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**The AMLI method: an algebraic multilevel iteration  
method for positive definite  
sparse matrices**

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**Report No. 9936 (July 1999)**

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# THE AMLI METHOD: AN ALGEBRAIC MULTILEVEL ITERATION METHOD FOR POSITIVE DEFINITE SPARSE MATRICES

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ABSTRACT. The algebraic multilevel iteration method, AMLI is a recursively defined method to construct spectrally equivalent preconditioners to a sequence of symmetric and positive definite matrices, corresponding to a number of levels with increasing degrees of freedom, such as arises for a sequence of nested finite element meshes. The matrix sequence is connected by the assumption that the Schur complement, for the corresponding two by two partitioning of the matrix on any level, is spectrally equivalent to the matrix on the next lower level with bounds which hold uniformly for any number of levels. It was originally presented for matrices for which there exists a hierarchical basis matrix form with an explicitly given transformation matrix between the standard form and the hierarchical form. This case allowed for arbitrary perturbations of the matrix block, corresponding to the added degrees of freedom, independent of the Schur complement.

In the present paper it is shown that for more general matrices, the spectral equivalence still holds if the perturbation of the above block diagonal matrix satisfies a certain spectral relation to the Schur complement. By solving the arising systems for this block with sufficient accuracy one can come arbitrary close to the condition number for the two-level method with exact such blocks.

## 1. INTRODUCTION

The computational complexity when solving large sparse systems of linear equations can grow rapidly with problem size unless a proper solution method is used. Ideally, we want a solution method whose complexity grows proportionally to the order  $n$  of the system, i.e. is of optimal order. To solve a linear system  $Ax = b$ , where  $A$  is symmetric and positive semidefinite, we shall consider the case where  $A$  is the final matrix  $A^{(J_0)}$  in a sequence of matrices  $\{A^{(k)}\}$ ,  $A^{(k)} \in L(\mathbb{R}^{n_k}, \mathbb{R}^{n_k})$ ,  $k = 0, 1, \dots, J_0$  for a number of  $J_0$  levels and  $n_k > n_{k-1}$ . On each level, the matrix is partitioned in a two by two block form

$$A^{(k)} = \begin{bmatrix} A_1^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_2^{(k)} \end{bmatrix} \left. \begin{array}{l} \} \\ \} \end{array} \right\} \begin{array}{l} n_k - n_{k-1} \\ n_{k-1} \end{array}$$

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1991 *Mathematics Subject Classification.* 65F10, 65N20, 65N30.

*Key words and phrases.* AMLI preconditioners, two-level preconditioner, Schur complement preconditioner.

of dimensions as indicated. In certain finite element applications the matrices are constructed for a sequence of nested meshes while in other applications the matrices must be constructed algebraically during a preprocessing phase. The matrices in the sequence are connected via the Schur complements

$$S_{A^{(k)}} = A_2^{(k)} - A_{21}^{(k)} A_1^{(k)-1} A_{12}^{(k)}$$

as the basic assumption made is that the Schur complement  $S_{A^{(k)}}$  on level  $k$  is spectrally equivalent to the matrix  $A^{(k-1)}$  on level  $k-1$ , with spectral equivalence bounds which hold uniformly for all levels.

The AMLI (algebraic multilevel iteration) method is a preconditioning method which can be used for various basic iteration methods, such as the conjugate gradient method. It was first presented in [8], [9]. In this method, the preconditioning matrices  $\{M^{(k)}\}$  are recursively defined via certain matrix polynomials in such a way that  $M^{(k)}$  becomes spectrally equivalent to  $A^{(k)}$ . In this recursion, besides some matrix-vector multiplications, the action of  $M^{(k)-1}$  involves a matrix polynomial of degree  $\nu_k$ , the action of which involves  $\nu_k$  actions of the preconditioner  $M^{(k-1)-1}$  on the previous level as well as  $\nu_k - 1$  actions of  $A^{(k-1)}$ . As it turns out, the spectral equivalence holds for sufficiently large values of  $\nu_k$ . Here  $\nu_k$  can vary between levels. However, for the cost of an action of  $M^{(j_0)}$  to be of  $O(n_{j_0})$ , i.e. of optimal order,  $\nu_k$  must also be limited from above. For important problems from finite element applications, there exist such  $\nu_k$  which satisfy both bounds, see e.g., [8], [18].

