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# ON A HYBRID METHOD OF CHARACTERISTICS AND CENTRAL DIFFERENCE METHOD FOR CONVECTION-DIFFUSION PROBLEMS

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# On a Hybrid Method of Characteristics and Central Difference Method for Convection–Diffusion Problems

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**Abstract**

A combined finite difference method and local method of characteristic line is presented for the numerical solution of singularly perturbed convection-diffusion problems. The method has discretization error estimates in maximum norm of second order. An analysis of the discretization errors, based on certain barrier function bounds and a discrete comparison principle, shows uniform in $\varepsilon$ convergence of the discretization error on layer adapted meshes. The advantages of this technique compared with some standardly used methods are illustrated by numerical experiments with outflow boundary layers.

## 1 Introduction

Consider the convection-diffusion problem: seek $u$ such that

$$\mathcal{L}u \equiv -\varepsilon\Delta u + \mathbf{v}\cdot\nabla u + c\,u = f, \quad (x,y)\in\Omega\subset\mathbb{R}^2 \tag{1}$$

with boundary conditions

$$u = g_1,\ (x,y)\in\Gamma_1, \quad \mathbf{n}\cdot\nabla u = g_2,\ (x,y)\in\Gamma_2 = \Gamma\setminus\Gamma_1, \tag{2}$$

where $\varepsilon > 0$, $c \geq 0$ and $\Omega$ is a polygonal domain with boundary $\Gamma$. Here

$$\mathbf{v}(x,y) = (v_1(x,y), v_2(x,y))$$

is assumed to be a smooth uniquely defined vector–field in $\Omega$ and $|\mathbf{v}| \geq v_0 > 0$, $\mathbf{n}$ is the outward pointing normal to $\Gamma$. If either $c > 0$ in $\Omega$ or surface $meas(\Gamma_1) > 0$, then the problem (1)-(2) has unique solution. We shall assume that the boundary and given functions are such that the solution is sufficiently smooth. It is further assumed that $\Gamma_- \subset \Gamma_1$. Here $\Gamma_-$ denotes the points $(x,y)\in\Gamma$, where $\mathbf{v}\cdot\mathbf{n} < 0$, i.e. the inflow part

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of the boundary. Further $\Gamma_+ = \{(x,y) \in \Gamma, \mathbf{v}\cdot\mathbf{n} > 0\}$ and $\Gamma_0 = \{(x,y) \in \Gamma, \mathbf{v}\cdot\mathbf{n} = 0\}$ denotes the outflow and characteristic part of the boundary.

As is well known this equation has both a convective and diffusive character and as $\varepsilon \to 0$ the convective behaviour becomes dominating, except in (boundary) layers. The boundary layers can arise at the outflow and characteristic boundaries and here the diffusion becomes significant. An efficient numerical solution method must have the same properties.

The layers may cause a degradation of the rate of convergence of the numerical solution methods because the solution has a singular behaviour (unbounded derivatives) there. The convective character is easily seen by considering the method of characteristics for the reduced equation,

$$\mathbf{v}\cdot\nabla u + cu = f \text{ in } \Omega, \quad u = g \text{ on } \Gamma_-.$$

Let $\mathbf{z}(t,s)$ be the parametric representation of the lines of characteristics defined by the vector-field through the point $(x_0,y_0)$ on $\Gamma_-$, i.e. we have $x = z_1(t,s)$, $y = z_2(t,s)$ for points on this line and

$$\frac{\mathrm{d}\mathbf{z}(t,s)}{\mathrm{d}t} = \mathbf{v}(x,y) = \mathbf{v}(\mathbf{z}(t)),\ t>0,\ \mathbf{z}(0,s) = (x_0,y_0) \in \Gamma_-.$$

Since the vector field is uniquely defined, no two characteristic lines may cross each other. Using the chain rule, we obtain

$$\frac{\mathrm{d}\hat{u}(t)}{\mathrm{d}t} = \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\frac{\mathrm{d}z_i}{\mathrm{d}t} = \mathbf{v}\cdot\nabla u,$$

where $\hat{u}(t) = u(\mathbf{z}(t))$. Hence

$$\frac{\mathrm{d}\hat{u}(t)}{\mathrm{d}t} + c\hat{u} = f(\mathbf{z}(t)),\ t>0,\ \hat{u}(0) = u(x_0,y_0),$$

so when the characteristic lines have been computed, the solution of the reduced equation can be computed as the solution of an initial value problem for an ordinary differential equation.

When $\varepsilon$ is small and $|\mathbf{v}| > 0$, the solution of (1) is close to the solution of the reduced equation except near the boundaries $\Gamma_0$ and $\Gamma_+$ where boundary layers occur because in general the solution of reduced equation does not satisfy the boundary conditions on $\Gamma_0$ and $\Gamma_+$.

There are a variety of physical processes in which boundary and interior layers in the solution may arise for certain parameter ranges. These problems may generally be characterized as singular perturbation problems. The primary objective in singular perturbation analysis of such problems is to develop approximations to the true solution that are uniformly valid with respect to the perturbation parameter.

Some examples of such perturbation problems are boundary layers in viscous fluid flow and concentration or thermal layers in mass and heat transfer problems. The steady convection–dominated flow with a Dirichlet boundary condition as the downstream or outflow condition is a frequently used test problem for assessing new numerical methods applicable to convection–dominated problems with layers.

Various finite difference and finite element methods have been proposed during the years, for a survey of some of them, see [1] and [11]. As is well known, for problems with layers the standard central difference method gives a numerical solution with significant unphysical wiggles. For a one space dimensional problem it can be seen (see for instance [1]), that the wiggles occur only for the odd numbered points (assuming that an even number of points have been used) but the solution at the even numbered points corresponds to a difference approximation with a heavy artificial diffusion. Hence, the solution at these points has a similar behaviour as the solution of an upwind difference scheme, i.e., a heavy smearing of the sharp gradients in the solution of the differential equation. Here we consider problems as defined by (1)–(2), discretized with the following difference method on a rectangular mesh. The basic idea of the method is to use a linear combination of the central difference (or standard Galerkin) method and a local method of characteristics, with a variable weight coefficient $\theta$ chosen such that the resulting numerical scheme has about the same ratios of diffusion and convection as the given problem at each point of the domain.

The method presented in the paper is applicable for more general problems. The idea of the method was originally presented in [1]. A preliminary version of the present paper has been published in [4].

The remainder of the paper is organised as follows: in Section 2 we present the hybrid method and discretization error estimates in 1D case which are illustrated with numerical tests in Section 3 and in Section 4 we do a similar presentation for 2D case. Some numerical results in a two–dimensional square are found in Section 5.

# 2 The difference scheme and discretization error estimates in 1D case

To illustrate the combined finite difference and local method of characteristics we consider first the 1D case. The convection-diffusion problem has then the following form

$$\mathcal{L}u \equiv -\varepsilon u'' + vu' + c\,u = f, \quad 0 < x < 1, \tag{3}$$

$$u(0) = 0, \quad u(1) = 1. \tag{4}$$

We assume that $v \geq v_0 > 0$ and $c \geq 0$ in $[0, 1]$ and that $v$ and $c$ are bounded $C^1$ functions, $v,\ c \in C^1[0, 1]$.

## 2.1 The Difference Scheme

Let $\theta$, $0 < \theta < 1$, be a weight coefficient. The difference method on an arbitrary mesh $\Omega_N = \{x_i, i = 0, 1, \ldots, N\}$ with variable step $h_i = x_i - x_{i-1},\ i = 1, \ldots, N$ ($N$ is the number of the mesh intervals) takes the form

$$
\begin{aligned}
L^N u_i^N = & -\frac{2\varepsilon}{h_i + h_{i+1}} \left( \frac{u_{i+1}^N - u_i^N}{h_{i+1}} - \frac{u_i^N - u_{i-1}^N}{h_i} \right) \\
& +\theta_i \frac{v(x_i)}{h_i + h_{i+1}} \left( h_i \frac{u_{i+1}^N - u_i^N}{h_{i+1}} + h_{i+1} \frac{u_i^N - u_{i-1}^N}{h_i} \right) + \theta_i c(x_i) u_i^N \\
& +(1-\theta_i) \left[ v(x_i - \tfrac{h_i}{2}) \frac{u_i^N - u_{i-1}^N}{h_i} + c(x_i - \tfrac{h_i}{2}) \frac{u_i^N + u_{i-1}^N}{2} \right] \\
& = f^N \equiv \theta_i f(x_i) + (1-\theta_i)\ f(x_i - \tfrac{h_i}{2}),
\end{aligned}
$$

for each interior mesh point $x_i$, $i = 1, 2, \ldots, N-1$. Here

- $u_i^N$ denotes the finite difference approximation of the solution $u$ at mesh point $x_i$, $i = 1, 2, \ldots, N-1$;
- $\theta_i = \dfrac{1}{1 + r_i}$, where $r_i = \dfrac{v(x_i)h}{2\varepsilon}$ is the local Peclet number, $h = \max\{h_i, h_{i+1}\}$;
- $1 - \theta_i = \dfrac{r_i}{1 + r_i} = \dfrac{v(x_i)h}{2\varepsilon + v(x_i)h}$.

The corresponding finite difference mesh is illustrated in Figure 1.

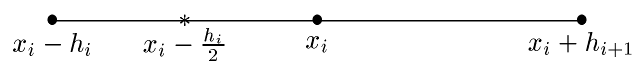



Figure 1: Finite-difference mesh in 1D case.

