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AN AMUSING IDENTITY

R.A. Kortram

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An amusing identity

R.A. Kortram

Abstract

We present a family of identities for integrals of square integrable functions. We also derive discrete analogues for square summable sequences.

Introduction

Let \( f \in L^2[0, \infty) \) and assume that

\[
x \to \frac{f(x)}{x} \in L^1[0,1].
\]

Since \( x \to \frac{1}{x} \in L^2[1, \infty) \) this assumption implies that

\[
x \to \frac{f(x)}{x} \in L^1[0, \infty).
\]

We shall show that for every \( \tau > 0 \) we have

\[
\int_0^\tau \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^\tau \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx + \left( \sqrt{\tau} \int_0^\infty \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) \, dt \right)^2
\]

(1)

and

\[
\int_0^\infty \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx.
\]

(2)

For the discrete analogue we consider sequences \( a_1, a_2, \ldots \in l^2 \). We define \( A_0 = 0, A_k = \sum_{n=1}^k a_n \) (\( k = 1, 2, \ldots \)) and \( \alpha_k = \sum_{n=k}^\infty \frac{a_n}{n} \). (Note that \( \alpha_k \) exists; both sequences \( a_1, a_2, \ldots \) and \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \) belong to \( l^2 \).)

We shall prove for every \( k \)

\[
\sum_{n=1}^k \alpha_n^2 = \sum_{n=1}^{k-1} \frac{1}{n(n+1)} A_n^2 + \left( \sqrt{k \alpha_k + \frac{1}{\sqrt{k}}} A_{k-1} \right)^2
\]

(3)

and

\[
\sum_{n=1}^\infty \alpha_n^2 = \sum_{n=1}^\infty \frac{1}{n(n+1)} A_n^2.
\]

(4)
Preliminaries

For the proofs of these identities we shall make use of some general results for integrable functions. In this section we state and prove these auxiliary results.

Lemma 1 Let \( f^2 \) be integrable on \([0, 1]\); then

\[
\lim_{a \to 0} \frac{1}{\sqrt{a}} \int_0^a f(t)dt = 0.
\]

**Proof:**

\[
|\int_0^a f(t)dt|^2 \leq \int_0^a dt \int_0^a \left| f^2(t) \right| dt = a \int_0^a \left| f^2(t) \right| dt,
\]

and the assertion follows immediately.

Lemma 2 Let \( f \in L^2 \), let \( a_1, a_2, \ldots \in l^2 \). Then we have

\[
\lim_{\tau \to \infty} \sqrt{\tau} \int_\tau^\infty \frac{f(t)}{t} dt = 0,
\]

and its discrete analogue

\[
\lim_{k \to \infty} \sqrt{k} \sum_{n=k}^\infty \frac{n}{a_n} = 0.
\]

**Proof:** The first assertion follows from

\[
\left| \int_\tau^\infty \frac{f(t)}{t} dt \right|^2 \leq \int_\tau^\infty \frac{1}{t^2} dt \cdot \int_\tau^\infty \left| f^2(t) \right| dt = \frac{1}{\tau} \int_\tau^\infty \left| f^2(t) \right| dt;
\]

the second follows from

\[
\left| \sum_{n=k}^\infty \frac{a_n}{n} \right|^2 \leq \sum_{n=k}^\infty \frac{1}{n^2} \cdot \sum_{n=k}^\infty \left| a_n \right|^2 \leq \frac{1}{k-1} \sum_{n=k}^\infty \left| a_n \right|^2.
\]

Lemma 3 Let \( f \in L^2 \), let \( a_1, a_2, \ldots \in l^2 \). Then we have

\[
\lim_{\tau \to \infty} \frac{1}{\sqrt{\tau}} \int_0^\tau f(t)dt = 0,
\]

and its discrete analogue

\[
\lim_{k \to \infty} \frac{1}{\sqrt{k}} \sum_{n=1}^k a_n = 0.
\]

**Proof:** Let \( \epsilon > 0 \) be given; choose \( b \) so large that

\[
\int_b^\infty \left| f^2(t) \right| dt < \frac{1}{4} \epsilon^2.
\]
For \( \tau > b \) we have
\[
\left| \int_0^{\tau} f(t) dt \right| \leq \int_0^b |f(t)| dt + \int_b^{\tau} |f(t)| dt \leq \int_0^b |f(t)| dt + \sqrt{\int_0^b f^2(t) dt} \cdot \sqrt{\int_b^{\tau} f^2(t) dt} \leq \int_0^b |f(t)| dt + \frac{1}{2} \sqrt{\tau - b}
\]
and the first assertion follows. The proof of the second assertion is completely analogous.