If there exists a transformation matrix

$$J^{(k)} = \begin{bmatrix} I_1^{(k)} & J_{12}^{(k)} \\ 0 & I_{k-1} \end{bmatrix},$$

where  $J_{12}^{(k)} \in L(\mathbb{R}^{n_{k-1}}, \mathbb{R}^{n_k - n_{k-1}})$  which takes  $A^{(k)}$  via  $\widehat{A}^{(k)} = J^T A^{(k)} J$  into a hierarchical matrix form

$$\widehat{A}^{(k)} = \begin{bmatrix} A_1^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & A^{(k-1)} \end{bmatrix},$$

then the required spectral equivalence holds with

$$(1.1) \quad (1-\gamma^2) \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2 \leq \mathbf{v}_2^T S_{\widehat{A}} \mathbf{v}_2 \leq \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2, \quad \text{for all } \mathbf{v}_2 \in \mathbb{R}^{n_{k-1}},$$

where  $\gamma$ ,  $\gamma < 1$ , is the constant in the strengthened Cauchy-Schwarz inequality,

$$\mathbf{v}^T \widehat{A}^{(k)} \mathbf{w} \leq \gamma \{ \mathbf{v}^T \widehat{A}^{(k)} \mathbf{v} \mathbf{w}^T \widehat{A}^{(k)} \mathbf{w} \}^{\frac{1}{2}}, \quad \forall \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ \mathbf{v}_2 \end{bmatrix}$$

It is readily seen that  $S_{A^{(k)}} = S_{\widehat{A}^{(k)}}$  so the required spectral equivalence holds also for the pair  $A^{(k-1)}$  and  $S_{A^{(k)}}$ ,

$$(1.2) \quad \eta^{-1} \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2 \leq \mathbf{v}_2^T S_A \mathbf{v}_2 \leq \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2, \quad \text{for all } \mathbf{v}_2 \in \mathbb{R}^{n_{k-1}},$$

for some  $\eta \geq 1$ , and  $\eta = \widehat{\eta} = (1 - \gamma^2)^{-1}$  in the above mentioned finite element applications.

In many applications for finite element matrices for a sequence of nested meshes, the constant  $\gamma$  does not depend on the level number. As shown in [9], the AMLI preconditioner can be readily derived for such matrices. To compensate for the use of

arbitrary perturbations of the first block  $A_1^{(k)}$ , in the preconditioner, one can perturb the corner blocks  $A_{12}^{(k)}$  and  $A_{21}^{(k)}$  with the matrices  $(A_1^{(k)} - B_1^{(k)})J_{12}^{(k)}$  and  $J_{12}^{(k)T}(A_1^{(k)} - B_1^{(k)})$ , respectively. Hence, in this method, the transformation matrix  $J_{12}^{(k)}$  is used explicitly in the preconditioner, which is viable in certain finite element applications where  $J_{12}^{(k)}$  has a simple, sparse form. This holds typically for finite element methods where there exists an equivalent hierarchical basis formulation.

However, as it turns out, for general matrices, besides the fundamental inequality (1.2), one must assume that a certain relation holds between the perturbations  $B_1^{(k)}$  to  $A_1^{(k)}$  and the Schur complement  $S_A$ . Such a version of the AMLI method was first presented by Notay [16] who used a special method, namely a modified (i.e. row sum compensated) incomplete factorization to construct the matrix  $B_1^{(k)}$ . A somewhat related approach was used earlier by Kuznetsov [14], but limited to difference matrices. In the present paper we show a general criteria which the matrices must satisfy to enable the construction of a spectrally equivalent AMLI method. Further, a sharper bound of the condition number is derived.