The scheme is a linear combination of a central difference scheme at $x_i$ and at $x_i - \frac{h_i}{2}$, with the exception that the central difference for the second order derivative is evaluated only at $x_i$. The scheme is a three point upwind scheme. When $\varepsilon \ll h$ it is dominated by the approximation at $x_i - \frac{h_i}{2}$ while when $\varepsilon \gg h$ it is dominated by the central difference approximation at $x_i$.

We assume that the mesh is uniform or varies smoothly, when $\varepsilon \geq h$.

## 2.2 Stability (boundness of $L_h^{-1}$)

**Lemma 1** *The operator $L^N$ is monotone if* $h_i \leq 2v_0 / \max_{x \in \Omega} c(x)$.

**Proof:** The operator $L^N u$ can be written in the form:

$$
L^N u(x_i) = \gamma_{i-1} u(x_{i-1}) + \gamma_i u(x_i) + \gamma_{i+1} u(x_{i+1}),
$$

where

$$\gamma_{i-1} = -\frac{2\varepsilon}{h_i(h_i+h_{i+1})} - \theta_i \frac{v(x_i)h_{i+1}}{h_i(h_i+h_{i+1})} - (1-\theta_i)\left[\frac{v(x_i - \frac{h_i}{2})}{h_i} - \frac{c(x_i - \frac{h_i}{2})}{2}\right],$$

$$\gamma_i = \frac{2\varepsilon}{h_i h_{i+1}} + \theta_i \frac{v(x_i)(h_{i+1}-h_i)}{h_i h_{i+1}} + \theta_i c(x_i) + (1-\theta_i)\left[\frac{v(x_i - \frac{h_i}{2})}{h_i} + \frac{c(x_i - \frac{h}{2})}{2}\right],$$

$$\gamma_{i+1} = -\frac{2\varepsilon}{h_{i+1}(h_i+h_{i+1})} + \theta_i \frac{v(x_i)h_i}{h_{i+1}(h_i+h_{i+1})}$$

for each interior mesh point $x_i$, $i = 1, 2, \ldots, N-1$.

It is seen that the coefficient $\gamma_i$ is positive and $\gamma_{i+1}$ is negative. The coefficient $\gamma_i$ is positive, because

$$\frac{2\varepsilon}{h_i h_{i+1}} + \theta_i \frac{v(x_i)(h_{i+1}-h_i)}{h_i h_{i+1}} = \frac{2\varepsilon}{h_i h_{i+1}}\left[1 + \frac{v(x_i)(h_{i+1}-h_i)}{2\varepsilon + v(x_i)h}\right]$$

$$\geq \frac{2\varepsilon}{h_i h_{i+1}}\left[1 - \frac{v(x_i)h}{2\varepsilon + v(x_i)h}\right] = \frac{2\varepsilon}{h_i h_{i+1}}[1-(1-\theta_i)] = \frac{2\varepsilon\theta_i}{h_i h_{i+1}} > 0.$$

For $\gamma_{i+1}$ holds

$$\gamma_{i+1} = \frac{-2\varepsilon + v(x_i)\theta_i h_i}{h_{i+1}(h_i+h_{i+1})} \leq \frac{-2\varepsilon\theta_i}{h_{i+1}(h_i+h_{i+1})} < 0. \tag{5}$$

The expression in the brackets in $\gamma_{i-1}$ is non-negative if one takes

$$0 < h_i \leq \frac{2v(x_i - \frac{h_i}{2})}{c(x_i - \frac{h}{2})}.$$

Hence, since $0 < \theta_i < 1$, the operator $L^N$ is of positive type if $c(x) \equiv 0$ in $(0,1)$ or if $c(x) \neq 0$ in $(0,1)$ and

$$\max_{1 \leq i \leq N} h_i \leq \frac{2v_0}{\max_{x\in\Omega} c(x)}\,. \tag{6}$$

Furthermore, the following inequality

$$\gamma_i + \gamma_{i-1} + \gamma_{i+1} \geq 0 \tag{7}$$

holds and it holds strictly at least for one point $x_i = ih$, $i = 0, 1, \ldots, N$, because in the interior points it holds

$$\gamma_i + \gamma_{i-1} + \gamma_{i+1} = \theta_i c(x_i) + (1-\theta_i)c(x_i - \tfrac{h_i}{2}) \geq 0, \quad i = 1, 2, \ldots, N-1 \tag{8}$$

and strict inequality holds for points next to the boundary points. Therefore the operator $L_h$ is of strongly positive type. Hence, it is monotone and the associated matrix is a M-matrix.

**Remark:** If one takes into account the first and second terms in $\gamma_{i-1}$ an even less restrictive bound on mesh sizes can be derived.

**Lemma 2** *[Barrier Lemma] Let $L$ be a discrete monotone operator defined on the mesh $\Omega_N = \{x_i,\ i = 0, 1, \dots, N\}$ and assume that $w$, called barrier vector, is such that $(Lw)_i \geq \alpha,\ i = 0, 1, \dots, N$, where $\alpha$ is a positive constant. Further, assume that $w$ is $\varepsilon$–uniformly bounded for this $\alpha$. Then $\|L^{-1}\| \leq \|w\|/\alpha$, uniformly in $\varepsilon$.*

With a barrier function $w = x|_{\Omega_N}$ a straightforward computation shows that

$$L^N w \geq v_0 e, \quad e = (1, 1, \dots, 1)^t \tag{9}$$

so by the Barrier Lemma,

$$\|(L^N)^{-1}\|_\infty \leq \frac{1}{v_0}\ \|w\|_\infty = \frac{1}{v_0}, \tag{10}$$

which holds uniformly in $\varepsilon$.

## 2.3 Truncation error estimate

To estimate the discretization error $\|u - u^N\|$ we will prove first the boundedness of the truncation error $L^N[u(x_i) - u_i^N] = L^N u(x_i) - f_i^N$. Assuming first $h = h_i = h_{i-1}$ and using a Taylor expansion we obtain the following result for the truncation error,

$$\begin{aligned}
L^N u(x_i) - f_i^N = & -\varepsilon[u''(x_i) + \tfrac{1}{12}h^2 u^{\mathsf{IV}}] + \theta_i v(x_i)[u'(x_i) + \tfrac{1}{6}h^2 u'''] + \theta_i c(x_i) u(x_i) \\
& + (1-\theta_i)\{v(x_i - \tfrac{h}{2})[u'(x_i - \tfrac{h}{2}) + \tfrac{1}{24}h^2 u'''] \\
& + c(x_i - \tfrac{h}{2})[u(x_i - \tfrac{h}{2}) + \tfrac{1}{8}h^2 u'']\} - \theta_i f(x_i) - (1-\theta_i) f(x_i - \tfrac{h}{2}) \\
= & -\varepsilon u''(x_i) + \theta_i[v(x_i)u'(x_i) + c(x_i)u(x_i) - f(x_i)] \\
& + (1-\theta_i)[v(x_i - \tfrac{h}{2})u'(x_i - \tfrac{h}{2}) + c(x_i - \tfrac{h}{2})u(x_i - \tfrac{h}{2}) \\
& - f(x_i - \tfrac{h}{2})] - \tfrac{1}{12}\varepsilon h^2 u^{\mathsf{IV}} + \tfrac{1}{6}\theta_i v(x_i) h^2 u''' \\
& + (1-\theta_i)[\tfrac{1}{24}v(x_i - \tfrac{h}{2})h^2 u''' + \tfrac{1}{8}c(x_i - \tfrac{h}{2})h^2 u''] \\
= & -\varepsilon u''(x_i) + \theta_i \varepsilon u''(x_i) + (1-\theta_i)\varepsilon u''(x_i - \tfrac{h}{2}) \\
& - \tfrac{1}{12}\varepsilon h^2 u^{\mathsf{IV}} + \tfrac{1}{6}\theta_i v(x_i) h^2 u''' \\
& + (1-\theta_i)[\tfrac{1}{24}v(x_i - \tfrac{h}{2})h^2 u''' + \tfrac{1}{8}c(x_i - \tfrac{h}{2})h^2 u''] \\
= & -\varepsilon(1-\theta_i)[u''(x_i) - u''(x_i - \tfrac{h}{2})] - \tfrac{1}{12}\varepsilon h^2 u^{\mathsf{IV}} + \tfrac{1}{6}\theta_i v(x_i) h^2 u''' \\
& + (1-\theta_i)[\tfrac{1}{24}v(x_i - \tfrac{h}{2})h^2 u''' + \tfrac{1}{8}c(x_i - \tfrac{h}{2})h^2 u'']
\end{aligned}$$

By the choice of $\theta_i$, the error arising from the term

$$-\varepsilon(1-\theta_i)[u''(x_i) - u''(x_i - \tfrac{h}{2})]$$

is

$$\begin{aligned}
-\frac{\varepsilon r_i}{1 + r_i}[u''(x_i) - u''(x_i - \tfrac{h}{2})] &= -\frac{\varepsilon v(x_i) h}{2\varepsilon + v(x_i) h}\frac{h}{2}u''' \\
&\leq \frac{1}{4}\min(v(x_i)h, 2\varepsilon) h |u'''| \leq \frac{1}{4}\max_{x \in [0,1]} v(x) h^2 |u'''|.
\end{aligned}$$

Above the derivatives $u''$, $u'''$, $u^{\mathsf{IV}}$ are evaluated at some intermediate points. With integral form of the remainder we obtain the following estimates for the truncation error

$$\begin{aligned}&|L^N u(x_i) - f_i^N| \\ &\quad \le C_1 \varepsilon h \int_{x_{i-1}}^{x_{i+1}} |u^{\mathsf{IV}}(\xi)|\ \mathrm{d}\xi + C_2 h \int_{x_{i-1}}^{x_{i+1}} |u'''(\xi)|\ \mathrm{d}\xi + C_3 h \int_{x_{i-1}}^{x_{i+1}} |u''(\xi)|\ \mathrm{d}\xi, \end{aligned} \tag{11}$$

where $C_1,\ C_2,\ C_3$ are positive constants.

