AN AMUSING IDENTITY

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Abstract

We present a family of identities for integrals of square integrable functions. We also derive discrete analogues for square summable sequences.

Introduction

Let \( f \in L^2[0, \infty) \) and assume that

\[
x \to \frac{f(x)}{x} \in L^1[0, 1].
\]

Since \( x \to \frac{1}{x} \in L^2[1, \infty) \) this assumption implies that

\[
x \to \frac{f(x)}{x} \in L^1[0, \infty).
\]

We shall show that for every \( \tau > 0 \) we have

\[
\int_0^\tau \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^\tau \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx + \left( \sqrt{\tau} \int_0^\infty \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) \, dt \right)^2
\]

and

\[
\int_0^\infty \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx.
\]

For the discrete analogue we consider sequences \( a_1, a_2, \ldots \in l^2 \). We define \( A_0 = 0 \), \( A_k = \sum_{n=1}^k a_n \) \((k = 1, 2, \ldots)\) and \( \alpha_k = \sum_{n=k}^\infty \frac{a_n}{n} \). (Note that \( \alpha_k \) exists; both sequences \( a_1, a_2, \ldots \) and \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \) belong to \( l^2 \).)

We shall prove for every \( k \)

\[
\sum_{n=1}^k \alpha_n^2 = \sum_{n=1}^{k-1} \frac{1}{n(n+1)} A_n^2 + \left( \sqrt{k} \alpha_k + \frac{1}{\sqrt{k}} A_{k-1} \right)^2
\]

and

\[
\sum_{n=1}^\infty \alpha_n^2 = \sum_{n=1}^\infty \frac{1}{n(n+1)} A_n^2.
\]
Preliminaries

For the proofs of these identities we shall make use of some general results for integrable functions. In this section we state and prove these auxiliary results.

**Lemma 1** Let $f^2$ be integrable on $[0, 1]$; then

$$\lim_{a \to 0} \frac{1}{\sqrt{a}} \int_0^a f(t) dt = 0.$$  

**Proof:**

$$\int_0^a |f(t)| dt \leq \int_0^a |f^2(t)| dt = a \int_0^a |f^2(t)| dt,$$

and the assertion follows immediately.

**Lemma 2** Let $f \in L^2$, let $a_1, a_2, \ldots \in l^2$. Then we have

$$\lim_{\tau \to \infty} \sqrt{\tau} \int_\tau^\infty \frac{f(t)}{t} dt = 0,$$

and its discrete analogue

$$\lim_{k \to \infty} \sqrt{k} \sum_{n=k}^\infty \frac{a_n}{n} = 0.$$  

**Proof:** The first assertion follows from

$$\left| \int_\tau^\infty \frac{f(t)}{t} dt \right|^2 \leq \int_\tau^\infty \frac{1}{t^2} dt \int_\tau^\infty |f^2(t)| dt = \frac{1}{\tau} \int_\tau^\infty |f^2(t)| dt;$$

the second follows from

$$\left( \sum_{n=k}^\infty \frac{a_n}{n} \right)^2 \leq \sum_{n=k}^\infty \frac{1}{n^2} \sum_{n=k}^\infty |a_n|^2 \leq \frac{1}{k-1} \sum_{n=k}^\infty |a_n|^2.$$  

**Lemma 3** Let $f \in L^2$, let $a_1, a_2, \ldots \in l^2$. Then we have

$$\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) dt = 0,$$

and its discrete analogue

$$\lim_{k \to \infty} \frac{1}{\sqrt{k}} \sum_{n=1}^k a_n = 0.$$  

**Proof:** Let $\varepsilon > 0$ be given; choose $b$ so large that

$$\int_b^\infty |f^2(t)| dt < \frac{1}{4} \varepsilon^2.$$
For $\tau > b$ we have
\[
\left| \int_0^{\tau} f(t)dt \right| \leq \int_0^b |f(t)|dt + \int_b^{\tau} |f(t)|dt \leq \int_0^b |f(t)|dt + \sqrt{\int_b^{\tau} |f(t)|dt} \cdot \sqrt{\int_0^{\tau} |f^2(t)|dt}
\leq \int_0^b |f(t)|dt + \sqrt{\tau - b} \cdot \sqrt{\int_0^{\tau} |f^2(t)|dt} \leq \int_0^b |f(t)|dt + \frac{1}{2} \epsilon \sqrt{\tau - b}
\]
and the first assertion follows. The proof of the second assertion is completely analogous.

