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Sándor Baran, Gyula Pap, Martien C. A. van Zuijlen

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UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

Estimation of the mean of stationary and nonstationary Ornstein–Uhlenbeck processes and sheets

Sándor Baran and Gyula Pap
Institute of Mathematics and Informatics, University of Debrecen
Pf. 12, H–4010 Debrecen, Hungary

Martien C. A. van Zuijlen
Department of Mathematics, University of Nijmegen
Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

Abstract

We consider the problem of estimating an unknown parameter m in case one observes in an interval (rectangle) stationary and nonstationary Ornstein-Uhlenbeck processes (sheets), which are shifted by m times a known deterministic function on the interval (rectangle). It turns out that the maximum likelihood estimator (MLE) has a normal distribution and for instance in case of the sheet this MLE is a weighted linear combination of the values at the vertices, integrals on the edges and the integral on the whole rectangle of the weighted observed process. We do not use partial stochastic differential equations, we apply direct discrete time approach instead. To make the transition from the discrete time to the contiuous time, a tool is developed, which might be of independent interest.

Key words: Wiener sheet, Ornstein-Uhlenbeck sheet, maximum likelihood estimation, Radon-Nikodym derivative.

1 Introduction

The stationary Ornstein-Uhlenbeck process $\{\widetilde{X}(s) : s \in \mathbb{R}\}$ is the stationary solution of the stochastic differential equation

$$\mathrm{d}\widetilde{X}(s) = -\alpha \widetilde{X}(s)\,\mathrm{d}s + \sigma\,\mathrm{d}W(s), \tag{1.1}$$

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where $\alpha > 0$, $\sigma > 0$ and $\{W(s) : s \in \mathbb{R}\}$ is a standard Wiener process. It is a zero mean Gaussian process with

$$\mathsf{E}\widetilde{X}(s_1)\widetilde{X}(s_2) = \frac{\sigma^2}{2\alpha}\mathrm{e}^{-\alpha|s_1-s_2|}.$$

We remark that the process $\{\widetilde{X}(s) : s \geqslant 0\}$ can also be represented as

$$\widetilde{X}(s) = \mathrm{e}^{-\alpha s}\Big(\widetilde{X}(0) + \sigma\int_0^s \mathrm{e}^{\alpha u}\,\mathrm{d}W(u)\Big),$$

where $\widetilde{X}(0)$ is a zero mean normal random variable with $\mathsf{E}\widetilde{X}^2(0) = \sigma^2/(2\alpha)$, independent of $\{W(s) : s \geqslant 0\}$. Consider the process $\widetilde{Y}(s) := \widetilde{X}(s) + m$, $s \in \mathbb{R}$. Let $[S_1, S_2] \subset (-\infty, \infty)$. Denote by $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ the measures generated on $C([S_1, S_2] \to \mathbb{R})$ by the processes $\widetilde{Y}$ and $\widetilde{X}$, respectively. Then $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ are equivalent and the Radon–Nikodym derivative has the form

$$\frac{\mathrm{d}\mathsf{P}_{\widetilde{Y}}}{\mathrm{d}\mathsf{P}_{\widetilde{X}}}(\widetilde{Y}) = \exp\left\{-\frac{\alpha m^2}{2\sigma^2}(2+\alpha(S_2-S_1)) + \frac{\alpha m}{\sigma^2}\left(\widetilde{Y}(S_1) + \widetilde{Y}(S_2) + \alpha\int_{S_1}^{S_2}\widetilde{Y}(s)\,\mathrm{d}s\right)\right\},$$

hence the maximum likelihood estimator (MLE) of m based on the observation of $\{\widetilde{Y}(s) : s \in [S_1, S_2]\}$ is given by

$$\widetilde{m} = \frac{\widetilde{Y}(S_1) + \widetilde{Y}(S_2) + \alpha\int_{S_1}^{S_2}\widetilde{Y}(s)\,\mathrm{d}s}{2+\alpha(S_2-S_1)},$$

and it has a normal distribution with mean m and variance $\sigma^2\alpha^{-1}(2+\alpha(S_2-S_1))^{-1}$ (see Grenander [5], Arató, M. [1]; it follows also from Theorem 2).

The process $\{X(s) : s \geqslant 0\}$ given by

$$\begin{cases} \mathrm{d}X(s) = -\alpha X(s)\mathrm{d}s + \sigma\mathrm{d}W(s),\\ X(0) = 0,\end{cases}$$

where $\alpha \in \mathbb{R}$, $\sigma > 0$, can be considered as the Ornstein-Uhlenbeck process with initial condition $X(0) = 0$. It can be represented as

$$X(s) = \sigma\int_0^s \mathrm{e}^{\alpha(u-s)}\,\mathrm{d}W(u).$$

Let $Y(s) := X(s)+m$, $s \geqslant 0$, and let $[S_1, S_2] \subset (0, \infty)$. Consider the measures P_Y and P_X generated on $C([S_1, S_2] \to \mathbb{R})$ by the processes Y and X, respectively. Then it can be shown that P_Y and P_X are equivalent and in case $\alpha \neq 0$ the Radon–Nikodym derivative has the form

$$\begin{aligned}\frac{\mathrm{d}\mathsf{P}_Y}{\mathrm{d}\mathsf{P}_X}(Y) = \exp\bigg\{ &-\frac{\alpha m^2}{2\sigma^2}(\coth(\alpha S_1) + 1 + \alpha(S_2-S_1))\\ &+\frac{\alpha m}{\sigma^2}\left(\coth(\alpha S_1)Y(S_1) + Y(S_2) + \alpha\int_{S_1}^{S_2} Y(s)\,\mathrm{d}s\right)\bigg\},\end{aligned}$$

hence the MLE of m based on the observation of $\{Y(s) : s \in [S_1, S_2]\}$ is

$$\widehat{m} = \frac{\coth(\alpha S_1)Y(S_1) + Y(S_2) + \alpha \int_{S_1}^{S_2} Y(s)\,\mathrm{d}s}{\coth(\alpha S_1) + 1 + \alpha(S_2 - S_1)},$$

and it has a normal distribution with mean m and variance $\sigma^2\alpha^{-1}(\coth(\alpha S_1) + 1 + \alpha(S_2 - S_1))^{-1}$. (It follows from Theorem 3.) In case $\alpha = 0$ the Radon–Nikodym derivative equals

$$\frac{\mathrm{d}\mathsf{P}_Y}{\mathrm{d}\mathsf{P}_X}(Y) = \exp\left\{-\frac{m^2}{2\sigma^2 S_1} + \frac{mY(S_1)}{\sigma^2 S_1}\right\},$$

hence the MLE of m based on the observation of $\{Y(s) : s \in [S_1, S_2]\}$ is $\widehat{m} = Y(S_1)$, and it has a normal distribution with mean m and variance $\sigma^2 S_1$. (It follows from Theorem 1.)

Arató, M. [1] studied also the complex–valued stationary Ornstein–Uhlenbeck process, that is, the stationary solution of the equation (1.1), where now $\alpha \in \mathbb{C}$ with $\mathsf{Re}\alpha > 0$, $\sigma > 0$ and $\{W(s) : s \in \mathbb{R}\}$ is a complex–valued standard Wiener process. Consider again the process $\widetilde{Y}(s) := \widetilde{X}(s) + m$, $s \in \mathbb{R}$, where $m \in \mathbb{C}$ is the unknown parameter. The complex–valued processes $\{\widetilde{Y}(s) : s \in \mathbb{R}\}$ and $\{\widetilde{X}(s) : s \geqslant 0\}$ can be considered as processes with values in $\mathbb{R}^2$, as well. Let $[S_1, S_2] \subset (-\infty, \infty)$. Let $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ the measures generated on $C([S_1, S_2] \to \mathbb{R}^2)$ by these processes, respectively. Then $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ are equivalent and the Radon–Nikodym derivative has the form

$$\begin{aligned}\frac{\mathrm{d}\mathsf{P}_{\widetilde{Y}}}{\mathrm{d}\mathsf{P}_{\widetilde{X}}}(\widetilde{Y}) = \exp\bigg\{ &-\frac{|m|^2}{2\sigma^2}(2\mathsf{Re}\alpha + |\alpha|^2(S_2 - S_1)) \\ &+ \frac{m}{\sigma^2}\left(\mathsf{Re}\alpha\big(\widetilde{Y}(S_1) + \widetilde{Y}(S_2)\big) + i\mathsf{Im}\alpha\big(\overline{\widetilde{Y}(S_2)} - \overline{\widetilde{Y}(S_1)}\big) + |\alpha|^2 \int_{S_1}^{S_2} \widetilde{Y}(s)\,\mathrm{d}s\right)\bigg\}.\end{aligned}$$

The MLE of m based on the observation of $\{\widetilde{Y}(s) : s \in [S_1, S_2]\}$ is given by

$$\widetilde{m} = \frac{\mathsf{Re}\alpha\big(\widetilde{Y}(S_1) + \widetilde{Y}(S_2)\big) + i\mathsf{Im}\alpha\big(\overline{\widetilde{Y}(S_2)} - \overline{\widetilde{Y}(S_1)}\big) + |\alpha|^2 \int_{S_1}^{S_2} \widetilde{Y}(s)\,\mathrm{d}s}{2\mathsf{Re}\alpha + |\alpha|^2(S_2 - S_1)},$$

and it has a normal distribution with mean m (see Arató, M. [1]). Complex–valued Ornstein–Uhlenbeck process with zero start can be handled similarly.

The stationary Ornstein-Uhlenbeck sheet $\{\widetilde{X}(s,t) : s, t \in \mathbb{R}\}$ is a zero mean Gaussian process with

$$\mathsf{E}\widetilde{X}(s_1, t_1)\widetilde{X}(s_2, t_2) = \frac{\sigma^2}{4\alpha\beta}\mathrm{e}^{-\alpha|s_2 - s_1| - \beta|t_2 - t_1|},$$

where $\alpha > 0$, $\beta > 0$, $\sigma > 0$. Consider the process $\widetilde{Y}(s,t) := \widetilde{X}(s,t) + m$, $s, t \in \mathbb{R}$. Arató, N.M. [2] proved by the help of partial stochastic differential equations that in case of $\alpha = \beta = 1$ the MLE of m based on the observation of $\{\widetilde{Y}(s,t) : s,t \in [0,T]\}$ is given by

$$\widetilde{m} = \frac{\widetilde{Y}(0,0) + \widetilde{Y}(0,T) + \widetilde{Y}(T,0) + \widetilde{Y}(T,T) + \int_{\partial G} \widetilde{Y} + \int_G \widetilde{Y}}{(2+T)^2},$$

where $G := [0,T]^2$ and ∂G denotes the boundary of G. (It follows also from Theorem 2.)

The random field

$$X(s,t) = \sigma \int_0^s \int_0^t \mathrm{e}^{\alpha(u-s)+\beta(v-t)} \,\mathrm{d}W(u,v), \qquad s,t \geqslant 0,$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\sigma > 0$ and $\{W(s,t) : s,t \geqslant 0\}$ is a standard Wiener sheet can be considered as the Ornstein-Uhlenbeck sheet with zero initial condition on the axes. We can consider the shifted random field $Y(s,t) := X(s,t) + m$, $s,t \geqslant 0$.

