)\big)(i) = \min\big(\alpha(i), \delta(i)\big)$.
Observe that for every $\gamma$ in $F_\delta$, $G_\delta(\gamma) = \gamma$.
$G_\delta$ is called a *retraction* of $\mathcal{N}$ onto $F_\delta$.
For every $\alpha$ in $\mathcal{N}$, $n$ in $\mathbb{N}$, we define: $\overline{\alpha}n := \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$.

**Lemma 5.6.2** *For every stump $\sigma$, for every infinite sequence $\delta$ of natural numbers, the set $\sigma \cap K_\delta$ is a finite set of finite sequences of natural numbers.*

**Proof:** We use the principle of induction on the set of stumps. Observe that the statement of the Lemma is obviously true in case $\sigma = \emptyset$. Assume that $\sigma$ is not the basic stump and that the statement of the Theorem holds for every immediate substump of $\sigma$.

Let $\delta$ be an infinite sequence of natural numbers.
Observe that $\sigma \cap K_\delta = \{\langle\,\rangle\} \cup \bigcup_{i \le \delta(0)} \langle i\rangle * (\sigma(i) \cap K_{\delta\circ S})$, therefore $\sigma \cap K_\delta$ is a finite subset of $\mathbb{N}^*$. □

**Theorem 5.6.3** (*Fan Theorem*)
*Let $\delta$ be an infinite sequence of natural numbers.*
*Let $P$ be a subset of $\mathbb{N}^*$ such that every $\gamma$ in $F_\delta$ has an initial segment in $P$.*
*There exists a finite subset $Q$ of $P$ such that every $\gamma$ in $F_\delta$ has an initial segment in $Q$.*

**Proof:** Assume that every $\gamma$ in $F_\delta$ has an initial segment in $P$.
Let $G_\delta$ be the retraction of $\mathcal{N}$ onto $F_\delta$ as defined in Section 5.6.1. Observe that for every $\gamma$ there exists $n$ such that $\overline{G_\delta(\gamma)}n$ belongs to $P$, therefore either $\overline{G_\delta(\gamma)}n \neq \overline{\gamma}n$ (and therefore: $\overline{\gamma}n \notin K_\delta$), or $\overline{\gamma}n$ belongs to $P$. Using Brouwer's Thesis, determine a stump $\sigma$ such that for every $\gamma$ there exists $n$ such that $\overline{\gamma}n$ belongs to $P$ or to $\mathbb{N}^* \backslash K_\delta$, and $\overline{\gamma}n$ belongs to $\sigma$. Consider the finite set $\sigma \cap K_\delta$ and determine for every finite sequence in this set of maximal length an initial segment in $P$. Let $Q$ consist of all initial segments obtained in this way. □

## 5.7

For every non-empty $t$ in $\mathbb{N}^*$ and every $c$ in $\mathbb{N}$, $c < \text{length}(t)$, we let $A_0(t,c)$ and $A_1(t,c)$ be the elements of $\mathbb{N}^*$ such that $t = A_0(t,0) * \langle t(c)\rangle * A_1(t,c)$.

**Theorem 5.7.1** *For every stump $\sigma$, for every at-most-binary relation $R$ on $\mathbb{N}$, if $\sigma$ secures that $R$ is almost full on $\mathbb{N}$, then $R^*$ is almost full on $\mathbb{N}^*$.*

**Proof:** We use the principle of induction on the set of stumps. If $\sigma = \emptyset$, then the statement of the Theorem is obviously true. Let us assume that $\sigma$ is a non-basic stump and that the statement of the Theorem has been proved for every one of its immediate substumps.
Let $R$ be an at-most-binary relation on $\mathbb{N}$ such that $\sigma$ secures that $R$ is almost full.
For every finite sequence $s$ of natural numbers we define the proposition $P(s)$ as follows:

$$P(s) := \quad \text{For every } \alpha : \mathbb{N} \to \mathbb{N}^* \text{ there exist } i, j \text{ such that } i < j \text{ and either } \alpha(i) R^* \alpha(j) \text{ or } s R^* \alpha(i).$$

We want to prove: for every finite sequence $s$ of natural numbers, $P(s)$, and do so by induction on length$(s)$.
Observe that $P(\langle\,\rangle)$ is trivially true.
Observe also that, for every $n$, the proposition $P(\langle n\rangle)$ is equivalent to the statement that $(R \cup R^n)^*$ is almost full on $\mathbb{N}^*$, and therefore true by the induction hypothesis.
Now assume that $s$ is a finite sequence of natural numbers of length at least 2 and that we proved $P\big(\text{Rem}(s)\big)$.

We want to prove $P(s)$. So assume $\alpha : \mathbb{N} \to \mathbb{N}^*$.
We define $\delta : \mathbb{N} \to \mathbb{N}$ as follows.
For each $i$, $\delta(i) := \text{length}(\alpha(i)) - 1$ if $\alpha(i) \neq \langle\,\rangle$ and $\delta(i) := 0$ if $\alpha(i) = \langle\,\rangle$. For every $\gamma$ in the fan $F_\delta$ we define functions $B_0(\alpha,\gamma)$ and $B_1(\alpha,\gamma)$ from $\mathbb{N}$ to $\mathbb{N}^*$ and a function $m(\alpha,\gamma)$ from $\mathbb{N}$ to $\mathbb{N}$ such that for every $i$, if $\alpha(i) \neq \langle\,\rangle$, then $(B_0(\alpha,\gamma))(i) := A_0(\alpha(i),\gamma(i))$ and $(m(\alpha,\gamma))(i) := (\alpha(i))(\gamma(i))$ and $(B_1(\alpha,\gamma))(i) := A_1(\alpha(i),\gamma(i))$ therefore $\alpha(i) = (B_0(\alpha,\gamma))(i) * \langle (m(\alpha,\gamma))(i)\rangle * (B_1(\alpha,\gamma))(i)$ and if $\alpha(i) = \langle\,\rangle$, then $(B_0(\alpha,\gamma))(i) = (B_1(\alpha,\gamma))(i) = \langle\,\rangle$ and $(m(\alpha,\gamma))(i) = 0$.
Observe that for every $\gamma$ in the fan $F_\delta$, every subsequence of $B_0(\alpha,\gamma)$ meets $(R \cup R^{s(0)})^*$, every subsequence of $m(\alpha,\gamma)$ meets $R$, and every subsequence of $B_1(\alpha,\gamma)$ meets $R^*$ or contains a member $b$ such that $\text{Rem}(s)R^*b$.
Applying Ramsey's Theorem we find $i,j$ such that $i < j$ and simultaneously:
$(B_0(\alpha,\gamma))(i)R^*(B_0(\alpha,\gamma))(j)$ or $\langle s(0)\rangle R^*(B_0(\alpha,\gamma))(i)$, and $(m(\alpha,\gamma))(i)R$
$(m(\alpha,\gamma))(j)$, and $(B_1(\alpha,\gamma))(i)R^*(B_1(\alpha,\gamma))(j)$ or $\text{Rem}(s)R^*B_1(\alpha,\gamma)(j)$.
Observe that of the infinite sequence $\gamma$ only the values $\gamma(i),\gamma(j)$ are involved in this fact; let us denote it by $C(\gamma(i),\gamma(j))$. Applying the Fan Theorem we find a natural number $N$ such that for every $\gamma$ in the fan $F_\delta$ there exist $i,j$ such that $i < j < N$ and $C(\gamma(i),\gamma(j))$.
We now consider the finite set of finite sequences $\{\overline{\gamma}N | \gamma \in F_\delta\}$. Let $c,d$ belong to this set. We say that $c$ is *earlier* than $d$ if there exists $i < n$ such that $c(i) < d(i)$ and for every $j < N$, $j \neq i$, $c(j) = d(j)$.
We say that $c$ is *safe* if for every $i < N$, if $c(i) < \delta(i)$, then some initial part of $\langle s(0), (\alpha(i))(c(i))\rangle$ belongs to $R$.
We define a proposition QED as follows:

QED:= There exist $i,j$ such that $i < j$ and either $\alpha(i)R^*\alpha(j)$ or $sR^*\alpha(i)$.

Now observe the following two facts: (i) $\overline{\delta}N$ is safe, and (ii) for every safe element $c$ of $\{\overline{\gamma}N | \gamma \in F_\delta\}$,either QED or there exists a safe element $d$ of $\{\overline{\gamma}N | \gamma \in F_\delta\}$ such that $d$ comes earlier than $c$.
We now prove (ii): suppose $c$ is safe. We may assume that for every $i < N$, $\alpha(i)$ is a *non-empty* finite sequence. Determine $i,j$ such that $i < j < N$ and $C(c(i),c(j))$. Now observe: either QED or $\langle s(0)\rangle R^*A_0(\alpha(i),c(i))$ or $\text{Rem}(s)R^*A_1(\alpha(i),c(i))$. Assume the latter, that is: $\text{Rem}(s)R^*A_1(\alpha(i),c(i))$; then, as $\text{Rem}(s)$ is non-empty and $c$ is safe,also some initial part of
$\langle s(0), (\alpha(i))(c(i))\rangle$ belongs to $R$, therefore $sR^*\alpha(i)$ and QED.
So either QED or $\langle s(0)\rangle R^*A_0(\alpha(i),c(i))$. Choose $k < c(i)$ such that some initial part of $\langle s(0), (\alpha(i))(k)\rangle$ belongs to $R$ and define the finite sequence $d$ of length $N$ by: $d(i) := k$ and for every $j < N$, if $j \neq i$, then $d(j) = c(j)$. $d$ belongs to $\{\overline{\gamma}N | \gamma \in F_\delta\}$ and $d$ is safe and $d$ comes earlier than $c$. It will be clear that we reach the conclusion QED within finitely many steps.
We conclude: for every $\alpha : \mathbb{N} \to \mathbb{N}^*$ there exist $i,j$ such that $i < j$ and either $\alpha(i)R^*\alpha(j)$ or $sR^*(i)$, that is: $P(s)$.
Clearly then, for every finite sequence $s$ of natural numbers, $P(s)$, and $R^*$ is almost full on $\mathbb{N}^*$. $\square$

**Corollary 5.8** (*Finite Sequence Theorem, sometimes called: Higman's Lemma*).
*Let $R$ be an at-most-binary relation on $\mathbb{N}$.*
*If $R$ is almost full on $\mathbb{N}$, then $R^*$ is almost full on $\mathbb{N}^*$.*

**Proof:** Use Brouwer's Thesis and apply Theorem 5.7.1. □

# 6 Vazsonyi's Conjecture for binary trees

## 6.1

We define the set $\mathcal{T}_{\{2\}}$ of binary trees by means of the following inductive definition:

(i) The empty set $\emptyset$ is a binary tree.
(ii) For all binary trees $T, U$, the ordered pair $\langle T, U\rangle$ is also a binary tree.
(iii) Every binary tree is obtained from the empty set by finitely many applications of step (ii).

(The above definition may be applied within any domain $V$ where we have a non-surjective one-to-one mapping $\langle\,\rangle$ from $V \times V$ into $V$. If we think of the set-theoretical Wiener-Kuratowski definition of ordered pair, we take for $V$ the collection of the hereditarily finite sets. But we might also start from the set $\mathbb{N}$ of natural numbers with a suitable pairing function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}\backslash\{0\}$, for instance: $\langle m, n\rangle := 2^m(2n+1)$. Binary trees then are natural numbers.)
It is convenient to think of an ordered pair $\langle T, U\rangle$ as a function on the set $\{0, 1\}$.
Every non-empty binary tree $T$ is of the form $T = \langle T(0), T(1)\rangle$.
We define a binary relation $\preccurlyeq$ on the set $\mathcal{T}_{\{2\}}$ of binary trees as follows, by induction:

| |
|---|
| For all binary trees $T, U$:<br>$T \preccurlyeq U$ ("*$T$ neatly embeds into $U$*") if and only if either $T = \emptyset$ or both $T$ and $U$ are non-empty and *either* $T \preccurlyeq U(0)$ *or* $T \preccurlyeq U(1)$ *or* both $T(0) \preccurlyeq U(0)$ and $T(1) \preccurlyeq U(1)$. |

We want to show, in this Section, that $\preccurlyeq$ is almost full on $\mathcal{T}_{\{2\}}$. This is a special case of Vazsonyi's Conjecture, mentioned in the Introduction.

## 6.2

We define a mapping $B$ on the set $\mathcal{T}_{\{2\}}$ that associates to every binary tree $T$ a finite subset $B(T)$ of $\{0, 1\}^*$:

(i) $B(\emptyset) := \{\langle\,\rangle\}$.
(ii) For every non-empty binary tree $T$, $B(T) := \{\langle\,\rangle\} \cup \langle 0\rangle * B\big(T(0)\big) \cup \langle 1\rangle * B\big(T(1)\big)$.

(We may be said to apply the definition from 6.1 in the domain $U$ consisting of the finite sets of finite sequences of natural numbers, where the pairing operation $\langle\,\rangle$ is defined by: $\langle T, U\rangle := \{\langle\,\rangle\} \cup \langle 0\rangle * T \cup \langle 1\rangle * U$.)
One may prove that for all binary trees $T, U$,

> $T$ embeds neatly into $U$ if and only if there exists a function $f$ from $B(T)$ to $B(U)$ such that for all $a$ in $B(T)$,
> $f(a) * \langle 0\rangle$ is an initial part of $f(a * \langle 0\rangle)$ and $f(a) * \langle 1\rangle$ is an initial part of $f(a * \langle 1\rangle)$.

## 6.3

We want to prove that the relation $\preccurlyeq$ is almost full on the set $\mathcal{T}_{\{2\}}$, that is,
for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist $i, j$ such that $i < j$ and $\alpha(i) \preccurlyeq \alpha(j)$.
For every binary tree $T$ we define the proposition $P(T)$ as follows:

$P(T) :=$ For every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist $i, j$ such that $i < j$ and either $\alpha(i) \preccurlyeq \alpha(j)$ or $T \preccurlyeq \alpha(i)$.

We intend to show: for every binary tree $T$, $P(T)$.
This obviously implies that $\preccurlyeq$ is almost full on $\mathcal{T}_{\{2\}}$.
We want to reach our goal by induction on $\mathcal{T}_{\{2\}}$.
It suffices to show: $P(\emptyset)$ and for every non-empty binary tree $T$, if both $P\big(T(0)\big)$ and $P\big(T(1)\big)$, then $P(T)$.
Observe that $P(\emptyset)$ is true.

## 6.4

Assume that $T$ is a non-empty binary tree and that we proved both $P\big(T(0)\big)$ and $P\big(T(1)\big)$. We wish to prove $P(T)$, that is:
for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist $i, j$ such that $i < j$ and either $\alpha(i) \preccurlyeq \alpha(j)$ or $T \preccurlyeq \alpha(i)$.
Our strategy for proving this is based upon the following observation:
For every binary tree $U$, if $T$ does not embed into $U$, then $T$ is non-empty and either $U$ is empty or $U$ is also non-empty and *either* $T(0)$ does not embed into $U(0)$ and $T$ does not embed into $U(1)$, *or* $T$ does not embed into $U(0)$ and $T(1)$ does not embed into $U(1)$.
We now let $A_0, A_1$, respectively be the set of all binary trees such that $T(0), T(1)$, respectively does not embed into $U$.
We now consider the set $(A_0 \biguplus A_1)^*$ consisting of all finite sequences of elements of the set $A_0 \biguplus A_1 = A_0 \times \{0\} \cup A_1 \times \{1\}$.
We define a so-called *evaluation mapping* $Ev$ from the set $(A_0 \biguplus A_1)^*$ to the set $\mathcal{T}_{\{2\}}$ of binary trees, as follows:

(i) $Ev(\emptyset) := \emptyset$.

(ii) For every non-empty finite sequence $s = \langle s(0)\rangle * \mathrm{Rem}(s)$ from $(A_0 \dot{\bigcup} A_1)^*$:
if $s(0)$ has the form $\langle U, 0\rangle$, then $Ev(s) := \langle U, Ev\big(\mathrm{Rem}(s)\big)\rangle$, and
if $s(0)$ has the form $\langle U, 1\rangle$, then $Ev(s) := \langle Ev\big(\mathrm{Rem}(s)\big), U\rangle$.

Remark that for every $U$ in $\mathcal{T}_{\{2\}}$, if not $T \preccurlyeq U$, then there exists $s$ in $(A_0 \dot{\bigcup} A_1)^*$ such that $Ev(s) = U$.
Now observe that by assumption $\preccurlyeq$ is almost full on both $A_0$ and $A_1$, therefore by Corollary 4.10.1, $\preccurlyeq + \preccurlyeq$ is almost full on $A_0 \dot{\bigcup} A_1$, and therefore, by the Finite Sequence Theorem 5.8, $(\preccurlyeq + \preccurlyeq)^*$ is almost full on $A_0 \dot{\bigcup} A_1$.