Lemma 4 Let \( I \) be either \([0,1]\) or \([0,\infty)\). The functions \( f \in L^2(I) \) such that \( x \to \frac{f(x)}{x} \in L^1(I) \) form a dense subset of \( L^2(I) \).

Proof: Let \( f \in L^2(I) \) and let \( f_k = \left( 1 - 1_{[0,\frac{1}{k}]} \right) f \). Then \( f_k \in L^2(I) \) and \( \frac{1}{k} f_k(x) \in L^1(I) \) and \( \|f - f_k\|_2 = \int_0^1 |f(x)|^2 dx \to 0 \).

Lemma 5 Let \( f, g \in L^2(0, \infty) \). Then we have
\[
\lim_{a \to \infty} \int_a^\infty \frac{g(t)}{t} dt \cdot \int_0^a f(t) dt = 0.
\]

Proof: We have
\[
\left| \int_a^\infty \frac{g(t)}{t} dt \cdot \int_0^a f(t) dt \right| \leq \sqrt{\int_a^\infty |g(t)|^2 dt} \cdot \sqrt{\int_a^\infty \frac{1}{t^2} dt} \leq \frac{1}{\sqrt{a}} \|g\|_2.
\]

Therefore
\[
\left| \int_a^\infty \frac{g(t)}{t} dt \cdot \int_0^a f(t) dt \right| \leq \|g\|_2 \cdot \frac{1}{\sqrt{a}} \left| \int_0^a f(t) dt \right|
\]
and the assertion follows from lemma 3.

**Historical remarks**

The identities (1), (2) and (3), (4) are inspired by classical inequalities proved by Hardy in the beginning of the 20th century (and refined by Landau) ([2], page 239). In order to obtain an appropriate setting for our identities we include simple proofs for (generalisations of) Hardy-Landau’s inequality.

**Theorem 1** Let \( f \in L^2(0, \infty) \) and let
\[
F(x) = \frac{1}{x} \int_0^x f(t) dt.
\]

Then \( F \in L^2(0, \infty) \) and \( \|F - f\| = \|f\| \).

In particular we have \( \|F\| \leq 2\|f\| \).
**Proof:** Assume first that $f$ is real valued. From

$$\frac{d}{dx} (xF^2(x)) = 2f(x)F(x) - F^2(x)$$

we see that for every $a > 0$

$$2 \int_0^a f(x)F(x)dx - \int_0^a F^2(x)dx = \left[ \frac{1}{x} \left( \int_0^x f(t)dt \right)^2 \right]_0^a,$$

and by an application of lemma 1

$$2 \int_0^a f(x)F(x)dx - \int_0^a F^2(x)dx = \frac{1}{a} \left( \int_0^a f(t)dt \right)^2. \quad (5)$$

Consequently

$$\int_0^a F^2(x)dx \leq 2 \int_0^a f(x)F(x)dx$$

and by Schwarz’s inequality

$$\int_0^a F^2(x)dx \leq 2 \sqrt{\int_0^a f^2(x)dx} \cdot \sqrt{\int_0^a F^2(x)dx},$$

i.e.

$$\int_0^a F^2(x)dx \leq 4 \int_0^a f^2(x)dx \quad (6)$$

This shows that $F \in L^2[0, \infty)$ (and that $\|F\| \leq 2||f\|$). Therefore we can let $a$ tend to infinity in (5) and we obtain from lemma 3:

$$2 \int_0^\infty f(x)F(x)dx - \int_0^\infty F^2(x)dx = 0,$$

i.e.