The purpose of the present paper is to derive the MLE of m based on the observation of $\{\widetilde{X}(s) + mh(s) : s \in [S_1,S_2]\}$, $\{X(s) + mh(s) : s \in [S_1,S_2]\}$, $\{\widetilde{X}(s,t) + mh(s,t) : s \in [S_1,S_2],\, t \in [T_1,T_2]\}$ or $\{X(s,t) + mh(s,t) : s \in [S_1,S_2],\, t \in [T_1,T_2]\}$. It turns out that, for instance in the case of the Ornstein-Uhlenbeck sheet, the MLE is a weighted linear combination of the values at the vertices, integrals on the edges and the integral on the whole rectangle of the weighted observed process. We do not use partial stochastic differential equations as in Arató, N.M. [2], we apply direct discrete time approach instead. To make the transition from the discrete time to the contiuous time, we will develop a tool (see Proposition 1), which might be of indpendent interest. Using appropriate representations of the above Ornstein–Uhlenbeck processes and sheets by the help of the Wiener process or the Wiener sheet, respectively, we derive the results by determining the MLE of m based on observation of the shifted Wiener process or the shifted Wiener sheet. The proofs are given in the Appendix. We remark that the results could also have been derived from the general Feldman–Hajek theorem (see, for example, Kuo [7]), but our direct approach seems to be essentially simpler.

2 Ornstein-Uhlenbeck processes

The stationary Ornstein-Uhlenbeck process $\{\widetilde{X}(s) : s \in \mathbb{R}\}$ can also be represented as

$$\widetilde{X}(s) = \frac{\sigma}{\sqrt{2\alpha}} \mathrm{e}^{-\alpha s} W(\mathrm{e}^{2\alpha s}), \qquad s \in \mathbb{R}, \tag{2.1}$$

where $\{W(s) : s \geqslant 0\}$ is a standard Wiener process. Hence the MLE $\widetilde{m}$ of m based on $\{\widetilde{X}(s) + mh(s) : s \in [S_1,S_2]\}$ can be obtained by determining the MLE of m based on $\{W(\mathrm{e}^{2\alpha s}) + m\sqrt{2\alpha}\sigma^{-1}\mathrm{e}^{\alpha s}h(s) : s \in [S_1,S_2]\}$ or

$$\left\{ W(u) + m\frac{\sqrt{2\alpha u}}{\sigma} h\left(\frac{\log u}{2\alpha}\right) : u \in \left[\mathrm{e}^{2\alpha S_1}, \mathrm{e}^{2\alpha S_2}\right] \right\}.$$

Let $[a_1,a_2] \subset (0,\infty)$. Consider the process $Z(u) := W(u) + mg(u)$, $u \in [a_1,a_2]$ with some function $g : [a_1,a_2] \to \mathbb{R}$. Denote by P_Z and P_W the measures generated on $C([a_1,a_2] \to \mathbb{R})$ by the processes Z and W, respectively. It is known that an absolutely continuous function is almost everywhere differentiable.

Theorem 1 *If g is absolutely continuous and $g' \in L^2([a_1,a_2])$, then the measures P_Z and P_W are equivalent and the Radon–Nikodym derivative of P_Z with respect to P_W equals*

$$\frac{\mathrm{d}\mathsf{P}_Z}{\mathrm{d}\mathsf{P}_W}(Z) = \exp\left\{-\frac{1}{2}(Am^2 - 2\zeta m)\right\},$$

where

$$A = \frac{g^2(a_1)}{a_1} + \int_{a_1}^{a_2} [g'(u)]^2 \, \mathrm{d}u, \qquad \zeta = \frac{g(a_1)Z(a_1)}{a_1} + \int_{a_1}^{a_2} g'(u) \, \mathrm{d}Z(u).$$

The maximum likelihood estimator of the shift parameter m based on the observations $\{Z(u) : u \in [a_1,a_2]\}$ has the form $\widetilde{m} = \zeta/A$ and it has a normal distribution with mean m and variance $1/A$.

Remark 1 Consider a partition $a_1 \leqslant u_1^{(M)} < u_2^{(M)} < \cdots < u_M^{(M)} \leqslant a_2$. The MLE of m based on the discrete sample $\{Z(u_i^{(M)}) : i = 1,\dots,M\}$ is $\widetilde{m}_M = \zeta_M/A_M$, where

$$A_M = \frac{g^2(u_1^{(M)})}{u_1^{(M)}} + \sum_{i=2}^{M} \frac{\left(g(u_i^{(M)}) - g(u_{i-1}^{(M)})\right)^2}{u_i^{(M)} - u_{i-1}^{(M)}},$$

$$\zeta_M = \frac{g(u_1^{(M)})Z(u_1^{(M)})}{u_1^{(M)}} + \sum_{i=2}^{M} \frac{\left(g(u_i^{(M)}) - g(u_{i-1}^{(M)})\right)\left(Z(u_i^{(M)}) - Z(u_{i-1}^{(M)})\right)}{u_i^{(M)} - u_{i-1}^{(M)}}.$$

We shall see in the proof of Theorem 1 that $\widetilde{m}_M$ converges to $\widetilde{m}$ in quadratic mean as $M \to \infty$ and $\max\limits_{2 \leqslant i \leqslant M} (u_i^{(M)} - u_{i-1}^{(M)}) \to 0$.

In case of $g(u) = 1$, $u \in [a_1,a_2]$, the MLE of m based on the observations $\{Z(u_i^{(M)}) : i = 1,\dots,M\}$ is simply $\widetilde{m}_N = Z(u_1^{(M)})$. The MLE of m based on the observation of $\{Z(u) : u \in [a_1,a_2]\}$ is $\widetilde{m} = Z(a_1)$.

Remark 2 The random variable ζ can also be expressed by the help of integral with respect to the Wiener process, namely

$$\zeta = Am + \frac{g(a_1)W(a_1)}{a_1} + \int_{a_1}^{a_2} g'(u) \, \mathrm{d}W(u).$$

Now let $[S_1, S_2] \subset (-\infty, \infty)$ and consider the process $\widetilde{Y}(s) := \widetilde{X}(s) + mh(s)$ with some function $h : [S_1, S_2] \to \mathbb{R}$. Applying Theorem 1 for the function

$$g(u) = \frac{\sqrt{2\alpha u}}{\sigma} h\left(\frac{\log u}{2\alpha}\right) \tag{2.2}$$

we obtain the following result. Denote by $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ the measures generated on $C([S_1, S_2] \to \mathbb{R})$ by the processes $\widetilde{Y}$ and $\widetilde{X}$, respectively.

Theorem 2 *If h is absolutely continuous and $h' \in L^2([S_1, S_2])$, then the measures $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ are equivalent and the Radon–Nikodym derivative of $\mathsf{P}_{\widetilde{Y}}$ with respect to $\mathsf{P}_{\widetilde{X}}$ equals*

$$\frac{\mathrm{d}\mathsf{P}_{\widetilde{Y}}}{\mathrm{d}\mathsf{P}_{\widetilde{X}}}(\widetilde{Y}) = \exp\left\{-\frac{\alpha}{2\sigma^2}(Am^2 - 2\zeta m)\right\},$$

where

$$A = h^2(S_1) + h^2(S_2) + \int_{S_1}^{S_2} \left(\alpha h^2(s) + \alpha^{-1}\left[h'(u)\right]^2\right) \mathrm{d}s,$$

$$\zeta = 2h(S_1)\widetilde{Y}(S_1) + \int_{S_1}^{S_2} \left(h(s) + \alpha^{-1}h'(s)\right)\left(\mathrm{d}\widetilde{Y}(s) + \alpha\widetilde{Y}(s)\,\mathrm{d}s\right).$$

The maximum likelihood estimator of the shift parameter m based on the observations $\{\widetilde{Y}(s) : s \in [S_1, S_2]\}$ has the form $\widetilde{m} = \zeta/A$ and it has a normal distribution with mean m and variance $\sigma^2/(\alpha A)$.

If, in addition, h is twice continuously differentiable then ζ can be written in the form

$$\zeta = h(S_1)\widetilde{Y}(S_1) + h(S_2)\widetilde{Y}(S_2) + \alpha^{-1}\left(h'(S_2)\widetilde{Y}(S_2) - h'(S_1)\widetilde{Y}(S_1)\right) + \int_{S_1}^{S_2} \left(\alpha h(s) - \alpha^{-1}h''(s)\right)\widetilde{Y}(s)\,\mathrm{d}s.$$

Particularly, if $h(s) = 1$, $s \in [S_1, S_2]$, then we obtain the result mentioned in the Introduction.

Next we consider the zero start Ornstein–Uhlenbeck process $\{X(s) : s \geqslant 0\}$. Let $[S_1, S_2] \subset (0, \infty)$ and consider the process $Y(s) := X(s) + mh(s)$ with some function $h : [S_1, S_2] \to \mathbb{R}$. In case $\alpha = 0$ we have simply $X(s) = \sigma W(s)$, $s \geqslant 0$, hence we can apply Theorem 1 for the function $g(u) = \sigma^{-1}h(u)$ in order to determine the MLE of m based on the observations $\{Y(s) : s \in [S_1, S_2]\}$.

In case $\alpha \neq 0$ the zero start Ornstein–Uhlenbeck process $\{X(s) : s \geqslant 0\}$ can be characterized as a zero mean Gaussian process with

$$\mathsf{E}X(s_1)X(s_2) = \frac{\sigma^2}{2\alpha}\left(\mathrm{e}^{-\alpha|s_1 - s_2|} - \mathrm{e}^{-\alpha(s_1+s_2)}\right),$$

hence it can be also represented as

$$X(s) = \begin{cases} \dfrac{\sigma}{\sqrt{2\alpha}} \, \mathrm{e}^{-\alpha s} \, W(\mathrm{e}^{2\alpha s} - 1) & \text{if } \alpha > 0, \\ \dfrac{\sigma}{\sqrt{-2\alpha}} \, \mathrm{e}^{-\alpha s} \, W(1 - \mathrm{e}^{2\alpha s}) & \text{if } \alpha < 0. \end{cases} \tag{2.3}$$

Let $[S_1, S_2] \subset (0, \infty)$ and consider the process $Y(s) := X(s) + mh(s)$ with some function $h : [S_1, S_2] \to \mathbb{R}$. Applying Theorem 1 for the function

$$g(u) = \begin{cases} \dfrac{\sqrt{2\alpha(u+1)}}{\sigma} \, h\left(\dfrac{\log(u+1)}{2\alpha}\right) & \text{if } \alpha > 0, \\ \dfrac{\sqrt{2\alpha(u-1)}}{\sigma} \, h\left(\dfrac{\log(1-u)}{2\alpha}\right) & \text{if } \alpha < 0, \end{cases} \tag{2.4}$$

on the interval $\left[\mathrm{e}^{2\alpha S_1} - 1, \mathrm{e}^{2\alpha S_2} - 1\right]$ or $\left[1 - \mathrm{e}^{2\alpha S_1}, 1 - \mathrm{e}^{2\alpha S_2}\right]$, respectively, we obtain the following result. Denote by P_Y and P_X the measures generated on $C([S_1, S_2] \to \mathbb{R})$ by the processes Y and X, respectively.

Theorem 3 *If $\alpha \neq 0$, and h is absolutely continuous and $h' \in L^2([S_1, S_2])$, then the measures P_Y and P_X are equivalent and the Radon–Nikodym derivative of P_Y with respect to P_X equals*

$$\frac{\mathrm{d}\mathsf{P}_Y}{\mathrm{d}\mathsf{P}_X}(Y) = \exp\left\{-\frac{\alpha}{2\sigma^2}(Am^2 - 2\zeta m)\right\},$$

where

$$A = \coth(\alpha S_1) h^2(S_1) + h^2(S_2) + \int_{S_1}^{S_2} \left(\alpha h^2(s) + \alpha^{-1} \left[h'(s)\right]^2\right) \mathrm{d}s,$$

$$\zeta = (1 + \coth(\alpha S_1)) h(S_1) Y(S_1) + \int_{S_1}^{S_2} \left(h(s) + \alpha^{-1} h'(s)\right) \left(\mathrm{d}Y(s) + \alpha Y(s)\, \mathrm{d}s\right).$$

The maximum likelihood estimator of the shift parameter m based on the observations $\{Y(s) : s \in [S_1, S_2]\}$ has the form $\widehat{m} = \zeta / A$ and it has a normal distribution with mean m and variance $\sigma^2/(\alpha A)$.