We now claim the following:

$$\boxed{\text{For all } s, t \text{ in } (A_0 \dot{\bigcup} A_1)^*, \text{ if } s(\preccurlyeq + \preccurlyeq)^* t, \text{ then } Ev(s) \preccurlyeq Ev(t).}$$

We justify this claim by induction on $\mathrm{length}(s) + \mathrm{length}(t)$.
Observe that for all $t$ in $(A_0 \dot{\bigcup} A_1)^*$, $\emptyset = Ev(\emptyset) \preccurlyeq Ev(t)$.
Assume now that $s, t$ are non-empty elements of $(A_0 \dot{\bigcup} A_1)^*$ and that we proved already: for all $u, v$ in $(A_0 \dot{\bigcup} A_1)^*$, such that $\mathrm{length}(u) + \mathrm{length}(v) < \mathrm{length}(s) + \mathrm{length}(t)$,
if $u(\preccurlyeq + \preccurlyeq)^* v$, then $Ev(u) \preccurlyeq Ev(v)$. Assume $s(\preccurlyeq + \preccurlyeq)^* t$. There are two cases to distinguish.
Case (i). $s(\preccurlyeq + \preccurlyeq)^*\mathrm{Rem}(t)$, therefore $Ev(s) \preccurlyeq Ev\big(\mathrm{Rem}(t)\big)$, and, as $Ev\big(\mathrm{Rem}(t)\big) \preccurlyeq Ev(t)$, also $Ev(s) \preccurlyeq Ev(t)$.
Case (ii). $s(0)(\preccurlyeq + \preccurlyeq)t(0)$ and $\mathrm{Rem}(s)(\preccurlyeq + \preccurlyeq)^*\mathrm{Rem}(t)$.
We may assume: $s(0) = \langle U, 0\rangle$ and $t(0) = \langle V, 0\rangle$. Then $U \preccurlyeq V$ and $Ev\big(\mathrm{Rem}(s)\big) \preccurlyeq Ev\big(\mathrm{Rem}(t)\big)$.
Now observe $Ev(s) = \langle U, Ev\big(\mathrm{Rem}(s)\big)\rangle$ and $Ev(t) = \langle V, Ev\big(\mathrm{Rem}(t)\big)\rangle$, therefore $Ev(s) \preccurlyeq Ev(t)$.

We now establish the proposition $P(T)$ as follows:
Let $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$. Determine $\beta : \mathbb{N} \to (A_0 \dot{\bigcup} A_1)^*$ such that for every $n$, if not $T \preccurlyeq \alpha(n)$, then $Ev\big(\beta(n)\big) = \alpha(n)$.
Determine $i, j$ such that $i < j$ and $\beta(i)(\preccurlyeq + \preccurlyeq)^*\beta(j)$.
Then either $T \preccurlyeq \alpha(i)$ or $T \preccurlyeq \alpha(j)$ or $Ev\big(\beta(i)\big) = \alpha(i)$ and $Ev\big(\beta(j)\big) = \alpha(j)$, therefore $\alpha(i) \preccurlyeq \alpha(j)$.

**Theorem 6.5** (*Vazsonyi's Conjecture for binary trees.*)
*$\preccurlyeq$ is almost full on $\mathcal{T}_{\{2\}}$, that is, for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$ there exist $i, j$ such that $i < j$ and $\alpha(i) \preccurlyeq \alpha(j)$.*

**Proof:** See Sections 6.3 and 6.4.
Observe that for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{2\}}$, $P\big(\alpha(0)\big)$, so there exist $i, j$ such that $0 < i < j$ and either $\alpha(0) \preccurlyeq \alpha(i)$ or $\alpha(i) \preccurlyeq \alpha(j)$. □

# 7 Higman's Theorem

## 7.1

Vazsonyi's Conjecture is also true for ternary trees.
How should one prove it?
The set $\mathcal{T}_{\{3\}}$ of ternary trees is defined as follows:

(i) The empty set $\emptyset$ belongs to $\mathcal{T}_{\{3\}}$.
(ii) For all $T_0, T_1, T_2$ in $\mathcal{T}_{\{3\}}$, the 3-sequence $\langle T_0, T_1, T_2\rangle$ belongs to $\mathcal{T}_{\{3\}}$.
(iii) Every element of $\mathcal{T}_{\{3\}}$ is obtained from the empty set by finitely many applications of step (ii).

We consider every non-empty element $T$ of $\mathcal{T}_{\{3\}}$ as a function with domain $3 = \{0, 1, 2\}$ and write: $T = \langle T(0), T(1), T(2)\rangle$.

We define a binary relation $\preccurlyeq$ on $\mathcal{T}_{\{3\}}$ as follows:
For all $T, U$ in $\mathcal{T}_{\{3\}}$: $T \preccurlyeq U$ ("$T$ embeds neatly into $U$") if and only if either $T = \emptyset$ or $U$ is non-empty and *either* there exists $i < 3$ such that $T \preccurlyeq U(i)$ *or* both $T, U$ are non-empty and for every $i < 3$, $T(i) \preccurlyeq U(i)$.
Suppose that we want to prove that $\preccurlyeq$ is almost full on $\mathcal{T}_{\{3\}}$ and try the approach of the proof of Theorem 5.8.
We then are led to consider binary trees.
For assume that $T$ is a non-empty ternary tree and that we got so far as to prove: for every $i < 3$, for every $\alpha : \mathbb{N} \to \mathcal{T}_{\{3\}}$ there exist $j, k$ such that $j < k$ and either $T(i) \preccurlyeq \alpha(j)$ or $\alpha(j) \preccurlyeq \alpha(k)$.
We then want to prove: for every $\alpha : \mathbb{N} \to T_{\{3\}}$ there exist $i, j$ such that $i < j$ and either $T \preccurlyeq \alpha(i)$ or $\alpha(i) \preccurlyeq \alpha(j)$, and study the set of all ternary trees $U$ such that $T$ does not embed into $U$. Let us call this set $\mathcal{T}_{\{3\}} \restriction T$. Observe that $T$ does not embed into $U$ if and only if either $U$ is empty or $U$ is non-empty and there exists $i < 3$ such that $T(i)$ does not embed into $U(i)$, and for all $j < 3$, if $j \neq i$, then $T$ does not embed into $U(j)$.
Therefore every member $X$ of $\mathcal{T}_{\{3\}} \restriction T$ is obtained from two earlier constructed members $U, V$ of $\mathcal{T}_{\{3\}} \restriction T$ in one of the following three ways: we choose $W$ such that $T(0)$ does not embed into $W$ and form $X := \langle W, U, V\rangle$, or we choose $W$ such that $T(1)$ does not embed into $W$ and form $X := \langle U, W, V\rangle$, or we choose $W$ such that $T(2)$ does not embed into $W$ and form $X := \langle U, V, W\rangle$.
We find it useful to consider the extra tree $W$ as a label. In this way the study of ternary trees leads to the study of labeled binary trees.
Further reflection brings one to consider at-most-binary trees rather than just binary trees.

## 7.2

Let $A$ be a non-empty finite set of natural numbers. We introduce the set $\mathcal{T}_A$ of $A$-ary trees as follows:

(i) The empty set $\emptyset$ belongs to $\mathcal{T}_A$.
(ii) For every $k$ in $A$, for all $T_0, T_1, \dots, T_{k-1}$ in $\mathcal{T}_A$ the $k$-sequence $\langle T_0, T_1, \dots, T_{k-1}\rangle$ belongs to $\mathcal{T}_A$.
(iii) Every element of $\mathcal{T}_A$ is obtained from the empty set by finitely many applications of step (ii).

Every non-empty element $T$ of $\mathcal{T}_A$ is a $k$-sequence
$T = \langle T(0), T(1), \dots, T(k-1)\rangle$ of elements of $\mathcal{T}_A$ where $k = \mathrm{Dom}(T)$ is an element of $A$.
We define the set $\mathcal{T}$ of *trees* by: $\mathcal{T} := \bigcup_{n\in\mathbb{N}} \mathcal{T}_{\{0,1,\dots,n\}}$.
We define a binary relation $\preccurlyeq$ on $\mathcal{T}$ as follows:
For all trees $T, U$:
$T \preccurlyeq U$ ("$T$ neatly embeds into $U$") if and only if either $T = \emptyset$ or $U$ is non-empty and *either* there exists $i \in \mathrm{Dom}(U)$ such that $T \preccurlyeq U(i)$, *or* both $T$ and $U$ are non-empty and $\mathrm{Dom}(T) = \mathrm{Dom}(U)$ and, for each $i$ in $\mathrm{Dom}(T), T(i) \preccurlyeq U(i)$.
We consider the mapping that we defined in Section 6.2 and extend it to a mapping $B$ that associates to every tree $T$ a finite subset $B(T)$ of the set $\mathbb{N}^*$ of finite sequences of natural numbers:

(i) $B(\emptyset := \{\langle\,\rangle\}$.
(ii) For every non-empty tree $T$:
$B(T) := \{\langle\,\rangle\} \cup \bigcup_{j<\mathrm{Dom}(T)} \langle j\rangle * B\big(T(j)\big)$.

One may prove the following:

For all trees $T, U$:
$T$ embeds neatly into $U$ if and only if there exists a function $f$ from $B(T)$ to $B(U)$ such that for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$, the following two conditions are fulfilled:

(i) If $s * \langle j\rangle$ belongs to $B(T)$, then $f(s) * \langle j\rangle$ is an initial part of $f(s * \langle j\rangle)$, and
(ii) $s * \langle j\rangle$ belongs to $B(T)$ if and only if $f(s) * \langle j\rangle$ belongs to $B(U)$.

Our aim, in this Section, is to prove that for each non-empty finite set $A$ of natural numbers, $\preccurlyeq$ is almost full on $\mathcal{T}_A$.
Observe that $\preccurlyeq$ is not almost full on $\mathcal{T}$.

## 7.3

Let $A$ be a non-empty set of natural numbers.
We introduce the set $\mathcal{LT}_A$ of $A$-ary labeled trees as follows:

(i) For every natural number $m$, the ordered pair $\langle m, \emptyset\rangle$ belongs to $\mathcal{LT}_A$. We sometimes call a pair $\langle m, \emptyset\rangle$ a *basic* labeled tree.
(ii) For every natural number $m$, for every $k$ in $A$, for every $k$-sequence $T = \langle T(0), T(1), \dots, T(k-1)\rangle$ of elements of $\mathcal{LT}_A$, the ordered pair $\langle m, T\rangle$ belongs to $\mathcal{LT}_A$.

(iii) Every member of $\mathcal{LT}_A$ is obtained by finitely many applications of step (ii) from trees of the form $\langle m, \emptyset \rangle$.

In this Section, the set $A$ will always be a finite set of natural numbers. Every element $T$ of $\mathcal{LT}_A$ is an ordered pair $\langle m, T \rangle$ where $m$ is a natural number and $T = \langle T(0), T(1), \dots, T(k-1) \rangle$ is a $k$-sequence of elements of $\mathcal{LT}_A$ and $k = \mathrm{Dom}(T)$ belongs to $A \cup \{0\}$. We define the set $\mathcal{LT}$ of labeled trees by $\mathcal{LT} := \bigcup_{n \in \mathbb{N}} \mathcal{LT}_{\{0,1,\dots,n\}}$. Let $R$ be an at-most-ternary relation on $\mathbb{N}$, that is, $R \subseteq \bigcup_{i<4} \mathbb{N}^i$. For each $k$ in $\mathbb{N}$ we let $R^k$ be the set of all elements $a$ of $\mathbb{N}^*$ such that $\langle k \rangle * a$ belongs to $R$. Observe that $R^k$ is an at-most-binary relation on $\mathbb{N}$.
We define a binary relation $\preccurlyeq_R$ on $\mathcal{LT}$ as follows:

> For all labeled trees $\langle m, T \rangle$ and $\langle m, U \rangle$:
> $\langle m, T \rangle \preccurlyeq_R \langle n, U \rangle$ ("$\langle m, T \rangle$ embeds neatly into $\langle n, U \rangle$ with respect to $R$") if and only if *either* there exists $k$ in $\mathbb{N}$ such that $\mathrm{Dom}(T) = \mathrm{Dom}(U) = \{0, 1, \dots, k-1\}$ and some initial part of $\langle m, n \rangle$ belongs to $R^k$ and for all $i < k$, $T(i) \preccurlyeq_R U(i)$, *or* there exists $i \in \mathrm{Dom}(U)$ such that $\langle m, T \rangle \preccurlyeq_R U(i)$.

We define a mapping $B$ that associates to every labeled tree $\langle m, T \rangle$ in $\mathcal{LT}$ a finite subset $B(T)$ of the set $\mathbb{N}^*$ of finite sequences of natural numbers:

(i) $B(\langle m, \emptyset \rangle) := \{\langle\, \rangle\}$.
(ii) For every labeled tree $\langle m, T \rangle$ such that $T$ is non-empty
$B(\langle m, T \rangle) := \{\langle\, \rangle\} \cup \bigcup_{j < \mathrm{Dom}(T)} \langle j \rangle * B\big(T(j)\big)$.

We also define a mapping $L$ that associates to every labeled tree $\langle m, T \rangle$ in $\mathcal{LT}$ a function $L(\langle m, T \rangle)$ from $B(T)$ to $\mathbb{N}$, a so-called *labeling* of $B(T)$:

(i) $\big(L(\langle m, T \rangle)\big)(\langle\, \rangle) := m$.
(ii) For every finite sequence $\langle j \rangle * s$ in $B(T)$, $\big(L(\langle m, T \rangle)\big)(\langle j \rangle * s) := \Big(L\big(T(j)\big)\Big)(s)$.

One may prove the following:

> For all labeled trees $\langle m, T \rangle$, $\langle n, U \rangle$, for every at-most-ternary relation $R$ on $\mathbb{N}$, $\langle m, T \rangle$ embeds neatly into $\langle n, U \rangle$ with respect to $R$ if and only if there exists a function $f$ from $B(\langle m, T \rangle)$ into $B(\langle n, U \rangle)$ such that
>
> (i) for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$,
> if $s * \langle j \rangle$ belongs to $B(\langle m, T \rangle)$, then $f(s) * \langle j \rangle$ is an initial part of $f(s * \langle j \rangle)$.
> (ii) for all $s$ in $\mathbb{N}^*$, $k$ in $\mathbb{N}$,
> if $s$ has $k$ immediate extensions in $B(\langle m, T \rangle)$ then $f(s)$ has $k$ immediate extensions in $B(\langle n, U \rangle)$ and some initial part of $\langle \big(L(\langle m, T \rangle)\big)(s), \big(L(\langle n, U \rangle)\big)\big(f(s)\big) \rangle$ belongs to $R^k$.

Actually, what we shall prove in this Section is the following extension of the statement presented as our aim at the end of Section 7.2:
For each non-empty finite set $A$ of natural numbers, for every at-most-ternary relation $R$ on $\mathbb{N}$, if for each $k$ in $A \cup \{0\}$ the at-most-binary relation $R^k$ is almost full on $\mathbb{N}$, then $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$.

## 7.4

Let $A$ be a non-empty finite set of natural numbers.
For every nonzero element $k$ of $A$ we want to define a so-called evaluation map $Ev_{A,k}$ from $\mathcal{LT}_{A\cup\{k-1\}}$ to $\mathcal{LT}_A$.
We first consider the case that $k-1$ belongs to $A$, and then the case that $k-1$ does not belong to $A$.

### 7.4.1

Suppose that $k-1$ belongs to $A$.
We let $f_{A,k}$ be a fixed one-to-one enumeration of the set $\mathbb{N} \times (\{0\} \cup \bigcup_{j<k} \mathcal{LT}_A \times \{j\})$. So, for every $n$, $f_{A,k}(n)$ either has the form $\langle m, 0\rangle$ where $m$ is a natural number, or the form $\langle m, V, j\rangle$ where $V$ belongs to $\mathcal{LT}_A$ and $j$ is some natural number less than $k$.
We now define the evaluation map $Ev_{A,k}$ from $\mathcal{LT}_{A\cup\{k-1\}} = \mathcal{LT}_A$ to $\mathcal{LT}_A$ as follows:

(i) Let $\ell$ be an element of $A$ that differs both from 0 and from $k-1$.
Then, for every $m$, for every $\ell$-sequence $U$ of elements of $\mathcal{LT}_A$, $Ev_{A,k}(\langle m, U\rangle) := \langle m, W\rangle$ where $W$ is an $\ell$-sequence of elements of $\mathcal{T}_A$ and, for each $j < \ell$, $W(j) := Ev_{A,k}\big(U(j)\big)$.

