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UNDERSTANDING AND USING BROUWER'S CONTINUITY PRINCIPLE

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In memory of Johan J. de Iongh (1915-1999)

Trying to learn to use words, and every attempt
Is a wholly new start, and a different kind of failure

T.S. Eliot, East Coker, V, 1940

Abstract

Brouwer's Continuity Principle distinguishes intuitionistic mathematics from other varieties of constructive mathematics, giving it its own flavour. We discuss the plausibility of this assumption and show how it is used. We explain how one may understand its consequences even if one hesitates to accept it as an axiom.

1 Brouwer's Continuity Principle

We let $\mathbb{N}$ be the set of all natural numbers. Its elements $0, 1, 2, \ldots$ are produced one by one. $\mathbb{N}$ is a never finished project that is executed step-by-step.
We let $\mathcal{N}$ be the set of all infinite sequences of natural numbers.
The acceptance of $\mathcal{N}$ as a totality has been a major step in the history of mathematical thinking, and led to the development of set theory. With Cantor's diagonal argument in mind, Brouwer probed the meaning of the words: "every *possible* infinite sequence of natural numbers" and found a way to sensibly use them.
An element α of $\mathcal{N}$ is a function from $\mathbb{N}$ to $\mathbb{N}$, $\alpha = \alpha(0), \alpha(1), \alpha(2), \ldots$ Every such element is produced step-by-step, and like the set $\mathbb{N}$ itself, it is a never finished project. The production process may consist in the evaluation of an algorithm, like:

The sequence $\mathbf{0}$ with the constant value 0:
$\mathbf{0}(0) = 0$, $\mathbf{0}(1) = 0$, $\mathbf{0}(2) = 0, \ldots$

or:

d, the decimal expansion of π:
$d(0) = 1$, $d(1) = 4$, $d(2) = 1, \ldots$

or:

example$_\mathbf{0}$:
for each n, **example$_\mathbf{0}$**$(n) = 1$ if there is no $i < n$ such that for each $j < 99$, $d(i+j) = 9$ but for each $j < 99$, $d(n+j) = 9$, and **example$_\mathbf{0}$**$(n) = 0$ otherwise.

Brouwer suggested also the following non-algorithmic project,

example$_\mathbf{1}$:
for each n, **example$_\mathbf{1}$**$(n) = 0$ as long as we do not find a proof that for each j, **example$_\mathbf{0}$**$(j) = 0$, and **example$_\mathbf{1}$**$(n) = 1$ otherwise.

example$_\mathbf{1}$ is unusual and strange; it depends on one's future mathematical experience and requires a subdivision of (part of) the mathematical future into stages $0, 1, 2, \ldots$. Even if one doubts that this could be done in a sensible way, the example may make clear that we are prepared to recognize all kinds of projects for constructing step-by-step an infinite sequence of natural numbers.

We even give up the idea that there must be a "rule" or a "secret" guiding the development of such a sequence. It suffices that we fulfil our intention to construct a next value every time, that is, we may create a sequence by freely choosing its successive values. The result of such a free step-by-step construction may be a disappointingly regular sequence like the sequence $\mathbf{0}$. There is no obligation to demonstrate one's freedom by avoiding every repetitive pattern of behaviour. We believe that every infinite sequence may be imagined to be the result of a free step-by-step construction.

We now discuss the Continuity Principle.

Let $R \subseteq \mathcal{N} \times \mathbb{N}$ be a relation between infinite sequences of natural numbers and natural numbers. If the pair $\langle \alpha, m \rangle$ belongs to R, we write $\alpha R m$ and say: "m is suitable for α".

Suppose we feel entitled to claim the following:

For every α in $\mathcal{N}$ there exists m in $\mathbb{N}$ such that $\alpha R m$.

This is a strong claim for the following two reasons:

(i) We take seriously the expression: "for every α in $\mathcal{N}$".
We in no way want to delimit the range of this quantifier and in particular allow infinite sequences that grow step-by-step by free choices.

(ii) We take seriously the expression: "there exists m in $\mathbb{N}$".
Given any α we must be able to construct and produce a natural number m suitable for α, and a natural number is a finite object.

Strong claims need strong evidence and thus may be seen to have strong implications. We argue as follows: if α is coming into existence step-by-step and we calculate a number m suitable for α, the construction of this number m will be completed when only finitely many values of α have been decided upon. The number m will be suitable not only for α itself but for every infinite sequence β that has these first finitely many values the same as α.

Let us summarize our conclusion, (we use $m, n, \ldots$ as variables over the set $\mathbb{N}$ of natural numbers):

1.1 *Brouwer's Continuity Principle:*
For every $R \subseteq \mathcal{N} \times \mathbb{N}$,
if $\forall \alpha \in \mathcal{N}\ \exists m\ [\alpha R m]$,
then $\forall \alpha \in \mathcal{N}\ \exists n\ \exists m\ \forall \beta \in \mathcal{N}$ [if for every $i < n$, $\alpha(i) = \beta(i)$, then $\beta R m$].

This axiom is called *Brouwer's principle for numbers* in Kleene and Vesley 1965 and $WC - \mathbb{N}$ in Troelstra and van Dalen 1988.