If, in addition, h is twice continuously differentiable then ζ can be written in the form

$$\zeta = \coth(\alpha S_1) h(S_1) Y(S_1) + h(S_2) Y(S_2) + \alpha^{-1}\left(h'(S_2) Y(S_2) - h'(S_1) Y(S_1)\right) + \int_{S_1}^{S_2} \left(\alpha h(s) - \alpha^{-1} h''(s)\right) Y(s)\, \mathrm{d}s.$$

Particularly, if $h(s) = 1$, $s \in [S_1, S_2]$, then we obtain the result mentioned in the Introduction.

3 Ornstein-Uhlenbeck sheets

First we study the MLE of the shift parameter of the standard Wiener sheet $\{W(s,t) : s,t \geqslant 0\}$. Let $[a_1,a_2]$, $[b_1,b_2] \subset (0,\infty)$. Consider the process $Z(s,t) := W(s,t) + mg(s,t)$ with some function $g : [a_1,a_2] \times [b_1,b_2] \to \mathbb{R}$. Denote by P_Z and P_W the measures generated on $C([a_1,a_2] \times [b_1,b_2] \to \mathbb{R})$ by the sheets Z and W, respectively.

Theorem 4 *If g is absolutely continuous and $\partial_1\partial_2 g \in L^2([a_1,a_2] \times [b_1,b_2])$, then the measures P_Z and P_W are equivalent and the Radon–Nikodym derivative of P_Z with respect to P_W equals*

$$\frac{\mathrm{d}\mathsf{P}_Z}{\mathrm{d}\mathsf{P}_W}(Z) = \exp\left\{-\frac{1}{2}(Am^2 - 2\zeta m)\right\},$$

where

$$A = \frac{g^2(a_1,b_1)}{a_1 b_1} + \int_{a_1}^{a_2} \frac{[\partial_1 g(u,b_1)]^2}{b_1}\,\mathrm{d}u + \int_{b_1}^{b_2} \frac{[\partial_2 g(a_1,v)]^2}{a_1}\,\mathrm{d}v + \int_{a_1}^{a_2}\int_{b_1}^{b_2} [\partial_1\partial_2 g(u,v)]^2\,\mathrm{d}u\mathrm{d}v,$$

$$\zeta = \frac{g(a_1,b_1)Z(a_1,b_1)}{a_1 b_1} + \int_{a_1}^{a_2} \frac{\partial_1 g(u,b_1)}{b_1}\,Z(\mathrm{d}u,b_1) + \int_{b_1}^{b_2} \frac{\partial_2 g(a_1,v)}{a_1}\,Z(a_1,\mathrm{d}v)$$
$$+ \int_{a_1}^{a_2}\int_{b_1}^{b_2} \partial_1\partial_2 g(u,v)\,Z(\mathrm{d}u,\mathrm{d}v).$$

The maximum likelihood estimator of the shift parameter m based on the observations $\{Z(s,t) : s \in [a_1,a_2],\, t \in [b_1,b_2]\}$ has the form $\widetilde{m} = \zeta/A$ and it has a normal distribution with mean m and variance $1/A$.

The stationary Ornstein-Uhlenbeck sheet $\{\widetilde{X}(s,t) : s,t \in \mathbb{R}\}$ can also be represented as

$$\widetilde{X}(s,t) = \frac{\sigma}{2\sqrt{\alpha\beta}}\,\mathrm{e}^{-\alpha s - \beta t}\,W(\mathrm{e}^{2\alpha s},\mathrm{e}^{2\beta t}), \qquad s,t \in \mathbb{R}. \tag{3.1}$$

Let $[S_1,S_2]$, $[T_1,T_2] \subset (-\infty,\infty)$ and consider the sheet $\widetilde{Y}(s,t) := \widetilde{X}(s,t) + mh(s,t)$ with some function $h : [S_1,S_2] \times [T_1,T_2] \to \mathbb{R}$. Applying Theorem 4 for the function

$$g(u,v) = \frac{2\sqrt{\alpha\beta uv}}{\sigma}\,h\left(\frac{\log u}{2\alpha}, \frac{\log v}{2\beta}\right) \tag{3.2}$$

we obtain the following result. Denote by $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ the measures generated on $C([S_1,S_2] \times [T_1,T_2] \to \mathbb{R})$ by the sheets $\widetilde{Y}$ and $\widetilde{X}$, respectively.

Theorem 5 *If h is absolutely continuous and $\partial_1\partial_2 h \in L^2([S_1,S_2] \times [T_1,T_2])$, then the measures $\mathsf{P}_{\widetilde{Y}}$ and $\mathsf{P}_{\widetilde{X}}$ are equivalent and the Radon–Nikodym derivative of $\mathsf{P}_{\widetilde{Y}}$ with respect to $\mathsf{P}_{\widetilde{X}}$ equals*

$$\frac{\mathrm{d}\mathsf{P}_{\widetilde{Y}}}{\mathrm{d}\mathsf{P}_{\widetilde{X}}}(\widetilde{Y}) = \exp\left\{-\frac{\alpha\beta}{2\sigma^2}(Am^2 - 2\zeta m)\right\},$$

where

$$A = h^2(S_1,T_1) + h^2(S_1,T_2) + h^2(S_2,T_1) + h^2(S_2,T_2)$$
$$+ \int_{S_1}^{S_2} \Big(\alpha\big(h^2(s,T_1) + h^2(s,T_2)\big) + \alpha^{-1}\big([\partial_1 h(s,T_1)]^2 + [\partial_1 h(s,T_2)]^2\big)\Big)\,\mathrm{d}s$$
$$+ \int_{T_1}^{T_2} \Big(\beta\big(h^2(S_1,t) + h^2(S_2,t)\big) + \beta^{-1}\big([\partial_2 h(S_1,t)]^2 + [\partial_2 h(S_2,t)]^2\big)\Big)\,\mathrm{d}t$$
$$+ \int_{S_1}^{S_2}\int_{T_1}^{T_2} \Big(\alpha\beta h^2(s,t) + \alpha^{-1}\beta\,[\partial_1 h(s,t)]^2 + \alpha\beta^{-1}\,[\partial_2 h(s,t)]^2 + \alpha^{-1}\beta^{-1}\,([\partial_1\partial_2 h(s,t)]^2\Big)\,\mathrm{d}s\mathrm{d}t,$$

$$\zeta = 4h(S_1,T_1)\widetilde{Y}(S_1,T_1) + 2\int_{S_1}^{S_2} \big(h(s,T_1) + \alpha^{-1}\partial_1 h(s,T_1)\big)\Big(\widetilde{Y}(\mathrm{d}s,T_1) + \alpha\widetilde{Y}(s,T_1)\,\mathrm{d}s\Big)$$
$$+ 2\int_{T_1}^{T_2} \big(h(S_1,t) + \beta^{-1}\partial_2 h(S_1,t)\big)\Big(\widetilde{Y}(S_1,\mathrm{d}t) + \beta\widetilde{Y}(S_1,t)\,\mathrm{d}t\Big)$$
$$+ \int_{S_1}^{S_2}\int_{T_1}^{T_2} \big(h(s,t) + \alpha^{-1}\partial_1 h(s,t) + \beta^{-1}\partial_2 h(s,t) + \alpha^{-1}\beta^{-1}\partial_1\partial_2 h(s,t)\big)$$
$$\Big(\widetilde{Y}(\mathrm{d}s,\mathrm{d}t) + \alpha\widetilde{Y}(s,\mathrm{d}t)\,\mathrm{d}s + \beta\widetilde{Y}(\mathrm{d}s,t)\,\mathrm{d}t + \alpha\beta\widetilde{Y}(s,t)\,\mathrm{d}s\mathrm{d}t\Big).$$

The maximum likelihood estimator of the shift parameter m *based on the observations* $\{\widetilde{Y}(s,t) : s \in [S_1,S_2],\, t \in [T_1,T_2]\}$ *has the form* $\widetilde{m} = \zeta/A$ *and it has a normal distribution with mean* m *and variance* $\sigma^2/(\alpha\beta A)$.

If, in addition, h *is twice continuously differentiable with respect to both of its coordinates then* ζ *can be written in the form*

$$\zeta = \big[(1-\alpha^{-1}\partial_1)(1-\beta^{-1}\partial_2)h(S_1,T_1)\big]\,\widetilde{Y}(S_1,T_1) + \big[(1-\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(S_1,T_2)\big]\,\widetilde{Y}(S_1,T_2)$$
$$+ \big[(1+\alpha^{-1}\partial_1)(1-\beta^{-1}\partial_2)h(S_2,T_1)\big]\,\widetilde{Y}(S_2,T_1) + \big[(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(S_2,T_2)\big]\,\widetilde{Y}(S_2,T_2)$$
$$+ \int_{S_1}^{S_2} \big[(\alpha-\alpha^{-1}\partial_1^2)(1-\beta^{-1}\partial_2)h(s,T_1)\big]\,\widetilde{Y}(s,T_1)\,\mathrm{d}s$$
$$+ \int_{S_1}^{S_2} \big[(\alpha-\alpha^{-1}\partial_1^2)(1+\beta^{-1}\partial_2)h(s,T_2)\big]\,\widetilde{Y}(s,T_2)\,\mathrm{d}s$$
$$+ \int_{T_1}^{T_2} \big[(1-\alpha^{-1}\partial_1)(\beta-\beta^{-1}\partial_2^2)h(S_1,t)\big]\,\widetilde{Y}(S_1,t)\,\mathrm{d}t$$
$$+ \int_{T_1}^{T_2} \big[(1+\alpha^{-1}\partial_1)(\beta-\beta^{-1}\partial_2^2)h(S_2,t)\big]\,\widetilde{Y}(S_2,t)\,\mathrm{d}t$$
$$+ \int_{S_1}^{S_2}\int_{T_1}^{T_2} \big[(\alpha-\alpha^{-1}\partial_1^2)(\beta-\beta^{-1}\partial_2^2)h(s,t)\big]\,\widetilde{Y}(s,t)\,\mathrm{d}s\mathrm{d}t.$$

Particularly, if $h(s,t)=1,\ s\in[S_1,S_2],\ t\in[T_1,T_2]$, then

$$
\begin{aligned}
A &= (2+\alpha(S_2-S_1))(2+\beta(T_2-T_1)),\\
\zeta &= \widetilde{Y}(S_1,T_1)+\widetilde{Y}(S_1,T_2)+\widetilde{Y}(S_2,T_1)+\widetilde{Y}(S_2,T_2)+\alpha\int_{S_1}^{S_2}(\widetilde{Y}(s,T_1)+\widetilde{Y}(s,T_2))\,\mathrm{d}s\\
&\quad+\beta\int_{T_1}^{T_2}(\widetilde{Y}(S_1,t)+\widetilde{Y}(S_2,t))\,\mathrm{d}t+\alpha\beta\int_{S_1}^{S_2}\int_{T_1}^{T_2}\widetilde{Y}(s,t)\,\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Especially, if $\alpha=\beta=1$ then we obtain the result mentioned in the Introduction.