(ii) For every $n, m$, for every $(k-1)$-sequence $U$ of elements of $\mathcal{LT}_A$, if $f_{A,k}(n) = \langle m, 0\rangle$, then $Ev_{A,k}(\langle n, U\rangle) := \langle m, W\rangle$ where $W$ is a $(k-1)$-sequence of elements of $\mathcal{LT}_A$ and for each $j < k-1$, $W(j) := Ev_{A,k}\big(U(j)\big)$.
For every $n$, for every $V$ in $\mathcal{LT}_A$, for every $j < k$, for every $(k-1)$-sequence $U$ of elements of $\mathcal{LT}_A$, if $f_{A,k}(n) = \langle m, V, j\rangle$, then $Ev_{A,k}(\langle n, U\rangle) := \langle m, W\rangle$ where $W$ is a $k$-sequence of elements of $\mathcal{LT}_A$, and for each $i$, if $i < j$, then $W(i) := Ev_{A,k}\big(U(i)\big)$, and, if $i > j$, then $W(i) := Ev_{A,k}\big(U(i-1)\big)$, and $W(j) := V$.

### 7.4.2

Suppose that $k-1$ does not belong to $A$. We now let $f_{A,k}$ be a fixed one-to-one enumeration of the set $\mathbb{N} \times \bigcup_{j<k} \mathcal{LT}_A \times \{j\}$. So, for every $n$, $f_{A,k}(n)$ has the form $\langle m, V, j\rangle$, where $V$ belongs to $\mathcal{LT}_A$ and $j$ is some natural number smaller than $k$.
We define the evaluation map $Ev_{A,k}$ from $\mathcal{LT}_{A\cup\{k-1\}}$ to $\mathcal{LT}_A$ as in Section 7.4.1.

### 7.4.3

The following observation will be important in the sequel:
for every finite set $A$ of natural numbers and every nonzero element $k$ of $A$:

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| For every $V$ in $\mathcal{LT}_A$ there exists at least one but only finitely many $U$ in $\mathcal{LT}_{A\cup\{k-1\}}$ such that $Ev_{A,k}(U) = V$. |

**Theorem 7.5** (*G. Higman, 1952*)
*For every finitary stump $\sigma$ such that $\sigma(0)$ is non-empty, for every at-most-ternary relation $R$ on $\mathbb{N}$, if for each $k$ in $A := Dom(\sigma) = \{i | i \in \mathbb{N} | \sigma(i) \neq \emptyset\}$ the stump $\sigma(k)$ secures that $R^k$ is almost full on $\mathbb{N}$, then $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$.*

**Proof:** We use the principle of induction on finitary stumps explained in Section 3.6.1.
Let $\sigma$ be a finitary stump and assume that the statement of the Theorem has been verified for every finitary stump $\tau$ that is easier than $\sigma$ in the sense of Section 3.6, that is, such that there exists $k$ such that $\tau(k)$ is an immediate substump of $\sigma(k)$ and for each $n > k$, $\tau(n) = \sigma(n)$.
Define $A := \mathrm{Dom}(\sigma)$.
Let $R$ be an at-most-ternary relation on $\mathbb{N}$ such that for each $k$ in $A$, $\sigma(k)$ secures that $R^k$ is almost full on $\mathbb{N}$. For every $A$-ary labeled tree $\langle m, T\rangle$ we define the proposition $P(\langle m, T\rangle)$ as follows:

> $P(\langle m, T\rangle)$ ("$\langle m, T\rangle$ has the property $P$") := For every $\alpha : \mathbb{N} \to \mathcal{LT}_A$ there exist $i, j$ such that $i < j$ and either $\alpha(i) \preccurlyeq_R \alpha(j)$ or $\langle m, T\rangle \preccurlyeq_R \alpha(i)$.

We now show that every $A$-ary labeled tree $\langle m, T\rangle$ has the property $P$.
We assume that $k$ is an element of $A$, and that we are given a tree of the form $\langle m, T\rangle$ where $T$ is a $k$-sequence of elements of $\mathcal{LT}_A$ such that for every $i < k$, $T(i)$ has the property $P$. We want to prove that $T$ itself has the property $P$.
We define an at-most-ternary relation $R'$ on $\mathbb{N}$ as follows.
For each $n_0, n_1$ in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^{k-1}$ if and only if *either* there exist $p_0, p_1$ such that $f_{A,k}(n_0) = \langle p_0, 0\rangle$ and $f_{A,k}(n_1) = \langle p_1, 0\rangle$ and some initial part of $\langle p_0, p_1\rangle$ belongs to $R^{k-1}$ *or* there exist $p_0, p_1$ in $\mathbb{N}$, $U_0, U_1$ in $\mathcal{LT}_A$ and $j_0, j_1$ in $\mathbb{N}$ such that $f_{A,k}(n_0) = \langle p_0, U_0, j_0\rangle$ and $f_{A,k}(n_1) = \langle p_1, U_1, j_1\rangle$,and some initial part of $\langle p_0, p_1\rangle$ belongs to $R^k$, $j_0 = j_1$ and $U_0 \preccurlyeq_R U_1$ or $T(j_0) \preccurlyeq_R U_0$. Further, for each $n_0, n_1$ in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^k$ if and only if either $\langle n_0, n_1\rangle$ belongs to $R^k$ or $\langle m, n_0\rangle$ belongs to $R^k$.
Finally, for each $i$ in $A$ such that $i$ differs from both $k$ and $k-1$ we define: $(R')^i = R^i$.
Observe that $(R')^{k-1}$ is an almost full at-most-binary relation on $\mathbb{N}$.
(One has to use: for each $i < k$, $T(i)$ has the property $P$ and the fact that both $R^k$ and $R^{k-1}$ are almost full at-most-binary relations on $\mathbb{N}$, and Ramsey's Theorem.)
We now distinguish two cases, the case $(\sigma(k))(m) \neq \emptyset$ and the case $(\sigma(k))(m) = \emptyset$.

Case (i). $(\sigma(k))(m) \neq \emptyset$.
We form a finitary stump $\tau$ such that $\mathrm{Dom}(\tau) = \mathrm{Dom}(\sigma) \cup \{k-1\}$ and $\tau(k) = (\sigma(k))(m)$ and for each $i > k$, $\tau(i) = \sigma(i)$ and for each $i$ in $\mathrm{Dom}(\tau)$, if $\tau(i) \neq \emptyset$, then $\tau(i)$ secures that $(R')^i$ is almost full on $\mathbb{N}$. Observe that $\tau$ is more easy than $\sigma$.
Applying the induction hypothesis, we conclude that $\preccurlyeq_R$ is almost full on $\mathcal{LT}_{A\cup\{k-1\}}$.
Now let $\alpha$ be a function from $\mathbb{N}$ to $\mathcal{LT}_A$.
We want to show: there exist $i, j$ such that $i < j$ and either $\langle m, T\rangle \preccurlyeq_R \alpha(i)$ or $\alpha(i) \preccurlyeq_R \alpha(j)$.
To this end we consider the set $F$ of all functions $\beta : \mathbb{N} \to \mathcal{LT}_A \cup \{k-1\}$ such that for each $i$, $Ev_{A,k}(\beta(i)) = \alpha(i)$.
It follows from Remark 7.4.3 that $F$ is a fan.
We use the induction hypothesis and the fan theorem and we determine a natural number $N$ such that for every $\beta$ in $F$ there exist $i, j$ such that $i < j < N$ and $\beta(i) \preccurlyeq_{R'} \beta(j)$.
Let $\langle n, U\rangle$ be an element of $\mathcal{LT}$.
We call $\langle n, U\rangle$ an ***analysis*** of its own evaluation $Ev_{A,k}(\langle n, U\rangle)$.
Let $\langle n, U\rangle$ be an element of $\mathcal{LT}_{A\cup\{k-1\}}$ such that $\mathrm{Dom}(U) = k$. We call $\langle n, U\rangle$ *disappointing* if and only if of some initial part $\langle m, n\rangle$ belongs to $R^k$. In general, we are unable to decide if $\langle n, U\rangle$ is disappointing or not.
Let $\langle n, U\rangle$ be an element of $\mathcal{LT}_{A\cup\{k-1\}}$ such that $\mathrm{Dom}(U) = k-1$. We call $\langle n, U\rangle$ *disappointing* if and only if there exist $p$ in $\mathbb{N}$, $U$ in $\mathcal{LT}_A$ and $j$ in $\mathbb{N}$ such that $f_{A,k}(n) = \langle p, V, j\rangle$ and $T(j) \preccurlyeq V$. In general, we are unable to decide if $\langle n, U\rangle$ is disappointing or not.
Now let $\langle n, U\rangle$ be an element of $\mathcal{LT}_{A\cup\{k-1\}}$ and $\langle p, W\rangle$ an element of $\mathcal{LT}_A$. We say that $\langle n, U\rangle$ is a ***disappointing analysis*** of $\langle p, W\rangle$ if $Ev_{A,k}(\langle n, U\rangle) = \langle p, W\rangle$ that is, $\langle n, U\rangle$ is an analysis of $\langle p, W\rangle$, and $\langle n, U\rangle$ contains a disappointing subtree.
We now observe the following:

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| For every $\langle p, W\rangle$ in $\mathcal{LT}_A$,<br>if every analysis $\langle n, U\rangle$ of $\langle p, W\rangle$ is a disappointing analysis of $\langle p, W\rangle$,<br>then $\langle m, T\rangle \preccurlyeq_R \langle p, W\rangle$ |

We leave the proof of this observation to the reader. He may prove it first for the case $\mathrm{Dom}(W) = k$, and then more generally.
We now enunciate a combinatorial principle:
Let $N$ be a natural number and let $A_0, A_1, \ldots, A_{N-1}$ be an $N$-sequence of finite sets. Let $P$ be a subset of $A_0 \times A_1 \times \cdots \times A_{N-1}$ and let, for each $i < N$, $B_i$ be a subset of $A_i$.
Assume that for every element $\langle a_0, a_1, \ldots, a_{N-1}\rangle$ of $A_0 \times A_1 \times \cdots \times A_{N-1}$ either $\langle a_0, a_1, \ldots, a_{N-1}\rangle$ belongs to $P$ or there exists $i < N$ such that $a_i$ belongs to $B_i$.
Then either there exists an element $\langle a_0, a_1, \ldots, a_{N-1}\rangle$ of $P$ or there exists $i < N$ such that $B_i$ coincides with $A_i$.
We leave the proof of this combinatorial principle to the reader.
We apply this principle and complete our proof.
We let $N$ be the natural number that we found from the Fan Theorem. For each $i < N$

we let $A_i$ be the set of all elements $\langle n, U\rangle$ of $\mathcal{LT}_{A\cup\{k-1\}}$ such that $Ev_{A,k}(\langle n, U\rangle) = \alpha(i)$.
We let $B_i$ be the set of all disappointing elements of $A_i$.
We let $P$ be the set of all elements $\langle\langle n_0, U_0\rangle, \ldots, \langle n_N, U_N\rangle\rangle$ of $A_0 \times A_1 \times \cdots \times A_{N-1}$ such that there exist $i, j$ such that $i < j < N$ and $Ev_{A,k}(\langle n_i, U_i\rangle) \preccurlyeq_R Ev_{A,k}(\langle n_j, U_j\rangle)$.
Observe that for all $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ in $\mathcal{LT}_{A\cup\{k-1\}}$, if $\langle n_0, U_0\rangle$ $\preccurlyeq_{R'}$ $\langle n_1, U_1\rangle$ then either $Ev_{A,k}(\langle n_0, U_0\rangle) \preccurlyeq_R Ev_{A,k}(\langle n_1, U_1\rangle)$ or one of the trees $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ contains a disappointing subtree.

We conclude:
Either there exists $\beta$ in $F$ and $i, j < N$ such that $Ev_{A,k}\big(\beta(i)\big) \preccurlyeq_R Ev_{A,k}\big(\beta(j)\big)$, that is, $\alpha(i) \preccurlyeq_R \alpha(j)$, or there exists $i < N$ such that for every $\beta$ in $F$, $\beta(i)$ is a disappointing analysis of $\alpha(i)$, therefore $\langle m, T\rangle \preccurlyeq_R \alpha(i)$.

Case (ii): $\sigma(k)(m) = \emptyset$. We conclude that the empty sequence belongs to $R^k$. We form the finitary stump $\tau$ such that $\mathrm{Dom}(\tau) = (\mathrm{Dom}(\sigma)\backslash\{k\}) \cup \{k-1\}$ and for every $i$ in $\mathrm{Dom}(\tau)$, $\tau(i) := \sigma(i)$. Observe that for every $i$ in $\mathrm{Dom}(\tau)$, $\tau(i)$ secures that $(R')^i$ is almost full on $\mathbb{N}$, and if $i > k$, then $\tau(i) = \sigma(i)$,so $\tau$ is more easy than $\sigma$.
We define the at-most-ternary relation $R'$ in almost the same way as in Case (i), that is, the definition of $(R')^{k-1}$ is as in Case (i), but we now set $(R')^k := \emptyset$.
The proof is virtually the same as in Case (i) and is left to the reader. □

**Corollary 7.6** *For every $n$, $\preccurlyeq$ is almost full on $\mathcal{T}_{\{0,1,\ldots,n-1\}}$.*

**Proof:** Apply Higman's Theorem to $\mathcal{LT}_{\{0,1,\ldots,n-1\}}$, where the relation $R$ coincides with the set $\{\langle i\rangle | i \in \mathbb{N}\}$, in particular, for every $i < n$,the empty sequence $\langle\,\rangle$ belongs to $R^i$. □

# 8 Vazsonyi's Conjecture and the Tree Theorem

We consider the set $\mathcal{T}$ of all finite trees, as we defined it in Section 7.2. Every finite tree $T$ is a finite sequence $T = \langle T(0), T(1), \ldots, T(k-1)\rangle$ of earlier-constructed finite trees.
The empty sequence $\langle\,\rangle = \emptyset$ is also a finite tree.
We define a binary relation $\sqsubseteq$ on the set $\mathcal{T}$, as follows:

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| For all $T, U$ in $\mathcal{T}$:<br>$T \sqsubseteq U$ ("$T$ embeds into $U$") if and only if either $T = \emptyset$ or $U$ is non-empty and *either* there exists $i \in \mathrm{Dom}(U)$ such that $T \sqsubseteq U(i)$, *or* both $T$ and $U$ are non-empty and $T(\sqsubseteq)^*U$, that is, there exists a strictly increasing function $h$ from $\mathrm{Dom}(T)$ to $\mathrm{Dom}(U)$ such that for all $i$ in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq U\big(h(i)\big)$. |

Observe that, for all finite trees $T, U$, $T$ embeds into $U$ if and only if there exists a mapping $f$ from $B(T)$ into $B(U)$ such that

(i) for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$,
if $s * \langle j\rangle$ belongs to $B(T)$, then $f(s * \langle j\rangle)$ is a proper extension of $f(s)$.

(ii) for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$,
if both $s * \langle j \rangle$ and $s * \langle j+1 \rangle$ belong to $B(T)$, then there exist $k_0, k_1$ such that $k_0 < k_1$ and $f(s) * \langle k_0 \rangle$ is an initial part of $f(s * \langle j \rangle)$ and $f(s) * \langle k_1 \rangle$ is an initial part of $f(s * \langle j+1 \rangle)$.

**Theorem 8.1** (*Vazsonyi's Conjecture*)
*$\sqsubseteq$ is almost full on $\mathcal{T}$, that is, for every $\alpha : \mathbb{N} \to \mathcal{T}$ there exist $i, j$ such that $i < j$ and $\alpha(i) \sqsubseteq \alpha(j)$.*

**Proof:** For every $T$ in $\mathcal{T}$ we define a proposition $P(T)$, as follows:

> $P(T)$ ("$T$ has property $P$") :=
> For every $\alpha : \mathbb{N} \to \mathcal{T}$ there exist $i, j$ such that $i < j$ and either $T \sqsubseteq \alpha(i)$ or $\alpha(i) \sqsubseteq \alpha(j)$.

