AN AMUSING IDENTITY

R.A. Kortram

Report No. 9942 (November 1999)
An amusing identity

R.A. Kortram

Abstract
We present a family of identities for integrals of square integrable functions. We also derive discrete analogues for square summable sequences.

Introduction
Let \( f \in L^2[0, \infty) \) and assume that
\[
x \to \frac{f(x)}{x} \in L^1[0,1].
\]
Since \( x \to \frac{1}{x} \in L^2[1,\infty) \) this assumption implies that
\[
x \to \frac{f(x)}{x} \in L^1[0,\infty).
\]
We shall show that for every \( r > 0 \) we have
\[
\int_0^r \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^r \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx + \left( \sqrt{r} \int_0^\infty \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{r}} \int_0^r f(t) \, dt \right)^2
\]
(1)
and
\[
\int_0^\infty \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx.
\]
(2)
For the discrete analogue we consider sequences \( a_1, a_2, \ldots \in l^2 \). We define \( A_0 = 0 \),
\[
A_k = \sum_{n=1}^k a_n \quad (k = 1, 2, \ldots)
\]
and \( \alpha_k = \sum_{n=k}^\infty \frac{a_n}{n} \). (Note that \( \alpha_k \) exists; both sequences \( a_1, a_2, \ldots \) and \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \) belong to \( l^2 \).)
We shall prove for every \( k \)
\[
\sum_{n=1}^k \alpha_n^2 = \sum_{n=1}^{k-1} \frac{1}{n(n+1)} A_n^2 + \left( \sqrt{k} \alpha_k + \frac{1}{\sqrt{k}} A_{k-1} \right)^2
\]
(3)
and
\[
\sum_{n=1}^\infty \alpha_n^2 = \sum_{n=1}^\infty \frac{1}{n(n+1)} A_n^2.
\]
(4)
Preliminaries

For the proofs of these identities we shall make use of some general results for integrable functions. In this section we state and prove these auxiliary results.

**Lemma 1** Let $f^2$ be integrable on $[0,1]$; then

$$\lim_{a \to 0} \frac{1}{\sqrt{a}} \int_0^a f(t) dt = 0.$$ 

**Proof:** $|\int_0^a f(t) dt|^2 \leq \int_0^a dt \int_0^a |f^2(t)| dt = a \int_0^a |f^2(t)| dt$, and the assertion follows immediately.

**Lemma 2** Let $f \in L^2$, let $a_1, a_2, \ldots \in l^2$. Then we have

$$\lim_{\tau \to \infty} \sqrt{\tau} \int_{-\tau}^{\tau} \frac{1}{t} f(t) dt = 0,$$

and its discrete analogue

$$\lim_{k \to \infty} \sqrt{k} \sum_{n=k}^{\infty} \frac{a_n}{n} = 0.$$ 

**Proof:** The first assertion follows from

$$\left( \int_{-\tau}^{\tau} \frac{1}{t} f(t) dt \right)^2 \leq \int_{-\tau}^{\tau} \frac{1}{t^2} dt \int_{-\tau}^{\tau} |f^2(t)| dt = \frac{1}{\tau} \int_{-\tau}^{\tau} |f^2(t)| dt;$$

the second follows from

$$\left( \sum_{n=k}^{\infty} \frac{a_n}{n} \right)^2 \leq \sum_{n=k}^{\infty} \frac{1}{n^2} \sum_{n=k}^{\infty} |a_n|^2 \leq \frac{1}{k-1} \sum_{n=k}^{\infty} |a_n|^2.$$ 

**Lemma 3** Let $f \in L^2$, let $a_1, a_2, \ldots \in l^2$. Then we have

$$\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) dt = 0,$$

and its discrete analogue

$$\lim_{k \to \infty} \frac{1}{\sqrt{k}} \sum_{n=1}^{k} a_n = 0.$$ 

**Proof:** Let $\varepsilon > 0$ be given; choose $b$ so large that

$$\int_b^{\infty} |f^2(t)| dt < \frac{1}{4} \varepsilon^2.$$
For $\tau > b$ we have

$$|\int_0^\tau f(t)\,dt| \leq \int_0^b |f(t)|\,dt + \int_b^\tau |f(t)|\,dt \leq \int_0^b |f(t)|\,dt + \sqrt{\int_b^\tau |f(t)|\,dt}$$

and the first assertion follows. The proof of the second assertion is completely analogous.