The Continuity Principle is a natural axiom, borne out by experience. We never were in a situation in which we found reason to affirm the premiss but hesitated to uphold the conclusion.
Brouwer, when using it, considers it as evident and after having mentioned the possibility of creating an infinite sequence by successive choices, offers no further apology.
In Heyting 1952/53, some explication is given, not very different from the one we sketched.
Kleene and Vesley 1965 prove the consistency of their formal system of intuitionistic analysis including the continuity principle by making use of a non-intended realizability interpretation.
The Continuity Principle changes the landscape of mathematics.
Brouwer is offering us a pair of spectacles and promising new vistas. We should not only study the spectacles but also put them on and describe what we see.

2 The continuity of real functions

The continuity of real functions is an immediate consequence of Brouwer's Continuity Principle, as we showed in Veldman 1982. We now repeat this argument.
We first formulate a generalization of the principle that easily follows from the principle itself.

2.1 Let X be a subset of $\mathcal{N}$. X will be called a *spread* if and only if the following two conditions are satisfied:
(i) For every finite sequence $s = \langle s(0), \ldots, s(n-1)\rangle$ of natural numbers one may decide if there exists α in X such that for each $i < n$, $\alpha(i) = s(i)$.
(ii) For every α in $\mathcal{N}$, if for each n in $\mathbb{N}$ there exists β in X such that for each $i < n$, $\alpha(i) = \beta(i)$, then α itself belongs to X.
A spread is a closed subset of Baire space $\mathcal{N}$ that satisfies the classically empty condition (i).

2.2 *Brouwer's Continuity Principle, general formulation:*
For every spread $X \subseteq \mathcal{N}$, for every $R \subseteq X \times \mathbb{N}$,
if $\forall \alpha \in X\ \exists m\ [\alpha R m]$,
then $\forall \alpha \in X\ \exists n\ \exists m\ \forall \beta \in X$ [if for every $i < n$, $\alpha(i) = \beta(i)$, then $\beta R m$].

One may prove 2.2 from 1.1 by defining a so-called retraction of $\mathcal{N}$ onto X, that is, a continuous function r from $\mathcal{N}$ onto X such that for every α in X, $r(\alpha) = \alpha$, and then arguing straightforwardly.

We have to say something on real numbers.

2.3 Let $q_0, q_1, \ldots$ be an enumeration of the set $\mathbb{Q}$ of the rational numbers. For the purposes of this paper, an element α of $\mathcal{N}$ will be called a *canonical real number* if and only if, for each n, $|q_{\alpha(n)} - q_{\alpha(n+1)}| < \frac{1}{2^{n+1}}$. We let $\mathbb{R}$ be the set of all canonical real numbers. Observe that $\mathbb{R}$ is a spread.

2.4 We now define the binary relation of real coincidence on the set $\mathbb{R}$.
Let α, β be canonical real numbers. We say that α *really coincides with* β, notation $\alpha =_{\mathbb{R}} \beta$, if and only if, for each n, $|\alpha(n) - \beta(n)| < \frac{1}{2^{n-1}}$. We don't go into the definition of the usual operations on the set $\mathbb{R}$. We want to make use of the following observation:

> For all canonical real numbers α, β, for every natural number n if $|\alpha - \beta| < \frac{1}{2^{n-1}}$, then there exists a canonical real number γ such that for each $i < n$, $\alpha(i) = \gamma(i)$ and $\gamma =_{\mathbb{R}} \beta$.

2.5 A *real function* f is an effective method that associates to every canonical real number α a canonical real number $f(\alpha)$ in such a way that, for all α, β in $\mathbb{R}$, if $\alpha =_{\mathbb{R}} \beta$, then $f(\alpha) =_{\mathbb{R}} f(\beta)$.

2.6 Theorem: (*Continuity of real functions*)
Let f be a real function.
Then $\forall \alpha \in \mathbb{R}\, \forall m\, \exists n\, \forall \beta \in \mathbb{R}$ [if $|\alpha - \beta| < \frac{1}{2^n}$, then $|f(\alpha) - f(\beta)| < \frac{1}{2^m}$].

Proof: Let f be a real function and α a canonical real number, and m a natural number.
Applying 2.2, we calculate n, such that for every γ in $\mathbb{R}$, if for each $i < n$, $\alpha(i) = \gamma(i)$, then $\big(f(\alpha)\big)(m+2) = \big(f(\gamma)\big)(m+2)$, therefore $|f(\alpha) - f(\gamma)| < \frac{1}{2^m}$.
Observe that, for every canonical real number β, if $|\alpha - \beta| < \frac{1}{2^n}$, then there exists a canonical real number γ such that for every $i < n$, $\alpha(i) = \gamma(i)$, and $\beta =_{\mathbb{R}} \gamma$, therefore $f(\beta) =_{\mathbb{R}} f(\gamma)$, and $|f(\alpha) - f(\beta)| < \frac{1}{2^m}$.
⊠

2.7 It is possible to generalize the result of Theorem 2.6.
Let D be a subset of $\mathbb{R}$. D will be called a *domain of continuity* if every real function from D to $\mathbb{R}$ is continuous, that is, for every function f from D to $\mathbb{R}$, if for all α, β in D such that $\alpha =_{\mathbb{R}} \beta$, also $f(\alpha) =_{\mathbb{R}} f(\beta)$, then $\forall \alpha \in D\, \forall m\, \exists n\, \forall \beta \in D$ [if $|\alpha - \beta| < \frac{1}{2^n}$, then $|f(\alpha) - f(\beta)| < \frac{1}{2^m}$].