In the remainder of the paper we survey in Section 2 the AMLI method when approximations  $B_1^{(k)}$  are used for the matrix blocks  $A_1^{(k)}$  in the preconditioner and compensated for as described above, using the transformation matrix.

In Section 3 we present the AMLI method for general matrices with no use of such a transformation matrix and show relations which the perturbations  $B_1^{(k)}$  to  $A_1^{(k)}$  and  $S_B$  and  $S_A$  must satisfy for the construction of spectrally equivalent preconditioners.

In Section 4 we shortly discuss variations of the method such as using variable polynomial degrees in the preconditioner, the use of a Lanczos method in a preprocessing phase to estimate the required eigenvalue information, and, finally, the use of inner iterations at certain stages to stabilize the method, in this way avoiding the need to use such eigenvalue information.

In the final section it is shown in the context of finite element matrices how matrix sequences and approximations  $B_1^{(k)}$  and  $S_{B^{(k)}}$  can be constructed which satisfy the assumptions made.

## 2. THE AMLI METHOD

The AMLI method as first proposed in Axelsson and Vassilevski [8], [9] will be described in this section. In this, original version a sequence of symmetric positive definite sparse matrices  $\{A^{(k)}\}_{k=0}^{J_0}$  where given and related variationally, i.e.,  $A^{(k-1)} = (I_{k-1}^k)^T A^{(k)} I_{k-1}^k$  where  $I_{k-1}^k = \begin{bmatrix} J_{12}^{(k)} \\ I_{k-1} \end{bmatrix}$ , where  $I_{k-1}$  stands for the identity operator at level  $k-1$  and  $J_{12}^{(k)}$  is typically an interpolation operator from the current coarse to the new components of the solution vector on the next finer level. The following two-level block partitioning of  $A^{(k)}$  will be used throughout the paper

$$A^{(k)} = \left[ \begin{array}{cc} A_1^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_2^{(k)} \end{array} \right] \begin{array}{l} \} n_k - n_{k-1} \\ \} n_{k-1} \end{array} .$$

Here  $n_k$  stands for the dimension of  $A^{(k)}$ . Note that  $A_2^{(k)}$  has the same dimension as  $A^{(k-1)}$ . We shall use the Schur complement  $S_{A^{(k)}} = A_2^{(k)} - A_{21}^{(k)} A_1^{(k)-1} A_{12}^{(k)}$ . Finally, we will need the transformed matrices  $\widehat{A}^{(k)} = J^T A^{(k)} J$ , where  $J = \begin{bmatrix} I_1^{(k)} & J_{12}^{(k)} \\ 0 & I_{k-1} \end{bmatrix} = \begin{bmatrix} I_1^{(k)} & \\ 0 & I_{k-1} \end{bmatrix}$ , where  $I_1^{(k)}$  is identity operator on the added vector spaces. It follows that  $\widehat{A}^{(k)}$  has the following two-level block form,

$$(2.1) \quad \widehat{A}^{(k)} = \begin{bmatrix} A_1^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & A^{(k-1)} \end{bmatrix},$$

where  $\widehat{A}_{12}^{(k)} = A_{12} + A_{11} J_{12}^{(k)}$ . Since the lower left  $(2, 2)$  block of  $\widehat{A}^{(k)}$  equals  $A^{(k-1)}$ , the transformed matrices are called two-level hierarchical basis (HB) matrices. Another observation is that  $S_{\widehat{A}^{(k)}} = S_{A^{(k)}}$ , where  $S_{\widehat{A}^{(k)}}$  is the Schur complement of  $\widehat{A}^{(k)}$ ,  $S_{\widehat{A}^{(k)}} \equiv A^{(k-1)} - \widehat{A}_{21}^{(k)} A_1^{(k)-1} \widehat{A}_{12}^{(k)}$ .