If the mesh is irregular and $h_i \neq h_{i+1}$, then for the estimate of the truncation error at $x_i$ we have

$$\begin{aligned}&|L^N u(x_i) - f_i^N| \\ &\quad \le C_1 \varepsilon \int_{x_{i-1}}^{x_{i+1}} |u'''(\xi)|\ \mathrm{d}\xi + C_2 h_i \int_{x_{i-1}}^{x_{i+1}} |u'''(\xi)|\ \mathrm{d}\xi + C_3 h_i \int_{x_{i-1}}^{x_{i+1}} |u''(\xi)|\ \mathrm{d}\xi. \end{aligned} \tag{12}$$

## 2.4 Discretization error estimate

The estimate of the discretization error includes the following steps. First, an estimate of the truncation error, second, a construction of a suitable barrier function $w^N$ and finally an estimate of the discretization error based on a discrete comparison principle introduced in [12].

To resolve the solution in the layer one can use a mesh refinement — a graded or a uniform mesh in the layer part. Consider the convection dominated case $\varepsilon \ll N^{-1}$. Set $\tau = \min\left\{\frac{1}{2}, \frac{2}{\beta}\varepsilon \ln N\right\}$, where $0 < \beta < v_0$, and denote

$$\begin{aligned}\Omega_1^N &= \{i(1-\tau)/(N/2),\, i = 0, \ldots, N/2\}, \\ \Omega_2^N &= \{x_i,\, i = N/2+1, \ldots, N\}.\end{aligned}$$

Then $\Omega^N = \Omega_1^N \cup \Omega_2^N$. The mesh points $x_i$ in $\Omega_2^N$ are defined by

- $x_i = 1 - \tau + \tau(i - N/2)/(N/2)$ for the piecewise uniform mesh;
- $x_i = 1 - \tau + \tau \ln(i + 1 - N/2)/\ln(N/2 + 1)$ for the logarithmically graded mesh.

We use the notation $\Omega_{\mathrm{S}}$ for the uniform mesh, called Shishkin mesh [13], and $\Omega_{\mathrm{L}}$ for the logarithmically graded mesh, which is of Bakhvalov type [5].

We consider here a mesh refinement method. To analyse the method we use the following splitting of the solution

$$u = g + z.$$

The smooth part $g$ satisfies the problem

$$\mathcal{L}g = f, \quad g(0) = u(0) = 0, \quad v(1)g'(1) + c(1)g(1) = f(1)$$

and it is readily seen that

$$|g^{(k)}| \leq C \quad \text{for} \quad 0 \leq k \leq q.$$

The layer part $z$ satisfies

$$\mathcal{L}z = 0, \quad z(0) = 0, \quad z(1) = u(1) - g(1)$$

and

$$|z^{(k)}| \leq C\varepsilon^{-k}\exp(-\tfrac{\beta(1-x)}{\varepsilon}) \quad \text{for} \quad 0 \leq k \leq q. \tag{13}$$

Let us consider each term of the right hand side of the following inequality separately,

$$\|u - u^N\| \leq \|g - g^N\| + \|z - z^N\|.$$

#### 2.4.1 Discretization error in the smooth part of the solution

For the discretization error estimate for the function $g$ an $O(N^{-2})$ estimate, uniformly in $\varepsilon$ can be readily derived. By the assumptions made, for the truncation error we have

$$|L^N[g(x_i) - g_i^N]| \leq C_1 N^{-1} \min\{\varepsilon, N^{-1}\} + C_2 N^{-2} \leq CN^{-2}. \tag{14}$$

Choosing a barrier function

$$w_i = C_w N^{-2}(1 + x_i)$$

and using the discrete comparison principle we obtain the following estimate for the discretization error

$$|g(x_i) - g_i^N| \leq w_i \leq CN^{-2} \quad \text{on} \quad \Omega_S. \tag{15}$$

#### 2.4.2 Discretization error in the layer part of the solution

For the layer part $z$ of the solution $u$ on the coarse part of the mesh $\Omega_1^N$ in the interval $(0, 1-\tau)$ we have

$$|z(x_i) - z_i^N| \leq CN^{-2} \quad \text{for} \quad 0 \leq i \leq N/2, \tag{16}$$

because the definition of $\tau$ implies that

$$\exp\left(-\tfrac{\beta(1-x_i)}{\varepsilon}\right) \leq \exp\left(-\tfrac{\beta\tau}{\varepsilon}\right) = N^{-2} \quad \text{for} \quad 0 \leq i \leq N/2.$$

Finally, we estimate $|z(x_i) - z_i^N|$ on the interval $(1-\tau, 1]$, where the mesh is fine. We estimate the truncation error first. After integrating (13) in the interval

$[x_{i-1}, x_{i+1}]$ we obtain the following estimate for the derivative $z^{(k)}$ in the remainder term,

$$\int_{x_{i-1}}^{x_{i+1}} |z^{(k)}(\xi)| \mathrm{d}\xi \leq \begin{cases} C\varepsilon^{-k+1} \sinh\left(\frac{\beta h_i}{\varepsilon}\right) \exp(-\frac{\beta(1-x_i)}{\varepsilon}), & i \neq N/2, \\ C\varepsilon^{-k+1} \exp(-\frac{\beta(1-x_i)}{\varepsilon}), & i = N/2. \end{cases} \tag{17}$$

From (12) it follows that the following bound of the truncation error holds,

$$|L^N[z(x_i) - z_i^N]| \leq C\varepsilon^{-2} h_i \sinh\left(\frac{\beta h_i}{\varepsilon}\right) \exp(-\frac{\beta(1-x_i)}{\varepsilon}), \quad N/2 < i < N, \tag{18}$$

for some positive constant $C$, independent on the mesh size and $\varepsilon$. In the above estimate the mesh size $h_i$ is assumed to be sufficiently fine and we have for the uniform mesh

$$\beta h_i \varepsilon^{-1} = \frac{\beta\tau\varepsilon^{-1}}{N/2} = 4N^{-1} \ln N \quad \text{for} \quad N/2 < i < N.$$

It follows that

$$\beta h_i \varepsilon^{-1} \sinh(\beta h_i \varepsilon^{-1}) \leq C_1 (\beta h_i \varepsilon^{-1})^2 \leq CN^{-2} (\ln N)^2 \quad \text{for} \quad N/2 < i < N.$$

The estimate (18) is equivalent with

$$|L^N[z(x_i) - z_i^N]| \leq C\varepsilon^{-1} N^{-2} (\ln N)^2 \exp(-\frac{\beta(1-x_i)}{\varepsilon}), \quad N/2 < i < N. \tag{19}$$

We choose a barrier function in the following way

$$w_i = C_w N^{-2} [1 + (\ln N)^2 \exp(-\frac{\beta(1-x_i)}{\varepsilon})], \quad \text{for} \quad N/2 < i < N. \tag{20}$$

**Lemma 3** *Assume that $v_0 > \beta > 0$. Then there is a constant $C$, independent of the mesh sizes and $\varepsilon$, such that*

$$L^N \left[\exp(-\frac{\beta(1-x_i)}{\varepsilon})\right] \geq C\varepsilon^{-1} \exp(-\frac{\beta(1-x_i)}{\varepsilon}), \quad N/2 < i < N. \tag{21}$$

**Proof:** In this case we have $\theta_i = 1$, $N/2 < i < N$ and $h_i = h_{i+1} = h$. If the variable $\mu$ is defined by $\mu = \beta h \varepsilon^{-1}$, then we have

$$\begin{aligned} & L^N \left[\exp(-\frac{\beta(1-x_i)}{\varepsilon})\right] \\ &= \exp(-\frac{\beta(1-x_i)}{\varepsilon}) \left\{ -\frac{\varepsilon}{h^2} [\exp(\mu) - 2 + \exp(-\mu)] + \frac{v(x_i)}{2h} [\exp(\mu) - \exp(-\mu)] + c(x_i) \right\} \\ &\geq \exp(-\frac{\beta(1-x_i)}{\varepsilon}) \left\{ -\frac{2\varepsilon}{h^2} [\cosh(\mu) - 1] + \frac{v_0}{h} \sinh(\mu) \right\} \\ &= \exp(-\frac{\beta(1-x_i)}{\varepsilon}) \left( -\frac{2\beta^2}{\varepsilon} \frac{\cosh(\mu) - 1}{\mu^2} + \frac{v_0 \beta}{\varepsilon} \frac{\sinh(\mu)}{\mu} \right) \\ &\geq \exp(-\frac{\beta(1-x_i)}{\varepsilon}) \frac{\beta^2}{\varepsilon} \left( \frac{\sinh \mu}{\mu} - 2 \frac{\cosh \mu - 1}{\mu^2} \right). \end{aligned}$$

Since by assumption $v_0 - \beta > 0$, then $\beta(v_0 - \beta) \geq c_1$, where $c_1$ is a positive constant ($\beta > 0$). The value of $\mu = 4N^{-1}\ln N$ is sufficiently small when $N$ is big. It is seen that statement (21) holds.

If $N/2 < i < N$ then $\exp\left(-\frac{\beta(1-x_i)}{\varepsilon}\right) \leq \exp(-\mu) \leq C \approx 1$. Hence, applying the discrete operator $L^N$ to the barrier function $w$, we find that

$$|z(x_i) - z_i^N| \leq w_i \leq C_w N^{-2}[1 + (\ln N)^2] \leq CN^{-2}(\ln N)^2. \tag{22}$$

We collect the results obtained from (15), (16) and (22) in the next theorem.