2.8 We let Perhaps ($\mathbb{Q}$) be the set of all canonical real numbers α that change their value at most one time, that is, for all i, k, if $i < k$ and $\alpha(i) \neq \alpha(i+1)$, then $\alpha(i+1) = \alpha(k)$.

2.9 Let $b_0, b_1, \ldots$ be an enumeration of the set $\mathbb{Q}_2$ of all binary rational numbers. We let Bin be the set of all canonical real numbers α such that for all i, k, if $b_i \leq \alpha(i)$, then $b_i \leq \alpha(i+k)$, and if $\alpha(i) \leq b(i)$, then $\alpha(i+k) \leq b(i)$.
Observe that every α in Bin has a binary development. Conversely, if α in $\mathbb{R}$ has a binary development, then there exists γ in Bin such that $\alpha =_{\mathbb{R}} \gamma$.

2.10 Theorem:
(i) Perhaps ($\mathbb{Q}$) is a domain of continuity.
(ii) Bin is a domain of continuity.

Proof: Observe that both Perhaps ($\mathbb{Q}$) and Bin are spreads and repeat the argument by which we established Theorem 2.6.
⊠

3 Strong counter-examples

Brouwer's attack upon classical logic started with the observation that upon his interpretation of the logical connectives we have no reason to affirm the principle of excluded third, $P \vee \neg P$.
For instance, we have no reason to affirm either: $\mathbf{example_0} = \mathbf{0}$ or: $\neg$ ($\mathbf{example_0} = \mathbf{0}$), therefore we have no reason to affirm: $\mathbf{example_0} = \mathbf{0} \vee \neg(\mathbf{example_0} = \mathbf{0})$.
This does not mean that the assumption $\mathbf{example_0} = \mathbf{0} \vee \neg(\mathbf{example_0} = \mathbf{0})$ leads to a contradiction; on the contrary, one may prove, in intuitionistic logic, for every proposition P, $\neg\neg(P \vee \neg P)$.
The Continuity Principle brings home to us that the assumption that we could decide, for *every* α in $\mathcal{N}$, if $\alpha = \mathbf{0}$ or $\neg(\alpha = \mathbf{0})$, is contradictory indeed.

3.1 Theorem: (*Absurdity of the Principle of Excluded Third*)
$\neg\forall\alpha \in \mathcal{N}\ [\alpha = \mathbf{0} \vee \neg(\alpha = \mathbf{0})]$.

Proof: Assume $\forall\alpha \in \mathcal{N}\ [\alpha = \mathbf{0} \vee \neq (\alpha = \mathbf{0})]$.
Applying the Continuity Principle we find n such that
either for every α in $\mathcal{N}$, if for every $i < n$, $\alpha(i) = 0$, then $\alpha = \mathbf{0}$,
or for every α in $\mathcal{N}$, if for every $i < n$, $\alpha(i) = 0$, then $\neg(\alpha = \mathbf{0})$.
Both alternatives are absurd.
⊠

This technique can be applied more generally. Once we have found an example that shows that we may be unable to take a decision of some given kind, it very often will be possible to derive a contradiction from the assumption that we should have a general method to take that kind of decision. The following Theorem offers a second example.

3.2 Theorem: (*Negative Continuity Theorem*)
There is no real function f such that, for each positive n, $f\left(\frac{1}{2^n}\right) = 0$ and $f(0) = 1$.

Proof: Assume there exists such a function.
Let $t : \mathbb{N} \to \mathbb{N}$ be such that for each positive n, $q_{t(n)} = \frac{1}{2^n}$. We now define a real number β as follows:
for each n, if for each $i \leq n$, $\mathbf{example_0}(i) = 0$, then $\beta(n) = t(n)$, and if there exists $i < n$ such that $\mathbf{example_0}(i) \neq 0$, then $\beta(n) = t(i_0)$ where i_0 is the least i such that $\mathbf{example_0}(i) \neq 0$.
Observe that, if $\mathbf{example_0} = \mathbf{0}$, then $\beta =_{\mathbb{R}} 0$ and $f(\beta) = 1$ and if $\mathbf{example_0} \neq \mathbf{0}$, then $f(\beta) = 0$.
So we are unable to calculate $f(\beta)$, even approximately, as we are unable to decide $f(\beta) > 0$ or $f(\beta) < 1$.

Generalizing this example, we may define, for every α in $\mathcal{N}$, a suitable canonical real number β with the property: if $\alpha = \mathbf{0}$, then $f(\beta) = 1$ and if $\alpha \neq \mathbf{0}$, then $f(\beta) = 0$. We may decide either $f(\beta) > 0$ or $f(\beta) < 1$, therefore either $\alpha = \mathbf{0}$ or $\alpha \neq \mathbf{0}$.