**Lemma 4** Let $I$ be either $[0,1]$ or $[0,\infty)$. The functions $f \in L^2(I)$ such that $x \to \frac{f(x)}{x}$ form a dense subset of $L^2(I)$.

**Proof:** Let $f \in L^2(I)$ and let $f_k = (1-1_{[0,\frac{1}{k}]})f$. Then $f_k \in L^2(I)$ and $\frac{1}{k}f_k(x) \in L^1(I)$ and $\|f-f_k\|^2 = \int_0^1 |f(x)|^2\,dx \to 0$.

**Lemma 5** Let $f,g \in L^2(0,\infty)$. Then we have

$$\lim_{a \to \infty} \int_a^\infty \frac{g(t)}{t}\,dt \cdot \int_0^a f(t)\,dt = 0.$$  

**Proof:** We have

$$\left|\int_a^\infty \frac{g(t)}{t}\,dt\right| \leq \sqrt{\int_a^\infty |g(t)|^2\,dt} \cdot \sqrt{\int_a^\infty \frac{1}{t^2}\,dt} \leq \frac{1}{\sqrt{a}} \|g\|_2.$$  

Therefore

$$\left|\int_a^\infty \frac{g(t)}{t}\,dt \cdot \int_0^a f(t)\,dt\right| \leq \frac{1}{\sqrt{a}} \|g\| \cdot \frac{1}{\sqrt{a}} \int_0^a f(t)\,dt$$

and the assertion follows from lemma 3.

**Historical remarks**

The identities (1), (2) and (3), (4) are inspired by classical inequalities proved by Hardy in the beginning of the 20th century (and refined by Landau) ([2], page 239). In order to obtain an appropriate setting for our identities we include simple proofs for (generalisations of) Hardy-Landau’s inequality.

**Theorem 1** Let $f \in L^2(0,\infty)$ and let

$$F(x) = \frac{1}{x} \int_0^x f(t)\,dt.$$  

Then $F \in L^2(0,\infty)$ and $\|F-f\| = \|f\|$.

In particular we have $\|F\| \leq 2\|f\|$.

3
Proof: Assume first that \( f \) is real valued. From

\[
\frac{d}{dx} (xF^2(x)) = 2f(x)F(x) - F^2(x)
\]

we see that for every \( a > 0 \)

\[
2 \int_0^a f(x)F(x)dx - \int_0^a F^2(x)dx = \left[ \frac{1}{x} \left( \int_0^x f(t)dt \right)^2 \right]_0^a,
\]

and by an application of lemma 1

\[
2 \int_0^a f(x)F(x)dx - \int_0^a F^2(x)dx = \frac{1}{a} \left( \int_0^a f(t)dt \right)^2. \quad (5)
\]

Consequently

\[
\int_0^a F^2(x)dx \leq 2 \int_0^a f(x)F(x)dx
\]

and by Schwarz’s inequality

\[
\int_0^a F^2(x)dx \leq 2 \sqrt{\int_0^a f^2(x)dx} \cdot \sqrt{\int_0^a F^2(x)dx},
\]

i.e.

\[
\int_0^a F^2(x)dx \leq 4 \int_0^a f^2(x)dx \quad (6)
\]

This shows that \( F \in L^2(0, \infty) \) (and that \( \|F\| \leq 2\|f\| \)). Therefore we can let \( a \) tend to infinity in (5) and we obtain from lemma 3:

\[
2 \int_0^\infty f(x)F(x)dx - \int_0^\infty F^2(x)dx = 0,
\]

i.e.

\[
\int_0^\infty (F(x) - f(x))^2 dx = \int_0^\infty f^2(x)dx. \quad (7)
\]

Thus we have

\[
\|F - f\| = \|f\|.
\]

In the general case where we do not assume \( f \) to be real valued, we apply (6) and (7) to \( \text{Re } f \) as well as to \( \text{Im } f \) and add the results.
Theorem 2 Let $a_1, a_2, a_3, \ldots \in l^2$ and let $A_k = \sum_{n=1}^{k} a_n$ ($k = 1, 2, \ldots$). Then $A_1, \frac{1}{2} A_2, \frac{1}{3} A_3, \ldots \in l^2$,

$$\sum_{n=1}^{k} \frac{1}{n^2} |A_n|^2 \leq 4 \sum_{n=1}^{k} |a_n|^2 \quad k = 1, 2, \ldots,$$

and

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} A_n - a_n \right|^2 \leq \sum_{n=1}^{\infty} |a_n|^2.$$