**Theorem 1** *Let $u$ be the solution of the convection-diffusion problem* (3)-(4) *and $u^N$ be its discrete solution obtained by applying the second order hybrid method on the Shishkin mesh $\Omega_S$. Let $v,\ c \in C^1[0,1]$ and $v \geq v_0 > \beta > 0$. Then the discretization error is globally uniformly bounded in $\varepsilon$ by*

$$\|u - u^N\| \leq CN^{-2}(\ln N)^2, \tag{23}$$

*where $N$ is the total number of points used.*

# 3 Numerical results in 1D case

We now demonstrate our error estimate numerically. In particular, we are interested in seeing the differences between the practical convergence behaviour of the method on a Shishkin and a logarithmically graded mesh.

In the test problem the coefficients are

$$v = 1 + x(1-x), \quad c = 0, \quad f = -\varepsilon u'' + vu' + cu,$$

where

$$u = x(1-x) + \frac{\exp(-1/\varepsilon) - \exp(-(1-x)/\varepsilon)}{\exp(-1/\varepsilon) - 1}.$$

The boundary conditions are $u(0) = 0, u(1) = 1$. For our tests we take $\varepsilon = 10^{-8}$ which is clearly sufficiently small to bring out the singularly perturbed nature of the problem.

We measure the accuracy in the discrete maximum norm $\|\cdot\|_\infty$

$$e^N = \|u - u^N\|_\infty = \max_i |u(x_i) - u^N(x_i)|$$

for different values of $N$. The rates of convergence are computed using

$$\text{rate}(N) = \log_2(e^{N/2}/e^N).$$

We also compute the value of $e^S = N^{-2}(\ln N)^2$ in our theoretical error bound (23) using Shishkin mesh and the relation $e^N/e^S$ which is the constant $C$ in the error estimate.

Table 1: Shishkin mesh.

| N | $e^N$ | $e^S = N^{-2}(\ln N)^2$ | $e^N/e^S$ | rate |
|---|---|---|---|---|
| $2^{10}$ | 0.399644e-4 | 0.457431e-4 | 0.8737 | 1.6963 |
| $2^{11}$ | 0.120876e-4 | 0.138487e-4 | 0.8728 | 1.7252 |
| $2^{12}$ | 0.359613e-5 | 0.412199e-5 | 0.8724 | 1.7490 |
| $2^{13}$ | 0.105509e-5 | 0.120966e-5 | 0.8722 | 1.7691 |
| $2^{14}$ | 0.305912e-6 | 0.350768e-6 | 0.8721 | 1.7862 |

Table 2: Logarithmically graded mesh.

| N | $e^N$ | $e^L = N^{-2}$ | $e^N/e^L$ | rate |
|---|---|---|---|---|
| $2^{10}$ | 0.116486e-4 | 0.953674e-6 | 12.214 | 1.9670 |
| $2^{11}$ | 0.266284e-5 | 0.238419e-6 | 11.169 | 1.9751 |
| $2^{12}$ | 0.751094e-6 | 0.596046e-7 | 12.601 | 1.9799 |
| $2^{13}$ | 0.189965e-6 | 0.139012e-7 | 12.748 | 1.9833 |
| $2^{14}$ | 0.479676e-7 | 0.372529e-8 | 12.876 | 1.9856 |

For the logarithmically graded mesh we compute $e^L = N^{-2}$ and the constant in this assumed bound, i.e.

$$\|u - u^N\| \leq CN^{-2}. \tag{24}$$

The numerical results are presented in Tables 1 and 2. It is clear from the first table that the theoretical results of Theorem 1 are very sharp. One can see from Table 2 that the accuracy of the method using a logarithmically graded mesh is nearly $CN^{-2}$.

# 4 The difference scheme and discretization error estimates in 2D case

## 4.1 Finite difference method

By analogy of the one-dimensional case we construct the combined finite difference and local method of characteristic line in the two-dimensional case. To simplify the presentation, we assume also that the mesh is uniform and $c$ is constant. A weighted combination of a finite difference approximation with weight factor $\theta$ and a local method of characteristics for the convective term with weight factor $(1-\theta)$, thereby approximating $\varepsilon\Delta u$ with $\varepsilon\Delta_h u_h$, is used.

Hence we let

$$
\begin{aligned}
L_h u_h(x_i, y_j) = & -\varepsilon \Delta_h u_h(x_i, y_j) \\
& + \theta\, v_1\,(x_i, y_j) \frac{u_h(x_i + h_x, y_j) - u_h(x_i - h_x, y_j)}{2h_x} \\
& + \theta\, v_2\,(x_i, y_j) \frac{u_h(x_i, y_j + h_y) - u_h(x_i, y_j - h_y)}{2h_y} \\
& + (1-\theta)\, |\mathbf{v}(x^\star, y^\star)|\, \frac{u_h(x_i, y_j) - u_h(x_\alpha, y_\alpha)}{l} \\
& + \theta\, c\, u_h(x_i, y_j) + (1-\theta)\, c\, \frac{u_h(x_i, y_j) + u_h(x_\alpha, y_\alpha)}{2} \\
= & f_h \equiv \theta f(x_i, y_j) + (1-\theta) f(x^\star, y^\star)
\end{aligned}
\tag{25}
$$

for each interior mesh point $(x_i, y_j) \in \Omega_h = \{(x_i, y_j), i = 1, \ldots, N_x, j = 1, \ldots, N_y\}$.

Here

- $u_h$ denotes the finite difference approximation of the solution $u$;
- $\Delta_h$ is the standard five–point difference approximation;
- $(x_\alpha, y_\alpha)$ is the intersection of the characteristic line at $(x_i, y_j)$, going backwards from $(x_i, y_j)$, with one of the lines: $x = x_i \pm h_x$, $y = y_j \pm h_y$;

  The corresponding finite difference mesh is illustrated in Figure 2.

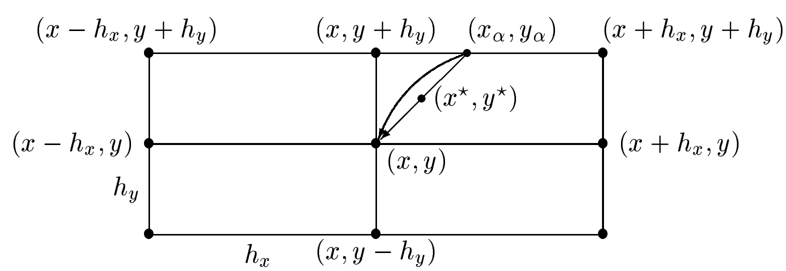



Figure 2: Finite-difference mesh in 2D case.

- $(x^\star, y^\star)$ is the midpoint on the straight line between $(x_i, y_j)$ and $(x_\alpha, y_\alpha)$, i.e.,

$$(x^\star, y^\star) = \left(\frac{x_i + x_\alpha}{2}, \frac{y_j + y_\alpha}{2}\right);$$

- $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$, $\theta = \dfrac{1}{1+r}$; $r = \dfrac{h|\mathbf{v}(x_i, y_j)|}{2\varepsilon}$, where the value of $h$ is chosen to be $h = \max\{h_x, h_y\}$;

- $l$ is the distance between $(x_i, y_j)$ and $(x_\alpha, y_\alpha)$.

Note that $\theta = \frac{2\varepsilon}{2\varepsilon + h|\mathbf{v}(x_i,y_j)|}$. Hence, when $\varepsilon \ll h$, we have $\theta = O(\varepsilon)$ and the difference approximation is dominated by the characteristic line method while when $h \le O(\varepsilon)$, we have $\theta = O(1)$ and the diffusion term difference approximation becomes significant.

The required approximate value of $u$ at $(x_\alpha, y_\alpha)$, where

$$x_\alpha = x_i + \alpha h_x, \quad 0 \le \alpha \le 1, \quad y_\alpha = y_j \pm h_y,$$

can be computed by *linear* or *quadratic* interpolation:

- *linear* interpolation:

$$u(x_\alpha, y_\alpha) \approx u_h(x_\alpha, y_\alpha) \equiv (1-\alpha) u_h(x_i, y_\alpha) + \alpha u_h(x_i + h_x, y_\alpha);$$

- *quadratic* interpolation:

$$u(x_\alpha, y_\alpha) \approx u_h(x_\alpha, y_\alpha) \equiv \frac{\alpha(\alpha-1)}{2}\, u_h(x_i - h_x, y_\alpha) + (1-\alpha^2)\, u_h(x_i, y_\alpha) + \frac{\alpha(1+\alpha)}{2}\, u_h(x_i + h_x, y_\alpha).$$

Similarly we let $\tilde{u}(x_\alpha, y_\alpha)$ denote the interpolated values of the exact solution, i.e.

$$\tilde{u}(x_\alpha, y_\alpha) \equiv (1-\alpha) u(x_i, y_\alpha) + \alpha u(x_i + h_x, y_\alpha),$$

$$\tilde{u}(x_\alpha, y_\alpha) \equiv \frac{\alpha(\alpha-1)}{2}\, u(x_i - h_x, y_\alpha) + (1-\alpha^2)\, u(x_i, y_\alpha) + \frac{\alpha(1+\alpha)}{2}\, u(x_i + h_x, y_\alpha),$$

respectively.