$$\int_0^\infty (F(x) - f(x))^2 dx = \int_0^\infty f^2(x)dx. \quad (7)$$

Thus we have

$$\|F-f\| = ||f||.$$

In the general case where we do not assume $f$ to be real valued, we apply (6) and (7) to $\text{Re } f$ as well as to $\text{Im } f$ and add the results.
Theorem 2 Let $a_1, a_2, a_3, \ldots \in l^2$ and let $A_k = \sum_{n=1}^{k} a_n$ $(k = 1, 2, \ldots)$. Then $A_1, \frac{1}{2}A_2, \frac{1}{3}A_3, \ldots \in l^2$, 

$$\sum_{n=1}^{k} \frac{1}{n^2} |A_n|^2 \leq 4 \sum_{n=1}^{k} |a_n|^2 \quad k = 1, 2, \ldots,$$

and

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} A_n - a_n \right|^2 \leq \sum_{n=1}^{\infty} |a_n|^2.$$

Proof: Assume first that all the numbers $a_1, a_2, a_3, \ldots$ are real. For every $n \in \{2, 3, 4, \ldots\}$ we have

$$\left( \frac{1}{n} A_n \right)^2 - \frac{2}{n} A_n a_n = \left( \frac{1}{n} A_n \right)^2 - \frac{2}{n} A_n (A_n - A_{n-1}) =$$

$$(1-2n) \left( \frac{1}{n} A_n \right)^2 + 2(n-1) \frac{1}{n} A_n \cdot \frac{1}{n-1} A_{n-1} \leq$$

$$(1-2n) \left( \frac{1}{n} A_n \right)^2 + (n-1) \left\{ \left( \frac{1}{n} A_n \right)^2 + \left( \frac{1}{n-1} A_{n-1} \right)^2 \right\} =$$

$$\frac{1}{n-1} A_{n-1}^2 - \frac{1}{n} A_n^2,$$

and for $n = 1$ we have

$$A_1^2 - 2A_1 a_1 = -A_1^2.$$

This shows that

$$\sum_{n=1}^{k} \left\{ \left( \frac{1}{n} A_n \right)^2 - \frac{2}{n} A_n a_n \right\} \leq -\frac{1}{k} A_k^2 \leq 0,$$

i.e.

$$\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2 \leq 2 \sum_{n=1}^{k} \frac{1}{n} A_n a_n,$$

and by Schwarz’s inequality

$$\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2 \leq 2 \sqrt{\sum_{n=1}^{k} a_n^2} \cdot \sqrt{\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2},$$

so

$$\sum_{n=1}^{k} \left( \frac{1}{n} A_n \right)^2 \leq 4 \sum_{n=1}^{k} a_n^2.$$
This shows that \( A_1, \frac{1}{2} A_2, \frac{1}{3} A_3, \ldots \in l^2 \).
Thus we can rewrite (8) and send \( k \) to infinity. We obtain
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} A_n - a_n \right)^2 \leq \sum_{n=1}^{\infty} a_n^2.
\]  
(10)

In the general case where we do not assume that the numbers \( a_1, a_2, a_3, \ldots \) are real, we apply (9) and (10) to \( \text{Re} \ a_n \) and \( \text{Im} \ a_n \) and add the results.

The identity for functions

Let \( f \in L^2[0, \infty) \) and assume that
\[
x \to \frac{f(x)}{x} \in L^1[0, \infty);
\]
then we have for every \( \tau > 0 \)
\[
\int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_{0}^{\tau} \left( \frac{1}{x} \int_{0}^{x} f(t) \, dt \right)^2 \, dx + \left( \frac{\sqrt{\tau}}{\sqrt{\tau}} \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2.
\]

Proof: Integration by parts shows that
\[
\int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right)^2 \, dx = \left[ x \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right) \right]_{0}^{\tau} + 2 \int_{0}^{\tau} x \cdot \frac{f(x)}{x} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right) \, dx.
\]