The hierarchical basis matrices admit the following strengthened Cauchy-inequality,

$$(2.2) \quad \mathbf{v}^T \widehat{A}^{(k)} \mathbf{w} \leq \gamma \left( \mathbf{v}^T \widehat{A}^{(k)} \mathbf{v} \right)^{\frac{1}{2}} \left( \mathbf{w}^T \widehat{A}^{(k)} \mathbf{w} \right)^{\frac{1}{2}}, \quad \text{for all } \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ \mathbf{v}_2 \end{bmatrix},$$

for some  $\gamma$ ,  $0 < \gamma < 1$ .

For finite element stiffness matrices  $A^{(k)}$  it turns out that the constant  $\gamma$  can be determined locally, elementwise, see [11], [3], [6], [15]. For such matrices, corresponding to triangulations obtained by successive refinement generating geometrically similar elements, one can prove that  $\gamma$  remains strictly less than one, independently of the refinement levels  $J_0 \geq 1$  (see e.g. [6]). Actually, in the analysis one needs the following relation,

$$(2.3) \quad (1-\gamma^2) \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2 \leq \mathbf{v}_2^T S_{A^{(k)}} \mathbf{v}_2 \leq \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2, \quad \text{for all } \mathbf{v}_2.$$

More generally, the following result holds.

**Lemma 2.1.** *Let  $A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$  be a spd matrix partitioned in blocks consistent with a vector partitioning  $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$ . Let  $\gamma$ ,  $0 < \gamma < 1$  be the smallest constant for which*

*$\mathbf{v}^T A \mathbf{w} \leq \gamma (\mathbf{v}^T A \mathbf{v})^{\frac{1}{2}} (\mathbf{w}^T A \mathbf{w})^{\frac{1}{2}}$ , for all  $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix}$ ,  $\mathbf{w} = \begin{bmatrix} 0 \\ \mathbf{v}_2 \end{bmatrix}$  holds. Then*

$$(i) \quad \gamma^2 = \sup_{\mathbf{v}_i} \{ \mathbf{v}_i^T A_{ij} A_j^{-1} A_{ji} \mathbf{v}_i / \mathbf{v}_i^T A_i \mathbf{v}_i \}, \quad i = 1 \text{ or } i = 2,$$

$$(iia) \quad \mathbf{v}_1^T S_A^{(1)} \mathbf{v}_1 \geq (1-\gamma^2) \mathbf{v}_1^T A_1 \mathbf{v}_1, \quad \text{for all } \mathbf{v}_1$$

$$(iib) \quad \mathbf{v}_2^T S_A^{(2)} \mathbf{v}_2 \geq (1-\gamma^2) \mathbf{v}_2^T A_2 \mathbf{v}_2, \quad \text{for all } \mathbf{v}_2$$

*where  $S_A^{(i)} = A_i - A_{ij} A_j^{-1} A_{ji}$ ,  $i \neq j$ ,  $i, j = 1, 2$  and the inequalities are sharp.*

**Proof.** For any  $\zeta$ ,  $\gamma \leq \zeta \leq \gamma^{-1}$  the arithmetic-geometric mean inequality  $2ab \leq \zeta a^2 + \zeta^{-1} b^2$ ,  $a, b \geq 0$ , with  $a = \mathbf{v}^T A \mathbf{v}$ ,  $b = \mathbf{w}^T A \mathbf{w}$ , with  $a = \mathbf{v}^T A \mathbf{v}$ ,  $b = \mathbf{w}^T A \mathbf{w}$  shows

that

$$\begin{aligned} (\mathbf{v}+\mathbf{w})^T A(\mathbf{v}+\mathbf{w}) &= \mathbf{v}^T A \mathbf{v} + 2\mathbf{v}^T A \mathbf{w} + \mathbf{w}^T A \mathbf{w} \\ &\geq (1-\zeta\gamma)\mathbf{v}^T A \mathbf{v} + (1-\zeta^{-1}\gamma)\mathbf{w}^T A \mathbf{w}. \end{aligned}$$