We use the following approximation for the value, $u(x^\star, y^\star)$

$$u(x^\star, y^\star) \approx u_h(x^\star, y^\star) \equiv [u_h(x_i, y_j) + u_h(x_\alpha, y_\alpha)]/2.$$

For points on $\Gamma_2$ we can use a similar difference approximation. It is readily seen that the resulting difference operator $L_h$ is of positive type and monotone when we use linear interpolation if

$$c = 0 \text{ or } h_x \le 2 \min_\Omega v_1(x,y)/c, \quad h_y \le 2 \min_\Omega v_2(x,y)/c \text{ when } c \ne 0. \tag{26}$$

(It is also of positive type when we use quadratic interpolation and $\alpha = 0$ or $\alpha = 1$ in which case linear and quadratic interpolation coincide.) When $L_h$ is monotone it holds that $L_h v > 0$ implies $v > 0$.

We mention that the main idea used in the above described difference scheme, namely, to make a linear combination of the local method of characteristic lines and the central difference method can also be used with a finite element method for a triangular or quadrilateral mesh, instead of the finite difference method. The method is also applicable for three space dimensional problems and for time-dependent problems $\frac{\partial u}{\partial t} + \mathcal{L}u = f$.

## 4.2 Discretization error estimate

Similarly to the one-dimensional case, we can derive a discretization error estimate in supremum norm. Consider first the truncation error. A crucial step in the derivation is the treatment of the truncation error term

$$\frac{(1-\theta)|\mathbf{v}(x^\star,y^\star)|}{l}[u(x_i,y_j)-\tilde{u}(x_\alpha,y_\alpha)]$$
$$=\frac{(1-\theta)|\mathbf{v}(x^\star,y^\star)|}{l}[u(x_i,y_j)-u(x_\alpha,y_\alpha)]+\frac{(1-\theta)|\mathbf{v}(x^\star,y^\star)|}{l}[u(x_\alpha,y_\alpha)-\tilde{u}(x_\alpha,y_\alpha)].$$

The second term is the interpolation error,

$$\frac{(1-\theta)|\mathbf{v}(x^\star,y^\star)|}{l}[u(x_\alpha,y_\alpha)-\tilde{u}(x_\alpha,y_\alpha)]=O(h^\nu) \tag{27}$$

where $\nu=1$ for linear and $\nu=2$ for quadratic interpolation and $h=\max\{h_x,h_y\}$.

The first term involves the direction derivative between $(x_i,y_j)$ and $(x_\alpha,y_\alpha)$. For a curved velocity field this direction is different from $\mathbf{v}(x^\star,y^\star)$ but the direction taken ($\hat{\mathbf{v}}(\hat{x},\hat{y})$, see below) is close so that

$$|\mathbf{v}(x^\star,y^\star)-\hat{\mathbf{v}}(\hat{x},\hat{y})|=O(h^2).$$

We have

$$(1-\theta)\ |\mathbf{v}(x^\star,y^\star)|[u(x_i,y_j)-u(x_\alpha,y_\alpha)]/l$$
$$=(1-\theta)\ |\mathbf{v}(x^\star,y^\star)|\left[\frac{\partial u}{\partial x}(x^\star,y^\star)\hat{v}_1+\frac{\partial u}{\partial y}(x^\star,y^\star)\hat{v}_2+O(h^2)\right] \tag{28}$$

where $\hat{v}_1=(x_i-x_\alpha)/l$, $\hat{v}_2=(y_i-y_\alpha)/l$.

Here $\hat{\mathbf{v}}=\mathbf{v}(\hat{x},\hat{y})/|\mathbf{v}(\hat{x},\hat{y})|$ for a point $(\hat{x},\hat{y})$, where the direction $(\hat{v_1},\hat{v_2})$ is tangential. This point is located at a distance $O(h^2)$ from $(x^\star,y^\star)$, see Figure 3. Therefore

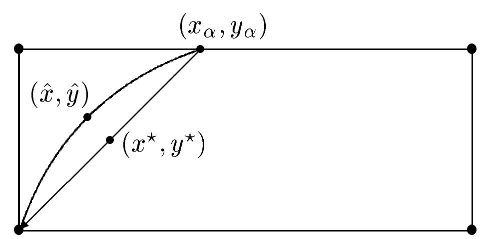



Figure 3: Points $(\hat{x},\hat{y})$ and $(x^\star,y^\star)$.

$$(1-\theta)|\mathbf{v}(x^\star,y^\star)|\,[u(x_i,y_j)-u(x_\alpha,y_\alpha)]\,/l$$
$$=(1-\theta)|\mathbf{v}(x^\star,y^\star)|\left[\left(\frac{\partial u}{\partial x}\hat{v}_1^\star+\frac{\partial u}{\partial y}\hat{v}_2^\star\right)+\frac{\partial u}{\partial x}(\hat{v}_1-\hat{v}_1^\star)+\frac{\partial u}{\partial y}(\hat{v}_2-\hat{v}_2^\star)+O(h^2)\right]$$
$$=(1-\theta)\mathbf{v}(x^\star,y^\star)\cdot\nabla u(x^\star,y^\star)+O(h^2),$$

where $\hat{v}_i^\star = \frac{v_i(x^\star,y^\star)}{|\mathbf{v}(x^\star,y^\star)|}$ and $|(\hat{x},\hat{y})-(x^\star,y^\star)| = O(h^2)$.

For the whole truncation error we have then

$$
\begin{aligned}
&L_h u - f_h \\
=& -\varepsilon\,\left[\Delta u(x_i,y_j) + \frac{1}{12}\left(h_x^2\frac{\partial^4 u}{\partial x^4} + h_y^2\frac{\partial^4 u}{\partial y^4}\right)\right] \\
&+\theta\left[v_1(x_i,y_j)\frac{\partial u}{\partial x}(x_i,y_j) + v_2(x_i,y_j)\frac{\partial u}{\partial y}(x_i,y_j) + \frac{1}{6}\left(h_x^2 v_1\frac{\partial^3 u}{\partial x^3} + h_y^2 v_2\frac{\partial^3 u}{\partial y^3}\right)\right] \\
&+(1-\theta)\ \mathbf{v}(x^\star,y^\star)\cdot\nabla u(x^\star,y^\star) + O(h^2) \\
&+\frac{(1-\theta)\ |\mathbf{v}(x^\star,y^\star)|}{l}\left[\tilde{u}(x_\alpha,y_\alpha) - u(x_\alpha,y_\alpha)\right] \\
&+\theta\ [c\ u(x_i,y_j) - f(x_i,y_j)] + (1-\theta)\ [c\ u(x^\star,y^\star) - f(x^\star,y^\star)] \\
=& -\frac{\varepsilon}{12}\left(h_x^2\frac{\partial^4 u}{\partial x^4} + h_y^2\frac{\partial^4 u}{\partial y^4}\right) + \frac{\theta}{6}\left(h_x^2 v_1\ \frac{\partial^3 u}{\partial x^3} + h_y^2 v_2\ \frac{\partial^3 u}{\partial y^3}\right) - \varepsilon\ \Delta u(x_i,y_j) \\
&+\theta\left[v_1(x_i,y_j)\frac{\partial u}{\partial x}(x_i,y_j) + v_2(x_i,y_j)\frac{\partial u}{\partial y}(x_i,y_j) + cu(x_i,y_j) - f(x_i,y_j)\right] \\
&+(1-\theta)\ \varepsilon\ \Delta u(x^\star,y^\star) + O(h^\nu) \\
=&O(h_x^2+h_y^2) - \varepsilon\ \Delta u(x_i,y_j) + \theta\ \varepsilon\ \Delta u(x_i,y_j) + (1-\theta)\ \varepsilon\ \Delta u(x^\star,y^\star) + O(h^\nu) \\
=&O(h_x^2+h_y^2) - \varepsilon\ (1-\theta)\left[\Delta u(x_i,y_j) - \Delta u(x^\star,y^\star)\right] + O(h^\nu).
\end{aligned}
\tag{29}
$$

The derivatives $\frac{\partial^4 u}{\partial x^4}$, $\frac{\partial^4 u}{\partial y^4}$, $\frac{\partial^3 u}{\partial x^3}$ and $\frac{\partial^3 u}{\partial y^3}$ are evaluated at some intermediate points.

By the choice of $\theta$, if $u \in C^4(\bar{\Omega})$ and $\mathbf{v}(x_i,y_j) \neq 0$ the error arising from the term

$$\varepsilon(1-\theta)\left[\Delta u(x_i,y_j) - \Delta u(x^\star,y^\star)\right]$$

is

$$\frac{\varepsilon\ r}{1+r}\left[\Delta u(x_i,y_j) - \Delta u(x^\star,y^\star)\right] = \frac{\varepsilon\ r}{1+r}O(h) = \frac{\varepsilon\ h}{2\varepsilon+h}O(h).$$

It follows that this term has $O(h^2)$ discretization error.

If $L_h$ is a monotone operator, the discretization error is then found to be

$$||u-u_h||_\infty \leq ||L_h^{-1}||_\infty ||L_h(u-u_h)||_\infty = ||L_h^{-1}||_\infty ||L_h u - f_h||_\infty.$$

When $|\mathbf{v}| \geq v_0 > 0$, the norm $||L_h^{-1}||_\infty$ is bounded uniformly in $\varepsilon$ and $h$, because

$$||L_h^{-1}||_\infty = ||L_h^{-1}\mathbf{e}||_\infty = ||\mathbf{z}||_\infty \quad \text{where } \mathbf{e}^t = (1,1,\ldots,1) \text{ and } L_h\mathbf{z} = \mathbf{e}.$$

One can see that for $\mathbf{z} = [(v_1 x + v_2 y)/|\mathbf{v}|^2]|_{\Omega_h}$ and $c = 0$ we have $L_h\mathbf{z} = \mathbf{e}$ and

$$||L_h^{-1}||_\infty = ||\mathbf{z}||_\infty \leq \frac{|v_1|+|v_2|}{|\mathbf{v}|^2}.$$

Clearly, this bound holds also for $c \geq 0$ with assumption (26). Hence, the discretization error behaves as $O(h)$ uniformly in $\varepsilon$ (and in $|\mathbf{v}|$) if $u \in C^4(\Omega)$ and $u$ does not depend (heavily) on $\varepsilon$. For points on the Neumann boundary we can use a similar method.