Next we consider the zero start Ornstein–Uhlenbeck sheet $\{X(s,t): s,t\geqslant 0\}$. If $\alpha\neq 0$ and $\beta\neq 0$ then it can be characterized as a zero mean Gaussian process with

$$
\mathsf{E}X(s_1,t_1)X(s_2,t_2)=\frac{\sigma^2}{4\alpha\beta}\left(\mathrm{e}^{-\alpha|s_1-s_2|}-\mathrm{e}^{-\alpha(s_1+s_2)}\right)\left(\mathrm{e}^{-\beta|t_1-t_2|}-\mathrm{e}^{-\beta(t_1+t_2)}\right),
$$

hence, for example, in case $\alpha>0$ and $\beta>0$ it can be also represented as

$$
X(s,t)=\frac{\sigma}{2\sqrt{\alpha\beta}}\,\mathrm{e}^{-\alpha s-\beta t}\,W(\mathrm{e}^{2\alpha s}-1,\mathrm{e}^{2\beta t}-1),\qquad s,t\geqslant 0. \tag{3.3}
$$

Let $[S_1,S_2],[T_1,T_2]\subset(0,\infty)$ and consider the process $Y(s,t):=X(s,t)+mh(s,t)$ with some function $h:[S_1,S_2]\times[T_1,T_2]\to\mathbb{R}$. Applying Theorem 4 for the function

$$
g(u,v)=\frac{2\sqrt{\alpha\beta(u+1)(v+1)}}{\sigma}\,h\left(\frac{\log(u+1)}{2\alpha},\frac{\log(v+1)}{2\beta}\right) \tag{3.4}
$$

we obtain the following result. Denote by P_Y and P_X the measures generated on $C([S_1,S_2]\times[T_1,T_2]\to\mathbb{R})$ by the processes Y and X, respectively.

Theorem 6 *If $\alpha\neq 0$ and $\beta\neq 0$, and h is absolutely continuous and $\partial_1\partial_2 h\in L^2([S_1,S_2]\times[T_1,T_2])$, then the measures P_Y and P_X are equivalent and the Radon–Nikodym derivative of P_Y with respect to P_X equals*

$$
\frac{\mathrm{d}\mathsf{P}_Y}{\mathrm{d}\mathsf{P}_X}(Y)=\exp\left\{-\frac{\alpha\beta}{2\sigma^2}(Am^2-2\zeta m)\right\},
$$

where

$$A = \coth(\alpha S_1)\coth(\beta T_1)h^2(S_1,T_1) + \coth(\alpha S_1)h^2(S_1,T_2) + \coth(\beta T_1)h^2(S_2,T_1) + h^2(S_2,T_2)$$
$$+ \int_{S_1}^{S_2}\Big(\alpha\big(h^2(s,T_1)+h^2(s,T_2)\big) + \alpha^{-1}\big([\partial_1 h(s,T_1)]^2 + [\partial_1 h(s,T_2)]^2\big)\Big)\,\mathrm{d}s$$
$$+ \int_{T_1}^{T_2}\Big(\beta\big(h^2(S_1,v)+h^2(S_2,t)\big) + \beta^{-1}\big([\partial_2 h(S_1,t)]^2 + [\partial_2 h(S_2,t)]^2\big)\Big)\,\mathrm{d}t$$
$$+ \int_{S_1}^{S_2}\int_{T_1}^{T_2}\Big(\alpha\beta h^2(s,t) + \alpha^{-1}\beta\,[\partial_1 h(s,t)]^2 + \alpha\beta^{-1}\,[\partial_2 h(s,t)]^2 + \alpha^{-1}\beta^{-1}\,([\partial_1\partial_2 h(s,t)]^2\Big)\,\mathrm{d}s\mathrm{d}t,$$

$$\zeta = (1+\coth(\alpha S_1))(1+\coth(\beta T_1))h(S_1,T_1)Y(S_1,T_1)$$
$$+ (1+\coth(\beta T_1))\int_{S_1}^{S_2}\big(h(s,T_1)+\alpha^{-1}\partial_1 h(s,T_1)\big)\,(Y(\mathrm{d}s,T_1)+\alpha Y(s,T_1)\,\mathrm{d}s)$$
$$+ (1+\coth(\alpha S_1))\int_{T_1}^{T_2}\big(h(S_1,t)+\beta^{-1}\partial_2 h(S_1,t)\big)\,(Y(S_1,\mathrm{d}t)+\beta Y(S_1,t)\,\mathrm{d}t)$$
$$+ \int_{S_1}^{S_2}\int_{T_1}^{T_2}\big(h(s,t)+\alpha^{-1}\partial_1 h(s,t)+\beta^{-1}\partial_2 h(s,t)+\alpha^{-1}\beta^{-1}\partial_1\partial_2 h(s,t)\big)$$
$$(Y(\mathrm{d}s,\mathrm{d}t)+\alpha Y(s,\mathrm{d}t)\,\mathrm{d}s+\beta Y(\mathrm{d}s,t)\,\mathrm{d}t+\alpha\beta Y(s,t)\,\mathrm{d}s\mathrm{d}t)\,.$$

The maximum likelihood estimator of the shift parameter m *based on the observations* $\{Y(t) : s \in [S_1,S_2],\, t \in [T_1,T_2]\}$ *has the form* $\widehat{m} = \zeta/A$ *and it has a normal distribution with mean* m *and variance* $\sigma^2/(\alpha\beta A)$.

If, in addition, h *is twice continuously differentiable then* ζ *can be written in the form*

$$\zeta = \big[(\coth(\alpha S_1)-\alpha^{-1}\partial_1)(\coth(\beta T_1)-\beta^{-1}\partial_2)h(S_1,T_1)\big]\,Y(S_1,T_1)$$
$$+ \big[(\coth(\alpha S_1)-\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(S_1,T_2)\big]\,Y(S_1,T_2)$$
$$+ \big[(1+\alpha^{-1}\partial_1)(\coth(\beta T_1)-\beta^{-1}\partial_2)h(S_2,T_1)\big]\,Y(S_2,T_1)$$
$$+ \big[(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(S_2,T_2)\big]\,Y(S_2,T_2)$$
$$+ \int_{S_1}^{S_2}\big[(\alpha-\alpha^{-1}\partial_1^2)(\coth(\beta T_1)-\beta^{-1}\partial_2)h(s,T_1)\big]\,Y(s,T_1)\,\mathrm{d}s$$
$$+ \int_{S_1}^{S_2}\big[(\alpha-\alpha^{-1}\partial_1^2)(1+\beta^{-1}\partial_2)h(s,T_2)\big]\,Y(s,T_2)\,\mathrm{d}s$$

$$+\int_{T_1}^{T_2}\big[(\coth(\alpha S_1)-\alpha^{-1}\partial_1)(\beta-\beta^{-1}\partial_2^2)h(S_1,t)\big]\,Y(S_1,t)\,\mathrm{d}t$$
$$+\int_{T_1}^{T_2}\big[(1+\alpha^{-1}\partial_1)(\beta-\beta^{-1}\partial_2^2)h(S_2,t)\big]\,Y(S_2,t)\,\mathrm{d}t$$
$$+\int_{S_1}^{S_2}\int_{T_1}^{T_2}\big[(\alpha-\alpha^{-1}\partial_1^2)(\beta-\beta^{-1}\partial_2^2)h(s,t)\big]\,Y(s,t)\,\mathrm{d}s\mathrm{d}t.$$

Particularly, if $h(s,t)=1,\ s\in[S_1,S_2],\, t\in[T_1,T_2],$ then

$$A=(\coth(\alpha S_1)+1+\alpha(S_2-S_1))(\coth(\beta T_1)+1+\beta(T_2-T_1)),$$
$$\zeta=\coth(\alpha S_1)\coth(\beta T_1)Y(S_1,T_1)+\coth(\alpha S_1)Y(S_1,T_2)+\coth(\beta T_1)Y(S_2,T_1)+Y(S_2,T_2)$$
$$+\alpha\int_{S_1}^{S_2}\big(\coth(\beta T_1)Y(s,T_1)+Y(s,T_2)\big)\,\mathrm{d}s+\beta\int_{T_1}^{T_2}\big(\coth(\alpha S_1)Y(S_1,t)+Y(S_2,t)\big)\,\mathrm{d}t$$
$$+\alpha\beta\int_{S_1}^{S_2}\int_{T_1}^{T_2}Y(s,t)\,\mathrm{d}s\mathrm{d}t.$$

4 Appendix

In order to determine Radon–Nikodym derivatives we have developed the following general method based on Section 2.3.2 in Arató [1].

Let Γ be an arbitrary index set and let $X\subset\mathbb{R}^\Gamma$ be a function space. Let $\mathcal{X}$ be a σ–algebra of subsets of X such that the cylinder sets

$$\{x\in X:(x(\gamma_1),\dots,x(\gamma_k))\in B\},\qquad k\in\mathbb{N},\quad \gamma_1,\dots,\gamma_k\in\Gamma,\quad B\in\mathcal{B}(\mathbb{R}^k)\quad (4.1)$$

generate $\mathcal{X}$. For example, if Γ is a separable metric space and $X=C(\Gamma)$ is the space of real–valued continuous functions on Γ with the uniform metric then the σ–algebra $\mathcal{X}=\mathcal{B}(C(\Gamma))$ of Borel sets is generated by the cylinder sets (4.1), see, e.g., Billingsley [4, Section 1.3]. Another example in case $\Gamma=[0,1]$ is the Skorokhod space $X=D[0,1]$ with the σ–algebra $\mathcal{X}=\mathcal{B}(D[0,1])$ of Borel sets, see, e.g., Billingsley [4, Theorem 14.5].

For a stochastic process $\{\xi_\gamma:\gamma\in\Gamma\}$ with trajectories in X on a probability space $(\Omega,\mathcal{A},\mathsf{P})$, the induced mapping $\xi:\Omega\to X$ is measurable since the inverse images of the cylinder sets (4.1) are in $\mathcal{A}$, because the mappings $\omega\mapsto(\xi_{\gamma_1}(\omega),\dots,\xi_{\gamma_k}(\omega))$ from Ω into $\mathbb{R}^k$ are Borel measurable. Let P_ξ denote the probability measure, generated by the process ξ on $(X,\mathcal{X})$. For a finite set $\Gamma'=\{\gamma_1,\dots,\gamma_k\}\subset\Gamma$, we denote by $\mathsf{P}_\xi^{\Gamma'}$ the probability measure, generated by the random variable $\xi(\Gamma'):=(\xi_{\gamma_1},\dots,\xi_{\gamma_k})$ on $(\mathbb{R}^k,\mathcal{B}(\mathbb{R}^k))$.

Proposition 1 *Let $\{\xi_\gamma:\gamma\in\Gamma\}$ and $\{\eta_\gamma:\gamma\in\Gamma\}$ be stochastic processes with trajectories in X. Suppose that there exists a measurable function $f:X\to\mathbb{R}$ such* *that* $\mathsf{E}f(\xi)=1$ *and that for any finite set* $\Gamma_0\subset\Gamma$, *there exists a sequence of finite subsets* Γ_n, $n=1,2,\ldots$ *with* $\Gamma_0\subset\Gamma_n\subset\Gamma$, $n=1,2,\ldots,$ *and with*

$$\frac{\mathrm{d}\mathsf{P}_\eta^{\Gamma_n}}{\mathrm{d}\mathsf{P}_\xi^{\Gamma_n}}(\xi(\Gamma_n))\xrightarrow{\mathsf{P}}f(\xi)\qquad \textit{as}\ \ n\to\infty.$$

Then P_η *is absolutely continuous with respect to* P_ξ *and* $\dfrac{\mathrm{d}\mathsf{P}_\eta}{\mathrm{d}\mathsf{P}_\xi}=f$.