Another integration by parts leads to
\[
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \left[ \int_{0}^{\tau} f(t) \, dt \, \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right]_{0}^{\tau} + 2 \int_{0}^{\tau} \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \, \frac{f(x)}{x} \, dx =
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \, \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \int_{0}^{\tau} \frac{1}{x} \, dx \left( \int_{0}^{x} f(t) \, dt \right)^2.
\]

Thus we have
\[
\int_{0}^{\tau} \left( \int_{x}^{\infty} \frac{f(t)}{t} \, dt \right)^2 \, dx - \int_{0}^{\tau} \left( \frac{1}{x} \int_{0}^{x} f(t) \, dt \right)^2 \, dx =
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \int_{0}^{x} \frac{f(t)}{t} \, dt + \int_{0}^{\tau} \frac{1}{x} \, dx \left( \int_{0}^{x} f(t) \, dt \right)^2 \, dx =
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \int_{0}^{\tau} \frac{1}{x} \, dx \left( \int_{0}^{x} f(t) \, dt \right)^2 \, dx =
\tau \left( \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt \right)^2 + 2 \int_{0}^{\tau} f(t) \, dt \int_{\tau}^{\infty} \frac{f(t)}{t} \, dt + \left[ \frac{1}{x} \left( \int_{0}^{x} f(t) \, dt \right)^2 \right]_{\tau}^{x}.
Lemma 1 implies that the third term gives no contribution at 0, so we are left with

\[
\tau \left( \int_0^\infty \frac{f(t)}{t} \, dt \right)^2 + 2 \int_0^\tau f(t) \, dt \int_0^\infty \frac{f(t)}{t} \, dt + \frac{1}{\tau} \left( \int_0^\tau f(t) \, dt \right)^2 = \\
\left( \sqrt{\tau} \int_0^\infty \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) \, dt \right)^2.
\]

Lemma 2 and 3 show that \( \lim_{\tau \to \infty} \) of this expression equals zero, hence

\[
\int_0^\infty \left( \int_x^\infty \frac{f(t)}{t} \, dt \right)^2 = \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx.
\]

If we apply the identities to Re \( f \) and to Im \( f \) and add them, we obtain

\[
\int_0^\infty \left| \int_x^\infty \frac{f(t)}{t} \, dt \right|^2 \, dx = \int_0^\infty \left| \frac{1}{x} \int_0^x f(t) \, dt \right|^2 \, dx + \left| \sqrt{\tau} \int_0^\tau \frac{f(t)}{t} \, dt + \frac{1}{\sqrt{\tau}} \int_0^\tau f(t) \, dt \right|^2
\]

(11)

and

\[
\int_0^\infty \left| \int_x^\infty \frac{f(t)}{t} \, dt \right|^2 \, dx = \int_0^\infty \left| \frac{1}{x} \int_0^x f(t) \, dt \right|^2 \, dx.
\]

(12)

Application of (1) to \( f \cdot 1_{[0,1]} \) with \( 0 < \tau < 1 \) and taking \( \lim_{\tau \to 1} \) leads to

\[
\int_0^1 \left( \int_x^1 \frac{f(t)}{t} \, dt \right)^2 \, dx = \int_0^1 \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^2 \, dx + \left( \int_0^1 f(t) \, dt \right)^2,
\]

and in a similar way as before to

\[
\int_0^1 \left| \int_x^1 \frac{f(t)}{t} \, dt \right|^2 \, dx = \int_0^1 \left| \frac{1}{x} \int_0^x f(t) \, dt \right|^2 \, dx + \left| \int_0^1 f(t) \, dt \right|^2.
\]

**Example**

Let \( a > 0 \) and let

\[
f : t \to \frac{1}{a+t}.
\]

Then the identity (2) leads to the well-known but nevertheless charming result

\[
\int_0^\infty \left( \frac{1}{x} \log \frac{a+x}{a} \right)^2 \, dx = \int_0^\infty \left( \frac{1}{a} \log \frac{a+x}{x} \right)^2 \, dx.
\]
The identity for sequences

The discrete analogues (3) and (4) of (1) and (2) are a bit more delicate. We have to replace integration by parts, with Abel’s partial summation formula:

\[ \sum_{k=1}^{m} b_k c_k = B_m c_m + \sum_{k=1}^{m-1} B_k (c_k - c_{k+1}), \]

where \( B_k = \sum_{n=1}^{k} b_n \quad k = 1, 2, \ldots. \)