We summarize next the above results

**Theorem 2** *Consider problem (1)–(2) with the assumptions made, in particular $|\mathbf{v}| \geq v_0 > 0$ in $\Omega$ and discretized as described, using a uniform mesh and linear interpolation. If $u \in C^4(\Omega)$ and $u$ does not have a boundary layer then the discretization error satisfies $||u - u_h||_\infty \leq O(h)$, uniformly in $\varepsilon$.*

**Remark:** The above derivation of the truncation error and the numerical results indicate that for quadratic interpolation the discretization error satisfies $||u - u_h||_\infty \leq O(h^2)$, uniformly in $\varepsilon$.

When $u$ has boundary layers they can be resolved using an anisotropic mesh adjusted to the boundary.

Since in general the operator $L_h$ is not of monotone type for quadratic interpolation, we can use in this case an approximation method involving two steps of a defect-correction method

$$L_h^{(0)} u_h^{(0)} = f_h, \quad L_h^{(0)} \delta_h^{(0)} = (L_h^{(0)} - L_h) u_h^{(0)}, \quad u_h^{(1)} = u_h^{(0)} + \delta_h^{(0)},$$

where $L_h^{(0)}$ and $L_h$ are the difference operators for linear and quadratic interpolation, respectively. Then it holds

$$L_h^{(0)}(u - u_h^{(1)}) = L_h u - f_h + (L_h^{(0)} - L_h)(u - u_h^{(0)}),$$

where $L_h u - f_h$ is the truncation error for quadratic interpolation. Since $||\left(L_h^{(0)}\right)^{-1}||$ is bounded (uniformly in $\varepsilon$) it holds

$$||u - u_h^{(1)}||_\infty \leq O(h^2).$$

The scheme is, therefore, of second order of accuracy uniformly in the singular perturbation parameter $\varepsilon$. In addition, this hybrid scheme has a very efficient property when solving the corresponding algebraic systems, as they can be more easily preconditioned by incomplete factorisation methods, for instance.

# 5 Numerical results in 2D case

The method described requires a numerical integration of the characteristic line in backward direction from each interior mesh point. This integration is typically done by three to five steps of a second order (explicit) Runge-Kutta method or one step of a fourth order Runge-Kutta method. Though this computation is of a relatively small amount compared to the remaining computations it requires some additional computations. The following alternative method approximates the characteristic line with the tangent line in $(x_i, y_j)$, see Figure 4. In this case the tangential direction, $\mathbf{v}(x_i, y_j)$ is taken at a point $(\hat{x}, \hat{y}) = (x_i, y_j)$ a distance $|(\hat{x}, \hat{y}) - (x^\star, y^\star)| = O(h)$ apart. The truncation error term corresponding to (30) is therefore $O(h)$, and the total error is $O(h)$ even if quadratic interpolation is used. The following numerical results demonstrate clearly these error orders, based on the results in Section 4.2 and the above conclusion regarding the use of the tangent approximation of the characteristic line. They also show a much higher accuracy than can be achieved by the upwind

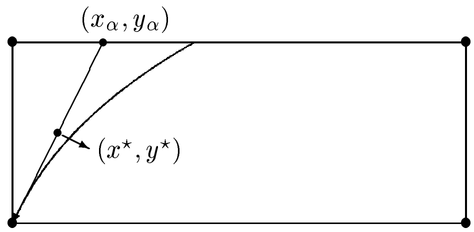



Figure 4: The tangential approximation of the characteristic line.

method, or the central difference method.

To illustrate the advantages of the hybrid method of characteristics we conduct numerical tests with different values of the parameters $\varepsilon$, $N_x$, $N_y$ and different velocity vectors. As we shall see, the numerical results confirm the uniform in $\varepsilon$ convergence. Due to the stabilisation effect of the characteristic line method there are no oscillations. We consider two examples. The first example (with a smooth solution on a uniform mesh) illustrates the uniform in $\varepsilon$ convergence and order of discretization error in maximum norm of the method. The second one (with two exponential layers along the outflow boundary) confirms these advantages on layer-fitted meshes.

### Test to illustrate the uniform in $\varepsilon$ behaviour of the discretization error

We consider the following example with a smooth solution:

**Example 1.** $f(x)$ is such that the exact solution is smooth,

$$u(x, y) = xy(1 - x)(1 - y)\ \exp(-xy),$$

i.e., there is no singular behaviour. The boundary conditions are $u(x, y) = 0$ on $\Gamma$. The velocity vector is variable $\mathbf{v} = [\mathrm{e}^{xy}, 1 + xy]$ and $c = 0.01$.

We compare the results obtained by: UM — standard upwind differences for $u_x$ and $u_y$; CD — standard central differences discretization; HLI — present method using linear interpolation for $u(x_\alpha, y_\alpha)$; HQI — present method using quadratic interpolation for $u(x_\alpha, y_\alpha)$; HLIT — the HLI method using tangential approximation; HQIT — the HQI method using tangential approximation of the velocity vector.

The results for

- maximal value of the exact discretization error at the meshpoints, i.e.

$$e_h = \|u - u_h\|_\infty = \max_{i,j} |u(x_i, y_j) - u_h(x_i, y_j)|$$

  for different values of $\varepsilon$ ranging from $\varepsilon = 10^{-9}$ to $\varepsilon = 1$ and $h$ on a uniform square mesh ($h = h_x = h_y$) are presented in Table 3;

- rate of convergence rate-conv $= \log_2(e_{2h}/e_h)$ for $\varepsilon = 1$ and $\varepsilon = 10^{-9}$ are shown in Table 4.

The discretization error of the HQI method is $O(h^2)$ uniformly in $\varepsilon$ if $u \in C^4(\Omega)$. While the discretization error of the HLIT, HQIT, HLI methods is $O(h)$ uniformly in $\varepsilon$, the discretization error of the CD method is $O(h^2)$, but still the actual errors of the CD method for viable choices of $h$ are bigger than the errors of the HLI method. The HQI method has the smallest errors and the fastest i.e., quadratic rate of convergence.

**Tests with boundary layers on a layer-fitted mesh**

We consider the singularly perturbed boundary value problem (1), (2), where $v_1 > \beta_1 > 0$, $v_2 > \beta_2 > 0$.

Due to the arising derivatives in the error estimates the rate of convergence is not uniform in the parameter $\varepsilon$ unless the mesh is properly refined in the layers. Making the best choice of such a layer-fitted mesh requires a sufficiently precise information how the derivatives of the exact solution depend on $\varepsilon$. For an outflow boundary layer we have

$$u^{(s)}(x,y) = \begin{cases} C_1 \varepsilon^{-s} \exp(-v_1(1-x)/\varepsilon) & \text{for a layer at } x = 1 \\ C_2 \varepsilon^{-s} \exp(-v_2(1-y)/\varepsilon) & \text{for a layer at } y = 1 \end{cases}, \quad s = 1, 2, \ldots .$$

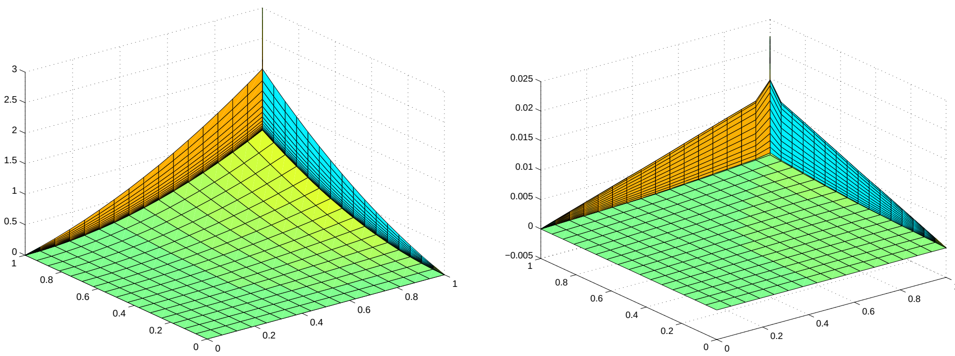
Like the one–dimensional case we must use a refined mesh which to be viable, must be anisotropic in the two-dimensional case. The mesh can be of uniform type, such as the Shishkin mesh with constant mesh size on each mesh line orthogonal to the boundary layer. Alternatively, it can be an exponentially graded mesh in the layer such as of Bakhvalov type for the same width. In both cases we choose $N/b$ mesh points on each such mesh line, so the fine mesh size is $h = \frac{\tau}{N/b}$ for the uniform mesh. The transition point between the coarse and the layer-fitted mesh has been chosen so that $\exp(-(1-x)/\varepsilon) = O(N^{-a})$ ($a = 1$ or $a = 2$), i.e., the singular part of the solution has the same order $O(N^{-a})$ as the discretization error we aim at, and can therefore be neglected outside a layer.