**Proof:** Assume first that all the numbers $a_1, a_2, a_3, \ldots$ are real. For every $n \in \{2, 3, 4, \ldots\}$ we have

$$\left(\frac{1}{n} A_n\right)^2 - \frac{2}{n} A_n a_n = \left(\frac{1}{n} A_n\right)^2 - \frac{2}{n} A_n (A_n - A_{n-1}) =$$

$$(1-2n) \left(\frac{1}{n} A_n\right)^2 + 2(n-1) \frac{1}{n} A_n \frac{1}{n-1} A_{n-1} \leq$$

$$(1-2n) \left(\frac{1}{n} A_n\right)^2 + (n-1) \left\{ \left(\frac{1}{n} A_n\right)^2 + \left(\frac{1}{n-1} A_{n-1}\right)^2 \right\} =$$

$$\frac{1}{n-1} A_{n-1}^2 - \frac{1}{n} A_n^2,$$

and for $n = 1$ we have

$$A_1^2 - 2 A_1 a_1 = -A_1^2.$$

This shows that

$$\sum_{n=1}^{k} \left\{ \left(\frac{1}{n} A_n\right)^2 - \frac{2}{n} A_n a_n \right\} \leq - \frac{1}{k} A_k^2 \leq 0,$$

i.e.

$$\sum_{n=1}^{k} \left(\frac{1}{n} A_n\right)^2 \leq 2 \sum_{n=1}^{k} \frac{1}{n} A_n a_n,$$

(8)

and by Schwarz’s inequality

$$\sum_{n=1}^{k} \left(\frac{1}{n} A_n\right)^2 \leq \sqrt{\sum_{n=1}^{k} a_n^2} \cdot \sqrt{\sum_{n=1}^{k} \left(\frac{1}{n} A_n\right)^2},$$

so

$$\sum_{n=1}^{k} \left(\frac{1}{n} A_n\right)^2 \leq 4 \sum_{n=1}^{k} a_n^2.$$

(9)
This shows that $A_1, \frac{1}{2}A_2, \frac{1}{3}A_3, \ldots \in l^2$.
Thus we can rewrite (8) and send $k$ to infinity. We obtain
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} A_n - a_n \right)^2 \leq \sum_{n=1}^{\infty} a_n^2.
\]
(10)
In the general case where we do not assume that the numbers $a_1, a_2, a_3, \ldots$ are real, we apply (9) and (10) to $\text{Re} \; a_n$ and $\text{Im} \; a_n$ and add the results.