Let \( a_1, a_2, a_3, \ldots \in I^2. \) Define \( A_k = \sum_{n=1}^{k} a_n \quad k = 1, 2, \ldots \) and

\[ \alpha_k = \sum_{n=k}^{\infty} \frac{a_n}{n}. \]

Then we have for every \( k \)

\[ \sum_{n=1}^{k} \alpha_n^2 = \sum_{n=1}^{k} \frac{1}{n(n+1)} A_n^2 + \left( \sqrt{k} \alpha_k + \frac{1}{\sqrt{k}} A_{k-1} \right)^2. \]

**Proof:** A first application of Abel’s partial summation formula leads to

\[ \sum_{n=1}^{k} \alpha_n^2 = k \alpha_k^2 + \sum_{n=1}^{k-1} n \left( \alpha_n^2 - \alpha_{n+1}^2 \right) = k \alpha_k^2 + \sum_{n=1}^{k-1} n (\alpha_n - \alpha_{n+1}) (\alpha_n + \alpha_{n+1}) = \]

\[ k \alpha_k^2 + \sum_{n=1}^{k} a_n (\alpha_n + \alpha_{n+1}). \]

A second application leads to

\[ k \alpha_k^2 + A_{k-1} (\alpha_{k-1} + \alpha_k) + \sum_{n=1}^{k-2} A_n (\alpha_n + \alpha_{n+1} - \alpha_{n+1} - \alpha_{n+2}) = \]

\[ k \alpha_k^2 + A_{k-1} \left( 2 \alpha_k + \frac{\alpha_{k-1}}{k-1} \right) + \sum_{n=1}^{k-2} A_n \left( \frac{a_n}{n} + \frac{a_{n+1}}{n+1} \right) = \]

\[ k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-1} (A_n + A_{n-1}) \frac{a_n}{n} = \]

\[ k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-2} 1 (A_n + A_{n-1}) (A_n - A_{n-1}) = \]

\[ k \alpha_k^2 + 2 \alpha_k A_{k-1} + \sum_{n=1}^{k-1} 1 (A_n^2 - A_{n-1}^2). \]
Thus we have
\[
\sum_{n=1}^{k} \alpha_n^2 - \sum_{n=1}^{k-1} \frac{1}{n(n+1)} A_n^2 = \left( k\alpha_k^2 + 2\alpha_k A_{k-1} + \sum_{n=1}^{k-1} \left( \frac{1}{n+1} A_n^2 - \frac{1}{n} A_{n-1}^2 \right) \right) = k\alpha_k^2 + 2\alpha_k A_{k-1} + \frac{1}{k} A_{k-1}^2 = \left( \sqrt{k}\alpha_k + \frac{1}{\sqrt{k}} A_{k-1} \right)^2
\]  
(3)

If we take \( \lim _{k \to \infty} \) we obtain
\[
\sum_{n=1}^{\infty} \alpha_n^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} A_n^2.
\]  
(4)

In the by now familiar way we obtain
\[
\sum_{n=1}^{k} |\alpha_n|^2 = \sum_{n=1}^{k-1} \frac{1}{n(n+1)} |A_n|^2 + \left| \sqrt{k}\alpha_k + \frac{1}{\sqrt{k}} A_{k-1} \right|^2
\]  
(13)

and
\[
\sum_{n=1}^{\infty} |\alpha_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} |A_n|^2.
\]  
(14)

### Generating functions

Let \( a_1, a_2, a_3, \ldots \in l^2 \), let \( A_k = \sum_{n=1}^{k} a_n \), and \( \alpha_k = \sum_{n=k}^{\infty} \frac{a_n}{n} \). We have shown in Theorem 2 that \( A_1, \frac{1}{2} A_2, \frac{1}{3} A_3, \ldots \in l^2 \), and it is a consequence of the previous section that \( \alpha_1, \alpha_2, \alpha_3, \ldots \in l^2 \). We shall present expressions for the generating functions of these sequences. Denote
\[
a(z) = \sum_{n=1}^{\infty} a_n z^{n-1}
\]
\[
A(z) = \sum_{n=1}^{\infty} \frac{1}{n} A_n z^{n-1}
\]
and
\[
\alpha(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}.
\]