Proof. We have to prove that for any $H\in\mathcal{X}$,

$$\mathsf{P}_\eta(H)=\int_H f(x)\,\mathsf{P}_\xi(\mathrm{d}x). \tag{4.2}$$

The left hand side is equal to $\mathsf{P}(\eta\in H)$ and the right hand side can be written in the form

$$\int_{\xi^{-1}(H)}f(\xi(\omega))\,\mathsf{P}(\mathrm{d}\omega)=\mathsf{E}\left[f(\xi)\mathbb{1}_H(\xi)\right].$$

Both sides of (4.2) are probability measures on $(X,\mathcal{X})$, and the σ–algebra $\mathcal{X}$ is generated by the algebra consisting of the cylinder sets (4.1), hence it is sufficient to show (4.2) for the cylinder sets (4.1), i.e., we have to show that for all finite subsets $\Gamma_0=\{\gamma_1,\ldots,\gamma_k\}\subset\Gamma$ and for all $B\in\mathcal{B}(\mathbb{R}^k)$,

$$\mathsf{P}(\eta(\Gamma_0)\in B)=\mathsf{E}\left[f(\xi)\mathbb{1}_B(\xi(\Gamma_0))\right]. \tag{4.3}$$

By the assumption, we may choose a sequence of finite subsets Γ_n, $n=1,2,\ldots$ with $\Gamma_0\subset\Gamma_n\subset\Gamma$, $n=1,2,\ldots,$ and with

$$\frac{\mathrm{d}\mathsf{P}_\eta^{\Gamma_n}}{\mathrm{d}\mathsf{P}_\xi^{\Gamma_n}}(\xi(\Gamma_n))\to f(\xi)\qquad \mathsf{P}\text{–a.s.}$$

By Fatou's lemma,

$$\liminf_{n\to\infty}\mathsf{E}\left[\frac{\mathrm{d}\mathsf{P}_\eta^{\Gamma_n}}{\mathrm{d}\mathsf{P}_\xi^{\Gamma_n}}(\xi(\Gamma_n))\mathbb{1}_B(\xi(\Gamma_0))\right]\geqslant\mathsf{E}\left[f(\xi)\mathbb{1}_B(\xi(\Gamma_0))\right].$$

Clearly, $\Gamma_0\subset\Gamma_n$ implies that the expectations on the left hand side are equal to $\mathsf{P}(\eta(\Gamma_0)\in B)$, thus we obtain

$$\mathsf{P}(\eta(\Gamma_0)\in B)\geqslant\mathsf{E}\left[f(\xi)\mathbb{1}_B(\xi(\Gamma_0))\right].$$

Applying this inequality for $\Omega\setminus B$ instead of B and using $\mathsf{E}f(\xi)=1$, we have

$$\begin{aligned}\mathsf{P}(\eta(\Gamma_0)\in B)&=1-\mathsf{P}(\eta(\Gamma_0)\in\Omega\setminus B)\leqslant 1-\mathsf{E}\left[f(\xi)\mathbb{1}_{\Omega\setminus B}(\xi(\Gamma_0))\right]\\&=\mathsf{E}\left[f(\xi)(1-\mathbb{1}_{\Omega\setminus B}(\xi(\Gamma_0)))\right]=\mathsf{E}\left[f(\xi)\mathbb{1}_B(\xi(\Gamma_0))\right],\end{aligned}$$

and we can conclude (4.3). □

For the proof of the Theorems 1 and 4 we need the following lemmas.

Lemma 1 *Let* $g : [a_1, a_2] \to \mathbb{R}$ *be an absolutely continuous function such that* $g' \in L^2([a_1, a_2])$. *Let* $a_1 = u_1^{(M)} < u_2^{(M)} < \cdots < u_M^{(M)} = a_2$, $M \in \mathbb{N}$, *be a sequence of partitions of the interval* $[a_1, a_2]$ *such that* $\lim_{M\to\infty} \max_{2\leqslant i\leqslant M} \Delta u_i^{(M)} = 0$, *where* $\Delta u_i^{(M)} := (u_i^{(M)} - u_{i-1}^{(M)})$. *For* $u \in [a_1, a_2]$ *and* $M \in \mathbb{N}$, *let*

$$g_M(u) := \sum_{i=2}^{M} \frac{\Delta g(u_i^{(M)})}{\Delta u_i^{(M)}} \mathbb{1}_{[u_{i-1}^{(M)}, u_i^{(M)})}(u),$$

where $\Delta g(u_i^{(M)}) := g(u_i^{(M)}) - g(u_{i-1}^{(M)})$. *Then*

$$g_M \to g' \qquad \text{in } L^2([a_1, a_2]) \text{ as } M \to \infty,$$

and

$$\lim_{M\to\infty} \sum_{i=2}^{M} \frac{\left(\Delta g(u_i^{(M)})\right)^2}{\Delta u_i^{(M)}} = \int_{a_1}^{a_2} |g'(u)|^2 \, \mathrm{d}u.$$

Proof. Obviously,

$$\|g_M\|^2 = \sum_{i=2}^{M} \frac{\left(\Delta g(u_i^{(M)})\right)^2}{\Delta u_i^{(M)}}$$

and

$$\langle g', g_M \rangle = \sum_{i=2}^{M} \frac{\Delta g(u_i^{(M)})}{\Delta u_i^{(M)}} \int_{u_{i-1}^{(M)}}^{u_i^{(M)}} g'(u) \, \mathrm{d}u = \sum_{i=2}^{M} \frac{\left(\Delta g(u_i^{(M)})\right)^2}{\Delta u_i^{(M)}},$$

hence $\|g' - g_M\|^2 = \langle g', g' \rangle - 2\langle g', g_M \rangle + \langle g_M, g_M \rangle = \|g'\|^2 - \|g_M\|^2$. Consequently, the two statements in the Theorem are equivalent, and $\|g_M\|^2 \leqslant \|g'\|^2$. Moreover,

$$\lim_{M\to\infty} g_M(u) = g'(u) \qquad \text{for almost every } u \in [a_1, a_2],$$

since if $u \in \left[u_{i-1}^{(M)}, u_i^{(M)}\right)$ such that $\exists\, g'(u)$ then

$$\begin{aligned} |g_M(u) - g'(u)| &= \left| \frac{\Delta g(u_i^{(M)})}{\Delta u_i^{(M)}} - g'(u) \right| \\ &\leqslant \left| \frac{g(u_i^{(M)}) - g(u)}{u_i^{(M)} - u} - g'(u) \right| + \left| \frac{g(u) - g(u_{i-1}^{(M)})}{u - u_{i-1}^{(M)}} - g'(u) \right| \to 0 \qquad \text{as } M \to \infty. \end{aligned}$$

Hence by Fatou's lemma,

$$\|g'\|^2 = \int_{a_1}^{a_2} (g'(u))^2 \, \mathrm{d}u \leqslant \liminf_{M\to\infty} \int_{a_1}^{a_2} (g_M(u))^2 \, \mathrm{d}u = \liminf_{M\to\infty} \|g_M\|^2,$$

thus we can conclude $\lim_{M\to\infty} \|g_M\|^2 = \|g'\|^2$. □

In a similar way one can prove the following lemma.

Lemma 2 *Let* $g : [a_1, a_2] \times [b_1, b_2] \to \mathbb{R}$ *be an absolutely continuous function such that* $\partial_1\partial_2 g \in L^2([a_1, a_2] \times [b_1, b_2])$. *Let* $a_1 = u_1^{(M)} < u_2^{(M)} < \cdots < u_M^{(M)} = a_2$, $M \in \mathbb{N}$, *and* $b_1 = v_1^{(N)} < v_2^{(N)} < \cdots < v_N^{(N)} = b_2$, $N \in \mathbb{N}$, *be sequences of partitions of the intervals* $[a_1, a_2]$ *and* $[b_1, b_2]$ *such that* $\lim_{M\to\infty} \max_{2\leqslant i\leqslant M} \Delta u_i^{(M)} = 0$ *and* $\lim_{N\to\infty} \max_{2\leqslant j\leqslant N} \Delta v_j^{(N)} = 0$. *For* $u \in [a_1, a_2]$, $v \in [b_1, b_2]$, *and* $M, N \in \mathbb{N}$, *let*

$$g_{M,N}(u, v) := \sum_{i=2}^{M}\sum_{j=2}^{N} \frac{\Delta_1\Delta_2 g(u_i^{(M)}, v_j^{(N)})}{\Delta u_i^{(M)}\Delta v_j^{(N)}} \mathbb{1}_{[u_{i-1}^{(M)}, u_i^{(M)}) \times [v_{j-1}^{(N)}, v_j^{(N)})}(u, v),$$

where $\Delta_1 g(u_i^{(M)}, v) := g(u_i^{(M)}, v) - g(u_{i-1}^{(M)}, v)$, $\Delta_2 g(u, v_j^{(N)}) := g(u, v_j^{(N)}) - g(u, v_{j-1}^{(N)})$. *Then*

$$g_{M,N} \to \partial_1\partial_2 g \qquad \textit{in} \ \ L^2([a_1, a_2] \times [b_1, b_2]) \ \ \textit{as} \ \ M, N \to \infty,$$

and

$$\lim_{M,N\to\infty} \sum_{i=2}^{M}\sum_{j=2}^{N} \frac{\left(\Delta_1\Delta_2 g(u_i^{(M)}, v_j^{(N)})\right)^2}{\Delta u_i^{(M)}\Delta v_j^{(N)}} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} |\partial_1\partial_2 g(u, v)|^2 \, \mathrm{d}u\mathrm{d}v.$$

Proof of Theorem 1. We will apply Proposition 1. First we note that the statement in Remark 2 may be proved by Itô's formula, and from this formula it is easy to show that ζ/A has a normal distribution with mean m and variance $1/A$, and that

$$\mathsf{E} \exp\left\{-\frac{1}{2}(Am^2 - 2\eta m)\right\} = 1,$$

where

$$\eta = \frac{g(a_1)W(a_1)}{a_1} + \int_{a_1}^{a_2} g'(u) \, \mathrm{d}W(u).$$

Hence it is sufficient to prove that for any sequence of partitions $a_1 = u_1^{(M)} < u_2^{(M)} < \cdots < u_M^{(M)} = a_2$, $M \in \mathbb{N}$, such that $\lim_{M\to\infty} \max_{2\leqslant i\leqslant M} \Delta u_i^{(M)} = 0$ we have

$$\frac{\mathrm{d}\mathsf{P}_Z^{(M)}}{\mathrm{d}\mathsf{P}_W^{(M)}}(W(u_1^{(M)}), \ldots, W(u_M^{(M)})) \xrightarrow{\mathsf{P}} \exp\left\{-\frac{1}{2}(Am^2 - 2\eta m)\right\} \qquad \text{as} \ \ M \to \infty, \tag{4.4}$$

where $\mathsf{P}_Z^{(M)}$ and $\mathsf{P}_W^{(M)}$ denote the probability measures generated by the random variables $(Z(u_1^{(M)}), \ldots, Z(u_M^{(M)}))$ and $(W(u_1^{(M)}), \ldots, W(u_M^{(M)}))$, respectively. The joint density of $(W(u_1^{(M)}), \ldots, W(u_M^{(M)}))$ has the form

$$f(x_1, \ldots, x_M) = c \exp\left\{-\frac{1}{2}\left(\frac{x_1^2}{u_1^{(M)}} + \sum_{i=2}^{M} \frac{(\Delta x_i)^2}{\Delta u_i^{(M)}}\right)\right\},$$

where $\Delta x_i := x_i - x_{i-1}$, and c is a norming constant. Consequently, the joint density of $(Z(u_1^{(M)}), \dots, Z(u_M^{(M)}))$ is

$$g(z_1, \dots, z_M) = c \exp\left\{-\frac{1}{2}\left(\frac{(z_1 - mg(u_1^{(M)}))^2}{u_1^{(M)}} + \sum_{i=2}^{M} \frac{(\Delta z_i - m\Delta g(u_i^{(M)}))^2}{\Delta u_i^{(M)}}\right)\right\},$$

where $\Delta z_i := z_i - z_{i-1}$. Hence

$$\frac{\mathrm{d}\mathsf{P}_Z^{(M)}}{\mathrm{d}\mathsf{P}_W^{(M)}}(W(u_1^{(M)}), \dots, W(u_M^{(M)})) = \exp\left\{-\frac{1}{2}(m^2 A_M - 2m\eta_M)\right\},$$

where

$$A_M = \frac{g^2(u_1^{(M)})}{u_1^{(M)}} + \sum_{i=2}^{M} \frac{\left(\Delta g(u_i^{(M)})\right)^2}{\Delta u_i^{(M)}},$$

$$\eta_M = \frac{g(u_1^{(M)})W(u_1^{(M)})}{u_1^{(M)}} + \sum_{i=2}^{M} \frac{\Delta g(u_i^{(M)})\Delta W(u_i^{(M)})}{\Delta u_i^{(M)}},$$

and $\Delta W(u_i^{(M)}) := W(u_i^{(M)}) - W(u_{i-1}^{(M)})$. Applying Lemma 1 we obtain $\lim\limits_{M\to\infty} A_M = A$ and $g_M \to g'$ in $L^2([a_1, a_2])$ as $M \to \infty$, which also implies $\eta_M \to \eta$ in L^2 as $M \to \infty$, hence we conclude in fact L^2–convergence in (4.4). $\square$