We want to prove that every finite tree has the property $P$ and use induction on the set $\mathcal{T}$ of finite trees.
It is obvious that the empty sequence $\emptyset$ has the property $P$, as $\emptyset$ embeds into every finite tree.
Now assume that $T = \langle T(0), T(1), \ldots, T(k-1) \rangle$ is a finite tree and that for every $j < k$, $T(j)$ has the property $P$.
Observe that, for every finite tree $U$, $T$ does not embed into $U$ if and only if *either* $\mathrm{Dom}(U) < k$ and for each $j$ in $\mathrm{Dom}(U)$, $T$ does not embed into $U(j)$, *or* $\mathrm{Dom}(U) \geq k$ and there exists a strictly increasing function $h$ from $k-1$ into $\mathrm{Dom}(U)$ such that for each $j < k-1$, $T$ does not embed into $U\big(h(j)\big)$, and for each $i \in \mathrm{Dom}(U)$, if $i < h(0)$, then $T(0)$ does not embed into $U(i)$, and for each $j < k-2$, if $h(j) < i < h(j+1)$, then $T(j+1)$ does not embed into $U(i)$, and if $i > h(k-2)$, then $T(k-1)$ does not embed into $U(i)$.
It follows that we have to consider labeled at-most-$(k-1)$-ary trees. For each $j < k$, we let $\mathcal{T} \restriction T(j)$ denote the set of all finite trees $U$ such that $T(j)$ does not embed into $U$. Observe that $T(j)$ has the property $P$, therefore $\sqsubseteq$ is almost full on $\mathcal{T} \restriction T(j)$, and $\sqsubseteq^*$ is almost full on $\big(\mathcal{T} \restriction T(j)\big)^*$.
We let $f$ be an enumeration of the set $\prod\limits_{j<k} \big(\mathcal{T} \restriction T(j)\big)^*$. So, for every $m$, $f(m)$ is a $k$-sequence of finite sequences of trees such that for each $j < \mathrm{length}(k)$, for each $i < \mathrm{length}\big(\big(f(m)\big)(j)\big)$, the tree $T(j)$ does not embed into $\big(\big(f(m)\big)(j)\big)(i)$.
We now consider the set $\mathcal{LT}_A$ where $A := \{0, 1, \ldots, k-1\}$. We define an evaluation map $Ev$ from the set $\mathcal{LT}_A$ to the set $\mathcal{T}$, as follows:

(i) For every natural number $m$, $Ev(\langle m, \emptyset \rangle) := \emptyset$.
(ii) Let $U$ be a finite sequence of elements of $\mathcal{LT}_A$ of length $< k-1$, and let $m$ be a natural number.
Then $Ev(\langle m, U \rangle) := W$ where $W$ is a finite tree such that $\mathrm{Dom}(W) = \mathrm{Dom}(U)$ and for each $j$ in $\mathrm{Dom}(W)$, $W(j) := Ev\big(U(j)\big)$.
(iii) Let $U$ be a finite sequence of elements of $\mathcal{LT}_A$ of length $k-1$, and let $m$ be a natural number. Consider $V := f(m)$. $V$ is a finite sequence of length $k$ and

for each $j < k$, $V(j)$ belongs to $\left(\mathcal{T} \upharpoonright T(j)\right)^*$. We define:
$Ev(\langle m, U\rangle) := V(0) * \langle Ev\left(U(0)\right)\rangle * V(1) * \langle Ev\left(U(1)\right)\rangle * \cdots * \langle Ev\left(U(k-2)\right)\rangle * V(k-1)$.

We define a ternary relation $R$ on the set $\mathbb{N}$ of natural numbers as follows: for each $i \neq k-1$, $R^i := \mathbb{N}\times\mathbb{N}$, and for all $m_0, m_1$ in $\mathbb{N}$, $\langle m_0, m_1\rangle$ belongs to $R^{k-1}$ if and only if for each $j < k$, $\left(f(m_0)\right)(j) \sqsubseteq^* f(m_1)(j)$.
We now make some remarks:

(i) $Ev$ is a surjective map from the set $\mathcal{LT}_A$ onto the set $\mathcal{T} \upharpoonright T$ of all finite trees $U$ such that $T$ does not embed into $U$.
(ii) For all finite sequences $U, V$ of elements of $\mathcal{LT}_A$, for all natural numbers $m, n$, if $\langle m, U\rangle \preccurlyeq_R \langle n, V\rangle$, then $Ev(\langle m, U\rangle) \sqsubseteq Ev(\langle n, V\rangle)$.
(iii) $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$.

((ii) follows may be proved by spelling out the definitions, and (iii) follows by Higman's Theorem from the fact that for each $i$, $R^i$ is almost full on $\mathbb{N}$. $R^{k-1}$ is almost full on $\mathbb{N}$ by Ramsey's Theorem, as, for each $j < k$, $\sqsubseteq^*$ is almost full on $\left(\mathcal{T} \upharpoonright T(j)\right)^*$.)

It is now easy to conclude: $\sqsubseteq$ is almost full on $\mathcal{T} \upharpoonright T$, that is, $T$ has the property $P$.
$\square$

## 8.2

Observe that the effort needed to prove Vazsonyi's Conjecture from Higman's Theorem is relatively small.
We now want to extend Theorem 8.1 to labeled trees.
Let $R$ be an at-most-ternary relation on the set $\mathbb{N}$ of natural numbers, and let $k$ be a natural number.
We define a binary relation $\sqsubseteq_{R,k}$ on the set $\mathcal{LT}$ of labeled finite trees as follows:

For all labeled trees $\langle m, T\rangle$, $\langle n, U\rangle$,
$\langle m, T\rangle \sqsubseteq_{R,k} \langle n, U\rangle$ ("The tree $\langle m, T\rangle$ embeds into the tree $\langle n, U\rangle$ with respect to $R$ up to $k$") if and only if *either* there exists $i$ in $\mathrm{Dom}(U)$ such that $\langle m, T\rangle \sqsubseteq_{R,k} U(i)$ *or* there exists $j < k$ such that $\mathrm{Dom}(T) = \mathrm{Dom}(U) = j$ and some initial part of $\langle m, n\rangle$ belongs to $R^j$ and for each $i$ in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq_{R,k} U(i)$ *or* both $\mathrm{Dom}(T) \geq k$ and $\mathrm{Dom}(U) \geq k$ and some initial part of $\langle m, n\rangle$ belongs to $R^k$ and $T(\sqsubseteq_{R,k})^*U$, that is, there is a strictly increasing function from $\mathrm{Dom}(T)$ to $\mathrm{Dom}(U)$ such that for each $i$ in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq_{R,k} U\left(h(i)\right)$.

One may prove the following:
For all labeled trees $\langle m, T\rangle$, $\langle n, U\rangle$, for every at-most-ternary relation $R$ on $\mathbb{N}$, for every $k$ in $\mathbb{N}$,
$\langle m, T\rangle$ embeds into $\langle n, U\rangle$ with respect to $R$ up to $k$ if and only if there exists a mapping $f$ from $B(\langle m, T\rangle)$ into $B(\langle n, U\rangle)$ such that

(i) for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$, if $s * \langle j \rangle$ belongs to $B(\langle m, T \rangle)$, then $f(s * \langle j \rangle)$ is a proper extension of $f(s)$.

(ii) For all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$, if both $s * \langle j \rangle$ and $s * \langle j+1 \rangle$ belong to $B(\langle m, T \rangle)$, then there exist $k_0, k_1$ such that $k_0 < k_1$ and $f(s) * \langle k_0 \rangle$ is an initial part of $f(s * \langle j \rangle)$ and $f(s) * \langle k_1 \rangle$ is an initial part of $f(s * \langle j+1 \rangle)$.

(iii) for all $s$ in $\mathbb{N}^*$, if $s$ has less than $k$ immediate extensions in $B(\langle m, T \rangle)$ then the number of immediate extensions of $s$ in $B(\langle m, T \rangle)$ is equal to the number of immediate extensions of $f(s)$ in $B(\langle n, U \rangle)$ (and therefore, in view of (ii), for each $j$ such that $s * \langle j \rangle$ belongs to $B(\langle m, T \rangle)$, $f(s) * \langle j \rangle$ is an initial part of $f(s * \langle j \rangle)$.)

(iv) For all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$ if $j < k$ and $s$ belongs to $B(\langle m, T \rangle)$ and has $j$ immediate extensions in $B(\langle m, T \rangle)$ then some initial part of $\langle \big(L(\langle m, T \rangle)\big)(s), \big(L(\langle n, U \rangle)\big)\big(f(s)\big) \rangle$ belongs to $R^j$; for all $s$ in $\mathbb{N}^*$, if $s$ has at least $k$ immediate extensions in $B(\langle m, T \rangle)$ then some initial part of $\langle \big(L(\langle m, T \rangle)\big)(s), \big(L(\langle n, U \rangle)\big)\big(f(s)\big) \rangle$ belongs to $R^k$.

We intend to prove the following statement (Kruskal's Tree Theorem):
For every co-finite set $A$ of natural numbers, for every at-most-ternary relation $R$ on $\mathbb{N}$,for every natural number $k$,
if for each $i \leq k$ such that $i$ belongs to $A \cup \{0\}$ the at-most-binary relation $R^i$ is almost full on $\mathbb{N}$, then the relation $\sqsubseteq_{R,k}$ is almost full on $\mathcal{LT}_A$.

### 8.3

Let $A$ be a co-finite set of natural numbers, and let $k$ be a nonzero natural number such that every natural number $n \geq k$ belongs to $A$.
For each nonzero natural number $i$ we want to define a so-called evaluation map $Ev^*_{A,k,i}$ from $\mathcal{LT}_{A \cup \{i-1\}}$ to $\mathcal{LT}_A$.
We distinguish several cases:

(i) $i \geq k$ and $i-1$ belongs to $A$.
(ii) $i = k$ and $k-1$ does not belong to $A$.
(iii) $i < k$ and $i-1$ belongs to $A$.
(iv) $i < k$ and $i-1$ does not belong to $A$.

#### 8.3.1

We first consider the case that $i \geq k$ and $i-1$ belongs to $A$. We let $f^*_{A,k,i}$ be an enumeration of the set $\mathbb{N} \times \Big(\{0\} \cup \big((\mathcal{LT}_A)^*\big)^i \times \{1\}\Big)$ so, for every $n$, $f^*_{A,k,i}(n)$ either has the form $\langle m, 0 \rangle$ where $m$ is a natural number, or the form $\langle m, V, 1 \rangle$ where $m$ is a natural number and $V$ is an $i$-sequence of elements of $(\mathcal{LT}_A)^*$. We now define the map $Ev^*_{A,k,i}$ from $\mathcal{LT}_A$ to $\mathcal{LT}_A$ as follows.

(i) Let $\ell$ be a element of $A$, $\ell \neq i-1$. Then, for every $n$, for every $\ell$-sequence $U$ of elements of $\mathcal{LT}_A$, $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle n, W\rangle$ where $W$ is an $\ell$-sequence of elements of $\mathcal{LT}_A$ and for each $j < \ell$, $W(j) := Ev^*_{A,k,i}\big(U(j)\big)$.

$(\mathrm{ii})_1$ For every $n, m$, for every $(i-1)$-sequence $U$ of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, 0\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where $W$ is an $(i-1)$-sequence of elements of $\mathcal{LT}_A$ such that for every $j < i-1$, $W(j) := Ev^*_{A,k,i}\big(U(j)\big)$.

For every $n, m$, for every $V$ from $\big((\mathcal{LT}_A)^*\big)^i$, for every $(i-1)$-sequence $U$ of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, V, 1\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where $W := V(0) * \langle Ev^*_{A,k,i}\big(U(0)\big)\rangle * \cdots * Ev^*_{A,k,i}\big(U(i-2)\big) * V(i-1)$.

### 8.3.2

We now consider the case that $i = k$ and $k-1$ does not belong to $A$. We let $f^*_{A,k,k}$ be an enumeration of the set $\mathbb{N} \times \big((\mathcal{LT}_A)^*\big)^k$. The definition of the map $Ev^*_{A,k,k}$ is almost the same as in case 8.3.1. We only replace $(\mathrm{ii})_1$ by:

$(\mathrm{ii})_2$ For every $n, m$, for every $V$ from $\big((\mathcal{LT}_A)^*\big)^k$, for every $(k-1)$-sequence $U$ of elements of $\mathcal{LT}_{A\cup\{k-1\}}$, if $f^*_{A,k,k}(n) = \langle m, V\rangle$ then $Ev^*_{A,k,k}(\langle n, U\rangle) := \langle m, W\rangle$ where
$W := V(0) * \langle Ev^*_{A,k,k}\big(U(0)\big)\rangle * \cdots * \langle Ev^*_{A,k,k}\big(U(k-2)\big)\rangle * V(k-1)$.

### 8.3.3

We then consider the case that $i < k$ and $i-1$ belongs to $A$. We let $f^*_{A,k,i}$ be an enumeration of the set $\mathbb{N} \times (\{0\} \cup \bigcup_{j<i} \mathcal{LT}_A \times \{j\})$. So, for every $n$, $f^*_{A,k,i}(n)$ has either the form $\langle m, 0\rangle$, where $m$ is a natural number, or the form $\langle m, V, j\rangle$ where $V$ belongs to $\mathcal{LT}_A$ and $m, j$ are natural numbers, $j < i$.
We define the map $Ev^*_{A,k,i}$ from $\mathcal{LT}_A$ to $\mathcal{LT}_A$ as follows:
(i) is as in Section 8.3.1,but we replace $(\mathrm{ii})_1$ by:

$(\mathrm{ii})_3$ For every $n, m$, for every $(i-1)$-sequence $U$ of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, 0\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where $W$ is an $(i-1)$-sequence of elements of $\mathcal{LT}_A$ such that for every $j < i-1$, $W(j) = Ev^*_{A,k,i}\big(U(j)\big)$.
For every $n, m, j$ such that $j < i$, for every $V$ in $\mathcal{LT}_A$, for every $(i-1)$-sequence $U$ of elements of $\mathcal{LT}_A$, if $f^*_{A,k,i}(n) = \langle m, V, j\rangle$, then $Ev^*_{A,k,i}(\langle n, U\rangle) := \langle m, W\rangle$ where $W$ is an $i$-sequence of elements of $\mathcal{LT}_A$ such that $W(j) := V$ and for every $\ell < j$, $W(\ell) := Ev^*_{A,k,i}\big(U(\ell)\big)$ and for every $\ell$ such that $j < \ell < i$, $W(\ell) := Ev^*_{A,k,i}\big(U(\ell-1)\big)$.

### 8.3.4

We finally consider the case that $i < k$ and $i-1$ does not belong to $A$. We now let $f^*_{A,k,i}$ be an enumeration of the set $\mathbb{N} \times \bigcup_{j<i}(\mathcal{LT}_A \times \{j\})$. The definition of the mapping $Ev^*_{A,k,i}$ is the same as in the previous Section 8.3.3.