The identity for functions
Let $f \in L^2[0, \infty)$ and assume that
\[
x \to \frac{f(x)}{x} \in L^1[0, \infty);
\]
then we have for every $\tau > 0$
\[
\int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} dt \right)^2 dx = \int_{0}^{\tau} \left( \frac{1}{x} \int_{0}^{x} f(t) dt \right)^2 dx + \left( \frac{1}{\sqrt{\tau}} \int_{\tau}^{\infty} f(t) dt + \frac{1}{\sqrt{\tau}} \int_{0}^{\tau} f(t) dt \right)^2.
\]
Proof: Integration by parts shows that
\[
\int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} dt \right)^2 dx = \left[ x \left( \int_{x}^{\infty} \frac{f(t)}{t} dt \right) \right]_0^\tau + 2 \int_{0}^{\tau} x \cdot \frac{f(x)}{x} \left( \int_{x}^{\infty} \frac{f(t)}{t} dt \right) dx = 
\]
\[
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} dt \right)^2 + 2 \int_{0}^{\tau} f(x) \int_{x}^{\infty} \frac{f(t)}{t} dt dx.
\]
Another integration by parts leads to
\[
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} dt \right)^2 + 2 \left[ \int_{0}^{\tau} f(t) dt \int_{\tau}^{\infty} \frac{f(t)}{t} dt \right]_0^\tau + 2 \int_{0}^{\tau} \int_{0}^{\tau} f(t) dt \cdot \frac{f(x)}{x} dx = 
\]
\[
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} dt \right)^2 + 2 \int_{0}^{\tau} f(t) dt \int_{0}^{\infty} \frac{f(t)}{t} dt + \int_{0}^{\tau} \frac{1}{x} \frac{d}{dx} \left( \int_{0}^{\tau} f(t) dt \right)^2 dx.
\]
Thus we have
\[
\int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} dt \right)^2 dx - \int_{0}^{\tau} \left( \frac{1}{x} \int_{0}^{x} f(t) dt \right)^2 dx = 
\]
\[
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} dt \right)^2 + 2 \int_{0}^{\tau} f(t) dt \int_{\tau}^{\infty} \frac{f(t)}{t} dt + \int_{0}^{\tau} \frac{1}{x} \frac{d}{dx} \left( \int_{0}^{\tau} f(t) dt \right)^2 dx = 
\]
\[
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} dt \right)^2 + 2 \int_{0}^{\tau} f(t) dt \int_{\tau}^{\infty} \frac{f(t)}{t} dt + \int_{0}^{\tau} \frac{1}{x} \left( \int_{0}^{\tau} f(t) dt \right)^2 dx = 
\]
\[
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} dt \right)^2 + 2 \int_{0}^{\tau} f(t) dt \int_{\tau}^{\infty} \frac{f(t)}{t} dt + \int_{0}^{\tau} \frac{1}{x} \left( \int_{0}^{\tau} f(t) dt \right)^2 \bigg|_0^\tau.
\]
Lemma 1 implies that the third term gives no contribution at 0, so we are left with
\[\tau \left( \int_\tau^\infty \frac{f(t)}{t} dt \right)^2 + 2 \int_0^\tau f(t) dt \int_\tau^\infty \frac{f(t)}{t} dt + \frac{1}{\tau} \left( \int_0^\tau f(t) dt \right)^2 = \left( \sqrt{\tau} \int_\tau^\infty \frac{f(t)}{t} dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) dt \right)^2.\]

Lemma 2 and 3 show that \(\lim_{\tau \to \infty}\) of this expression equals zero, hence
\[\int_0^\infty \left( \int_x^\infty \frac{f(t)}{t} dt \right)^2 = \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) dt \right)^2 \, dx.\]

If we apply the identities to \(\text{Re } f\) and to \(\text{Im } f\) and add them, we obtain
\[\int_0^\infty \left| \int_x^\infty \frac{f(t)}{t} dt \right|^2 \, dx = \int_0^\infty \left| \frac{1}{x} \int_0^x f(t) dt \right|^2 \, dx + \left| \sqrt{\tau} \int_\tau^\infty \frac{f(t)}{t} dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) dt \right|^2 = \int_0^\infty \left| \int_x^\infty \frac{f(t)}{t} dt \right|^2 \, dx = \int_0^\infty \left| \frac{1}{x} \int_0^x f(t) dt \right|^2 \, dx.\] (11)

and
\[\int_0^\infty \left| \int_x^\infty \frac{f(t)}{t} dt \right|^2 \, dx = \int_0^\infty \left| \frac{1}{x} \int_0^x f(t) dt \right|^2 \, dx.\] (12)

Application of (1) to \(f \cdot 1_{[0,1]}\) with \(0 < \tau < 1\) and taking \(\lim_{\tau \to 1}\) leads to
\[\int_0^1 \left( \int_x^1 \frac{f(t)}{t} dt \right)^2 \, dx = \int_0^1 \left( \frac{1}{x} \int_0^x f(t) dt \right)^2 \, dx + \left( \int_0^1 f(t) dt \right)^2,\]
and in a similar way as before to
\[\int_0^1 \left| \int_x^1 \frac{f(t)}{t} dt \right|^2 \, dx = \int_0^1 \left| \frac{1}{x} \int_0^x f(t) dt \right|^2 \, dx + \left| \int_0^1 f(t) dt \right|^2.\]

**Example**

Let \(a > 0\) and let
\[f : t \to \frac{1}{a + t}.\]

Then the identity (2) leads to the well-known but nevertheless charming result
\[\int_0^\infty \left( \frac{1}{x} \log \frac{a+x}{a} \right)^2 \, dx = \int_0^\infty \left( \frac{1}{x} \log \frac{a+x}{x} \right)^2 \, dx.\]
The identity for sequences