We obtain a contradiction by Theorem 3.1.
⊠

3.3 We have included Theorem 3.2 for historical reasons.
In the first Section of Brouwer 1927, Brouwer proves the Negative Continuity Theorem, but he proves it in the weak sense only: there cannot be such a function because if we had one, then we could solve some problem for which we do not have a solution. Brouwer fails to distinguish carefully between this weak negation and the stronger one, which says that the negated proposition would lead to a contradiction. If one wants to prove the Negative Continuity Theorem in this strong sense, one needs the Continuity Principle.
But why should one prove the Negative Continuity Theorem rather than the Continuity Theorem 2.6 itself, if one is prepared to use the Continuity Principle? (The Negative Continuity Theorem of course easily follows from Theorem 2.6.)
Brouwer probably felt doubtful about the argument we used for proving Theorem 2.6. He does not state a Continuity Theorem and, in Brouwer 1927, goes on to prove, by a far more complicated argument, that every real function from the closed interval $[0, 1]$ to $\mathbb{R}$ must be uniformly continuous.
It is remarkable that something like our argument for Theorem 2.6 is used in Heyting 1952/53, but Charles Parsons, writing, in 1967, an introductory note to Brouwer 1927, explicitly but unconvincingly disputes the correctness of this reasoning.

4 Brouwer's first application

Brouwer 1918 contains a formulation of the Continuity Principle together with the following application:

4.1 Theorem:
There is no injective function from $\mathcal{N}$ to $\mathbb{N}$.

Proof: Let f be a function from $\mathcal{N}$ to $\mathbb{N}$.
Determine n such that for each α in $\mathcal{N}$, if for each $i < n$, $\alpha(i) = 0$, then $f(\alpha) = f(\mathbf{0})$.

Obviously, f is not injective.
$\boxtimes$

4.2 Cantor's argument is valid intuitionistically and does not depend on the Continuity Principle:

> For every function $f : \mathbb{N} \to \mathcal{N}$ there exists α in $\mathcal{N}$ such that for each n, $f(n) \neq \alpha$. Define α as follows: for each n, $\alpha(n) := \big(f(n)\big)(n) + 1$.

This argument does not prove Theorem 4.1.

4.3 Brouwer 1918 introduces the following notion:

Let X, Y be sets. We define: X is *smaller than* Y, or: Y is *greater than* X, notation $X \sqsubset Y$, if and only there exists an injective function from X to Y and there does not exist an injective function from Y to X.

Brouwer observes that $\sqsubset$ is a transitive relation between sets and that $\mathbb{N}$ is smaller than $\mathcal{N}$.
There is a slight ambiguity in the notion "smaller than", as we may take the notion of an injective function in (at least) two senses, whenever a constructive inequality or apartness relation is present.
For the theorems we want to prove in this paper it does not make any difference which interpretation we choose.

4.4 We want to show that Theorem 4.1 admits of a vast extension.
We let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers. We let $*$ denote the binary operation of concatenation of finite sequences and also the operation of concatenating a finite and an infinite sequence of natural numbers.
Let T be the set of all α in $\mathcal{N}$ that assume no other values than $0, 1$, and assume the value 1 at most one time.
Here is an infinite sequence of elements of T:

$$\mathbf{0},\ \langle 1 \rangle * \mathbf{0},\ \langle 0, 1 \rangle * \mathbf{0},\ \langle 0, 0, 1 \rangle * \mathbf{0},\ \ldots$$

This sequence is an injective function from $\mathbb{N}$ into T.
Remark that $\mathbf{example_0}$ belongs to T but we do not know where it occurs in the above sequence.
Observe that T is a spread and use the argument from the proof of Theorem 4.1 in order to see that there is no injective function from T into $\mathbb{N}$. (In particular, the above sequence is not an enumeration of T.) So $\mathbb{N}$ is smaller than T.

4.5 For every s in $\mathbb{N}^*$, every subset X of $\mathcal{N}$ we define: $s * X := \{s * \alpha \mid \alpha \in X\}$. For all subsets X, Y of $\mathcal{N}$ we define: $X \oplus Y := \langle 0 \rangle * X \cup \langle 1 \rangle * Y$. For every subset X of $\mathcal{N}$ we define a sequence $0 \cdot X,\ 1 \cdot X,\ 2 \cdot X, \ldots$ of subsets of $\mathcal{N}$, as follows: $0 \cdot X := \emptyset$ and, for each n, $(n + 1) \cdot X := (n \cdot X) \oplus X$. For each α in $\mathcal{N}$, n in $\mathbb{N}$, we define: $\overline{\alpha}(n) := \langle \alpha(0), \ldots \alpha(n-1) \rangle$. If confusion is unlikely to arise, we write $\overline{\alpha}n$ rather than $\overline{\alpha}(n)$.

4.6 Theorem:
For each n, $n \cdot T \sqsubset (n+1) \cdot T$.

Proof: We first show: $T \sqsubset T \oplus T$. Observe that T is a subset of $T \oplus T$.
Assume now that f is an injective function from $T \oplus T$ into T. Observe that $f(\mathbf{0}) = \mathbf{0}$. For assume, for some m, $\big(f(\mathbf{0})\big)(m) = 1$. Calculate n such that for all α in T, if for all $i < n$, $\alpha(i) = 0$, then $\big(f(\alpha)\big)(m) = \big(f(\mathbf{0})\big)(m)$, and $f(\alpha) = f(\mathbf{0})$. So f is not injective. For similar reasons, $f(\langle 1 \rangle * \mathbf{0}) = \mathbf{0}$. So f is not injective.
Observe that, for each n, $n \cdot T$ is a subset of $(n+1) \cdot T$. Assume now that for some n, f is an injective function from $(n+1) \cdot T$ into $n \cdot T$. Observe, by repeating the above argument, that f has to map each one of the sequences $\langle 1 \rangle * \mathbf{0}, \ldots, \overline{\mathbf{0}}n * \langle 1 \rangle * \mathbf{0}$ onto one of the sequences $\langle 1 \rangle * \mathbf{0}, \ldots, \overline{\mathbf{0}}(n-1) * \langle 1 \rangle * \mathbf{0}$, and so cannot be injective.
$\boxtimes$