**Theorem 8.4** (*Tree Theorem, J.B. Kruskal,1960*).
*For every finitary stump $\sigma$ such that $\sigma(0)$ is non-empty, for every at-most-ternary relation $R$ on $\mathbb{N}$, for each co-finite subset $A$ of $\mathbb{N}$, for every natural number $k$ in $A$, if for every $i$ in $A$ such that $i \leq k$, $\sigma(i)$ secures that $R^i$ is almost full on $\mathbb{N}$, then $\sqsubseteq_{R,k}$ is almost full on $\mathcal{LT}_A$.*

**Proof:** We use the second principle of induction on finitary stumps, that we introduced in Section 3.6.2. Let $\sigma$ be a finitary stump and assume that the statement of the Theorem has been verified for every finitary stump $\tau$ that is more facile than $\sigma$ in the sense of Section 3.6.2, that is, either there exists $k$ such that $\tau(k)$ is an immediate substump of $\sigma(k)$ and for every $n > k$, $\tau(n) = \sigma(n)$, or $\tau\big(i(\tau)\big)$ is an immediate substump of $\sigma\big(i(\sigma)\big)$ where for each nontrivial finitary stump $\psi$, $i(\psi)$ is the greatest natural number $j$ such that $\psi(j)$ is non-empty.
Let $A$ be a co-finite subset of $\mathbb{N}$, and $k$ an element of $A$, and $R$ an at-most-ternary relation on $\mathbb{N}$ such that for every $i$ in $A$ such that $i \leq k$, $\sigma(i)$ secures that $R^i$ is almost full on $\mathbb{N}$.
For every $A$-ary labeled tree $\langle m, T\rangle$ we define the proposition $P(\langle m, T\rangle)$, to be pronounced as: "$\langle m, T\rangle$ has the property $P$", as follows:

$$P(\langle m,T\rangle) := \quad \text{For every } \alpha : \mathbb{N} \to \mathcal{LT}_A \text{ there exist } i, j \text{ such that } i < j \text{ and either } \alpha(i) \sqsubseteq_{R,k} \alpha(j) \text{ or } \langle m, T\rangle \sqsubseteq_{R,k} \alpha(i).$$

We wish to prove that every $A$-ary labeled tree $\langle m, T\rangle$ has the property $P$ and do so by induction.
So assume that $\langle m, T\rangle$ is an $A$-ary labeled tree, and that for every $j$ in $\mathrm{Dom}(T)$, $T(j)$ has the property $P$.
(In particular, $T$ might be the empty sequence).
We consider $i := \mathrm{Dom}(T)$ and distinguish the cases $i < k$ and $i \geq k$.
Let us first study the case $i < k$.
Reminding ourselves of the proof of Higman's Theorem we easily see how to treat this case.
We define an at-most-ternary relation $R'$ on $\mathbb{N}$ as follows.
For all $n_0, n_1$ in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^{i-1}$ if and only if *either* there exist $p_0, p_1$ such that $f^*_{A,k,i}(n_0) = \langle p_0, 0\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, 0\rangle$ and some initial part of $\langle p_0, p_1\rangle$ belongs to $R^{i-1}$,
*or* there exist $p_0, p_1, j_0, j_1$ in $\mathbb{N}$ and $V_0, V_1$ in $\mathcal{LT}_A$ such that $f^*_{A,k,i}(n_0) = \langle p_0, V_0, j_0\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, V_1, j_1\rangle$ and $j_0 = j_1 < i$ and some initial part of $\langle p_0, p_1\rangle$ belongs to $R^i$, and either $V_0 \sqsubseteq_{R,k} V_1$ or $T(j_0) \sqsubseteq_{R,k} V_0$.
For every finite sequence $s$ of natural numbers of length at most 2, $s$ belongs to $(R')^i$ if and only if $s$ belongs to $R^i$ or $\langle m\rangle * s$ belongs to $R^i$.
Finally, for each $\ell$ in $A$ such that $\ell$ differs from both $i$ and $i-1$, we define $(R')^\ell := R^\ell$.
Observe that $(R')^{i-1}$ is an almost full at-most-binary relation on $\mathbb{N}$. Observe that also $(R')^i$ is an almost full on $\mathbb{N}$, so, for every $\ell$ in $A \cup \{i-1\}$, if $\ell \leq k$, then $(R')^\ell$ is almost full.

We now distinguish two cases: $\big(\sigma(i)\big)(m) \neq \emptyset$ and $\big(\sigma(i)\big)(m) \neq \emptyset$. The treatment of the two cases is largely the same. Let us first assume $\big(\sigma(i)\big)(m) \neq \emptyset$. We form a finitary stump $\tau$ such that $\tau(i) = \big(\sigma(i)\big)(m)$, and for all $\ell$ in $A \cup \{i-1\}$,if $\ell \leq k$, then $\tau(\ell)$ secures that $(R')^\ell$ is almost full on $\mathbb{N}$, and if $i < \ell \leq k$, then $\tau(\ell) = \sigma(\ell)$, and if $\ell > k$, then $\sigma(\ell) = \emptyset$.
Observe that $\tau$ is more facile than $\sigma$.
Applying the induction hypothesis, we conclude that $\sqsubseteq_{R',k}$ is almost full on $\mathcal{LT}_{A\cup\{i-1\}}$.
Before completing the argument let us consider how we handle the case $\big(\sigma(i)\big)(m) = \emptyset$. We then form a finitary stump $\tau$ such that $\tau(i) = \emptyset$, and for all $\ell$ in $(A\backslash\{i\}) \cup \{i-1\}$, if $\ell \leq k$, then $\tau(\ell)$ secures that $(R')^\ell$ is almost full on $\mathbb{N}$, and if $i < \ell \leq k$, then $\tau(\ell) = \sigma(\ell)$, and if $\ell > k$, then $\sigma(\ell) = \emptyset$. Again $\tau$ is more facile than $\tau$, and we may conclude that $\sqsubseteq_{R',k}$ is almost full on $\mathcal{LT}_{(A\backslash\{i\})\cup\{i-1\}}$.
The argument is now completed -for both cases- as in the proof of Higman's Theorem.
Let $\langle n, V\rangle$ be an element of $\mathcal{LT}_{A\cup\{i-1\}}$.
We call $\langle n, V\rangle$ *disappointing* if *either* $\mathrm{Dom}(V) = i$ and some initial segment of $\langle m, n\rangle$ belongs to $R^i$, *or* $\mathrm{Dom}(V) = i-1$ and there exist $p$ in $\mathbb{N}$, $U$ in $\mathcal{LT}_A$ and $j < i$ such that $f^*_{A,k,i}(n) = \langle p, U, j\rangle$ and $T(j) \sqsubseteq_{A,k} U$.
We shall call $\langle n, V\rangle$ an ***analysis*** of its own evaluation $Ev^*_{A,k,i}(\langle n, V\rangle)$.
We first observe that for all $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ in $\mathcal{LT}_{A\cup\{i-1\}}$ (or $\mathcal{LT}_{(A\backslash\{i\}\cup\{i-1\}}$, respectively):

> If $\langle n_0, U_0\rangle \sqsubseteq_{R',k} \langle n_1, U_1\rangle$, then either $Ev^*_{A,k,i}(\langle n_0, U_0\rangle) \sqsubseteq_{R,k} Ev^*_{A,k,i}(\langle n_1, U_1\rangle)$ or one of the trees $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ has a disappointing subtree.

We now show that $\langle m, T\rangle$ has the property $P$.
Let $\alpha : \mathbb{N} \to \mathcal{LT}_A$. We consider the fan $F$ consisting of all functions $\beta : \mathbb{N} \to \mathcal{LT}_{A\cup\{i-1\}}$ (or $\mathcal{LT}_{A\backslash\{i\}\cup\{i-1\}}$, respectively) such that for every $n$, $Ev^*_{A,k,i}\big(\beta(n)\big) = \alpha(n)$.
Using the Fan Theorem we determine a natural number $N$ such that for every $\beta$ in $F$ there exist $p, q$ such that $p < q < N$ and $\beta(p) \sqsubseteq_{R',k} \beta(q)$. Consider the finite set $B := \{\overline{\beta}N | \beta \in F\}$. For each sequence $b$ in $B$ we determine $p_b$, $q_b$ such that $p_b < q_b < N$ and $b(p_b) \sqsubseteq_{R',k} b(q_b)$ or one of the trees $b(p_b)$, $b(q_b)$ has a disappointing subtree. Inspection of the set of pairs $\{\langle b(p_b), b(q_b)\rangle | b \in B\}$ will lead us to find either $p, q$ such that $p < q < N$ and $\alpha(p) \sqsubseteq_{R,k} \alpha(q)$ or some $p$ such that $p < N$ and for every $b \in B$, $b(p)$ contains a disappointing subtree, therefore every analysis of $\alpha(p)$ contains a disappointing subtree, therefore $\langle m, T\rangle \sqsubseteq_{R,k} \alpha(p)$.

We now study the case $i \geq k$.
We again distinguish two subcases: $\big(\sigma(k)\big)(m) \neq \emptyset$ and $\big(\sigma(k)\big)(m) = \emptyset$. Let us first assume that $\big(\sigma(k)\big)(m) \neq \emptyset$.
We define an at-most-ternary relation $R'$ on $\mathbb{N}$ as follows:
For all $n_0, n_1$ in $\mathbb{N}$, $\langle n_0, n_1\rangle$ belongs to $(R')^{i-1}$ if and only if *either* there exist $p_0, p_1$ such that $f^*_{A,k,i}(n_0) = \langle p_0, 0\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, 0\rangle$ and some initial part of $\langle p_0, p_1\rangle$ belongs to $R^{i-1}$,
*or* there exist $p_0, p_1$ in $\mathbb{N}$, and $V_0, V_1$ in $(\mathcal{LT}^*_a)^i$ such that $f^*_{A,k,i}(n_0) = \langle p_0, V_0, 1\rangle$ and $f^*_{A,k,i}(n_1) = \langle p_1, V_1, 1\rangle$ and $\langle p_0, p_1\rangle$ belongs to $R^k$ and for each $j < i$, there exists a strictly increasing function from $\mathrm{Dom}\big(V_0(j)\big)$ to $\mathrm{Dom}\big(V_1(j)\big)$ such that for each $\ell$ in

$\mathrm{Dom}\big(V_0(j)\big)$, $\big(V_0(j)\big)(\ell) \sqsubseteq_{R,k} \big(V_1(j)\big)\big(h(\ell)\big)$ or $T(j) \sqsubseteq_{R,k} \big(V_0(j)\big)(\ell)$.
For every finite sequence $s$ of natural numbers of length at most 2, $s$ belongs to $(R')^i$ if and only if $\langle m\rangle * s$ belongs to $R^i$.
Finally, for each $\ell$ such that $\ell < k$, we define $(R')^\ell := R^\ell$, and for each $\ell$ such that $k \le \ell < i-1$, we define $(R')^\ell := R^k$. Observe that $(R')^{i-1}$ is almost full on $\mathbb{N}$. (When proving this one has to use the fact that for each $j < i$, $T(j)$ has the property $P$, and the Finite Sequence Theorem, and Ramsey's Theorem.) Observe that for each $\ell$ in $A \cup \{i-1\}$, if $\ell \le i$, then $(R')^\ell$ is almost full on $\mathbb{N}$.
We consider the finitary stump $\tau$ such that for all $\ell < k$, $\tau(\ell) = \sigma(\ell)$, and for all $\ell$ such that $k \le \ell < i$, $\tau(\ell) = \sigma(k)$ and $\tau(i) = \big(\sigma(i)\big)(m)$ and for all $\ell > i$, $\tau(\ell) = \emptyset$.
Observe that, for each $\ell$ in $A$ such that $\ell \le i$, $\tau(\ell)$ secures that $(R')^\ell$ is almost full on $\mathbb{N}$, and that $\tau$ is more facile than $\sigma$.
We may assume, therefore, that $\sqsubseteq_{R',i}$ is almost full on $\mathcal{LT}_A$.
Let $\langle n, U\rangle$ be an element of $\mathcal{LT}_{A\cup\{i-1\}}$.
We call $\langle n, U\rangle$ *disappointing* if *either* $\mathrm{Dom}(u) \ge i$ and some initial part of $\langle m, n\rangle$ belongs to $R^k$ *or* $\mathrm{Dom}(U) = i-1$ and there exist $p$ in $\mathbb{N}$ and $V$ in $\big((\mathcal{LT}_A)^*\big)^k$ such that $f^*_{A,k,i}(n)$ equals either $\langle p, V\rangle$ or $\langle p, V, 1\rangle$ and for some $j < i$, for some $q < \mathrm{Dom}\big(V(j)\big)$, $T(j) \sqsubseteq_{R,k} \big(V(j)\big)(q)$.
We make two observations:

(i) Assume that $\langle n, U\rangle$ belongs to $\mathcal{LT}_A$ and that $\mathrm{Dom}(U) \ge i$. If every $\langle p, W\rangle$ in $\mathcal{LT}_{A\cup\{i-1\}}$ such that $Ev^*_{A,k,i}(\langle p, W\rangle) = \langle n, U\rangle$, that is, every analysis of $\langle n, U\rangle$, is disappointing, then $\langle m, T\rangle \sqsubseteq_{R,k} \langle n, U\rangle$.
(ii) Assume that $\langle n, U\rangle$ belongs to $\mathcal{LT}_A$. If every analysis of $\langle n, U\rangle$ contains a disappointing subtree, then $\langle m, T\rangle \sqsubseteq_{R,k} \langle n, U\rangle$.

We also need the following remark:

For all $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ in $\mathcal{LT}_{A\cup\{i-1\}}$, if $\langle n_0, U_0\rangle \sqsubseteq_{R',k} \langle n_1, U_1\rangle$ then *either* $Ev^*_{A,k,i}(\langle n_0, U_0\rangle) \sqsubseteq_{R,k} Ev^*_{A,k,i}(\langle n_1, U_1\rangle)$ *or* one of the trees $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ contains a disappointing subtree.

We now prove that $\langle m, T\rangle$ has the property $P$, as follows.
Let $\alpha : \mathbb{N} \to \mathcal{LT}_A$.
Consider the fan $F$ consisting of all functions $\beta : \mathbb{N} \to \mathcal{LT}_{A\cup\{i-1\}}$ such that for every $n$, $Ev^*_{A,k,i}\big(\beta(n)\big) = \alpha(n)$.
Using the Fan Theorem we determine a natural number $N$ such that for every $\beta$ in $F$ there exist $p, q$ such that $p < q < N$ and $\beta(p) \sqsubseteq_{R',k} \beta(q)$.
Reasoning as in the first part of this proof, we conclude:
either there exist $p, q$ such that $p < q < N$ and $\alpha(p) \sqsubseteq_{R,k} \alpha(q)$, or for some $p < N$, every analysis of $\alpha(p)$ contains a disappointing subtree, therefore $\langle m, T\rangle \sqsubseteq_{R,k} \alpha(p)$.

Let us now consider the case $\big(\sigma(k)\big)(m) = \emptyset$. We conclude that the empty sequence $\langle\,\rangle$ belongs to $R^k$.
We define $B := \{\ell | \ell \in A | \ell \le i\}$ and consider $\mathcal{LT}_B$.
We define an at-most-ternary relation $R'$ on $\mathbb{N}$ exactly as in the previous case $\big(\sigma(k)\big)(m) \ne \emptyset$.
Observe that for each $\ell$ in $B$, $(R')^\ell$ is almost full on $\mathbb{N}$.

Applying Higman's Theorem, we conclude that $\preccurlyeq_{R'}$ is almost full on $\mathcal{LT}_B$.
We need the following observations:

(i) For all $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ in $\mathcal{LT}_B$, if $\langle n_0, U_0\rangle \preccurlyeq_{R'} \langle n_1, U_1\rangle$, then $\langle n_0, U_0\rangle \sqsubseteq_{R',k} \langle n_1, U_1\rangle$ and *either* $Ev^*_{A,k,i}(\langle n_0, U_0\rangle) \sqsubseteq_{R,k} Ev^*_{A,k,i}(\langle n_1, U_1\rangle)$ *or* one of the trees $\langle n_0, U_0\rangle$, $\langle n_1, U_1\rangle$ contains a disappointing subtree.

(ii) For all $\langle p, W\rangle$ in $\mathcal{LT}_B$, if every analysis $\langle n, U\rangle$ of $\langle p, W\rangle$ that belongs to $\mathcal{LT}_B$ has a disappointing subtree, then $\langle m, T\rangle \preccurlyeq_{R,k} \langle p, W\rangle$.

The proof that $\langle m, T\rangle$ has the property $P$ is from here on almost the same as in the case $\big(\sigma(k)\big)(m) \neq \emptyset$ and is left to the reader. □

# 9 Minimal-bad-sequence arguments

We show how some of the results proved in this paper are obtained more easily by the minimal-bad-sequence argument due to Nash-Williams. We freely use classical logic in this Section. In Section 10 we shall discuss the problem if we could do something similar constructively.

## 9.1 *The Finite Sequence Theorem*

### 9.1.1

$$\boxed{\leq^* \text{ is almost full on } \mathbb{N}^*.} \qquad \text{(Cf. Thm 5.2)}$$

Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. We say $\alpha$ is *bad* if $\alpha$ does not meet $\leq^*$. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathbb{N}^*$.
We define $\alpha_0 : \mathbb{N} \to \mathbb{N}^*$ in such a way that $\alpha_0$ is bad and for each $i$, for each $\alpha : \mathbb{N} \to \mathbb{N}^*$, if for each $j < i$,$\alpha(j) = \alpha_0(j)$, but $\alpha(i) = \mathrm{Rem}\big(\alpha_0(i)\big)$, then $\alpha$ is good, that is, $\alpha$ meets $\leq^*$.
$\alpha_0$ is called a *minimal bad sequence.* Observe that for each $i$, $\alpha_0(i) \neq \langle\,\rangle$. We consider the sequences $\gamma : \mathbb{N} \to \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}^*$ such that for each $i$, $\gamma(i) = \big(\alpha_0(i)\big)(0)$ and $\beta(i) = \mathrm{Rem}\big(\alpha_0(i)\big)$, so $\alpha(i) = \langle\gamma(i)\rangle * \beta(i)$. We claim that for every strictly increasing $\delta : \mathbb{N} \to \mathbb{N}$ the sequence $\beta \circ \delta$ meets $\leq^*$.