Let $N_x = N_y = N$ be an even integer. The number of mesh points used are in total $N^2$ and are distributed with $((1 - b^{-1})N)^2$ points in the layer free, interior part of the domain, $(1 - b^{-1})b^{-1}N^2$ in the two layer parts along $x = 1$ and $y = 1$, respectively, and $(b^{-1}N)^2$ points in the corner at $x = 1$, $y = 1$. Efficient values of $b$ in distributing the error about equal in the interior and layer parts turns out to be 2, 3/2, 4/3. For $b = 3/2$, there are 1/9 of the total number of points in the layer free part, 2/9 in the two layer parts and 4/9 in the corner. In all tests for a problem with a layer, we have taken the parameter $b = 2$, which means $(N/2)^2$ points in the layer free part of the domain.

**Example 2.** $f(x)$ is such that the exact solution is

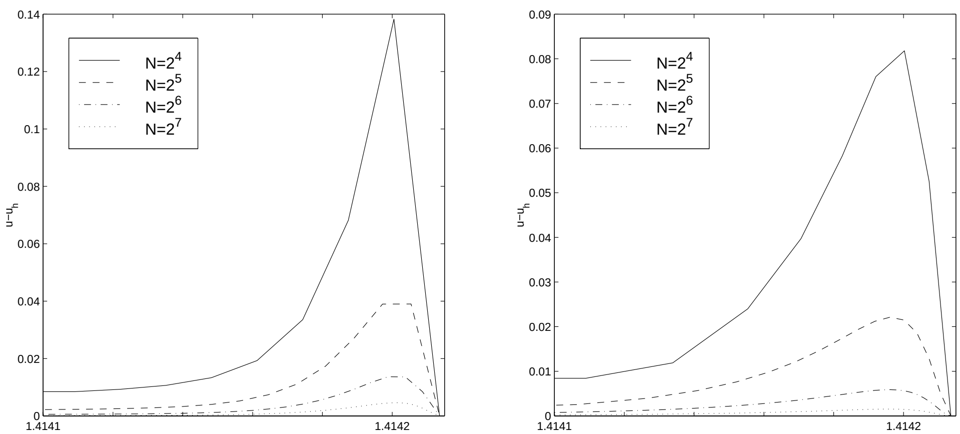
$$u(x, y) = x^2 y^2 + xy\,(\exp(-(1-x)/\varepsilon) + \exp(-(1-y)/\varepsilon)). \tag{30}$$

The velocity vector is $\mathbf{v} = (3 - y, 3 - x)$ and $c = 0$. The boundary conditions are $u(x, y)$ on $\Gamma$.



(a) Exact solution (b) Discretization error

Figure 5: Example 2 on fitted mesh $\Omega_{\mathrm{L}}$ with $N = 2^5$, $\varepsilon = 10^{-5}$.




(a) Shishkin mesh (b) Logarithmically graded mesh

Figure 6: Zoomed diagonal cross–section (with $y = x$) of errors on layer-fitted meshes, $\varepsilon = 10^{-5}$.

In this case, two exponential layers are present along the outflow boundary at

$x = 1$ and $y = 1$, see Figure 5(a). These layers cause serious instabilities in the central difference or standard Galerkin finite element scheme on a uniform mesh resulting in an oscillatory numerical solution, see e.g. [7], where the streamline upwind diffusion method on a Shishkin mesh is used in order to overcome these difficulties. We handle the layers by using a fine anisotropic mesh, which is defined by a tensor product of two one-dimensional meshes.

We restrict our attention to the convection-dominated case with $\varepsilon \ll N^{-1}$. Set $\tau_x = \min\left\{\frac{1}{2}, \frac{2}{\beta_1}\varepsilon \ln N\right\}$, $\tau_y = \min\left\{\frac{1}{2}, \frac{2}{\beta_2}\varepsilon \ln N\right\}$, and divide each of the subintervals $[0, 1-\tau_x]$ and $[1-\tau_x, 1]$ into $N/2$ subintervals. The coarse meshsizes are defined by

$$H_x = 2(1-\tau_x)/N, \quad H_y = 2(1-\tau_y)/N.$$

Denote

$$\Omega_{1,x} = \{x_i = iH_x,\ i = 0, \ldots, N/2\},$$
$$\Omega_{2,x} = \{x_i,\ i = N/2, \ldots, N\}.$$

Then $\Omega_x = \Omega_{1,x} \cup \Omega_{2,x}$. Similarly $\Omega_y$ is defined. The mesh in $\Omega$ is $\Omega_{xy} = \Omega_x \times \Omega_y$. For the Shishkin mesh $\Omega_{\mathrm{S}}$ we subdivide $\Omega_{2,x}$ and $\Omega_{2,y}$ by an equidistant mesh. The second layer-fitted mesh $\Omega_{\mathrm{L}}$ is somewhat more complicated. By analogy of the one dimensional case we apply a logarithmically graded mesh near the boundaries $x = 1$ and $y = 1$.

All numerical results described below are obtained by the hybrid finite difference method with quadratic interpolation using a Shishkin mesh and a logarithmically graded mesh near the layer. The pointwise error of the second mesh is plotted in Figure 5(b). We see that there are no oscillations. In the layer-fitted mesh we choose $\theta = 1$. The calculated norms $||u - u^N||_\infty$ ($u^N$ denotes the finite difference approximation of the solution $u$) are shown in Table 5 for $\varepsilon \in [10^{-8}, 10^{-4}]$ and $N = 2^3, 2^4, 2^5, 2^6, 2^7$. The convergence holds uniformly in $\varepsilon$ as one can see from the table. The numerical results in Table 5 and Figure 6 confirm that the finite difference solution approximates $u$ to almost $O(N^{-2})$ order in $L_\infty$-norm on a graded layer-fitted mesh. In this example, the solution decays rapidly (exponentially) and is smooth outside the layers. Since $\theta = 1$ in the layer, the HLI and HQI methods are essentially equal and give here a nearly quadratic rate of convergence in the layer part.

We consider finally the maximum errors outside the layer part for $\varepsilon = 10^{-8}$. The numerical results in Table 6 show the order of convergence almost $O(N^{-1})$ for HLI method and $O(N^{-2})$ for HQI method in approximating $u$ outside the layer part. The rate of convergence using a graded mesh is better than the rate of convergence using a Shishkin mesh as one can see from the Tables 5 and 6. The errors outside the layer part are much smaller than in the layer part which shows that it would be more efficient, at least for this particular example, to distribute the meshpoints such as $\frac{1}{16}N^2$ points outside the layers and the remaining $\frac{15}{16}N^2$ points in the layer parts.

## 6 Conclusions

It has been shown that the hybrid method of characteristics and central difference approximation has significant advantages over other commonly used methods, such as the central difference or upwind methods. It gives solutions free of unphysical wiggles and a high accuracy, in particular if it is combined with a mesh resolution in the layer part of the solution. Quadratic interpolation and logarithmically graded meshes give second order of convergence in maximum norm on the whole domain, including the layer, which holds uniformly in the singular perturbation parameter $\varepsilon$. The arising algebraic systems are (nearly) M-matrices and can therefore be solved readily by incomplete factorization preconditioned iterative methods, for instance.

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## References

[1] O.Axelsson, A Survey of Numerical Methods for Convection-Diffusion Equations. In Proceedings of XIV National Summer School on Numerical Solution Methods, Varna, 29.8–3.9 (1988) [Also published as Report No 8822 (1988) Dept. of Mathematics, University of Nijmegen, The Netherlands]

[2] O.Axelsson, G.F.Carey: On the numerical solution of two-point singularly perturbed boundary value problems, Comp. Math. Appl. Mech. Eng. **50** (1985), 217–229

[3] O.Axelsson, I.Gustafsson, A modified upwind scheme for convective transport equation and the use of a conjugate gradient method for the solution of nonsymmetric system of equations. J.Inst.Math.Appl., **23** (1979), 321–337

[4] O.Axelsson, R.Marinova, Combined Finite Difference and Local Method of Characteristic Line Approximation for Convection-Diffusion Problems, *Workshop on the numerical and computational methods for convection-dominated and singular perturbed problems*, Lozenetz, Bulgaria, August 27-31 (1998) (proceedings to appear)

[5] A.Bakhvalov, On the optimization of methods for solving boundary value problems with boundary layers. J. Vychisl. Math. i Math. Fysika, **9** (1969), 841–859 (in Russian)

[6] J.H.Miller, E.O'Riordan, G.I.Shishkin, Fitted numerical methods for singular perturbation problems. World Scientific, Singapore, (1996)

[7] M.Nikolova, O.Axelsson, Uniform in $\varepsilon$ Convergence of Finite Element Method for Convection-Diffusion Equations Using a priori Chosen Meshes. CWI Quarterly **10** (3&4) (1997), 253–276

[8] M.Nikolova, O.Axelsson, Uniform in $\varepsilon$ Convergence of Defect-Correction Method for Convection-Diffusion Problems. Report #9905, Department of Mathematics, University of Nijmegen, The Netherlands, February 1999.

[9] M.Stynes, H.Roos, The midpoint upwind scheme. Appl. Num. Math., **23** (1997), 361–374

[10] H.Roos, T.Skalický, A Comparison of the Finite Element Method on Shishkin and Gartland-Type Meshes for Convection-Diffusion Problems. CWI Quarterly **10** No 3&4 (1997), 277–300

[11] H.Roos, M.Stynes, L.Tobiska, Numerical methods for singularly perturbed differential equations. Springer, Heidelberg 1996

[12] A.Samarskii, *Theory of the Finite Difference Schemes.* Nauka, Moskow 1977 ( in Russian).