4.7 For every subset X of $\mathcal{N}$, we let $\overline{X}$ be the set of all α in $\mathcal{N}$ such that for each n in $\mathbb{N}$ there exists β in X such that $\overline{\beta}n = \overline{\alpha}n$.
$\overline{X}$ is called the *closure of* X.
$\overline{X}$ will be a spread provided we are able to decide for each s in $\mathbb{N}^*$, if there exists α in X such that, for some n, $\overline{\alpha}n = s$.
We now define a sequence $T_0, T_1, T_2, \ldots$ of subsets of $\mathcal{N}$, as follows:
$T_0 := \{\mathbf{0}\}$ and for each m, $T_{m+1} := \overline{\bigcup_{n \in \mathbb{N}} \overline{\mathbf{0}}n * \langle 1 \rangle * T_m}$
Observe that each T_m is a spread and that T_1 coincides with the set T introduced in Section 4.4. For each m, T_m is the set of all α in $\mathcal{N}$ that assume no other values than $0, 1$ and that assume the value 1 at most m times. For each n, m, $n \cdot T_m$ is a subset of T_{m+1}.

4.8 Theorem:
For each n, m, $\quad n \cdot T_m \sqsubset (n+1) \cdot T_m \sqsubset T_{m+1}$.

Proof: We show that T_2 is smaller than $T_2 \oplus T_2$ and leave the rest of the proof to the reader.
Observe that T_2 is a subset of $T_2 \oplus T_2$, and that, for each n, $n \cdot T_1$ is a subset of T_2, so, in view of Theorem 4.6, $n \cdot T_1 \sqsubset T_2$.
Assume that f is an injective function from $T_2 \oplus T_2$ into T_2. Observe that $f(\mathbf{0}) = \mathbf{0}$. For assume, for some m, $\big(f(\mathbf{0})\big)(m) = 1$. Calculate n such that for each α in T_2, if for all $i < n$, $\alpha(i) = 0$ then $\overline{\big(f(\alpha)\big)}(m+1) = \overline{\big(f(\mathbf{0})\big)}(m+1)$. Now define a mapping g from T_2 into $\mathcal{N}$ as follows: for each β in T_2, $f(\beta)$ is the sequence γ such that $f(\overline{\mathbf{0}}n * \beta) = \overline{f(\mathbf{0})}(m+1) * \gamma$. Observe that g embeds T_2 into T_1. Contradiction.
For similar reasons $f(\langle 1 \rangle * \mathbf{0}) = \mathbf{0}$. So f is not injective.
$\boxtimes$

4.9 Theorem 4.8 admits of a further refinement.
Consider the countable ordinal $\omega^{\cdot\omega}$. It consists of all polynomials $\omega^{\cdot n_0} \cdot p_0 + \omega^{\cdot n_1} \cdot p_1 + \cdots + \omega^{\cdot n_{k-1}} \cdot p_{k-1}$ where $\langle n_0, n_1, \ldots, n_{k-1} \rangle$ is a strictly decreasing sequence of natural numbers and $\langle p_0, p_1, \ldots, p_{k-1} \rangle$ is a finite sequence of natural numbers of the

same length k.
We may associate a subset of $\mathcal{N}$ to any such polynomial by defining: $\mathbb{T}(\omega^{\cdot n_0} \cdot p_0 + \omega^{\cdot n_1} \cdot p_1 + \cdots + \omega^{\cdot n_{k-1}} \cdot p_{k-1}) := p_0 \cdot T_{n_0} \oplus p_1 \cdot T_{n_1} \oplus \ldots \oplus p_{k-1} \cdot T_{n_{k-1}}$. Let $<$ denote the usual ordering on $\omega^{\cdot\omega}$.

4.10 Theorem:
For all α, β in $\omega^{\cdot\omega}$, if $\alpha < \beta$, then $\mathbb{T}(\alpha) \sqsubset \mathbb{T}(\beta)$.

Proof: We leave the straightforward proof to the reader.
$\boxtimes$

4.11 These considerations may of course be extended further into the transfinite. Suppose we have a sequence $U_0, U_1, \ldots$ of spreads such that for each n, $U_n \sqsubset U_{n+1}$. Define $V := \overline{\bigcup_{n\in\mathbb{N}} \overline{\mathbf{0}}(n) * \langle 1\rangle * U_n}$ and observe: for each n, $U_n \sqsubset V$ and $n \cdot V \sqsubset (n+1) \cdot V$.

5 A model-theoretic observation

5.1 We consider sentences in the first-order-language of equality.
Let X, Y be sets. We define: X ***differs elementarily from*** Y if there exists a sentence φ in the first-order-language of equality such that $\langle X\rangle \models \varphi$ and $\langle Y\rangle \models \neg\varphi$.
The sets $\mathbb{N}$ and $\mathcal{N}$ are elementarily different: consider the sentence $\forall x \forall y\, [x = y \vee \neg(x = y)]$ and recall Theorem 3.1. (When interpreting such a sentence, we use Tarski's truth definition, but of course understand connectives and quantifiers intuitionistically.)
The sets T and $\mathcal{N}$ are also elementarily different.
Observe that for all α in T, either $\alpha = \langle 1\rangle * \mathbf{0}$ or not: $\alpha = \langle 1\rangle * \mathbf{0}$ and that for every β in $\mathcal{N}$, not for all α in $\mathcal{N}$, either $\alpha = \beta$ or not: $\alpha = \beta$. So the sentence $\exists x\ \forall y\ [x = y \vee \neg(x = y)]$ is true in the structure $\langle T\rangle$ but not in the structure $\langle \mathcal{N}\rangle$.