For suppose $\delta : \mathbb{N} \to \mathbb{N}$ is strictly increasing.
Consider the sequence: $\alpha_0(0)\alpha_0(1), \ldots, \alpha_0(\delta(0) - 1)$, $\beta \circ \delta(0)$, $\beta \circ \delta(1), \ldots$.
This sequence meets $\leq^*$. There are several possibilities.

(i) There exists $i, j$ such that $i < j < \delta(0)$ and $\alpha_0(i) \leq^* \big(\alpha_0(j)\big)$.
This will not happen, as $\alpha_0$ is bad.

(ii) There exist $i, j$ such that $i < \delta(0) \leq \delta(j)$ and $\alpha_0(i) \leq^* \beta \circ \delta(j)$. Then also $\alpha_0(i) \leq^* \alpha_0\big(\delta(j)\big)$.
This will not happen, as $\alpha_0$ is bad.

(iii) There exist $i, j$ such that $i < j$ and $\beta \circ \delta(i) \leq^* \beta \circ \delta(j)$.
Then $\beta \circ \delta$ meets $\leq^*$.

We now determine $\delta : \mathbb{N} \to \mathbb{N}$ such that $\delta$ is strictly increasing and for each $i$, $\gamma(\delta(i)) \leq \gamma(\delta(i+1))$. We calculate $i, j$ such that $i < j$ and $\beta(\delta(i)) \leq^* \beta(\delta(j))$ and conclude: $\alpha_0(i) \leq^* \alpha_0(j)$.
Contradiction, as $\alpha_0$ is bad.
We conclude that there is no bad sequence, therefore, every $\alpha : \mathbb{N} \to \mathbb{N}^*$ will meet $\leq^*$.

**9.1.2**

| For every binary relation $R$ on $\mathbb{N}$: If $R$ is almost full on $\mathbb{N}$, then $R^*$ is almost full on $\mathbb{N}^*$. | (Cf. Thm 5.4) |
|---|---|

Suppose $R$ is almost full on $\mathbb{N}$.
Let $\alpha : \mathbb{N} \to \mathbb{N}^*$. We say $\alpha$ is *bad* if $\alpha$ does not meet $R^*$. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathbb{N}^*$.
Determine $\alpha_0 : \mathbb{N} \to \mathbb{N}^*$ such that $\alpha_0$ is bad and for every $i$ in $\mathbb{N}$, there is no bad $\alpha : \mathbb{N} \to \mathbb{N}^*$ such that $\overline{\alpha}(i+1) = \overline{\alpha}_0 i * \langle \mathrm{Rem}(\alpha_0(i)) \rangle$. Observe that for every $i$, $\alpha_0(i) \neq \langle \rangle$.
Determine $\gamma : \mathbb{N} \to \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}^*$ such that for every $i$, $\alpha_0(i) = \langle \gamma(i) \rangle * \beta(i)$.
Arguing as in Section 9.1.1, we prove that for every increasing $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and $\beta(\delta(i)) R^* \beta(\delta(j))$.
Also, for every strictly increasing $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and $\gamma(\delta(i)) \, R \, \gamma(\delta(j))$.
Using Ramsey's Theorem, we conclude that there exist $i, j$ such that $i < j$ and both $\gamma(i) R \gamma(j)$ and $\beta(i) R^* \beta(j)$, therefore $\alpha_0(i) R^* \alpha_0(j)$. So $\alpha_0$ is not bad. Contradiction.
We conclude that there is no bad $\alpha : \mathbb{N} \to \mathbb{N}^*$, that is, every $\alpha : \mathbb{N} \to \mathbb{N}^*$ meets $R^*$.

## 9.2 *Higman's Theorem*

**9.2.1**

| $\preccurlyeq$ is almost full on $\mathcal{T}_2$. | (Cf. Thm 6.5) |
|---|---|

Let $\alpha$ be a function from the set $\mathbb{N}$ of natural numbers to the set $\mathcal{T}_2$ of strictly binary trees. We say $\alpha$ is *bad* if $\alpha$ does not meet $\preccurlyeq$. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathbb{N}^*$.
Determine $\alpha_0 : \mathbb{N} \to \mathcal{T}_2$ such that $\alpha_0$ is bad and for every $i$, for every $\alpha : \mathbb{N} \to \mathcal{T}_2$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then $\alpha$ is good, that is, $\alpha$ meets $\preccurlyeq$. Observe that for each $i$, $\alpha_0(i)$ is non-empty.
Consider the sequence

$$\big(\alpha_0(0)\big)(0), \big(\alpha_0(0)\big)(1), \big(\alpha_0(1)\big)(0), \big(\alpha_0(1)\big)(1), \ldots$$

consisting of the immediate subtrees of the elements of $\alpha_0$, in their natural order.
Let us call this sequence $\beta$. So, for every $i$, $\beta(2i) = \big(\alpha_0(i)\big)(0)$ and $\beta(2i+1) = \big(\alpha_0(i)\big)(1)$.
We claim that for every strictly increasing sequence $\delta$, the sequence $\beta \circ \delta$ meets $\preccurlyeq$.

For suppose $\delta : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Consider $\delta(0)$. Calculate $i_0$ such that $\delta(0) = 2i_0$ or $\delta(0) = 2i_0 + 1$.
Consider the sequence: $\alpha_0(0)$, $\alpha_0(1), \dots, \alpha_0(i_0 - 1)$, $\beta \circ \delta(0)$, $\beta \circ \delta(1), \dots$.
This sequence meets $\preccurlyeq$. There are several possibilities.

(i) There exist $i, j$ such that $i < j < i_0$ and $\alpha_0(i) \preccurlyeq \alpha_0(j)$.
This will not happen, as $\alpha_0$ is bad.

(ii) There exist $i, j$ such that $i < \delta(0)$ and $\alpha_0(i) \preccurlyeq \beta \circ \delta(j)$. Calculate $i_1$ such that $\delta(j) = 2i_1$ or $\delta(j) = 2i_1 + 1$.
Then $\alpha_0(i) \preccurlyeq \beta \circ \delta(j) \preccurlyeq \alpha_0(i_1)$, and, as $\beta \circ \delta(j)$ is a proper subtree of $\alpha_0(i_1)$, $i < i_1$ and $\alpha_0(i) \preccurlyeq \alpha_0(i_1)$.
This will not happen, as $\alpha_0$ is bad.

(iii) There exists $i, j$ such that $i < j$ and $\beta \circ \delta(i) \preccurlyeq \beta \circ \delta(j)$.
Then $\beta \circ \delta$ meets $\preccurlyeq$.

Observe that the sequences $(\alpha_0(0))(0)$, $(\alpha_0(1))(0)$, $(\alpha_0(2))(0), \dots$ and $(\alpha_0(0))(1)$, $(\alpha_0(1))(1)$, $(\alpha_0(2))(1), \dots$ are subsequences of $\beta$.
Using Ramsey's Theorem, we determine $i, j$ such that $i < j$ and both $(\alpha_0(0))$ $(\alpha_0(i))(0) \preccurlyeq (\alpha_0(j))(0)$ and $(\alpha_0(i))(1) \preccurlyeq (\alpha_0(j))(1)$, therefore $\alpha_0(i) \preccurlyeq \alpha_0(j)$.
Contradiction, as $\alpha_0$ is bad.
We conclude that there is no bad sequence, therefore every $\alpha : \mathbb{N} \to \mathcal{T}_2$ will meet $\preccurlyeq$.

### 9.2.2

| For every finite subset $A$ of $\mathbb{N}$ containing 0, for every at-most-ternary relation $R$ on $\mathbb{N}$, if, for each $k$ in $A$, $R^k$ is almost full on $\mathbb{N}$, then $\preccurlyeq_R$ is almost full on $\mathcal{LT}_A$. | (Cf. Thm 7.5) |
|---|---|

Let $A$ be a finite subset of $\mathbb{N}$ containing 0, and $R$ an at-most-ternary relation on $\mathbb{N}$ such that for each $k$ in $A$, $R^k$ is almost full on $\mathbb{N}$. Let $\alpha : \mathbb{N} \to \mathcal{LT}_A$. We say that $\alpha$ is bad if $\alpha$ does not meet $\preccurlyeq_R$. Suppose that there exists at least one bad sequence $\alpha : \mathbb{N} \to \mathcal{LT}_A$.
We determine a sequence $\alpha_0 : \mathbb{N} \to \mathcal{LT}_A$ such that $\alpha_0$ is bad and for every $i$, for every $\alpha : \mathbb{N} \to \mathcal{LT}_A$, if for every $j < i$, $\alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then $\alpha$ meets $\preccurlyeq_R$.
We determine $\gamma : \mathbb{N} \to \mathbb{N}$ and $\tau : \mathbb{N} \to \bigcup_{k \in A} (\mathcal{LT}_A)^k$ such that for every $i$, $\alpha_0(i) = \langle \gamma(i), \tau(i) \rangle$.
We let $\beta : \mathbb{N} \to \mathcal{LT}_A$ be an enumeration of the set $\{(\tau(i))(j) | i \in \mathbb{N},\ j \in \mathrm{Dom}(\tau(i))\}$.
Arguing as in Section 9.2.1 we prove that for every strictly increasing $\delta : \mathbb{N} \to \mathbb{N}$ there exist $i, j$ such that $i < j$ and $\beta(i) \preccurlyeq_R \beta(j)$.
We determine a strictly increasing function $\delta : \mathbb{N} \to \mathbb{N}$ and $k$ in $A$ such that for every $i$, $\mathrm{Dom}\Big(\tau\big(\delta(i)\big)\Big) = k$.
Using Ramsey's Theorem, we find $i, j$ such that $i < j$ and for each $p < k$, $\Big(\tau\big(\delta(i)\big)\Big)$

$(p) \preccurlyeq_R \Big(\tau\big(\delta(j)\big)\Big)(p)$ and $\langle\gamma\big(\delta(i)\big), \gamma\big(\delta(j)\big)\rangle$ belongs to $R^k$, therefore: $\alpha_0\big(\delta(i)\big) \preccurlyeq_R \alpha_0\big(\delta(j)\big)$.
Contradiction, as $\alpha_0$ is bad.
We conclude that there is no bad sequence, therefore every $\alpha : \mathbb{N} \to \mathcal{LT}_A$ will meet $\preccurlyeq_R$.

## 9.3 *Kruskal's Theorem*

### 9.3.1

| Vazsonyi's Conjecture: $\sqsubseteq$ is almost full on $\mathcal{T}$. | (Cf. Thm 8.1) |
|---|---|

Let $\alpha$ be a function from the set $\mathbb{N}$ of natural numbers to the set $\mathcal{T}$ of finite trees. We say $\alpha$ is *bad* if $\alpha$ does not meet $\sqsubseteq$. Suppose there exists at least one bad $\alpha : \mathbb{N} \to \mathcal{T}$.
Determine $\alpha_0 : \mathbb{N} \to \mathcal{T}$ such that $\alpha_0$ is bad and for every $i$, for every $\alpha : \mathbb{N} \to \mathcal{T}_2$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then $\alpha$ meets $\sqsubseteq$.
Let $B$ be the set $\{\big(\alpha_0(i)\big)(j) | i \in \mathbb{N}, j \in \mathrm{Dom}\big(\alpha_0(i)\big)\}$ be the set of all immediate subtrees of the trees $\alpha_0(0), \alpha_0(1), \ldots$.
Arguing as in the previous Sections, we prove that $\sqsubseteq$ is almost full on $B$.
Using the Finite Sequence Theorem we conclude that $\sqsubseteq^*$ is almost full on $B^*$.
Observe that $B^*$ is a subset of $\mathcal{T}$ and that the trees $\alpha_0(0), \alpha_0(1), \ldots$ belong to $B^*$.
So there exist $i, j$ such that $i < j$ and $\alpha_0(i) \sqsubseteq^* \alpha_0(j)$, that is $\alpha_0(i) \sqsubseteq \alpha_0(j)$.
Contradiction, as $\alpha_0$ is bad.
We conclude that there is no bad sequence, therefore every $\alpha : \mathbb{N} \to \mathcal{T}$ will meet $\sqsubseteq$.

### 9.3.2

| For every at-most-ternary relation $R$ on $\mathbb{N}$, for every $k$,<br>if, for every $i \leq k$, $R^i$ is almost full on $\mathbb{N}$,<br>then $\sqsubseteq_{R,k}$ is almost full on $\mathcal{LT}_{\mathbb{N}}$. | (Cf. Thm 8.4) |
|---|---|

Suppose that $R$ is an at-most-ternary relation on $\mathbb{N}$, and $k$ is a natural number and for every $i \leq k$, $R^i$ is almost full on $\mathbb{N}$. Let $\alpha$ be a function from the set $\mathbb{N}$ of natural numbers to the set $\mathcal{LT}_{\mathbb{N}}$ of labeled finite trees. We say $\alpha$ is *bad* if $\alpha$ does not meet $\sqsubseteq_{R,k}$.
Assume that there exists at least one bad $\alpha : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$. We determine $\alpha_0 : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$ such that for every $i$, for every $\alpha : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$, if, for each $j < i$, $\alpha(j) = \alpha_0(j)$ and $\alpha(i)$ is an immediate subtree of $\alpha_0(i)$, then $\alpha$ meets $\sqsubseteq_{R,k}$.
We determine $\gamma : \mathbb{N} \to \mathbb{N}$ and $\tau : \mathbb{N} \to (\mathcal{LT}_{\mathbb{N}})^*$ such that for every $i$, $\alpha_0(i) = \langle\gamma(i), \tau(i)\rangle$.
Let $B := \{\big(\tau(i)\big)(j) | i \in \mathbb{N}, | j \in \mathrm{Dom}\big(\tau(i)\big)\}$ be the set of all immediate subtrees of the elements of $\alpha_0$. Arguing as before, we conclude that $\sqsubseteq_{R,k}$ is almost full on $B$.
We now distinguish two cases:

(i) There exists a strictly increasing function $\delta : \mathbb{N} \to \mathbb{N}$ and a natural number $n_0 < k$ such that for every $i$, $\mathrm{Dom}\Big(\tau\big(\delta(i)\big)\Big) = n_0$. Applying Ramsey's Theorem we

find $i, j$ such that $i < j$ and some initial part of $\langle \gamma(\delta(i)), \gamma(\delta(j)) \rangle$ belongs to $R^{n_0}$, and for each $q < n_0$, $\Big(\tau\big(\delta(i)\big)\Big)(q) \sqsubseteq_{R,k} \Big(\tau\big(\delta(j)\big)\Big)(q)$, therefore $\alpha_0\big(\delta(i)\big) \sqsubseteq_{R,k} \alpha_0\big(\delta(j)\big)$.
Contradiction, as $\alpha_0$ is bad.

(ii) There exists a strictly increasing function $\delta : \mathbb{N} \to \mathbb{N}$ such that for every $i$, $\mathrm{Dom}\big(\tau(i)\big) \geq k$.
Applying the Finite Sequence Theorem we find $i, j$ such that $i < j$ and some initial part of $\langle \gamma(\delta(i)), \gamma(\delta(j)) \rangle$ belongs to $R^k$ and
$\tau\big(\delta(i)\big)(\sqsubseteq_{R,k})^* \tau\big(\delta(j)\big)$ therefore $\alpha_0\big(\delta(i)\big) \sqsubseteq_{R,k} \alpha_0\big(\delta(j)\big)$.
Contradiction, as $\alpha_0$ is bad.

We conclude that there is no bad sequence, therefore every $\alpha : \mathbb{N} \to \mathcal{LT}_{\mathbb{N}}$ will meet $\sqsubseteq_{R,k}$.

## 9.4 *Extending Kruskal's Theorem*

We discuss a famous extension of Kruskal's Theorem found by H. Friedman. For each nonzero natural number $k$ we introduce a subset $\mathcal{LT}_k$ of the set $\mathcal{LT}$ of labeled trees as follows:

(i) For each $j > k$, the ordered pair $\langle j, \emptyset \rangle$ belongs to $\mathcal{LT}_k$.
(ii) For each $j \leq k$, for each non-empty finite sequence $T$ of elements $\mathcal{LT}_k$, the ordered pair $\langle j, T \rangle$ belongs to $\mathcal{LT}_k$.
(iii) Clauses (i), (ii) produce all elements of $\mathcal{LT}_k$.