Proof of Theorem 2. Using the representation (2.1) and applying Theorem 1 for the function g in (2.2) on the interval $[a_1, a_2] = \left[\mathrm{e}^{2\alpha S_1}, \mathrm{e}^{2\alpha S_2}\right]$ we obtain that the statement in Theorem 2 holds with

$$A = 2h^2\left(\frac{\log a_1}{2\alpha}\right) + 2\int_{a_1}^{a_2}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\sqrt{u}\, h\left(\frac{\log u}{2\alpha}\right)\right)\right]^2 \mathrm{d}u,$$

$$\zeta = 2h\left(\frac{\log a_1}{2\alpha}\right)\widetilde{Y}\left(\frac{\log a_1}{2\alpha}\right) + 2\int_{a_1}^{a_2}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\sqrt{u}\, h\left(\frac{\log u}{2\alpha}\right)\right)\right]\mathrm{d}\left(\sqrt{u}\,\widetilde{Y}\left(\frac{\log u}{2\alpha}\right)\right).$$

We have

$$\begin{aligned} A &= 2h^2(S_1) + \frac{1}{2\alpha^2}\int_{a_1}^{a_2} u^{-1}\left[\alpha h\left(\frac{\log u}{2\alpha}\right) + h'\left(\frac{\log u}{2\alpha}\right)\right]^2 \mathrm{d}u \\ &= 2h^2(S_1) + \alpha^{-1}\int_{S_1}^{S_2} (\alpha h(s) + h'(s))^2\, \mathrm{d}s \\ &= h^2(S_1) + h^2(S_2) + \int_{S_1}^{S_2}\left(\alpha h^2(s) + \alpha^{-1}\left[h'(s)\right]^2\right)\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} \zeta &= 2h(S_1)\widetilde{Y}(S_1) + \int_{a_1}^{a_2} u^{-1/2}\left[h\left(\frac{\log u}{2\alpha}\right) + \alpha^{-1}h'\left(\frac{\log u}{2\alpha}\right)\right]\mathrm{d}\left(\sqrt{u}\,\widetilde{Y}\left(\frac{\log u}{2\alpha}\right)\right) \\ &= 2h(S_1)\widetilde{Y}(S_1) + \int_{S_1}^{S_2}\left(h(s) + \alpha^{-1}h'(s)\right)\left(\mathrm{d}\widetilde{Y}(s) + \alpha\widetilde{Y}(s)\,\mathrm{d}s\right). \end{aligned}$$

The last statement in the Theorem can be proved by partial integration:

$$\int_{S_1}^{S_2} h'(s)\,\mathrm{d}\widetilde{Y}(s) = \Big[h'(s)\widetilde{Y}(s)\Big]_{s=S_1}^{s=S_2} - \int_{S_1}^{S_2} h''(s)\widetilde{Y}(s)\,\mathrm{d}s,$$

from which we obtain the formula for ζ given in the theorem. □

Proof of Theorem 3. We give the proof in case $\alpha > 0$. Now we use the representation (2.3) and apply Theorem 1 for the function g in (2.4) on the interval $[a_1, a_2] = \big[\mathrm{e}^{2\alpha S_1} - 1, \mathrm{e}^{2\alpha S_2} - 1\big]$. We obtain the statement in Theorem 3 with

$$A = 2\frac{a_1+1}{a_1} h^2\left(\frac{\log(a_1+1)}{2\alpha}\right) + 2\int_{a_1}^{a_2}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\sqrt{u+1}\,h\left(\frac{\log(u+1)}{2\alpha}\right)\right)\right]^2 \mathrm{d}u,$$

$$\zeta = 2\frac{a_1+1}{a_1} h\left(\frac{\log(a_1+1)}{2\alpha}\right) Y\left(\frac{\log(a_1+1)}{2\alpha}\right)$$
$$+ 2\int_{a_1}^{a_2}\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\sqrt{u+1}\,h\left(\frac{\log(u+1)}{2\alpha}\right)\right)\right] \mathrm{d}\left(\sqrt{u+1}\,Y\left(\frac{\log(u+1)}{2\alpha}\right)\right).$$

We have

$$\begin{aligned} A &= 2\frac{\mathrm{e}^{2\alpha S_1}}{\mathrm{e}^{2\alpha S_1}-1} h^2(S_1) + \frac{1}{2\alpha^2}\int_{a_1}^{a_2} (u+1)^{-1}\left[\alpha h\left(\frac{\log(u+1)}{2\alpha}\right) + h'\left(\frac{\log(u+1)}{2\alpha}\right)\right]^2 \mathrm{d}u \\ &= 2\frac{\mathrm{e}^{2\alpha S_1}}{\mathrm{e}^{2\alpha S_1}-1} h^2(S_1) + \alpha^{-1}\int_{S_1}^{S_2} (\alpha h(s) + h'(s))^2\,\mathrm{d}s \\ &= \coth(\alpha S_1) h^2(S_1) + h^2(S_2) + \int_{S_1}^{S_2}\left(\alpha h^2(s) + \alpha^{-1}\left[h'(s)\right]^2\right)\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} \zeta = {} & 2\frac{\mathrm{e}^{2\alpha S_1}}{\mathrm{e}^{2\alpha S_1}-1} h(S_1) Y(S_1) \\ & + \int_{a_1}^{a_2} (u+1)^{-1/2}\left[h\left(\frac{\log(u+1)}{2\alpha}\right) + \alpha^{-1} h'\left(\frac{\log(u+1)}{2\alpha}\right)\right] \mathrm{d}\left(\sqrt{u+1}\,Y\left(\frac{\log(u+1)}{2\alpha}\right)\right) \\ = {} & (1+\coth(\alpha S_1)) h(S_1) Y(S_1) + \int_{S_1}^{S_2}\left(h(s) + \alpha^{-1}h'(s)\right)\left(\mathrm{d}Y(s) + \alpha Y(s)\,\mathrm{d}s\right), \end{aligned}$$

and the last statement in Theorem 3 can be proved by partial integration. □

Proof of Theorem 4. Similar to the proof of Theorem 1. By Itô's formula it is easy to show that ζ/A has a normal distribution with mean m and variance $1/A$, and that

$$\mathsf{E}\exp\left\{-\frac{1}{2}(Am^2 - 2\eta m)\right\} = 1,$$

where

$$\eta = \frac{g(a_1,b_1)W(a_1,b_1)}{a_1 b_1} + \int_{a_1}^{a_2}\frac{\partial_1 g(u,b_1)}{b_1}\,W(\mathrm{d}u,b_1) + \int_{b_1}^{b_2}\frac{\partial_2 g(a_1,v)}{a_1}\,W(a_1,\mathrm{d}v)$$
$$+ \int_{a_1}^{a_2}\int_{b_1}^{b_2}\partial_1\partial_2 g(u,v)\,W(\mathrm{d}u,\mathrm{d}v).$$

Taking arbitrary sequences of partitions $a_1 = u_1^{(M)} < u_2^{(M)} < \cdots < u_M^{(M)} = a_2$, $M \in \mathbb{N}$, and $b_1 = v_1^{(N)} < v_2^{(N)} < \cdots < v_N^{(N)} = b_2$, $N \in \mathbb{N}$, such that $\lim\limits_{M\to\infty} \max\limits_{2\leqslant i\leqslant M} \Delta u_i^{(M)} = 0$ and $\lim\limits_{N\to\infty} \max\limits_{2\leqslant j\leqslant N} \Delta v_j^{(N)} = 0$, we will prove that for $M, N \to \infty$ we have

$$\frac{\mathrm{d}\mathsf{P}_Z^{(M,N)}}{\mathrm{d}\mathsf{P}_W^{(M,N)}}(W(u_i^{(M)}, v_j^{(N)}; 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N)) \xrightarrow{\mathsf{P}} \exp\left\{-\frac{1}{2}(Am^2 - 2\eta m)\right\}, \qquad (4.5)$$

where $\mathsf{P}_Z^{(M,N)}$ and $\mathsf{P}_W^{(M,N)}$ denote the probability measures generated by the random variables $(Z(u_i^{(M)}, v_j^{(N)}); 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N)$ and $(W(u_i^{(M)}, v_j^{(N)}); 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N)$, respectively. The joint density of $(W(u_i^{(M)}, v_j^{(N)}) : 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N)$ has the form

$$\begin{aligned}
&f(x_{i,j}; 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N)\\
&= c\exp\left\{-\frac{1}{2}\left(\frac{x_{1,1}^2}{u_1^{(M)}v_1^{(N)}} + \sum_{i=2}^{M}\frac{(\Delta_1 x_{i,1})^2}{v_1^{(N)}\Delta u_i^{(M)}} + \sum_{j=2}^{N}\frac{(\Delta_2 x_{1,j})^2}{u_1^{(M)}\Delta v_j^{(N)}} + \sum_{i=2}^{M}\sum_{j=2}^{N}\frac{(\Delta_1\Delta_2 x_{i,j})^2}{\Delta u_i^{(M)}\Delta v_j^{(N)}}\right)\right\},
\end{aligned}$$

where $\Delta_1 x_{i,j} := x_{i,j} - x_{i-1,j}$, $\Delta_2 x_{i,j} := x_{i,j} - x_{i,j-1}$, and c is a norming constant. Consequently, the joint density of $(Z(u_i^{(M)}, v_j^{(N)}) : 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N)$ is

$$\begin{aligned}
&g(z_{i,j}; 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N)\\
&= c\exp\Bigg\{-\frac{1}{2}\Bigg(\frac{(z_{1,1} - mg(u_1^{(M)}, v_1^{(N)}))^2}{u_1^{(M)}v_1^{(N)}} + \sum_{i=2}^{M}\frac{(\Delta_1 z_{i,1} - m\Delta_1 g(u_i^{(M)}, v_1^{(N)}))^2}{v_1^{(N)}\Delta u_i^{(M)}}\\
&\qquad + \sum_{j=2}^{N}\frac{(\Delta_2 z_{1,j} - m\Delta_2 g(u_1^{(M)}, v_j^{(N)}))^2}{u_1^{(M)}\Delta v_j^{(N)}}\\
&\qquad + \sum_{i=2}^{M}\sum_{j=2}^{N}\frac{(\Delta_1\Delta_2 z_{i,j} - m\Delta_1\Delta_2 g(u_i^{(M)}, v_j^{(N)}))^2}{\Delta u_i^{(M)}\Delta v_j^{(N)}}\Bigg)\Bigg\}.
\end{aligned}$$