5.2 Theorem:
For all α, β in $\omega^{\cdot\omega}$, if $\alpha < \beta$, then $\mathbb{T}(\alpha)$ differs elementarily from $\mathbb{T}(\beta)$.

Proof: We define a sequence $\mathrm{Undec}_0, \mathrm{Undec}_1, \ldots$ of formulas in the first-order-language of equality as follows:
$\mathrm{Undec}_0(x) := \neg\forall x_1\ [x = x_1 \vee \neg(x = x_1)]$, and for each n:
$\mathrm{Undec}_{n+1}(x) := \mathrm{Undec}_n(x) \wedge \neg\forall x_{n+2}\ [\mathrm{Undec}_n(x_{n+2}) \to (x = x_{n+2} \vee \neg(x = x_{n+2}))]$.
(We assume that $x, x_0, x_1, \ldots$ are different individual variables of our language.)
For any formula $\varphi = \varphi(x)$ we use $\exists! x[\varphi(x)]$ as an abbreviation of the formula $\exists x\ [\varphi(x) \wedge \forall y\ [\varphi(y) \to y = x]]$.
Observe that the sentence $\exists! x[\mathrm{Undec}_0(x)]$ is true in $\langle T_1\rangle$ but not in $\langle 2 \cdot T_1\rangle$, observe that the sentence $\exists x\ \exists y\ [x \neq y \wedge \mathrm{Undec}_0(x) \wedge \mathrm{Undec}_0(y) \wedge \forall z\ [\mathrm{Undec}_0(z) \to (z = x \vee z = y)]$ is true in $\langle 2 \cdot T_1\rangle$ but not in $\langle 3 \cdot T_1\rangle$, observe that, for each n, the set of all α in T_{n+1} satisfying the formula Undec_0 coincides with T_n, observe that for each n, the sentence $\exists x\ [\mathrm{Undec}_n(x)]$ is true in T_{n+1} but not in T_n and now complete the proof yourself.
$\boxtimes$

Some intuitionistic model theory is developed in Veldman and Jansen 1990 and Veldman and Waaldijk 1996.

6 Beginning the Borel hierarchy

6.1 Let X be a subset of the set $\mathbb{R}$ of canonical real numbers. We define: X is *basic open* if and only if there exists q in $\mathbb{Q}$ and n in $\mathbb{N}$ such that $X = \{\alpha | \alpha \in \mathbb{R}| \ |\alpha - q| < \frac{1}{2^n}\}$.
X is *open* or $\mathbf{\Sigma}^0_1$ if and only if there exists a sequence $X_0, X_1, \ldots$ of basic open sets such that $X = \bigcup_{n \in \mathbb{N}} X_n$.
X is *closed* or $\mathbf{\Pi}^0_1$ if and only if there exists an open set Y such that $X = \mathbb{R} \backslash Y$.
X is *positively Borel* if X may be obtained from open and closed sets by means of the repeated application of the operations of countable union and countable intersection.
X is $\mathbf{\Sigma}^0_2$ if and only if there exists a sequence $X_0, X_1, \ldots$ of closed sets such that $X = \bigcup\limits_{n \in \mathbb{N}} X_n$.
X is $\mathbf{\Pi}^0_2$ if and only if there exists a sequence $X_0, X_1, \ldots$ of open sets such that $X = \bigcap\limits_{n \in \mathbb{N}} X_n$.
The study of the positively Borel sets starts with the observation that neither one of the classes $\mathbf{\Pi}^0_2$, $\mathbf{\Sigma}^0_2$ includes the other.

6.2 We let **Rat** be the set of all canonical real numbers α such that there exists a real rational number q with the property: for each n, $|q_{\alpha(n)} - q| < \frac{1}{2^{n-1}}$.
We let **PosIrr** be the set of all canonical real numbers α such that for each n, $q_{\alpha(2n)} < q_{\alpha(2n+2)} < q_{\alpha(2n+3)} < q_{\alpha(2n+1)}$ and either $q_n < q_{\alpha(2n)}$ or $q_{\alpha(2n+1)} < q_n$.
Observe that for every α in **PosIrr**, every q in $\mathbb{Q}$ there exists n such that $|\alpha - q| > \frac{1}{2^n}$, that is, α is *positively irrational.*
Conversely, every positively irrational real number really coincides with an element of **PosIrr**.
Observe that **PosIrr** is a spread.
Observe that **Rat** is $\mathbf{\Sigma}^0_2$ and that the set of all positively irrational canonical real numbers is a $\mathbf{\Pi}^0_2$ subset of $\mathbb{R}$.

6.3 Theorem:
For every sequence $G_0, G_1, \ldots$ of open sets,
if $\mathbf{Rat} \subseteq \bigcap\limits_{n \in \mathbb{N}} G_n$, there exists α in **PosIrr** such that $\alpha \in \bigcap\limits_{n \in \mathbb{N}} G_n$.