Let $R$ be a binary relation on $\mathbb{N}$.
For each nonzero natural number $k$ we define a binary relation $\sqsubseteq^{\#}_{k,R}$ on the set $\mathcal{LT}_k$ as follows:

For all $\langle m, T \rangle$, $\langle n, U \rangle$ in $\mathcal{LT}_k$, $\langle m, T \rangle \sqsubseteq^{\#}_{k,R} \langle n, U \rangle$ if and only if *either* $T = U = \emptyset$ and $\langle m, n \rangle$ belongs to $R$,
*or* $m = n \leq k$ and both $T, U$ are non-empty and $T(\sqsubseteq^{\#}_{k,R})^* U$ that is, there exists a strictly increasing function $h$ from $\mathrm{Dom}(T)$ to $\mathrm{Dom}(U)$ such that for each $i$ in $\mathrm{Dom}(T)$, $T(i) \sqsubseteq^{\#}_{k,R} U\big(h(i)\big)$,
*or* $m \leq n < k$ and for some $i$ in $\mathrm{Dom}(U)$, $\langle m, T \rangle \sqsubseteq^{\#}_{k,R} U\big(h(i)\big)$.

Let $\langle m, T \rangle$ be some labeled tree.
An element $s$ of $B(\langle m, T \rangle)$ is called an *interior* point of $B(\langle m, T \rangle)$ if $s * \langle 0 \rangle$ belongs to $B(\langle m, T \rangle)$. An element $s$ of $B(\langle m, T \rangle)$ is called an *endpoint* of $B(\langle m, T \rangle)$ if it is not an interior point of $B(\langle m, T \rangle)$.

One may prove the following:
For every binary relation $R$ on $\mathbb{N}$, for every nonzero natural number $k$, for all $\langle m, T \rangle$, $\langle n, U \rangle$ in $\mathcal{LT}_k$, $\langle m, T \rangle \sqsubseteq^{\#}_{k,R} \langle n, U \rangle$ if and only if there exists a mapping from $B(\langle m, T \rangle)$ into $B(\langle n, U \rangle)$ such that

(i) for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$, if $s * \langle j \rangle$ belongs to $B(\langle m, T \rangle)$ then $f(s * \langle j \rangle)$ is a proper extension of $f(s)$.
(ii) for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$, if both $s * \langle j \rangle$ and $s * \langle j+1 \rangle$ belong to $B(\langle m, T \rangle)$, then there exist $k_0, k_1$ such that $k_0 < k_1$ and $f(s) * \langle k_0 \rangle$ is an initial part of $f(s * \langle j \rangle)$, and $f(s) * \langle k_1 \rangle$ is an initial part of $f(s * \langle j+1 \rangle)$.
(iii) for all $s$ in $\mathbb{N}^*$, if $s$ is an endpoint of $B(\langle m, T \rangle)$, then $f(s)$ is an endpoint of $B(\langle n, U \rangle)$ and the ordered pair $\langle (\ell(\langle m, T \rangle)), \big(\ell(\langle n, U \rangle)\big)\big(f(s)\big) \rangle$ belongs to $R$.
(iv) for all $s$ in $\mathbb{N}^*$, if $s$ is an interior point of $B(\langle m, T \rangle)$, then $\big(\ell(\langle m, T \rangle)\big)(s) = \big(\ell(\langle n, U \rangle)\big)\big(f(s)\big)$.
(v) For all $t$ in $\mathbb{N}^*$, if $t$ is an initial part of $f(\langle\rangle)$, then $\big(\ell(\langle n, U \rangle)\big)(t) \geq m$. Also, for all $s$ in $\mathbb{N}^*$, $j$ in $\mathbb{N}$, if both $s$ and $s * \langle j \rangle$ belong to $B(\langle m, T \rangle)$, then for every $t$ in $\mathbb{N}^*$, if $t$ properly extends $f(s)$ and is a proper initial part of $f(s * \langle j \rangle)$, then $\big(\ell(\langle n, U \rangle)\big)(t) \geq \big(\ell(\langle n, U \rangle)\big)\big(f(s * \langle j \rangle)\big)$.

Condition (v) is often called *Friedman's gap condition.*
Observe that for all trees $\langle 0, T \rangle$, $\langle 0, U \rangle$ in $\mathcal{LT}_1$, if there exists a subtree $\langle j, V \rangle$ of $\langle 0, U \rangle$ such that $\langle 0, T \rangle \sqsubseteq^{\#}_{k,R} \langle j, V \rangle$ then $\langle 0, T \rangle \sqsubseteq^{\#}_{k,R} \langle 0, U \rangle$. This statement is not true for trees $\langle 1, T \rangle$, $\langle 0, U \rangle$ in $\mathcal{LT}_1$.
Friedman's extension of Kruskal's Theorem says the following:

> For every nonzero natural number $k$, for every binary relation $R$ on $\mathbb{N}$, if $R$ is almost full on $\mathbb{N}$, then $\sqsubseteq^{\#}_{k,R}$ is almost full on $\mathcal{LT}_k$.

This theorem is proved by induction on $k$, by a repeated minimal-bad-sequence-argument. I do not see how to replace this minimal-bad-sequence argument by a constructively valid argument and am unable to decide if Friedman's extension of Kruskal's Theorem is intuitionistically true. This seems to be the most important question arising from this paper.
We now sketch Friedman's argument for the case $k = 1$. We have to make some preparations.
We let $\mathcal{LT}_1^0$ be the set of all trees of the form $\langle 0, T \rangle$ where $T$ is a non-empty finite sequence of elements of $\mathcal{LT}_1$.
Let $A$ be a subset of $\mathcal{LT}_1$. We let $\mathcal{LT}_1[A]$ be the subset of $\mathcal{LT}_1$ that is given by the following definition:

(i) Every element of $A$ belongs to $\mathcal{LT}_1[A]$.
(ii) For every non-empty finite sequence $T$ of elements of $\mathcal{LT}_1[A]$ the ordered pair $\langle 1, T \rangle$ belongs to $\mathcal{LT}_1[A]$.
(iii) Every element of $\mathcal{LT}_1[A]$ is produced from elements of $A$ by repeated applications of step (ii).

The following statement is easily proved from Kruskal's Theorem:
For every subset $A$ of $\mathcal{LT}$,
If $\sqsubseteq^{\#}_{1,R}$ is almost full on $A$, then $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1[A]$.

Let $\alpha$ be a function from $\mathbb{N}$ to $\mathcal{LT}_1$.
We say $\alpha$ is *bad* if $\alpha$ does not meet $\sqsubseteq^{\#}_{1,R}$.
We now first prove that $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1^0$, as follows. Suppose that there exists at least one bad $\alpha : \mathbb{N} \to \mathcal{LT}_1^0$. Determine $\alpha_0 : \mathbb{N} \to \mathcal{LT}_1^0$ such that $\alpha_0$ is bad and for each $i$, for each $\alpha : \mathbb{N} \to \mathcal{LT}_1^0$, if for each $j < i$, $\alpha(j) = \alpha_0(j)$, and $\alpha(i)$ is a proper subtree of $\alpha_0(i)$, then $\alpha$ meets $\sqsubseteq^{\#}_{1,R}$. Let $A$ be the set of all elements of $\mathcal{LT}_1^0$ that are a proper subtree of one of the trees $\alpha_0(0), \alpha_0(1), \dots$. Reasoning as in earlier such cases,we conclude that $\sqsubseteq^{\#}_{1,R}$ is almost full on $A$.
Let $B$ be the set of all basic trees in $\mathcal{LT}_1$, that is of all trees $\langle j, \emptyset\rangle$, where $j > 1$. As $R$ is almost full on $\mathbb{N}$, $\sqsubseteq^{\#}_{1,R}$ is almost full on $B$.
Using Ramsey's Theorem we conclude that $\sqsubseteq^{\#}_{1,R}$ is almost full on $A \cup B$.
Using the remark we just made, we conclude that $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1[A \cup B]$.
We now reconsider $\alpha_0$. Observe that there exists $\tau : \mathbb{N} \to (\mathcal{LT}_1[A \cup B])^*$ such that for each $i$, $\alpha_0(i) = \langle 0, \tau(i)\rangle$.
As, by the Finite Sequence Theorem, $\tau$ meets $(\sqsubseteq^{\#}_{1,R})^*$, $\alpha_0$ will meet $\sqsubseteq^{\#}_{1,R}$.
Contradiction, as $\alpha_0$ is bad.
We conclude that there is no bad $\alpha : \mathbb{N} \to \mathcal{LT}_1^0$.
Therefore $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1^0$.
It now follows easily that $\sqsubseteq^{\#}_{1,R}$ is almost full on $\mathcal{LT}_1$ as a whole.
It suffices to remark that $\mathcal{LT}_1$ coincides with $\mathcal{LT}_1[\mathcal{LT}_1^0]$.

# 10 The Principle of Open Induction

## 10.1

We consider the set $\mathcal{N}$ of all infinite sequences of natural numbers.
Let $A$ be a subset of the set $\mathbb{N}^*$ of all finite sequences of natural numbers. As in Section 4 we let $A^{\#}$ be the set of all $\alpha$ in $\mathcal{N}$ such that there exists $n$ such that $\langle \alpha(0), \alpha(1), \dots, \alpha(n-1)\rangle$ belongs to $A$. $\alpha$ is $A$-*good* if $\alpha$ belongs to $A^{\#}$, $\alpha$ is $A$-*bad* otherwise.
Let $\alpha, \beta$ be elements of $\mathcal{N}$.
We define: $\alpha$ *comes before* $\beta$, notation $\alpha < \beta$, if and only if there exists $i$ such that for all $j < i$, $\alpha(j) = \beta(j)$, and $\alpha(i) < \beta(i)$.
A typical case of the argument used again and again in the previous Section is the following

| |
|---|
| *Minimal Bad Sequence Principle*<br>For every subset $A$ of $\mathbb{N}^*$,<br>if there exists $\alpha$ such that $\alpha$ does not belong to $A^{\#}$,<br>then there exists $\alpha$ such that $\alpha$ does not belong to $A^{\#}$<br>while every $\beta$ coming before $\alpha$ does belong to $A^{\#}$. |

This principle is false when read as it stands and interpreted constructively.
It would imply that every inhabited subset $A$ of the set $\mathbb{N}$ of natural numbers has a least element.

## 10.2

We consider the following contrapositive formulation of the Minimal Bad Sequence Principle:

> *Open Induction Principle*
> For every subset $A$ of $\mathbb{N}^*$,
> if every $\alpha$ belongs to $A^\#$ as soon as every $\beta$ coming before $\alpha$ belongs to $A^\#$, then every $\alpha$ belongs to $A^\#$.

If we should accept this principle as an axiom of intuitionistic analysis, we could retain, after a slight revision,the arguments and results in Section 9.
Unfortunately, we do not see why the principle is true.

## 10.3

It is useful to compare the Open Induction Principle with the well-known Principle of Induction on Monotone Bars.
Let $P$ be a subset of $\mathbb{N}^*$.
$P$ is called a *bar* if and only if for every $\alpha$ there exists $n$ such that
$\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$ belongs to $P$, that is, $\mathcal{N}$ coincides with $P^\#$.
$P$ is called *monotone* if and only if for every $s$ in $\mathbb{N}^*$, $i$ in $\mathbb{N}$, if $s$ belongs to $P$, then $s * \langle i\rangle$ belongs to $P$.
$P$ is called *hereditary* if and only if for every $s$ in $\mathbb{N}^*$, if, for every $i$, $s * \langle i\rangle$ belongs to $P$, then $s$ belongs to $P$.

> *Principle of Induction on Monotone Bars*
> For every subset $P$ of $\mathbb{N}^*$,
> if $P$ is a monotone bar and an hereditary subset of $\mathbb{N}^*$,
> then the empty sequence $\langle\,\rangle$ belongs to $P$.

This principle may be proved from Brouwer's Thesis.
Brouwer's Thesis guarantees:
For every subset $P$ of $\mathbb{N}^*$, if $P$ is a bar, then there exists a stump $\sigma$ such that $P \cap \sigma$ is a bar.
So it suffices to prove, by induction on the set **Stp** of stumps:
For every stump $\sigma$, for every subset $P$ of $\mathbb{N}^*$.
If $P \cap \sigma$ is a bar and $P$ is a monotone and hereditary subset of $\mathbb{N}^*$, then the empty sequence $\langle\,\rangle$ belongs to $P$.
We leave the straightforward proof to the reader.

## 10.4

Let $T$ be a subset of $\mathbb{N}^*$.
$T$ is a *frame* if and only if the empty sequence $\langle\,\rangle$ belongs to $T$ and for every $s$ in $\mathbb{N}^*$, $s$ belongs to $T$ if and only if there exists $i$ such that $s * \langle i \rangle$ belongs to $T$.
Let $T \subseteq \mathbb{N}^*$ be a frame. We let $[T]$ be the set of all $\alpha$ in $\mathcal{N}$ such that for every $n$, $\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$ belongs to $T$.
Let $T \subseteq \mathbb{N}^*$ be a frame and let $P$ be a subset of $T$.
$P$ is called a *bar in* $[T]$ if and only if for every $\alpha$ in $[T]$ there exists $n$ such that $\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$ belongs to $P$.
$P$ is called *monotone in* $T$ if and only if for every $s$ in $P$, $i \in \mathbb{N}$, if $s * \langle i \rangle$ belongs to $T$, then $s * \langle i \rangle$ belongs to $P$.
$P$ is called *hereditary in* $T$ if and only if for every $s$ in $T$, if, for every $i$ such that $s * \langle i \rangle$ belongs to $T$, $s * \langle i \rangle$ belongs to $P$, then $s$ belongs to $P$.

> *Principle of Induction on Monotone Bars in Decidable Frames*
> Let $T$ be a decidable subset of $\mathbb{N}^*$ and a frame.
> For every subset $P$ of $T$:
> If $P$ is a bar in $[T]$, and monotone in $T$ and hereditary in $T$, then the empty sequence $\langle\,\rangle$ belongs to $P$.

One may prove this extension of the Principle of Induction on Monotone Bars from the principle itself without much difficulty.

Let $T$ be a decidable subset of $\mathbb{N}^*$ and a frame.
It is useful to define a function $R$ from $\mathbb{N}^*$ to the frame $T$, as follows: $R(\langle\,\rangle) := \langle\,\rangle$ and for each $s$ in $\mathbb{N}^*$, $i$ in $\mathbb{N}$, if $s * \langle i \rangle$ belongs to $T$, then $R(s * \langle i \rangle) := s * \langle i \rangle$, and if $s * \langle i \rangle$ does not belong to $T$, then $R(s * \langle i \rangle) := R(s) * \langle i_0 \rangle$ where $i_0$ is the least $j$ such that $R(s) * \langle j \rangle$ belongs to $T$.
Let $P$ be a subset of $T$. Observe that, if $P$ is a bar in $[T]$ then $(\mathbb{N}^* \setminus T) \cup P$ is a bar in $[\mathbb{N}^*] = \mathcal{N}$.

A decidable frame is called a *spread direction* in Brouwer 1954.
Observe that for every $s$ in $T$, $R(s) = s$. One might call $R$ a *retraction* of $\mathbb{N}^*$ onto $T$.

## 10.5

It has been asked if the principle that we obtain by removing the condition of decidability from the Principle of Induction on Monotone Bars in Decidable Frames, is acceptable as an axiom of intuitionistic analysis.

*Principle of Induction on Monotone Bars in Frames*
Let $T \subseteq \mathbb{N}^*$ be a frame.
For every subset $P$ of $T$,
if $P$ is a bar in $[T]$, and monotone in $T$ and hereditary in $T$, then the empty sequence $\langle\,\rangle$ belongs to $P$.

Before going into this question we first show that this principle entails the principle of Open Induction.

**Theorem 10.5.1** *The Principle of Induction on Monotone Bars in Frames implies the Open Induction Principle.*

**Proof:** Let $A$ be a subset of $\mathbb{N}^*$ such that, for every $\alpha$ in $\mathcal{N}$, if every $\beta$ coming before $\alpha$ has an initial part in $A$, then $\alpha$ has an initial part in $A$. Let $\alpha$ belong to $\mathcal{N}$ and $s$ to $\mathbb{N}^*$. We define:

$\alpha$ *comes before* $s$ if and only if there exists $i < \mathrm{Dom}(s)$ such that $\alpha(i) < s(i)$ and for all $j < i$, $\alpha(j) = s(j)$.)

Let $T$ be the set of all $s$ in $\mathbb{N}^*$ such that every $\alpha$ coming before $s$ has an initial part in $A$.

Observe that $T$ is a frame, and that every $\alpha$ in $[T]$ has an initial part in $A$.

We let $P$ be the set of all $s$ in $T$ such that every $\alpha$ in $\mathcal{N}$ that has $s$ as an initial part has an initial part in $A$.