Letting here  $\zeta = \gamma$  shows that

$$\mathbf{v}^T S_A^{(1)} \mathbf{v} = \inf_{\mathbf{w}} (\mathbf{v}+\mathbf{w})^T A(\mathbf{v}+\mathbf{w}) \geq (1-\gamma^2)\mathbf{v}^T A \mathbf{v}.$$

which is (ia). Similarly letting  $\zeta = \gamma^{-1}$  (iib) follows. The relation in (i) and the sharpness of the estimates follows by considering  $\tilde{\mathbf{v}}_i^T \tilde{A}_{ij} \tilde{\mathbf{v}}_j \leq \gamma \{\tilde{\mathbf{v}}_i^T \tilde{\mathbf{v}}_i \tilde{\mathbf{v}}_j^T \tilde{\mathbf{v}}_j\}^{\frac{1}{2}}$ , where  $\tilde{\mathbf{v}}_i = A_i^{-\frac{1}{2}} \mathbf{v}_i$ ,  $\tilde{A}_{ij} = A_i^{-\frac{1}{2}} A_{ij} A_j^{-\frac{1}{2}}$  and repeating the above for the matrix  $\tilde{A} = \begin{bmatrix} I_1 & \tilde{A}_{12} \\ \tilde{A}_{21} & I_2 \end{bmatrix}$ .

**2.1. Definition of the original AMLI method.** The AMLI method is a preconditioning method which is used as a preconditioner in a Chebyshev or conjugate gradient method. One can define the AMLI method using the hierarchical basis matrices. However, the latter are less sparse than the standard basis matrices and their direct use in the AMLI method is therefore less efficient. The method will therefore be presented for the standard basis matrix.

As we shall see the form of the preconditioner is similar to the exact matrix factorization

$$A^{(k)} = \begin{bmatrix} A_1^{(k)} & 0 \\ A_{21}^{(k)} & S \end{bmatrix} \begin{bmatrix} I_1^{(k)} & A_1^{(k)-1} A_{12}^{(k)} \\ 0 & I_{k-1} \end{bmatrix}$$

where  $S = S_{A^{(k)}}$ . Here we must approximate  $S$  (which is normally a full matrix) and mostly  $A_1^{(k)}$  is approximated also. The approximation of  $S$  will be recursively defined.

Let  $P_\nu(t)$  be a given polynomial of degree  $\nu \geq 1$  such that  $0 \leq P_\nu(t) < 1$ ,  $0 < t \leq 1$  and normalized at the origin,  $P_\nu(0) = 1$ . Further, let  $B_1^{(k)}$  be an approximation of  $A_1^{(k)}$ , spectrally related to it in a form to be shown later.

**Definition 2.1.** • Let  $M^{(0)} = A^{(0)}$ .

- For  $k = 1, 2, \dots, J_0$ , assuming that  $M^{(k-1)}$  has been defined, one first defines  $\tilde{M}^{(k-1)}$ ,

$$\tilde{M}^{(k-1)} = \begin{cases} (I - P_\nu(M^{(k-1)-1} S)) S^{-1}, & \text{in version (i)} \\ (I - P_\nu(M^{(k-1)-1} A^{(k-1)})) A^{(k-1)-1}, & \text{in version (ii)} \end{cases}$$

- Then

$$M^{(k)} = \begin{bmatrix} B_1^{(k)} & 0 \\ \tilde{A}_{21}^{(k)} & \tilde{M}^{(k-1)} \end{bmatrix} \begin{bmatrix} I_1^{(k)} & B_1^{(k)-1} \tilde{A}_{12}^{(k)} \\ 0 & I_{k-1} \end{bmatrix}$$

where

$$(2.4) \quad \begin{aligned} \tilde{A}_{12}^{(k)} &= A_{12}^{(k)} + (A_1^{(k)} - B_1^{(k)}) J_{12}^{(k)}, \\ \tilde{A}_{21}^{(k)} &= A_{21}^{(k)} + J_{12}^{(k)T} (A_1^{(k)} - B_1^{(k)}) \end{aligned}$$

It follows that the preconditioner  $M^{(k)}$  is only implicitly (recursively) defined.