**Lemma 4** Let $I$ be either $[0,1]$ or $[0,\infty)$. The functions $f \in L^2(I)$ such that $x \to \frac{f(x)}{x}$ are dense subset of $L^2(I)$.

**Proof:** Let $f \in L^2(I)$ and let $f_k = \left(1 - \frac{1}{k}, \frac{1}{k}\right) f$. Then $f_k \in L^2(I)$ and $\frac{1}{k} f_k(x) \in L^1(I)$ and $\|f - f_k\|_2 = \frac{1}{k} \int |f(x)|^2 dx \to 0$.

**Lemma 5** Let $f, g \in L^2(0, \infty)$. Then we have
\[
\lim_{a \to \infty} \int_a^\infty \frac{g(t)}{t} dt \cdot \int_0^{a} f(t)dt = 0.
\]

**Proof:** We have
\[
\left| \int_a^\infty \frac{g(t)}{t} dt \cdot \int_0^{a} f(t)dt \right| \leq \sqrt{\int_a^\infty \frac{g^2(t)}{t} dt} \cdot \sqrt{\int_0^a \frac{1}{t^2} dt} \leq \frac{1}{\sqrt{a}} \|g\|_2^2.
\]
Therefore
\[
\left| \int_a^\infty \frac{g(t)}{t} dt \cdot \int_0^{a} f(t)dt \right| \leq \|g\|_2 \cdot \frac{1}{\sqrt{a}} \int_0^a |f(t)| dt
\]
and the assertion follows from lemma 3.

**Historical remarks**

The identities (1), (2) and (3), (4) are inspired by classical inequalities proved by Hardy in the beginning of the 20th century (and refined by Landau) ([2], page 239). In order to obtain an appropriate setting for our identities we include simple proofs for (generalisations of) Hardy-Landau’s inequality.

**Theorem 1** Let $f \in L^2(0, \infty)$ and let
\[
F(x) = \frac{1}{x} \int_0^x f(t)dt.
\]
Then $F \in L^2(0, \infty)$ and $\|F - f\| = \|f\|$. In particular we have $\|F\| \leq 2\|f\|$.
**Proof:** Assume first that \( f \) is real valued. From

\[
\frac{d}{dx} (xF^2(x)) = 2f(x)F(x) - F^2(x)
\]

we see that for every \( a > 0 \)

\[
2 \int_0^a f(x)F(x)dx - \int_0^a F^2(x)dx = \left[ \frac{1}{x} \left( \int_0^x f(t)dt \right)^2 \right]_0^a,
\]

and by an application of lemma 1

\[
2 \int_0^a f(x)F(x)dx - \int_0^a F^2(x)dx = \frac{1}{a} \left( \int_0^a f(t)dt \right)^2. \tag{5}
\]

Consequently

\[
\int_0^a F^2(x)dx \leq 2 \int_0^a f(x)F(x)dx
\]

and by Schwarz's inequality

\[
\int_0^a F^2(x)dx \leq 2 \sqrt{\int_0^a f^2(x)dx} \cdot \sqrt{\int_0^a F^2(x)dx},
\]

i.e.

\[
\int_0^a F^2(x)dx \leq 4 \int_0^0 f^2(x)dx \tag{6}
\]