The discrete analogues (3) and (4) of (1) and (2) are a bit more delicate. We have to replace integration by parts, with Abel’s partial summation formula:

\[ \sum_{k=1}^{m} b_k c_k = B_m c_m + \sum_{k=1}^{m-1} B_k \left( c_k - c_{k+1} \right), \]

where \[ B_k = \sum_{n=1}^{k} b_n \quad k = 1, 2, \ldots \]

Let \( a_1, a_2, a_3, \ldots \in l^2 \). Define \( A_k = \sum_{n=1}^{k} a_n \quad k = 1, 2, \ldots \) and

\[ \alpha_k = \sum_{n=k}^{\infty} \frac{a_n}{n}. \]

Then we have for every \( k \)

\[ \sum_{n=1}^{k} \alpha_n^2 = \sum_{n=1}^{k} \frac{1}{n(n+1)} A_n^2 + \left( \alpha_k \right)^2. \]

**Proof:** A first application of Abel’s partial summation formula leads to

\[ \sum_{n=1}^{k} \alpha_n^2 = k \alpha_k^2 + \sum_{n=1}^{k-1} \left( \alpha_n^2 - \alpha_{n+1}^2 \right) = k \alpha_k^2 + \sum_{n=1}^{k-1} \left( \alpha_n - \alpha_{n+1} \right) \left( \alpha_n + \alpha_{n+1} \right) = k \alpha_k^2 + \sum_{n=1}^{k} \left( \alpha_n + \alpha_{n+1} \right). \]

A second application leads to

\[ k \alpha_k^2 + A_{k-1} \left( \alpha_{k-1} + \alpha_k \right) + \sum_{n=1}^{k-2} A_n \left( \alpha_n + \alpha_{n+1} - \alpha_{n+1} - \alpha_{n+2} \right) = \]

\[ k \alpha_k^2 + A_{k-1} \left( 2 \alpha_k + \frac{\alpha_{k-1}}{k-1} \right) + \sum_{n=1}^{k-2} A_n \left( \frac{a_n}{n} + \frac{a_{n+1}}{n+1} \right) = k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-1} \left( A_n + A_{n-1} \right) \frac{a_n}{n} = \]

\[ k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-1} \frac{1}{n} \left( A_n + A_{n-1} \right) \left( A_n - A_{n-1} \right) = \]

\[ k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-1} \frac{1}{n} \left( A_n^2 - A_{n-1}^2 \right). \]
Thus we have
\[
\sum_{n=1}^{k} \alpha_n^2 = \sum_{n=1}^{k-1} \frac{1}{n(n+1)} A_n^2 = k\alpha_k^2 + 2\alpha_k A_{k-1} + \sum_{n=1}^{k-1} \left( \frac{1}{n+1} A_n^2 - \frac{1}{n} A_{n-1}^2 \right) =
\]
\[
k\alpha_k^2 + 2\alpha_k A_{k-1} + \frac{k}{k} A_{k-1}^2 = (\sqrt{k}\alpha_k + \frac{1}{\sqrt{k}} A_{k-1})^2
\]  
(3)

If we take \( \lim_{k \to \infty} \) we obtain
\[
\sum_{n=1}^{\infty} \alpha_n^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} A_n^2.
\]  
(4)

In the by now familiar way we obtain
\[
\sum_{n=1}^{k} |\alpha_n|^2 = \sum_{n=1}^{k-1} \frac{1}{n(n+1)} |A_n|^2 + |\sqrt{k}\alpha_k + \frac{1}{\sqrt{k}} A_{k-1}|^2
\]  
(13)

and
\[
\sum_{n=1}^{\infty} |\alpha_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} |A_n|^2.
\]  
(14)

Generating functions

Let \( a_1, a_2, a_3, \ldots \in l^2, \) let \( A_k = \sum_{n=1}^{k} a_n, \) and \( \alpha_k = \sum_{n=k}^{\infty} \frac{a_n}{n}. \) We have shown in Theorem 2 that \( A_1, \frac{1}{2} A_2, \frac{1}{3} A_3, \ldots \in l^2, \) and it is a consequence of the previous section that \( \alpha_1, \alpha_2, \alpha_3, \ldots \in l^2. \) We shall present expressions for the generating functions of these sequences. Denote
\[
a(z) = \sum_{n=1}^{\infty} a_n z^{n-1}
\]
and
\[
A(z) = \sum_{n=1}^{\infty} \frac{1}{n} A_n z^{n-1}
\]

and
\[
\alpha(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}.
\]