Here  $\widetilde{M}^{(k-1)}$  is an approximation of  $S$  and of  $A^{(k-1)}$  in version (i) and version (ii), respectively. In the following we shall only consider version (ii).

The reason for perturbing the off-diagonal block matrices as done in (2.4) is that in this way

$$\widehat{M}^{(k)} \equiv J^{(k)T} M^{(k)} J^{(k)}$$

takes the form

$$(2.5) \quad \widehat{M}^{(k)} = \begin{bmatrix} B_1^{(k)} & \widehat{A}_{12}^{(k)} \\ \widehat{A}_{21}^{(k)} & \widetilde{M}^{(k-1)} + \widehat{A}_{21}^{(k)} B_1^{(k)-1} \widehat{A}_{12}^{(k)} \end{bmatrix},$$

which follows from an elementary computation. Hence  $\widehat{M}^{(k)}$  can be considered as a preconditioner to  $\widehat{A}^{(k)}$  and the extreme eigenvalues of  $M^{(k)-1} A^{(k)}$  equal those of  $\widehat{M}^{(k)-1} \widehat{A}^{(k)}$ , since

$$\begin{aligned} \sup_{\mathbf{v}} \frac{\mathbf{v}^T A^{(k)} \mathbf{v}}{\mathbf{v}^T M^{(k)} \mathbf{v}} &= \sup_{\widehat{\mathbf{v}}} \frac{\widehat{\mathbf{v}}^T \widehat{A}^{(k)} \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^T \widehat{M}^{(k)} \widehat{\mathbf{v}}}, \\ \inf_{\mathbf{v}} \frac{\mathbf{v}^T A^{(k)} \mathbf{v}}{\mathbf{v}^T M^{(k)} \mathbf{v}} &= \inf_{\widehat{\mathbf{v}}} \frac{\widehat{\mathbf{v}}^T \widehat{A}^{(k)} \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^T \widehat{M}^{(k)} \widehat{\mathbf{v}}}. \end{aligned}$$

Since the off-diagonal blocks in  $\widehat{M}^{(k)}$  equal those in  $\widehat{A}^{(k)}$  the estimate of the extreme eigenvalues of  $\widehat{M}^{(k)-1} A^{(k)}$  can be readily done.

**Remark 2.1.** Restrictions on  $\nu$ .

Since  $P_\nu(0) = 1$ , it is readily seen that each action of  $M^{(k-1)-1}$  requires  $\nu-1$  actions of  $S$  or  $A^{(k-1)}$  and  $\nu$  actions of  $M^{(k-1)-1}$ . Therefore, to get an optimal order,  $O(n)$ ,  $n = n_{J_0}$  of computational complexity of each action of  $M^{(J_0)}$  the polynomial degree chosen must be bounded above and related to the ratio  $n_k/n_{k-1}$  of the degrees of freedom. For a uniform recursive triangulation of a plane domain where on each new level each triangle is divided in four parts, it holds  $\nu < 4$ . On the other hand, as we shall see, to get an optimal order condition number,  $\text{cond}(M^{(k)-1} A^{(k)}) = O(1)$  which is bounded as  $k \rightarrow \infty$ , we must choose  $\nu$  sufficiently large. As will be seen later, in many important applications, both conditions can be met.

**2.2. Analysis of the spectral condition number.** Let  $B_1^{(k)}$  be a spectrally equivalent preconditioner to  $A_1^{(k)}$  satisfying the inequality

$$(2.6) \quad \mathbf{v}_1^T A_1^{(k)} \mathbf{v}_1 \leq \mathbf{v}_1^T B_1^{(k)} \mathbf{v}_1 \leq (1+b) \mathbf{v}_1^T A_1^{(k)} \mathbf{v}_1, \quad \text{for all } \mathbf{v}_1 \in \mathbb{R}^{n_k - n_{k-1}},$$

for some  $b \geq 0$ .