This shows that \( F \in L^2[0, \infty) \) (and that \( ||F|| \leq 2||f|| \)). Therefore we can let \( a \) tend to infinity in (5) and we obtain from lemma 3:

\[
2 \int_0^\infty f(x)F(x)dx - \int_0^\infty F^2(x)dx = 0,
\]

i.e.

\[
\int_0^\infty (F(x) - f(x))^2 dx = \int_0^\infty f^2(x)dx. \tag{7}
\]

Thus we have

\[
||F - f|| = ||f||.
\]

In the general case where we do not assume \( f \) to be real valued, we apply (6) and (7) to \( \text{Re } f \) as well as to \( \text{Im } f \) and add the results.
Theorem 2 Let \( a_1, a_2, a_3, \ldots \in l^2 \) and let \( A_k = \sum_{n=1}^{k} a_n \) \((k = 1, 2, \ldots)\). Then \( A_1, 1/2 A_2, 1/3 A_3, \ldots \in l^2 \),

\[
\sum_{n=1}^{k} \frac{1}{n^2} |A_n|^2 \leq 4 \sum_{n=1}^{k} |a_n|^2 \quad k = 1, 2, \ldots,
\]

and

\[
\sum_{n=1}^{\infty} \left| \frac{1}{n} A_n - a_n \right|^2 \leq \sum_{n=1}^{\infty} |a_n|^2.
\]

Proof: Assume first that all the numbers \( a_1, a_2, a_3, \ldots \) are real. For every \( n \in \{2, 3, 4, \ldots\} \) we have

\[
\left( \frac{1}{n} A_n \right)^2 - \frac{2}{n} A_n a_n = \left( \frac{1}{n} A_n \right)^2 - \frac{2}{n} A_n (A_n - A_{n-1}) = \\
(1 - 2n) \left( \frac{1}{n} A_n \right)^2 + 2(n-1) \frac{1}{n} A_n \frac{1}{n-1} A_{n-1} \leq \\
(1 - 2n) \left( \frac{1}{n} A_n \right)^2 + (n-1) \left\{ \left( \frac{1}{n} A_n \right)^2 + \left( \frac{1}{n-1} A_{n-1} \right)^2 \right\} = \\
\frac{1}{n-1} A_{n-1}^2 - \frac{1}{n} A_n^2,
\]

and for \( n = 1 \) we have

\[ A_1^2 - 2 A_1 a_1 = -A_1^2. \]

This shows that

\[
\sum_{n=1}^{k} \left\{ \left( \frac{1}{n} A_n \right)^2 - \frac{2}{n} A_n a_n \right\} \leq -\frac{1}{k} A_k^2 \leq 0,
\]

i.e.

\[
\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2 \leq 2 \sum_{n=1}^{k} \frac{1}{n} A_n a_n,
\]

and by Schwarz’s inequality

\[
\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2 \leq 2 \sqrt{\sum_{n=1}^{k} a_n^2} \cdot \sqrt{\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2},
\]

so

\[
\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2 \leq 4 \sum_{n=1}^{k} a_n^2.
\]
This shows that $A_1, A_2, A_3, \ldots \in l^2$.
Thus we can rewrite (8) and send $k$ to infinity. We obtain
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} A_n - a_n \right)^2 \leq \sum_{n=1}^{\infty} a_n^2. \] (10)

In the general case where we do not assume that the numbers $a_1, a_2, a_3, \ldots$ are real, we apply (9) and (10) to $\text{Re} a_n$ and $\text{Im} a_n$ and add the results.