Observe that $P$ is a bar in $[T]$ and that $P$ is monotone in $T$. $P$ is also hereditary in $T$. For suppose $s$ belongs to $T$, and for every $i$, if $s * \langle i \rangle$ belongs to $T$, then $s * \langle i \rangle$ belongs to $P$. Then $s * \langle 0 \rangle$ belongs to $T$, and therefore to $P$, so $s * \langle 1 \rangle$ belongs to $T$, therefore to $P$, so $s * \langle 2 \rangle$ belongs to $T$, therefore to $P, \ldots$. We conclude that, for every $i$, $s * \langle i \rangle$ belongs to $P$, so $s$ belongs to $P$. Using the Principle of Induction on Monotone Bars in Frames we conclude that the empty sequence belongs to $P$, that is, every $\alpha$ has an initial part in $A$. □

If one is prepared to accept Brouwer's Thesis and its consequence, the Principle of Induction on Monotone Bars in Decidable Frames, what reason could one have not to accept the more general Principle of Induction in Frames?

We should reflect on the meaning of the statement:

$$P \text{ is a bar in } [T]$$

which says:

For every $\alpha$ in $\mathcal{N}$, **if,** for every $m$, $\overline{\alpha}m$ belongs to $T$,
**then** there exists $n$ such that $\overline{\alpha}n$ belongs to $P$.

This is a weak statement.

Given some sequence $\alpha$, we only have to produce a natural number $n$ such that $\overline{\alpha}n$ belongs to $P$, if we are first given a proof that for every $m$, $\overline{\alpha}m$ belongs to $T$.

We are unable to conclude from this that we have a bar in $\mathcal{N}$ itself, that is, given some $\alpha$ in $\mathcal{N}$, in general we will be unable to calculate $n$ such that, if for every $m$, $\overline{\alpha}m$ belongs to $T$, then $\overline{\alpha}n$ belongs to $P$.

(A logical scheme enabling one to conclude draw a conclusion $\exists x[A \rightarrow B]$ from a hypothesis of the form $A \rightarrow \exists x[B]$ is sometimes called an *independence-of-premiss* scheme. There is no constructive justification for such schemes).

## 10.6

In view of the hesitations expressed in the previous Section, the following result, due to Thierry Coquand, is very surprising.

**Theorem 10.6.1** (*Open Induction for Cantor space, Th. Coquand, 1997*).
*For every subset $A$ of $\{0,1\}^*$, if every $\alpha$ in $\mathcal{C}$ belongs to $A^\#$ as soon as every $\beta$ in $\mathcal{C}$ coming before $\alpha$ belongs to $A^\#$, then every $\alpha$ in $\mathcal{C}$ belongs to $A^\#$.*

**Proof:** Let $A$ be a decidable subset $A$ of $\{0,1\}^*$ such that every $\alpha$ in $\mathcal{C}$ belongs to $A^\#$ as soon as every $\beta$ in $\mathcal{C}$ coming before $\alpha$ belongs to $A^\#$.
Let $a = \langle a(0), \ldots, a(n-1)\rangle$ belong to $\{0,1\}^*$.
We call $a$ *$A$-safe* if and only if every $\beta < a$ belongs to $A^\#$.
Observe that the set of all $\beta$ in $\mathcal{C}$ such that $\beta < a$ coincides with a fan.
Therefore, $a$ is $A$-safe if and only if there exists $n$ in $\mathbb{N}$ such that for every $\beta < a$ there exists $m \leq n$ such that $\overline{\beta}m$ belongs to $A$.
As a consequence, the set of all $A$-safe members of $\{0,1\}^*$ is enumerable. We let $f$ be a function from $\mathbb{N}$ to $\{0,1\}^*$ enumerating this set.
We now build a function $F$ from $\mathbb{N}^*$ to $\{0,1\}^*$ as follows:

(i) $F(\langle\,\rangle) = \langle\,\rangle$.
(ii) For all $a$ in $\mathbb{N}^*$, for all $i$ in $\mathbb{N}$,
if $f(i) \neq F(a) * \langle 1\rangle$, then $f(a * \langle i\rangle) = F(a) * \langle 0\rangle$, and
if $f(i) = F(a) * \langle 1\rangle$, then $f(a * \langle i\rangle) = F(a) * \langle 1\rangle$.

One easily verifies that for each $a$ in $\mathbb{N}^*$, $F(a)$ is an $a$-safe member of $\{0,1\}^*$.
We now consider the function $G$ from $\mathcal{N}$ to $\mathcal{C}$ that is defined by: for all $\alpha$ in $\mathcal{N}$, $n$ in $\mathbb{N}$, $\overline{G(\alpha)}n = F(\overline{\alpha}n)$.
Observe that for every $\alpha$ in $\mathcal{N}$, every $\beta$ in $\mathcal{C}$ coming before $F(\alpha)$ has an initial part in $A$, therefore $F(\alpha)$ itself has an initial part in $A$.
We now define a subset $P$ of $\mathbb{N}^*$ as follows:
For all $a$ in $\mathbb{N}^*$, $a$ belongs to $P$ if and only if for all $\alpha$ in $\mathcal{C}$, if $F(a)$ is an initial part of $\alpha$, then $\alpha$ belongs to $A^\#$.
Observe that $P$ is a bar in $\mathcal{N}$.
Observe that $P$ is a monotone subset of $\mathbb{N}^*$.
Observe finally that $P$ is a hereditary subset of $\mathbb{N}^*$.
Assume that $a$ belongs to $\mathbb{N}^*$, and that for every $n$, $a * \langle n\rangle$ belongs to $P$, that is, for every $\alpha$ in $\mathcal{C}$, if $F(a * \langle n\rangle)$ is an initial part of $\alpha$, then $\alpha$ belongs to $A^\#$.
Therefore, every $\alpha$ in $\mathcal{C}$ such that $F(a) * \langle 0\rangle$ is an initial part of $\alpha$ belongs to $A^\#$.
Also, every $\alpha$ in $\mathcal{C}$ coming before $F(a)$ belongs to $A^\#$, as $F(a)$ is $A$-safe.
We conclude that every $\alpha$ coming before $F(a) * \langle 1\rangle$ belongs to $A^\#$, therefore $F(a) * \langle 1\rangle$ is $A$-safe. We now calculate $i$ such that $f(i) = F(a) * \langle 1\rangle$ and observe: $F(a * \langle i\rangle) = F(a) * \langle 1\rangle$, and $a * \langle i\rangle$ belongs to $P$, therefore every $\alpha$ such that $F(a) * \langle 1\rangle$ is an initial part of $\alpha$ belongs to $A^\#$.
We conclude that every $\alpha$ such that $F(a)$ is an initial part of $\alpha$ belongs to $A^*$, that is, $a$ belongs to $P$.
Using the principle of Induction on Monotone Bars we conclude that the empty sequence $\langle\,\rangle$ belongs to $P$, therefore every $\alpha$ in $\mathcal{C}$ belongs to $A^\#$. □

### 10.6.2

We now spend some thought on the question if not something like the above argument would give us the Open Induction Principle in general. The above argument hinges on the fact that the set of all $A$-safe elements of $\{0,1\}^*$ is enumerable. Now suppose $A$ is a decidable subset of $\mathbb{N}^*$ rather than $\{0,1\}^*$ and consider the set of all $s$ in $\mathbb{N}^*$ such that every $\beta$ coming before $s$ belongs to $A^\#$. This set is a co-analytical subset of $\mathcal{N}$, and such sets in general are not enumerable. In order to produce systematically all $A$-safe elements of $\mathbb{N}^*$ we would need a survey of the set of stumps. Such a survey is not possible in the following precise sense: there does not exist a continuous function from Baire space $\mathcal{N}$ to the set of all decidable subsets of $\mathbb{N}^*$ (a set that may be identified with Cantor space $\mathcal{C}$) such that its range coincides with the set of stumps (this is a consequence of the so-called Boundedness Theorem, see Veldman 200?). Therefore, Open Induction in general is still far away.

### 10.6.3

The Principle of Open Induction for Cantor space extends to a Principle of Open Induction for the real closed interval $[0,1]$ and the set $[0,\infty)$ in the following way.
Let $\mathbb{Q}$ be the set of rational numbers and let $\rho : \mathbb{N} \to \mathbb{Q}$ be some enumeration of $\mathbb{Q}$.
Let $J : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijective map, a so-called pairing function, with inverse functions $K, L$, such that for every $n$, $n = J\big(K(n), L(n)\big)$. For every natural $m$ we define: $m' := \rho\big(K(m)\big)$ and $m'' := \rho\big(L(m)\big)$. Let $\alpha$ belong to $\mathcal{N}$. $\alpha$ is called a *real number* if and only if for each $n$ $\alpha(n)' \leq \alpha(n+1)' \leq \alpha(n+1)'' \leq \alpha(n)''$ and for each $p$ there exists $n$ such that $\alpha(n)'' - \alpha(n)' \leq \frac{1}{p}$.
Let $\alpha, \beta$ be real numbers. $\alpha$ *really coincides with* $\beta$ if and only if for each $n$, $\alpha(n)' \leq \beta(n)''$ and $\beta(n)' \leq \alpha(n)''$. We denote the set of real numbers by $\mathbb{R}$. Let $X, Y$ be subsets of $\mathbb{R}$. We say that $X$ *really coincides* with $Y$ if and only if every member of $X$ really coincides with a member of $Y$ and every member of $Y$ really coincides with a member of $X$.
$[0,1]$ is the set of all real numbers $\alpha$ such that for each $n$, $0 \leq \alpha(n)''$ and $\alpha(n)' \leq 1$.
We now consider the set $\{0,1,2\}^*$ of all finite sequences of $0,1,2$. We construct a mapping $H$ from $\{0,1,2\}^*$ to $\mathbb{N}$ as follows. We let $H(\langle\,\rangle)$ be the natural number $m$ such that $m' = 0$ and $m'' = 1$. Assume that $s$ belongs to $\{0,1,2\}^*$ and that we defined $H(a)$. For each $i < 3$ we now define $H(a * \langle i\rangle)$, in such a way that
$H(s*\langle 0\rangle)' = H(s)'$, $H(s*\langle 1\rangle)' = H(s)' + \frac{1}{4}(H(s)'' - H(s)'')$ and
$H(s*\langle 2\rangle)' = H(s)' * +\frac{1}{2}(H(s)'' - H(s)')$ and for each $i < 3$
$H(s*\langle i\rangle)'' - H(s*\langle i\rangle)' = \frac{1}{2}(H(s)'' - H(s)')$.
We define a mapping $h$ from the fan $\{0,1,2\}^{\mathbb{N}}$ to $[0,1]$ as follows: for every $\alpha, n$, $\big(h(\alpha)\big)(n) = H(\overline{\alpha}n)$.
Observe that, for every $\alpha$ in $\{0,1,2\}^{\mathbb{N}}$, $h(\alpha)$ belongs to $[0,1]$, and that for every $\beta$ in $[0,1]$ there exists $\alpha$ in $\{0,1,2\}^{\mathbb{N}}$ such that $h(\alpha)$ really coincides with $\beta$.
Let $A$ be a subset of $\mathbb{N}$. We let $A^\flat$ be the set of all real numbers $\alpha$ such that there exists $n, m$ such that $m$ belongs to $A$ and $m' < \alpha(n)' \leq \alpha(n)'' < m''$. We might call $A^\flat$ the (real) open set determined by $A$.

Let $\alpha, \beta$ be real numbers. We say that $\alpha$ is *really smaller* than $\beta$, notation $\alpha < \beta$, if and only if there exists $n$ such that $\alpha(n)'' < \beta(n)'$. Let $B$ be a subset of $[0,1]$, or $[0,\infty)$, respectively.
We say that $B$ is *progressive* if and only if every $\alpha$ in $[0,1]$,(or $[0,\infty)$, respectively) belongs to $B$ as soon as every $\beta$, $\beta < \alpha$ belongs to $B$.
We let $[0,\infty)$ be the set of all real numbers $\alpha$ such that for each $n$, $0 \leq \alpha(n)''$.

**Theorem 10.6.3.1** (*Open Induction for* $[0,1]$ *and* $[0,\infty)$*, Th. Coquand, 1997*)

(i) *For every decidable subset $A$ of $\mathbb{N}$,*
*if $A^\flat$ is a progressive subset of $[0,1]$, then $A^\flat$ really coincides with $[0,1]$.*
(ii) *For every decidable subset $A$ of $\mathbb{N}$,*
*if $A^\flat$ is a progressive subset of $[0,\infty)$ then $A^\flat$ really coincides with $[0,\infty)$.*

**Proof:**

(i) Let $A$ be a decidable subset of $\mathbb{N}$ such that $A^\flat$ is a progressive subset of $[0,1]$. We let $B$ be the set of all finite sequences $s$ in $\{0,1,2\}^*$ such that there exists $m \leq \text{length}(s)$, $m$ in $A$ and $m' < H(s) < H(s)'' < m''$.
Observe that $B$ is a decidable subset of $\{0,1,2\}^*$ and that $B^\#$ consists of all $\alpha$ in $[0,1]$ such that $\alpha$ belongs to $A^b$. A moment's reflection shows that every $\alpha$ in $\{0,1,2\}^{\mathbb{N}}$ belongs to $B^\#$ as soon as every $\beta$ in $\{0,1,2\}^{\mathbb{N}}$ coming before $\alpha$ belongs to $B^\#$, therefore, by an obvious extension of Theorem 10.6.1, $B^\#$ coincides with $\{0,1,2\}^{\mathbb{N}}$, and $A^\flat$ really coincides with $[0,1]$.
(ii) Let $A$ be a decidable subset of $\mathbb{N}$ such that $A^\flat$ is a progressive subset of $[0,\infty)$. One proves, by induction, using (i), that for each $n$, $A^\flat$ is a progressive subset of $[n, n+1)$, and every member of $[n, n+1)$ belongs to $A^\flat$. Therefore, $A^\flat$ really coincides with $[0,\infty)$.

□

# 11 Concluding remarks

This paper was elicited by a purported intuitionistic proof of Kruskal's Theorem given by Thierry Coquand. He used the principle of Open Induction explained in Section 10. I felt dissatisfied with this proof as it exceeds the bounds of intuitionistic analysis as formalized in Kleene and Vesley 1965. I had the impression that the original proofs given by Higman and Kruskal were more constructive notwithstanding the fact that these authors freely use classical logic.
I wrote this paper in order to verify this impression in detail. I discussed these matters with Thierry Coquand when visiting him in Göteborg in February 1997. He then discovered the two special cases of principle of Open Induction mentioned and proved in Section 10. Some years before we had exchanged views on possible intuitionistic versions of Ramsey's Theorem, see Veldman and Bezem 1993, and Coquand 1994. Kruskal's Theorem is of course a Ramseyan Theorem, so it was natural that we should study its constructive content.
J.H. Gallier, in his survey paper Gallier 1991 mentions the finding of a constructive

proof of Kruskal's Theorem as a major problem. Various people were searching constructive proofs of Ramseyan theorems, see for instance Murthy and Russell, and Richman and Stolzenberg 1993.
In the latter paper the Finite Sequence Theorem is proved for decidable relations on $\mathbb{N}$.

Kruskal's Theorem plays an important role in proof theory. There are deep connections with proof theoretic ordinals and the project of Reverse Mathematics initiated by H. Friedman, see Simpson 1985. It seems that ordinals made their entry in the discussions about Kruskal's Theorem in Schmidt 1979. Monica Seisenberger succeeded in reconstructing a constructive proof, avoiding ordinals, from the ordinal-theoretic proof in Rathjen and Weiermann 1993, see Seisenberger 2000. She restricts herself to the case of decidable relations on $\mathbb{N}$. Another difference with the present paper is that she avoids Brouwer's Thesis, thereby following a line recommended by P. Martin-Löf, see Martin - Löf 1970.
Rather than invoking Brouwer's Thesis one might *define* a relation $R$ to be *almost full* or *unavoidable* if and only if there exists a stump $\sigma$ such that every finite sequence of natural numbers not belonging to $\sigma$ meets $R$. This of course is a difference in style mainly, the problem of how to prove Kruskal's Theorem remains the same.
The question if the generalized principle of induction on monotone bars mentioned in Section 10 is intuitionistically acceptable was raised already by G. Kreisel in Kreisel 1963.
Such an extension would enable one to give an intuitionistic consistency proof for classical analysis. H. Luckhardt defended the extension as a natural one in Luckhardt 1973.
The extension is also discussed by A.S. Troelstra in Troelstra 1980. He carefully distinguishes between various possible formulations of the extension.
It seems that Friedman's extension of Kruskal's Theorem, mentioned in Section 9 came to be thought of in connection with the large project of proving the Graph Minor Theorem, see Robertson and Seymour 1990.
The special case of this extension discussed in Schütte and Simpson 1985 is provable intuitionistically as well as classically.

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