For  $P_\nu$  we take a shifted and scaled Chebyshev polynomial,

$$P_\nu(t) = \frac{T_\nu\left(\frac{1+\alpha-2t}{1-\alpha}\right) + 1}{T_\nu\left(\frac{1+\alpha}{1-\alpha}\right) + 1}$$

where

$$T_\nu(x) = \frac{1}{2} \left[ \left( x + \sqrt{x^2 - 1} \right)^\nu + \left( x - \sqrt{x^2 - 1} \right)^\nu \right]$$



i.e.,  $T_\nu$  is the Chebyshev polynomial of the first kind. Further  $\alpha$ ,  $\alpha > 0$  is a lower bound of the eigenvalues of  $M^{(k-1)^{-1}}A^{(k-1)}$ . As will be seen, the upper bound of these eigenvalues is bounded by the unit number. Therefore

$$\alpha_{k-1} = \min_{\mathbf{v}_2} \frac{\mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2}{\mathbf{v}_2^T M^{(k-1)} \mathbf{v}_2} = \lambda_{k-1}^{-1}$$

where

$$\lambda_k = \max_v \frac{\mathbf{v}^T A^{(k)} \mathbf{v}}{\mathbf{v}^T M^{(k)} \mathbf{v}}.$$

By an elementary computation

$$T_\nu \left( \frac{1+\alpha}{1-\alpha} \right) = \frac{1}{2}(\rho + \rho^{-1}),$$

$$1 - P_\nu(\alpha) = \frac{T_\nu \left( \frac{1+\alpha}{1-\alpha} \right) - 1}{T_\nu \left( \frac{1+\alpha}{1-\alpha} \right) + 1} = \left( \frac{1-\rho}{1+\rho} \right)^2$$

where  $\rho = \left( \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} \right)^\nu$ , or  $\alpha = \left( \frac{1-\rho^{\frac{1}{\nu}}}{1+\rho^{\frac{1}{\nu}}} \right)^2$

Here

$$P_\nu(\alpha) = \max_{\alpha \leq t \leq 1} P_\nu(t) = \min_{\tilde{P}_\nu \in \pi_\nu^+} \max_{\alpha \leq t \leq 1} \tilde{P}_\nu(t)$$

where  $\pi_\nu^+$  denotes the set of all polynomials of degree  $\nu$  which are nonnegative on the interval  $[\alpha, 1]$  and normalized at the origin.

**Remark 2.2.** *It can be seen from the analysis to follow that the condition number of  $M^{(J_0)^{-1}}A^{(J_0)}$  is the same for the alternate choice  $\tilde{P}_\nu(t) = T_\nu \left( \frac{\beta+\tilde{\alpha}-2t}{\beta-\tilde{\alpha}} \right) / T_\nu \left( \frac{\beta+\tilde{\alpha}}{\beta-\tilde{\alpha}} \right)$  where  $0 < \tilde{\alpha} \leq t \leq \beta$  and  $\beta$  is the maximum eigenvalue of  $M^{(k-1)^{-1}}A^{(k-1)}$ . Here  $\tilde{\alpha} > \alpha$  but  $\beta > 1$  and, as it turns out,  $\beta/\tilde{\alpha} = 1/\alpha$ .*

Since  $\widehat{M}^{(k)} = J^T M^{(k)} J$ , it follows from (2.5) that  $\widehat{M}^{(k)}$  can be regarded as a preconditioner to  $\widehat{A}^{(k)}$  and  $\widehat{M}^{(k)} - \widehat{A}^{(k)}$  takes a block diagonal form

$$\widehat{M}^{(k)} - \widehat{A}^{(k)} = \begin{bmatrix} B_1^{(k)} - A_1^{(k)} & 0 \\ 0 & \widetilde{M}^{(k-1)} - A^{(k-1)} + \widehat{A}_{21}^{(k)} B_1^{(k)-1} \widehat{A}_{12}^{(k)} \end{bmatrix}.$$

Although the preconditioner  $M^{(k)}$  does not involve matrices in the hierarchical basis, the condition number of  $M^{(k)^{-1}}A^{(k)}$  can now be estimated via the hierarchical matrix.