**The identity for functions**

Let $f \in L^2[0, \infty)$ and assume that
\[ x \to \frac{f(x)}{x} \in L^1[0, \infty); \]
then we have for every $\tau > 0$
\[ \int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_{0}^{\tau} \left( \frac{1}{x} \int_{0}^{x} f(t) \, dt \right)^2 \, dx + \left( \sqrt{\tau} \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{\tau}} \int_{0}^{\tau} f(t) \, dt \right)^2. \]

**Proof:** Integration by parts shows that
\[ \int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right)^2 \, dx = \left[ x \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right) \right]_{x=0}^{x=x} + 2 \int_{0}^{\tau} x \cdot \frac{f(x)}{x} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right) \, dx = \tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(x) \int_{x}^{\infty} \frac{f(t)}{t} \, dt \, dx. \]

Another integration by parts leads to
\[ \tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \int_{0}^{\tau} \frac{1}{x} \left( \int_{0}^{x} f(t) \, dt \right)^2 \, dx. \]

Thus we have
\[ \int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right)^2 \, dx - \int_{0}^{\tau} \left( \frac{1}{x} \int_{0}^{x} f(t) \, dt \right)^2 \, dx = \tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \int_{0}^{\tau} \frac{1}{x} \left( \int_{0}^{x} f(t) \, dt \right)^2 \, dx = \tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \int_{0}^{\tau} \frac{1}{x} \left( \int_{0}^{x} f(t) \, dt \right)^2 \, dx = \tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \frac{1}{x} \left( \int_{0}^{x} f(t) \, dt \right)^2 \right]_{0}^{\tau}. \]
Lemma 1 implies that the third term gives no contribution at 0, so we are left with
\[
\tau \left( \int_0^\infty \frac{f(t)}{t} \, dt \right)^2 + 2 \int_0^\tau f(t) \, dt \int_\tau^\infty \frac{f(t)}{t} \, dt + \frac{1}{\tau} \left( \int_0^\tau f(t) \, dt \right)^2 =
\]
\[
\left( \sqrt{\tau} \int_\tau^\infty \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) \, dt \right)^2.
\]

Lemma 2 and 3 show that \( \lim_{\tau \to \infty} \) of this expression equals zero, hence
\[
\int_0^\infty \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 = \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx.
\]

If we apply the identities to Re \( f \) and to Im \( f \) and add them, we obtain
\[
\int_0^\tau \left| \int_x^\infty \frac{f(t)}{t} \, dt \right|^2 \, dx = \int_0^\tau \left| \frac{1}{x} \int_0^x f(t) \, dt \right|^2 \, dx + \left| \sqrt{\tau} \int_\tau^\infty \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) \, dt \right|^2
\]
and
\[
\int_0^\infty \left| \int_x^\infty \frac{f(t)}{t} \, dt \right|^2 \, dx = \int_0^\infty \left| \frac{1}{x} \int_0^x f(t) \, dt \right|^2 \, dx.
\]

Application of (1) to \( f \cdot 1_{[0,1]} \) with \( 0 < \tau < 1 \) and taking \( \lim_{\tau \to 1} \) leads to
\[
\int_0^1 \left( \int_x^1 \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^1 \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx + \left( \int_0^1 f(t) \, dt \right)^2,
\]
and in a similar way as before to
\[
\int_0^1 \left| \int_x^1 \frac{f(t)}{t} \, dt \right|^2 \, dx = \int_0^1 \left| \frac{1}{x} \int_0^x f(t) \, dt \right|^2 \, dx + \left| \int_0^1 f(t) \, dt \right|^2.
\]

**Example**

Let \( a > 0 \) and let
\[
f : t \to \frac{1}{a+t}.
\]

Then the identity (2) leads to the well-known but nevertheless charming result
\[
\int_0^\infty \left( \frac{1}{x} \log \frac{a+x}{a} \right)^2 \, dx = \int_0^\infty \left( \frac{1}{a} \log \frac{a+x}{x} \right)^2 \, dx.
\]
The identity for sequences

The discrete analogues (3) and (4) of (1) and (2) are a bit more delicate. We have to replace integration by parts, with Abel’s partial summation formula:

\[
\sum_{k=1}^{m} b_k c_k = B_m c_m + \sum_{k=1}^{m-1} B_k (c_k - c_{k+1}),
\]

where \( B_k = \sum_{n=1}^{k} b_n \quad k = 1, 2, \ldots \).