From
\[
(zA(z))' = \sum_{n=1}^{\infty} A_n z^{n-1}
\]
we deduce that
\[
(1-z) (zA(z))' = a(z)
\]
hence
\[ A(z) = \frac{1}{z} \int_{0}^{\infty} \frac{a(w)}{1-w} \, dw. \]

Since \( a_1, a_2, a_3, \ldots \in l^2 \) and \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \in l^2 \), we have \( a_1, \frac{1}{2}a_2, \frac{1}{3}a_3, \ldots \in l^1 \); therefore by Abel’s theorem
\[ \alpha_1 = \lim_{z \to 1} \sum_{n=1}^{\infty} \frac{a_n}{n} z^n = \lim_{z \to 1} z \int_{0}^{z} a(w) \, dw \]
which we denote as \( \int_{0}^{1} a(w) \, dw \). Starting from
\[ \alpha(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1} \]
we see that
\[ (1-z)\alpha(z) = \alpha_1 - \sum_{n=2}^{\infty} \frac{a_{n-1}}{n-1} z^{n-1} = \alpha_1 - \int_{0}^{z} a(w) \, dw = \int_{z}^{1} a(w) \, dw, \]
i.e.
\[ \alpha(z) = \frac{1}{1-z} \int_{z}^{1} a(w) \, dw. \]

**Application**

Define the Cesàro operator \( T \) on \( L^2[0, \infty) \) by
\[ Tf(x) = \frac{1}{x} \int_{0}^{x} f(t) \, dt. \]

As a consequence of theorem 1 we see that \( T \) is continuous, and that \( ||T|| \leq 2 \). For every \( g \in L^2[0, \infty) \) such that
\[ x \to \frac{g(x)}{x} \in L^1[0, \infty) \]
we have
\[ \langle f, T^* g \rangle = \langle Tf, g \rangle = \int_{0}^{\infty} \frac{1}{x} \int_{0}^{x} f(t) \, dt \cdot \overline{g(x)} \, dx = \]
(by partial integration, and an application of lemma 5)
\[ \left[ -\int_{x}^{\infty} \frac{g(t)}{t} \, dt \cdot \int_{x}^{\infty} f(t) \, dt \right] + \int_{0}^{\infty} f(x) \int_{x}^{\infty} \frac{g(t)}{t} \, dt \, dx = \left\langle f, \int_{x}^{\infty} \frac{g(t)}{t} \, dt \right\rangle. \]

Therefore we have
\[ T^* g(x) = \int_{x}^{\infty} \frac{g(t)}{t} \, dt. \]
It follows from (12) that for every \( f \in L^2[0, \infty) \) such that \( x \to \frac{f(x)}{x} \in L^1(0,1] \) we have
\[
\|Tf\| = \|T^*f\|.
\]
Since this collection of functions is dense in \( L^2 \) we have for all \( f \in L^2 \) this equality, i.e. \( T \) is a normal operator.

**Application**

As a corollary of (14) we mention the following result that was proved in [1].

The Cesàro operator on \( l^2 \) has a positive commutator.

**Proof:** The Cesàro operator \( T \) on \( l^2 \) is defined by
\[
T(a_1, a_2, a_3, \ldots) = \left( A_1, \frac{1}{2} A_2, \frac{1}{3} A_3, \ldots \right).
\]
A simple computation shows that its adjoint \( T^* \) satisfies
\[
T^*(a_1, a_2, a_3, \ldots) = (a_1, a_2, a_3, \ldots).
\]

It follows from (14) that
\[
\|T^*(a_1, a_2, a_3, \ldots)\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n(n-1)} A_n^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} |A_n|^2 = \|T(a_1, a_2, \ldots)\|^2.
\]
We thus have
\[
\langle TT^*a, a \rangle \leq \langle T^*Ta, a \rangle
\]
where \( a = a_1, a_2, a_3, \ldots \) i.e.
\[
\langle (T^*T - TT^*)a, a \rangle \geq 0,
\]
so \( T^*T - TT^* \) is positive.

**References**