From
\[
(zA(z))' = \sum_{n=1}^{\infty} A_n z^{n-1}
\]
we deduce that
\[
(1-z) (zA(z))' = a(z)
\]
hence
\[ A(z) = \frac{1}{z} \int_0^z \frac{a(w)}{1-w} \, dw. \]

Since \( a_1, a_2, a_3, \ldots \in l^2 \) and \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \in l^2 \), we have \( a_1, \frac{1}{2}a_2, \frac{1}{3}a_3, \ldots \in l^1 \); therefore by Abel’s theorem
\[ \alpha_1 = \lim_{z \to 1} \sum_{n=1}^{\infty} \frac{a_n}{n} z^n = \lim_{z \to 1} \int_0^z a(w) \, dw \]
which we denote as \( \int_0^1 a(w) \, dw \). Starting from
\[ \alpha(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1} \]
we see that
\[ (1-z)\alpha(z) = \alpha_1 - \sum_{n=2}^{\infty} \frac{a_{n-1}}{n-1} z^{n-1} = \alpha_1 - \int_0^z a(w) \, dw = \int_z^1 a(w) \, dw, \]
i.e.
\[ \alpha(z) = \frac{1}{1-z} \int_z^1 a(w) \, dw. \]

**Application**

Define the Cesaro operator \( T \) on \( L^2[0, \infty) \) by
\[ T f(x) = \frac{1}{x} \int_0^x f(t) \, dt. \]

As a consequence of theorem 1 we see that \( T \) is continuous, and that \( ||T|| \leq 2 \). For every \( g \in L^2[0, \infty) \) such that
\[ x \to \frac{g(x)}{x} \in L^1[0, \infty) \]
we have
\[ \langle f, T^* g \rangle = \langle Tf, g \rangle = \int_0^\infty \frac{1}{x} \int_0^x f(t) \, dt \cdot g(x) \, dx = \]
(by partial integration, and an application of lemma 5)
\[ \left[ - \int_x^\infty \frac{g(t)}{t} \, dt \cdot \int_0^x f(t) \, dt \right] + \int_0^\infty f(x) \int_x^\infty \frac{g(t)}{t} \, dt \, dx = \langle f, \int_x^\infty \frac{g(t)}{t} \, dt \rangle. \]

Therefore we have
\[ T^* g(x) = \int_x^\infty \frac{g(t)}{t} \, dt. \]
It follows from (12) that for every $f \in L^2[0, \infty)$ such that $x \to \frac{|f(x)|}{x} \in L^1(0,1]$ we have
\[
\|Tf\| = \|T^*f\|.
\]
Since this collection of functions is dense in $L^2$ we have for all $f \in L^2$ this equality, i.e. $T$ is a normal operator.

Application

As a corollary of (14) we mention the following result that was proved in [1].

The Cesàro operator on $l^2$ has a positive commutator.

Proof: The Cesàro operator $T$ on $l^2$ is defined by
\[
T(a_1, a_2, a_3, \ldots) = \left( A_1, \frac{1}{2}A_2, \frac{1}{3}A_3, \ldots \right).
\]

A simple computation shows that its adjoint $T^*$ satisfies
\[
T^*(a_1, a_2, a_3, \ldots) = (\alpha_1, \alpha_2, \alpha_3, \ldots).
\]

It follows from (14) that
\[
\|T^*(a_1, a_2, a_3, \ldots)\|^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n(n-1)} A_n^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} |A_n|^2 = \|T(a_1, a_2, \ldots)\|^2.
\]

We thus have
\[
\langle TT^*a, a \rangle \leq \langle T^*Ta, a \rangle
\]
where $a = a_1, a_2, a_3, \ldots$ i.e.
\[
\langle (T^*T - TT^*)a, a \rangle \geq 0,
\]
so $T^*T - TT^*$ is positive.

References