Proof: Let $G_0, G_1, \ldots$ be a sequence of open sets such that $\mathbf{Rat} \subseteq \bigcap\limits_{n \in \mathbb{N}} G_n$. We construct step by step a canonical real number α satisfying the condition: for each n, $q_{\alpha(2n)} < q_{\alpha(2n+2)} < q_{\alpha(2n+3)} < q_{\alpha(2n+1)}$. Let n be a natural number and suppose we defined already
$\alpha(0), \alpha(1), \ldots, \alpha(2n-2), \alpha(2n-1)$, such that $q_{\alpha(2n-2)} < q_{\alpha(2n-1)}$.
We then define $q_{\alpha(2n)}$ and $q_{\alpha(2n+1)}$ in such a way that
$q_{\alpha(2n-2)} < q_{\alpha(2n)} < q_{\alpha(2n+1)} < q_{\alpha(2n-1)}$ *and* either $q_n < q_{\alpha(2n)}$ or $q_{\alpha(2n+1)} < q_n$ *and*

the open interval $(q_{\alpha(2n)}, q_{\alpha(2n+1)})$ is a subset of G_n *and* $q_{\alpha(2n+1)} - q_{\alpha(2n)} < \frac{1}{2^{2n}}$.
It will be clear that α is positively irrational and belongs to every G_n.
⊠

6.4 Theorem:
For every sequence $F_0, F_1, \ldots$ of closed sets,
if **PosIrr** $\subseteq \bigcup_{n \in \mathbb{N}} F_n$, there exists α in **Rat** such that $\alpha \in \bigcup_{n \in \mathbb{N}} F_n$.

Proof: Let $F_0, F_1, \ldots$ be a sequence of closed sets such that **PosIrr** $\subseteq \bigcup_{n \in \mathbb{N}} F_n$. Let α_0 be some element of **PosIrr**. We recall that **PosIrr** is a spread, and apply the Continuity Principle. Determine m, n such that for all α in **PosIrr**, if $\overline{\alpha}(2m) = \overline{\alpha}_0(2m)$, then α belongs to F_n. Now every positively irrational number from the open interval $(q_{\alpha_0(2m-2)}, q_{\alpha_0(2m-1)})$ will belong to the closed set F_n, and the closed interval $[q_{\alpha_0(2m-2)}, q_{\alpha_0(2m-1)}]$ will be a subset of F_n and F_n will contain many rational numbers.
⊠

6.5 Observe that Theorem 6.4 depends on the Continuity Principle, and that Theorem 6.3 does not but is proved by a straightforward "*Baire Category*" argument. Theorem 6.3 shows that there exist $\mathbf{\Sigma}^0_2$ sets that are not $\mathbf{\Pi}^0_2$. The Continuity Principle enables one to prove that there are easier examples.

6.6 Theorem:
The set $[0,1] \cup [1,2]$ is $\mathbf{\Sigma}^0_2$ but not $\mathbf{\Pi}^0_2$.

Proof: The set $[0,1] \cup [1,2]$ is obviously $\mathbf{\Sigma}^0_2$, so it suffices to show that it is not $\mathbf{\Pi}^0_2$. Let $G_0, G_1, \ldots$ be a sequence of open sets such that $[0,1] \cup [1,2] = \bigcap_{n \in \mathbb{N}} G_n$. Observe that for each n, $[0,2] \subseteq G_n$, therefore $[0,2] = [0,1] \cup [1,2]$. Therefore, for every α in $\mathbb{R}$, either $\alpha \leq 0$ or $0 \leq \alpha$.
Consider the element α_0 of $\mathbb{R}$ defined by: for all n, $\alpha_0(n) = \left(-\frac{1}{2}\right)^n$. Applying the Continuity Principle we determine n such that either for all α in $\mathbb{R}$, if $\overline{\alpha}_0 n = \overline{\alpha} n$, then $\alpha \leq 0$, or for all α in $\mathbb{R}$, $\overline{\alpha}_0 n = \overline{\alpha} n$, then $0 \leq \alpha$.
Both alternatives are absurd.
⊠

6.7 The problem of establishing the second level of the Borel hierarchy was considered by Brouwer around 1924, see Brouwer 1991, and Veldman 200? One of his proofs is somewhat circuitous and not wholly correct.

7 Borel hierarchy theorems

7.1 We now consider Baire space $\mathcal{N}$. A subset X of $\mathcal{N}$ is called *basic open* if and only if there exists s in $\mathbb{N}^*$ such that X is the set of all α in $\mathcal{N}$ with the property:

for some n, $\overline{\alpha}n = s$. We define *open, closed* and *positively Borel* subsets of $\mathcal{N}$ like the corresponding subsets of $\mathbb{R}$ in Section 6.1.

7.2 Let X, Y be subsets of $\mathcal{N}$. We define: X *reduces to* Y if and only if there exists a continuous function f from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, α belongs to X if and only if $f(\alpha)$ belongs to Y.
We also define: X is *strongly irreducible to* Y if and only if for every continuous function f from $\mathcal{N}$ to $\mathcal{N}$, if f maps X into Y, then f also maps some element of $\mathcal{N}\backslash X$ into Y.
We now state without proof:

7.3 Theorem: (*Intuitionistic Borel Hierarchy Theorem*)
For every positively Borel set P there exists a positively Borel set Q such that Q is strongly irreducible to P.