Let \( a_1, a_2, a_3, \ldots \in \ell^2 \). Define \( A_k = \sum_{n=1}^{k} a_n \quad k = 1, 2, \ldots \) and

\[
\alpha_k = \sum_{n=k}^{\infty} \frac{a_n}{n}.
\]

Then we have for every \( k \)

\[
\sum_{n=1}^{k} \alpha_n^2 = \sum_{n=1}^{k} \frac{1}{n(n+1)} A_n^2 + \left( \sqrt{k} \alpha_k + \frac{1}{\sqrt{k}} A_{k-1} \right)^2.
\]

**Proof:** A first application of Abel’s partial summation formula leads to

\[
\sum_{n=1}^{k} \alpha_n^2 = k \alpha_k^2 + \sum_{n=k}^{k-1} n \left( \alpha_n^2 - \alpha_{n+1}^2 \right) = k \alpha_k^2 + \sum_{n=1}^{k-1} \left( a_n - a_{n+1} \right) \left( a_n + a_{n+1} \right) =
\]

\[
= k \alpha_k^2 + \sum_{n=1}^{k-1} a_n (a_n + a_{n+1}).
\]

A second application leads to

\[
k \alpha_k^2 + A_{k-1} \left( \alpha_{k-1} + \alpha_k \right) + \sum_{n=1}^{k-2} A_n \left( a_n + a_{n+1} - a_{n+1} - \alpha_n - \alpha_{n+2} \right) =
\]

\[
k \alpha_k^2 + A_{k-1} \left( 2 \alpha_k + \frac{\alpha_{k-1}}{k-1} \right) + \sum_{n=1}^{k-2} A_n \left( \frac{a_n}{n} + \frac{a_{n+1}}{n+1} \right) =
\]

\[
k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-2} \left( A_n + A_{n-1} \right) \frac{a_n}{n} =
\]

\[
k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-2} \frac{1}{n} \left( A_n + A_{n-1} \right) \left( A_n - A_{n-1} \right) =
\]

\[
k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-2} \frac{1}{n} \left( A_n^2 - A_{n-1}^2 \right).
\]
Thus we have
\[
\sum_{n=1}^{k} a_n^2 - \sum_{n=1}^{k-1} \frac{1}{n(n+1)} A_n^2 = \frac{k}{k+1} + 2a_k A_{k-1} + \sum_{n=1}^{k-1} \left( \frac{1}{n+1} A_n^2 - \frac{1}{n} A_{n+1}^2 \right) = \frac{k}{k+1} A_{k-1}^2 + \frac{1}{k} A_k^2 + \left( \sqrt{k} a_k + \frac{1}{k} A_{k-1} \right)^2
\]

(3)

If we take \( \lim_{k \to \infty} \) we obtain

\[
\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} A_n^2.
\]

(4)

In the by now familiar way we obtain

\[
\sum_{n=1}^{k} |a_n|^2 = \sum_{n=1}^{k-1} \frac{1}{n(n+1)} |A_n|^2 + \left| \sqrt{k} a_k + \frac{1}{k} A_{k-1} \right|^2
\]

(13)

and

\[
\sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} |A_n|^2.
\]

(14)

**Generating functions**

Let \( a_1, a_2, a_3, \ldots \in l^2 \), let \( A_k = \sum_{n=1}^{k} a_n \), and \( \alpha_k = \sum_{n=k}^{\infty} \frac{a_n}{n} \). We have shown in Theorem 2 that \( A_1, \frac{1}{2} A_2, \frac{1}{3} A_3, \ldots \in l^2 \), and it is a consequence of the previous section that \( \alpha_1, \alpha_2, \alpha_3, \ldots \in l^2 \). We shall present expressions for the generating functions of these sequences. Denote

\[
a(z) = \sum_{n=1}^{\infty} a_n z^{n-1}
\]
and

\[
a(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} z^{n-1}
\]