[13] G.Shishkin, Grid approximation of singularly perturbed elliptic and parabolic equations. Second doctoral thesis, Keldysh Institute, Russian Academy of Science, Moskow (1990) (in Russian)

Table 3: Norms $e_h = \|u - u_h\|_\infty$ using a uniform square mesh — Example 1, $c = 0.01$, $\mathbf{v} = [e^{xy}, 1 + xy]$.

| $\varepsilon$ | Method | $h = 2^{-3}$ | $h = 2^{-4}$ | $h = 2^{-5}$ | $h = 2^{-6}$ | $h = 2^{-7}$ |
|---|---|---|---|---|---|---|
| 1 | UM | 0.38411e-2 | 0.19121e-2 | 0.95389e-3 | 0.47641e-3 | 0.23810e-3 |
| $10^{-3}$ | | 0.24235e-1 | 0.13179e-1 | 0.69025e-2 | 0.35370e-2 | 0.17916e-2 |
| $10^{-5}$ | | 0.24296e-1 | 0.13216e-1 | 0.69224e-2 | 0.35476e-2 | 0.17970e-2 |
| $10^{-7}$ | | 0.24297e-1 | 0.13217e-1 | 0.69226e-2 | 0.35477e-2 | 0.17970e-2 |
| $10^{-9}$ | | 0.24297e-1 | 0.13217e-1 | 0.69226e-2 | 0.35477e-2 | 0.17970e-2 |
| 1 | CD | 0.33768e-3 | 0.86073e-4 | 0.21510e-4 | 0.53831e-5 | 0.13458e-5 |
| $10^{-3}$ | | 0.16132e-1 | 0.19677e-2 | 0.38903e-3 | 0.92173e-4 | 0.20807e-4 |
| $10^{-5}$ | | 0.14878 | 0.22212e-1 | 0.25781e-2 | 0.33213e-3 | 0.38890e-4 |
| $10^{-7}$ | | 0.18522 | 0.43000e-1 | 0.10265e-1 | 0.22935e-2 | 0.40943e-3 |
| $10^{-9}$ | | 0.18568 | 0.43425e-1 | 0.10669e-1 | 0.26526e-2 | 0.65930e-3 |
| 1 | HLIT | 0.29869e-3 | 0.91828e-4 | 0.24968e-4 | 0.65085e-5 | 0.16606e-5 |
| $10^{-3}$ | | 0.17618e-2 | 0.40886e-3 | 0.20286e-3 | 0.99032e-4 | 0.46435e-4 |
| $10^{-5}$ | | 0.18701e-2 | 0.46119e-3 | 0.21175e-3 | 0.10686e-3 | 0.53499e-4 |
| $10^{-7}$ | | 0.18712e-2 | 0.46177e-3 | 0.21184e-3 | 0.10694e-3 | 0.53578e-4 |
| $10^{-9}$ | | 0.18712e-2 | 0.46178e-3 | 0.21184e-3 | 0.10694e-3 | 0.53579e-4 |
| 1 | HQIT | 0.30614e-3 | 0.93694e-4 | 0.25418e-4 | 0.66188e-5 | 0.16879e-5 |
| $10^{-3}$ | | 0.17904e-2 | 0.42058e-3 | 0.14190e-3 | 0.68560e-4 | 0.31782e-4 |
| $10^{-5}$ | | 0.18988e-2 | 0.47309e-3 | 0.15046e-3 | 0.75128e-4 | 0.37465e-4 |
| $10^{-7}$ | | 0.19000e-2 | 0.47367e-3 | 0.15055e-3 | 0.75197e-4 | 0.37528e-4 |
| $10^{-9}$ | | 0.19000e-2 | 0.47368e-3 | 0.15055e-3 | 0.75198e-4 | 0.37529e-4 |
| 1 | HLI | 0.30491e-3 | 0.93044e-4 | 0.25147e-4 | 0.65324e-5 | 0.16643e-5 |
| $10^{-3}$ | | 0.17803e-2 | 0.41283e-3 | 0.90634e-4 | 0.45360e-4 | 0.22185e-4 |
| $10^{-5}$ | | 0.18892e-2 | 0.46536e-3 | 0.11379e-3 | 0.47792e-4 | 0.24391e-4 |
| $10^{-7}$ | | 0.18903e-2 | 0.46594e-3 | 0.11408e-3 | 0.47817e-4 | 0.24419e-4 |
| $10^{-9}$ | | 0.18903e-2 | 0.46595e-3 | 0.11409e-3 | 0.47818e-4 | 0.24420e-4 |
| 1 | HQI | 0.31042e-3 | 0.94645e-4 | 0.25564e-4 | 0.66384e-5 | 0.16911e-5 |
| $10^{-3}$ | | 0.17954e-2 | 0.42107e-3 | 0.95398e-4 | 0.19679e-4 | 0.36363e-5 |
| $10^{-5}$ | | 0.19044e-2 | 0.47405e-3 | 0.11889e-3 | 0.29641e-4 | 0.73765e-5 |
| $10^{-7}$ | | 0.19055e-2 | 0.47466e-3 | 0.11919e-3 | 0.29789e-4 | 0.74463e-5 |
| $10^{-9}$ | | 0.19055e-2 | 0.47466e-3 | 0.11919e-3 | 0.29790e-4 | 0.74470e-5 |

Table 4: Rate of convergence (rate-conv $= \log_2(e_{2h}/e_h)$) — Example 1.

| $\varepsilon$ | Meth. | $h=2^{-3}$ | $h=2^{-4}$ | $h=2^{-5}$ | $h=2^{-6}$ | $h=2^{-7}$ |
|---|---|---|---|---|---|---|
| 1 | UM | 1.012 | 1.006 | 1.003 | 1.002 | 1.001 |
| $10^{-9}$ | | 0.815 | 0.878 | 0.933 | 0.964 | 0.981 |
| 1 | CD | 1.962 | 1.972 | 2.001 | 1.998 | 2.000 |
| $10^{-9}$ | | 2.324 | 2.096 | 2.025 | 2.008 | 2.008 |
| 1 | HLIT | 1.916 | 1.702 | 1.879 | 1.940 | 1.971 |
| $10^{-9}$ | | 1.901 | 2.019 | 1.124 | 0.986 | 0.997 |
| 1 | HQIT | 1.877 | 1.708 | 1.882 | 1.941 | 1.971 |
| $10^{-9}$ | | 1.935 | 2.004 | 1.654 | 1.001 | 1.003 |
| 1 | HLI | 1.883 | 1.712 | 1.888 | 1.945 | 1.973 |
| $10^{-9}$ | | 1.941 | 2.020 | 2.030 | 1.255 | 0.969 |
| 1 | HQI | 1.856 | 1.714 | 1.888 | 1.945 | 1.973 |
| $10^{-9}$ | | 1.953 | 2.005 | 1.994 | 2.000 | 2.000 |

Table 5: Norms $e^N = \|u - u^N\|_\infty$ using layer-fitted meshes ($\Omega_S$—Shishkin type mesh, $\Omega_L$—logarithmically graded mesh), Example 2, $c = 0$, $\mathbf{v} = (3-y, 3-x)$, rate=$\max_\varepsilon \ln(e^{N/2}/e^N)/\ln 2$, $\theta = 1$ in the layer, $\beta_1 = \beta_2 = 3/4$.

| Mesh | $\varepsilon$ | $N=2^3$ | $N=2^4$ | $N=2^5$ | $N=2^6$ | $N=2^7$ |
|---|---|---|---|---|---|---|
| $\Omega_S$ | $10^{-4}$ | 0.370729764 | 0.138206803 | 0.039358691 | 0.013653866 | 0.004602242 |
| | $10^{-5}$ | 0.370944266 | 0.138262406 | 0.039374863 | 0.013661813 | 0.004604718 |
| | $10^{-6}$ | 0.370965721 | 0.138267976 | 0.039376487 | 0.013662614 | 0.004604969 |
| | $10^{-7}$ | 0.370967867 | 0.138268533 | 0.039376649 | 0.013662692 | 0.004605004 |
| | $10^{-8}$ | 0.370968079 | 0.138268589 | 0.039376666 | 0.013662699 | 0.004604932 |
| | | | rate=1.43 | rate=1.81 | rate=1.53 | rate=1.57 |
| $\Omega_L$ | $10^{-4}$ | 0.381428647 | 0.081741794 | 0.022101680 | 0.005871118 | 0.001527280 |
| | $10^{-5}$ | 0.381619982 | 0.081797325 | 0.022124934 | 0.005879709 | 0.001530634 |
| | $10^{-6}$ | 0.381639123 | 0.081802886 | 0.022127266 | 0.005880573 | 0.001530972 |
| | $10^{-7}$ | 0.381641038 | 0.081803442 | 0.022127499 | 0.005880655 | 0.001531033 |
| | $10^{-8}$ | 0.381641230 | 0.081803498 | 0.022127508 | 0.005880697 | 0.001531191 |
| | | | rate=2.22 | rate=1.89 | rate=1.91 | rate=1.94 |

Table 6: Norms $e_N = \|u - u_N\|_\infty$ of the coarse part $\Omega_{1,x}$ of the Shishkin and logarithmically graded mesh, Example 2.

| Method | $N=2^3$ | $N=2^4$ | $N=2^5$ | $N=2^6$ | $N=2^7$ |
|---|---|---|---|---|---|
| HLI | 0.0175781190 | 0.0059814419 | 0.00171661196 | 0.001167675 | 0.0006761714 |
| | | rate=1.56 | rate=1.80 | rate=0.56 | rate= 0.79 |
| HQI | 0.0175781191 | 0.0059814423 | 0.0017166124 | 0.0004582399 | 0.0001182852 |
| | | rate=1.56 | rate=1.80 | rate=1.91 | rate=1.95 |