Hence

$$\frac{\mathrm{d}\mathsf{P}_Z^{(M,N)}}{\mathrm{d}\mathsf{P}_W^{(M,N)}}(W(u_i^{(M)}, v_j^{(N)}); 1\leqslant i\leqslant M,\ 1\leqslant j\leqslant N) = \exp\left\{-\frac{1}{2}(m^2 A_{M,N} - 2m\eta_{M,N})\right\},$$

where

$$A_{M,N} = \frac{g^2(u_1^{(M)}, v_1^{(N)})}{u_1^{(M)} v_1^{(N)}} + \sum_{i=2}^{M} \frac{\left(\Delta_1 g(u_i^{(M)}, v_1^{(N)})\right)^2}{v_1^{(N)} \Delta u_i^{(M)}} + \sum_{j=2}^{N} \frac{\left(\Delta_2 g(u_1^{(M)}, v_j^{(N)})\right)^2}{u_1^{(M)} \Delta v_j^{(N)}}$$
$$+ \sum_{i=2}^{N} \sum_{j=2}^{N} \frac{\left(\Delta_1 \Delta_2 g(u_i^{(M)}, v_j^{(N)})\right)^2}{\Delta u_i^{(M)} \Delta v_j^{(N)}},$$

$$\eta_{M,N} = \frac{g(u_1^{(M)}, v_1^{(N)}) W(u_1^{(M)}, v_1^{(N)})}{u_1^{(M)} v_1^{(N)}} + \sum_{i=2}^{M} \frac{\Delta_1 g(u_i^{(M)}, v_1^{(N)}) \Delta_1 W(u_i^{(M)}, v_1^{(N)})}{v_1^{(N)} \Delta u_i^{(M)}}$$
$$+ \sum_{j=2}^{N} \frac{\Delta_2 g(u_1^{(M)}, v_j^{(N)}) \Delta_2 W(u_1^{(M)}, v_j^{(N)})}{u_1^{(M)} \Delta v_j^{(N)}}$$
$$+ \sum_{i=2}^{M} \sum_{j=2}^{N} \frac{\Delta_1 \Delta_2 g(u_i^{(M)}, v_j^{(N)}) \Delta_1 \Delta_2 W(u_i^{(M)}, v_j^{(N)})}{\Delta u_i^{(M)} \Delta v_j^{(N)}}.$$

Applying Lemma 2 we obtain $\lim_{M,N \to \infty} A_{M,N} = A$ and $g_{M,N} \to \partial_1 \partial_2 g$ in $L^2([a_1, a_2] \times [b_1, b_2])$ as $M, N \to \infty$, which also implies $\eta_{M,N} \to \eta$ in L^2 as $M, N \to \infty$, hence we conclude in fact L^2–convergence in (4.5). □

Proof of Theorem 5. Using the representation (3.1) and applying Theorem 4 for the function g in (3.2) on the rectangular $[a_1, a_2] \times [b_1, b_2] = \left[e^{2\alpha S_1}, e^{2\alpha S_2}\right] \times \left[e^{2\beta T_1}, e^{2\beta T_2}\right]$ we obtain that the statement in Theorem 5 holds with

$$A = 4h^2 \left(\frac{\log a_1}{2\alpha}, \frac{\log b_1}{2\beta}\right) + 4 \int_{a_1}^{a_2} \left[\frac{\partial}{\partial u} \left(\sqrt{u}\, h \left(\frac{\log u}{2\alpha}, \frac{\log b_1}{2\beta}\right)\right)\right]^2 \mathrm{d}u$$
$$+ 4 \int_{b_1}^{b_2} \left[\frac{\partial}{\partial v} \left(\sqrt{v}\, h \left(\frac{\log a_1}{2\alpha}, \frac{\log v}{2\beta}\right)\right)\right]^2 \mathrm{d}v$$
$$+ 4 \int_{a_1}^{a_2} \int_{b_1}^{b_2} \left[\frac{\partial^2}{\partial u \partial v} \left(\sqrt{uv}\, h \left(\frac{\log u}{2\alpha}, \frac{\log v}{2\beta}\right)\right)\right]^2 \mathrm{d}u \mathrm{d}v,$$

$$\zeta = 4h \left(\frac{\log a_1}{2\alpha}, \frac{\log b_1}{2\beta}\right) \widetilde{Y} \left(\frac{\log a_1}{2\alpha}, \frac{\log b_1}{2\beta}\right)$$
$$+ 4 \int_{a_1}^{a_2} \left[\frac{\partial}{\partial u} \left(\sqrt{u}\, h \left(\frac{\log u}{2\alpha}, \frac{\log b_1}{2\beta}\right)\right)\right] \mathrm{d} \left(\sqrt{u}\, \widetilde{Y} \left(\frac{\log u}{2\alpha}, \frac{\log b_1}{2\beta}\right)\right)$$
$$+ 4 \int_{b_1}^{b_2} \left[\frac{\partial}{\partial v} \left(\sqrt{v}\, h \left(\frac{\log a_1}{2\alpha}, \frac{\log v}{2\beta}\right)\right)\right] \mathrm{d} \left(\sqrt{v}\, \widetilde{Y} \left(\frac{\log a_1}{2\alpha}, \frac{\log v}{2\beta}\right)\right)$$
$$+ 4 \int_{a_1}^{a_2} \int_{b_1}^{b_2} \left[\frac{\partial^2}{\partial u \partial v} \left(\sqrt{uv}\, h \left(\frac{\log u}{2\alpha}, \frac{\log v}{2\beta}\right)\right)\right] \mathrm{d} \left(\sqrt{uv}\, \widetilde{Y} \left(\frac{\log u}{2\alpha}, \frac{\log v}{2\beta}\right)\right).$$

We have

$$
\begin{aligned}
A &= 4h^2(S_1,T_1) + \frac{1}{\alpha^2}\int_{a_1}^{a_2} u^{-1}\left[(\alpha+\partial_1)h\left(\frac{\log u}{2\alpha},\frac{\log b_1}{2\beta}\right)\right]^2 \mathrm{d}u \\
&\quad + \frac{1}{\beta^2}\int_{b_1}^{b_2} v^{-1}\left[(\beta+\partial_2)h\left(\frac{\log a_1}{2\alpha},\frac{\log v}{2\beta}\right)\right]^2 \mathrm{d}v \\
&\quad + \frac{1}{4\alpha^2\beta^2}\int_{a_1}^{a_2}\int_{b_1}^{b_2} u^{-1}v^{-1}\left[(\alpha+\partial_1)(\beta+\partial_2)h\left(\frac{\log u}{2\alpha},\frac{\log v}{2\beta}\right)\right]^2 \mathrm{d}u\mathrm{d}v \\
&= 4h^2(S_1,T_1) + 2\alpha^{-1}\int_{S_1}^{S_2}\left[(\alpha+\partial_1)h(s,T_1)\right]^2\,\mathrm{d}s \\
&\quad + 2\beta^{-1}\int_{T_1}^{T_2}\left[(\beta+\partial_2)h(S_1,t)\right]^2\,\mathrm{d}t + \alpha^{-1}\beta^{-1}\int_{S_1}^{S_2}\int_{T_1}^{T_2}\left[(\alpha+\partial_1)(\beta+\partial_2)h(s,t)\right]^2\,\mathrm{d}s\mathrm{d}t \\
&= 4h^2(S_1,T_1) + 2\int_{S_1}^{S_2}\Big(\alpha h^2(s,T_1)+\alpha^{-1}[\partial_1 h(s,T_1)]^2\Big)\,\mathrm{d}s + 2\left[h^2(s,T_1)\right]_{s=S_1}^{s=S_2} \\
&\quad + 2\int_{T_1}^{T_2}\Big(\beta h^2(S_1,t)+\beta^{-1}[\partial_2 h(S_1,t)]^2\Big)\,\mathrm{d}t + 2\left[h^2(S_1,t)\right]_{t=T_1}^{t=T_2} + \Big[\left[h^2(s,t)\right]_{s=S_1}^{s=S_2}\Big]_{t=T_1}^{t=T_2} \\
&\quad + \int_{S_1}^{S_2}\int_{T_1}^{T_2}\Big(\alpha\beta h^2(s,t)+\alpha^{-1}\beta\,[\partial_1 h(s,t)]^2+\alpha\beta^{-1}\,[\partial_2 h(s,t)]^2+\alpha^{-1}\beta^{-1}\,[\partial_1\partial_2 h(s,t)]^2\Big)\,\mathrm{d}s\mathrm{d}t \\
&\quad + \int_{S_1}^{S_2}\Big[\alpha h^2(s,t)+\alpha^{-1}\,[\partial_1 h(s,t)]^2\Big]_{t=T_1}^{t=T_2}\mathrm{d}s + \int_{T_1}^{T_2}\Big[\beta h^2(s,t)+\beta^{-1}\,[\partial_2 h(s,t)]^2\Big]_{s=S_1}^{s=S_2}\mathrm{d}t,
\end{aligned}
$$

hence

$$
\begin{aligned}
A &= h^2(S_1,T_1)+h^2(S_1,T_2)+h^2(S_2,T_1)+h^2(S_2,T_2) \\
&\quad + \int_{S_1}^{S_2}\Big(\alpha\big(h^2(s,T_1)+h^2(s,T_2)\big)+\alpha^{-1}\big([\partial_1 h(s,T_1)]^2+[\partial_1 h(s,T_2)]^2\big)\Big)\,\mathrm{d}s \\
&\quad + \int_{T_1}^{T_2}\Big(\beta\big(h^2(S_1,t)+h^2(S_2,t)\big)+\beta^{-1}\big([\partial_2 h(S_1,t)]^2+[\partial_2 h(S_2,t)]^2\big)\Big)\,\mathrm{d}t \\
&\quad + \int_{S_1}^{S_2}\int_{T_1}^{T_2}\Big(\alpha\beta h^2(s,t)+\alpha^{-1}\beta\,[\partial_1 h(s,t)]^2+\alpha\beta^{-1}\,[\partial_2 h(s,t)]^2+\alpha^{-1}\beta^{-1}\,[\partial_1\partial_2 h(s,t)]^2\Big)\,\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\zeta &= 4h(S_1,T_1)\widetilde{Y}(S_1,T_1) + 2\int_{a_1}^{a_2} u^{-1/2}(1+\alpha^{-1}\partial_1)h\left(\frac{\log u}{2\alpha},\frac{\log b_1}{2\beta}\right)\mathrm{d}\left(\sqrt{u}\,\widetilde{Y}\left(\frac{\log u}{2\alpha},\frac{\log b_1}{2\beta}\right)\right)\\
&\quad + 2\int_{b_1}^{b_2} v^{-1/2}(1+\beta^{-1}\partial_2)h\left(\frac{\log a_1}{2\alpha},\frac{\log v}{2\beta}\right)\mathrm{d}\left(\sqrt{v}\,\widetilde{Y}\left(\frac{\log a_1}{2\alpha},\frac{\log v}{2\beta}\right)\right)\\
&\quad + \int_{a_1}^{a_2}\int_{b_1}^{b_2}(uv)^{-1/2}(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h\left(\frac{\log u}{2\alpha},\frac{\log v}{2\beta}\right)\mathrm{d}\left(\sqrt{uv}\,\widetilde{Y}\left(\frac{\log u}{2\alpha},\frac{\log v}{2\beta}\right)\right)\\
&= 4h(S_1,T_1)\widetilde{Y}(S_1,T_1) + 2\int_{S_1}^{S_2}(1+\alpha^{-1}\partial_1)h(s,T_1)\left(\widetilde{Y}(\mathrm{d}s,T_1)+\alpha\widetilde{Y}(s,T_1)\,\mathrm{d}s\right)\\
&\quad + 2\int_{T_1}^{T_2}(1+\beta^{-1}\partial_2)h(S_1,t)\left(\widetilde{Y}(S_1,\mathrm{d}t)+\beta\widetilde{Y}(S_1,t)\,\mathrm{d}t\right)\\
&\quad + \int_{S_1}^{S_2}\int_{T_1}^{T_2}(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(s,t)\\
&\qquad\qquad \left(\widetilde{Y}(\mathrm{d}s,\mathrm{d}t)+\alpha\widetilde{Y}(s,\mathrm{d}t)\,\mathrm{d}s+\beta\widetilde{Y}(\mathrm{d}s,t)\,\mathrm{d}t+\alpha\beta\widetilde{Y}(s,t)\,\mathrm{d}s\mathrm{d}t\right).
\end{aligned}
$$