From

\[
(zA(z))' = \sum_{n=1}^{\infty} A_n z^{n-1}
\]

we deduce that

\[
(1-z) (zA(z))' = a(z)
\]
hence
\[ A(z) = \frac{1}{z} \int_{0}^{z} \frac{a(w)}{1-w} \, dw. \]

Since \( a_1, a_2, a_3, \ldots \in l^2 \) and \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \in l^2 \), we have \( a_1, \frac{1}{2} a_2, \frac{1}{3} a_3, \ldots \in l^1 \); therefore by Abel's theorem
\[ \alpha_1 = \lim_{r \to 1} \sum_{n=1}^{\infty} \frac{a_n}{n} z^n = \lim_{r \to 1} \int_{0}^{z} a(w) \, dw, \]
which we denote as \( \int_{0}^{1} a(w) \, dw \). Starting from
\[ \alpha(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1} \]
we see that
\[ (1-z) \alpha(z) = \alpha_1 - \sum_{n=2}^{\infty} \frac{a_{n-1}}{n-1} z^{n-1} = \alpha_1 - \int_{0}^{z} a(w) \, dw = \int_{z}^{1} a(w) \, dw, \]
i.e.
\[ \alpha(z) = \frac{1}{1-z} \int_{z}^{1} a(w) \, dw. \]

Application
Define the Cesàro operator \( T \) on \( L^2[0, \infty) \) by
\[ T f(x) = \frac{1}{x} \int_{0}^{x} f(t) \, dt. \]
As a consequence of theorem 1 we see that \( T \) is continuous, and that \( \|T\| \leq 2 \). For every \( g \in L^2[0, \infty) \) such that
\[ x \to \frac{g(x)}{x} \in L^1[0, \infty) \]
we have
\[ \langle f, T^* g \rangle = \langle T f, g \rangle = \int_{0}^{\infty} \frac{1}{x} \int_{0}^{x} f(t) \, dt \cdot \overline{g(x)} \, dx = \]
(by partial integration, and an application of lemma 5)
\[ \left[ - \int_{x}^{\infty} \frac{g(t)}{t} \, dt \cdot \int_{0}^{x} f(t) \, dt \right] + \int_{0}^{\infty} f(x) \int_{x}^{\infty} \frac{g(t)}{t} \, dt \, dx = \left\langle f, \int_{x}^{\infty} \frac{g(t)}{t} \, dt \right\rangle. \]
Therefore we have
\[ T^* g(x) = \int_{x}^{\infty} \frac{g(t)}{t} \, dt. \]
It follows from (12) that for every $f \in L^2[0, \infty)$ such that $x \to \frac{|x|}{x} \in L^1(0,1)$ we have
$$\|Tf\| = \|T^*f\|. $$

Since this collection of functions is dense in $L^2$ we have for all $f \in L^2$ this equality, i.e. $T$ is a normal operator.

**Application**

As a corollary of (14) we mention the following result that was proved in [1].

The Cesàro operator on $l^2$ has a positive commutator.

**Proof:** The Cesàro operator $T$ on $l^2$ is defined by

$$T(a_1, a_2, a_3, \ldots) = \left( A_1 \frac{1}{2} A_2, \frac{1}{3} A_3, \ldots \right).$$

A simple computation shows that its adjoint $T^*$ satisfies

$$T^*(a_1, a_2, a_3, \ldots) = (a_1, a_2, a_3, \ldots).$$

It follows from (14) that

$$\|T^*(a_1, a_2, a_3, \ldots)\|^2 = \sum_{n=1}^{\infty} |a_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n(n-1)} A_n^2 = \frac{1}{n^2} |A_n|^2 = \|T(a_1, a_2, \ldots)\|^2.$$

We thus have

$$\langle TT^* a, a \rangle \leq \langle T^* T a, a \rangle$$

where $a = a_1, a_2, a_3, \ldots$ i.e.

$$\langle (T^* T - TT^*) a, a \rangle \geq 0,$$

so $T^* T - TT^*$ is positive.

**References**