7.4 The Continuity Principle plays a key rôle in the proof of Theorem 7.3. We may elucidate this rôle as follows.
For very many positively Borel set P one may construct a continuous function h from $\mathcal{N}$ to $\mathcal{N}$ such that $P = \mathrm{Ran}(h)$.
Suppose we are given Borel sets P, Q and continuous functions h_0, h_1 from $\mathcal{N}$ to $\mathcal{N}$, such that $P = \mathrm{Ran}(h_0)$ and $Q = \mathrm{Ran}(h_1)$, and also a continuous function f from $\mathcal{N}$ to $\mathcal{N}$ mapping P into Q.

$$\begin{tikzcd} \mathcal{N} \arrow[r, dotted, "g"] \arrow[d, "h_0"] & \mathcal{N} \arrow[d, "h_1"] \\ \mathcal{N} \arrow[r, "f"'] & \mathcal{N} \end{tikzcd}$$

Observe that for every α there exists β such that $f\big(h_0(\alpha)\big) = h_1(\beta)$. The Continuity Principle leads one to suspect: there exists a continuous function g from $\mathcal{N}$ to $\mathcal{N}$ such that $h_1 \circ g = f \circ h_0$. There is a stronger form of the Continuity Principle that gives exactly this conclusion, but in the actual proof, our version of the Principle suffices. In any case, the fact that f maps P into Q is taken very seriously: we require strong evidence for it.
The above picture may help one to see the meaning of the Intuitionistic Borel Hierarchy Theorem, which is very different from the meaning of the Hierarchy Theorem proved by Borel and Lebesgue.

7.5 Cantor space $\mathcal{C}$ is the set of all α in $\mathcal{N}$ that assume no other values than $0, 1$. We introduce two subsets of Cantor space, **Finite** and **Almostfinite**. We studied these sets in Veldman 1995. **Finite** is the set of all α in $\mathcal{C}$ such that $\exists n\, \forall m > n\, [\alpha(m) = 0]$. **Almostfinite** is the set of all α in $\mathcal{C}$ such that for every strictly increasing sequence γ of natural numbers there exists n such $\alpha\big(\gamma(n)\big) = 0$. Observe that α belongs to **Finite** if and only if α is the characteristic function of a finite subset $\mathbb{N}$.

7.6 Theorem: (*Intuitionistic Borel Hierarchy Theorem, Second Version*)

For every positively Borel set P there exists a positively Borel set Q such that Q is strongly irreducible to P and $\mathbf{Finite} \subseteq Q \subseteq \mathbf{Almostfinite}$.

From a classical point of view, the result is surprising as, with classical logic, the sets **Finite** and **Almostfinite** coincide.
Almostfinite is a subset of **Notnotfinite**, that is, the set of all α in $\mathcal{C}$ such that $\neg\neg(\alpha \in \mathbf{Finite})$, the statement that these two sets coincide is equivalent with the (implausible) generalized Markov principle.
It follows from the Theorem that the set **Almostfinite** is not positively Borel.
The Theorem establishes the Borel hierarchy and at the same time shows in a spectacular way the expressiveness of intuitionistic logic.

7.7 Let α, β belong to Cantor space $\mathcal{C}$. We define: α *admits* β if and only if, for each n, $\alpha(\overline{\beta}n) = 0$. (We are assuming that finite sequences of natural numbers are coded by natural numbers and do not distinguish between a finite sequence of natural numbers and its code number.)
As in Veldman 1999, we let $\mathbf{Share}(T)$ be the set of all α in $\mathcal{C}$ that admit a member of T, and in this sense, *share* a member with T. Observe that $\overline{\mathbf{Share}(T)}$ is a spread, as for every α in $\mathcal{C}$, α belongs to $\overline{\mathbf{Share}(T)}$ if and only if for each n there exists γ in T such that $\alpha(\overline{\gamma}n) = 0$. Observe that for every α in $\overline{\mathbf{Share}(T)}$, if either $\exists n\, [\alpha(\overline{\mathbf{0}}n) \neq 0]$ or $\neg\exists n\, [\alpha(\overline{\mathbf{0}}n) \neq 0]$, then α belongs to $\mathbf{Share}(T)$, so in any case, $\neg\neg(\alpha \in \mathbf{Share}(T))$, therefore $\overline{\mathbf{Share}(T)}$ coincides with $\mathbf{Share}(T)^{\neg\neg}$, the double complement of $\mathbf{Share}(T)$ in C.

7.8 Theorem: (*Intuitionistic Borel Hierarchy Theorem, Third Version*)
For every positively Borel set P there exists a positively Borel set Q such that Q is irreducible to P and $\mathbf{Share}(T) \subseteq Q \subseteq \overline{\mathbf{Share}(T)}$.

Observe that we are a bit more careful than in Theorem 7.6, as we do not require that Q is strongly irreducible to P. It follows from Theorem 7.8 that the set $\mathbf{Share}(T)$ is not positively Borel. Observe that, classically, **Almostfinite** is $\mathbf{\Sigma}^0_2$ and $\mathbf{Share}(T)$ is $\mathbf{\Pi}^0_1$.
Theorems 7.6 and 7.8 are fascinating consequences of the Continuity Principle. We intend to prove them in a future paper. Theorem 7.3 is proved in Veldman 200?

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