The last statement can be proved by partial integration:

$$
\begin{aligned}
\int_{S_1}^{S_2}(1+\alpha^{-1}\partial_1)h(s,T_1)\widetilde{Y}(\mathrm{d}s,T_1) &= \Big[(1+\alpha^{-1}\partial_1)h(s,T_1)\widetilde{Y}(s,T_1)\Big]_{s=S_1}^{s=S_2}\\
&\quad - \int_{S_1}^{S_2}(\partial_1+\alpha^{-1}\partial_1^2)h(s,T_1)\widetilde{Y}(s,T_1)\,\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\int_{T_1}^{T_2}(1+\beta^{-1}\partial_2)h(S_1,t)\widetilde{Y}(S_1,\mathrm{d}t) &= \Big[(1+\beta^{-1}\partial_2)h(S_1,t)\widetilde{Y}(S_1,t)\Big]_{t=T_1}^{t=T_2}\\
&\quad - \int_{T_1}^{T_2}(\partial_2+\beta^{-1}\partial_2^2)h(S_1,t)\widetilde{Y}(S_1,t)\,\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{S_1}^{S_2}\int_{T_1}^{T_2}(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(s,\mathrm{d}t)\,\mathrm{d}s\\
&\qquad = \int_{S_1}^{S_2}\Big[(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(s,t)\Big]_{t=T_1}^{t=T_2}\mathrm{d}s\\
&\qquad\quad - \int_{S_1}^{S_2}\int_{T_1}^{T_2}(1+\alpha^{-1}\partial_1)(\partial_2+\beta^{-1}\partial_2^2)h(s,t)\widetilde{Y}(s,t)\,\mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

$$\int_{S_1}^{S_2}\int_{T_1}^{T_2}(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(\mathrm{d}s,t)\,\mathrm{d}t$$
$$=\int_{T_1}^{T_2}\Big[(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(s,t)\Big]_{s=S_1}^{s=S_2}\mathrm{d}t$$
$$-\int_{S_1}^{S_2}\int_{T_1}^{T_2}(\partial_1+\alpha^{-1}\partial_1^2)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(s,t)\,\mathrm{d}s\mathrm{d}t,$$

$$\int_{S_1}^{S_2}\int_{T_1}^{T_2}(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(\mathrm{d}s,\mathrm{d}t)$$
$$=\Big[\Big[(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(s,t)\Big]_{s=S_1}^{s=S_2}\Big]_{t=T_1}^{t=T_2}$$
$$-\int_{S_1}^{S_2}\Big[(\partial_1+\alpha^{-1}\partial_1^2)(1+\beta^{-1}\partial_2)h(s,t)\widetilde{Y}(s,t)\Big]_{t=T_1}^{t=T_2}\mathrm{d}s$$
$$-\int_{T_1}^{T_2}\Big[(1+\alpha^{-1}\partial_1)(\partial_2+\beta^{-1}\partial_2^2)h(s,t)\widetilde{Y}(s,t)\Big]_{s=S_1}^{s=S_2}\mathrm{d}t$$
$$+\int_{S_1}^{S_2}\int_{T_1}^{T_2}(\partial_1+\alpha^{-1}\partial_1^2)(\partial_2+\beta^{-1}\partial_2^2)h(s,t)\widetilde{Y}(s,t)\,\mathrm{d}s\mathrm{d}t,$$

from which we obtain the formula given in the theorem. □

Proof of Theorem 6. We give the proof in case $\alpha>0$, $\beta>0$. Now we use the representation (3.3) and apply Theorem 4 for the function g in (3.4) on the rectangular

$$[a_1,a_2]\times[b_1,b_2]=\left[\mathrm{e}^{2\alpha S_1}-1,\mathrm{e}^{2\alpha S_2}-1\right]\times\left[\mathrm{e}^{2\beta T_1}-1,\mathrm{e}^{2\beta T_2}-1\right].$$

We obtain

$$\zeta=4\frac{(a_1+1)(b_1+1)}{a_1b_1}h\left(\frac{\log(a_1+1)}{2\alpha},\frac{\log(b_1+1)}{2\beta}\right)Y\left(\frac{\log(a_1+1)}{2\alpha},\frac{\log(b_1+1)}{2\beta}\right)$$
$$+4\frac{b_1+1}{b_1}\int_{a_1}^{a_2}\frac{\partial}{\partial u}\left(\sqrt{u+1}\,h\left(\frac{\log(u+1)}{2\alpha},\frac{\log(b_1+1)}{2\beta}\right)\right)$$
$$\mathrm{d}\left(\sqrt{u+1}\,Y\left(\frac{\log(u+1)}{2\alpha},\frac{\log(b_1+1)}{2\beta}\right)\right)$$
$$+4\frac{a_1+1}{a_1}\int_{b_1}^{b_2}\frac{\partial}{\partial v}\left(\sqrt{v+1}\,h\left(\frac{\log(a_1+1)}{2\alpha},\frac{\log(v+1)}{2\beta}\right)\right)$$
$$\mathrm{d}\left(\sqrt{v+1}\,Y\left(\frac{\log(a_1+1)}{2\alpha},\frac{\log(v+1)}{2\beta}\right)\right)$$
$$+4\int_{a_1}^{a_2}\int_{b_1}^{b_2}\frac{\partial^2}{\partial u\partial v}\left(\sqrt{(u+1)(v+1)}\,h\left(\frac{\log(u+1)}{2\alpha},\frac{\log(v+1)}{2\beta}\right)\right)$$
$$\mathrm{d}\left(\sqrt{(u+1)(v+1)}\,Y\left(\frac{\log(u+1)}{2\alpha},\frac{\log(v+1)}{2\beta}\right)\right),$$

$$
\begin{aligned}
A = {} & 4\frac{(a_1+1)(b_1+1)}{a_1 b_1} h^2\left(\frac{\log(a_1+1)}{2\alpha}, \frac{\log(b_1+1)}{2\beta}\right) \\
& + 4\frac{b_1+1}{b_1}\int_{a_1}^{a_2}\left[\frac{\partial}{\partial u}\left(\sqrt{u+1}\, h\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(b_1+1)}{2\beta}\right)\right)\right]^2 \mathrm{d}u, \\
& + 4\frac{a_1+1}{a_1}\int_{b_1}^{b_2}\left[\frac{\partial}{\partial v}\left(\sqrt{v+1}\, h\left(\frac{\log(a_1+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right)\right)\right]^2 \mathrm{d}v, \\
& + 4\int_{a_1}^{a_2}\int_{b_1}^{b_2}\left[\frac{\partial^2}{\partial u \partial v}\left(\sqrt{(u+1)(v+1)}\, h\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right)\right)\right]^2 \mathrm{d}u\mathrm{d}v.
\end{aligned}
$$

We have

$$
\begin{aligned}
A = {} & 4\frac{\mathrm{e}^{2\alpha S_1 + 2\beta T_1}}{(\mathrm{e}^{2\alpha S_1}-1)(\mathrm{e}^{2\beta T_1}-1)} h^2(S_1, T_1) \\
& + \frac{\mathrm{e}^{2\beta T_1}}{\alpha^2(\mathrm{e}^{2\beta T_1}-1)}\int_{a_1}^{a_2}(u+1)^{-1}\left[(\alpha h + \partial_1)h\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(b_1+1)}{2\beta}\right)\right]^2 \mathrm{d}u \\
& + \frac{\mathrm{e}^{2\alpha S_1}}{\beta^2(\mathrm{e}^{2\alpha S_1}-1)}\int_{b_1}^{b_2}(v+1)^{-1}\left[(\beta h + \partial_2)h\left(\frac{\log(a_1+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right)\right]^2 \mathrm{d}v \\
& + \frac{1}{4\alpha^2\beta^2}\int_{a_1}^{a_2}\int_{b_1}^{b_2}(u+1)^{-1}(v+1)^{-1}\left[(\alpha h + \partial_1)(\beta h + \partial_2)h\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right)\right]^2 \mathrm{d}u\mathrm{d}v \\
= {} & \frac{4\mathrm{e}^{2\alpha S_1 + 2\beta T_1}}{(\mathrm{e}^{2\alpha S_1}-1)(\mathrm{e}^{2\beta T_1}-1)} h^2(S_1, T_1) + \frac{2\mathrm{e}^{2\beta T_1}}{\alpha(\mathrm{e}^{2\beta T_1}-1)}\int_{S_1}^{S_2}\left[(\alpha+\partial_1)h(s, T_1)\right]^2 \mathrm{d}s \\
& + \frac{2\mathrm{e}^{2\alpha S_1}}{\beta(\mathrm{e}^{2\alpha S_1}-1)}\int_{T_1}^{T_2}\left[(\beta+\partial_2)h(S_1, t)\right]^2 \mathrm{d}t + (\alpha\beta)^{-1}\int_{S_1}^{S_2}\int_{T_1}^{T_2}\left[(\alpha+\partial_1)(\beta+\partial_2)h(s,t)\right]^2 \mathrm{d}s\mathrm{d}t,
\end{aligned}
$$

which equals the formula given in the theorem.

Moreover,

$$
\begin{aligned}
\zeta = & \frac{4\mathrm{e}^{2\alpha S_1+2\beta T_1}}{(\mathrm{e}^{2\alpha S_1}-1)(\mathrm{e}^{2\beta T_1}-1)} h(S_1,T_1)Y(S_1,T_1) \\
& + \frac{2\mathrm{e}^{2\beta T_1}}{\alpha^2(\mathrm{e}^{2\beta T_1}-1)} \int_{a_1}^{a_2} (u+1)^{-1/2}(1+\alpha^{-1}\partial_1)h\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(b_1+1)}{2\beta}\right) \\
& \qquad \mathrm{d}\left(\sqrt{u+1}\,Y\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(b_1+1)}{2\beta}\right)\right) \\
& + \frac{2\mathrm{e}^{2\alpha S_1}}{\beta^2(\mathrm{e}^{2\alpha S_1}-1)} \int_{b_1}^{b_2} (v+1)^{-1/2}(1+\beta^{-1}\partial_2)h\left(\frac{\log(a_1+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right) \\
& \qquad \mathrm{d}\left(\sqrt{v+1}\,Y\left(\frac{\log(a_1+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right)\right) \\
& + \int_{a_1}^{a_2}\int_{b_1}^{b_2} (u+1)^{-1/2}(v+1)^{-1/2}(1+\alpha^{-1}\partial_1)(1+\beta^{-1}\partial_2)h\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right) \\
& \qquad \mathrm{d}\left(\sqrt{(u+1)(v+1)}\,Y\left(\frac{\log(u+1)}{2\alpha}, \frac{\log(v+1)}{2\beta}\right)\right),
\end{aligned}
$$

which equals the formula given in the theorem. The last statement can be proved by partial integration. □

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