Since by (2.6),

$$(2.7) \quad \mathbf{v}_1^T B_1^{(k)-1} \mathbf{v}_1 \leq \mathbf{v}_1^T A_1^{(k)-1} \mathbf{v}_1$$

and since

$$\widetilde{M}^{(k-1)} - A^{(k-1)} = A^{(k-1)} P_\nu(M^{(k-1)^{-1}}A^{(k-1)}) [I - P_\nu(M^{(k-1)^{-1}}A^{(k-1)})]^{-1},$$

it follows by the choice of  $P_\nu$  that  $\mathbf{v}^T \widehat{M}^{(k-1)} \mathbf{v} \geq \mathbf{v}^T A^{(k-1)} \mathbf{v}$  so  $\mathbf{v}^T \widehat{M}^{(k)} \mathbf{v} \geq \mathbf{v}^T \widehat{A}^{(k)} \mathbf{v}$ , and hence also  $\mathbf{v}^T M^{(k)} \mathbf{v} \geq \mathbf{v}^T A^{(k)} \mathbf{v}$ . To find an upper bound  $\lambda_k \mathbf{v}^T A^{(k)} \mathbf{v} \geq \mathbf{v}^T M^{(k)} \mathbf{v}$ , i.e. equivalently  $\lambda_k \mathbf{v}^T \widehat{A}^{(k)} \mathbf{v} \geq \mathbf{v}^T \widehat{M}^{(k)} \mathbf{v}$  we use

$$\mathbf{v}_2^T \widehat{A}_{21}^{(k)} B_1^{(k)-1} \widehat{A}_{12}^{(k)} \mathbf{v}_2 \leq \mathbf{v}_2^T \widehat{A}_{21}^{(k)} A_1^{(k)-1} \widehat{A}_{12}^{(k)} \mathbf{v}_2 \leq \gamma^2 \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2, \text{ for all } \mathbf{v}_2 \in \mathbb{R}^{n_{k-1}},$$

which follows from (2.7) and Lemma 2.1.

Further

$$\begin{aligned} \lambda_k &= 1 + \sup_v \frac{v^T (M^{(k)} - A^{(k)}) v}{v^T A^{(k)} v} = 1 + \sup_{\widehat{\mathbf{v}}} \frac{\widehat{\mathbf{v}}^T (\widehat{M}^{(k)} - \widehat{A}^{(k)}) \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^T \widehat{A}^{(k)} \widehat{\mathbf{v}}} \\ &\leq 1 + \sup_v \frac{\widehat{\mathbf{v}}_1^T (B_1^{(k)} - A_1^{(k)}) \widehat{\mathbf{v}}_1 + (d_\nu + \gamma^2) \mathbf{v}_2^T A^{(k-1)} \mathbf{v}_2}{\widehat{\mathbf{v}}^T \widehat{A}^{(k)} \widehat{\mathbf{v}}} \end{aligned}$$

where  $d_\nu = P_\nu(\alpha)/(1 - P_\nu(\alpha))$ ,

$$P_\nu(\alpha) = \max_t P_\nu(t), \lambda_{k-1}^{-1} \leq t \leq 1.$$

Lemma 2.1 now shows that

$$\lambda_k \leq 1 + \frac{b + d_\nu + \gamma^2}{1 - \gamma^2} = \frac{1 + b + d_\nu}{1 - \gamma^2} = \frac{b + (1 - P_\nu(\alpha))^{-1}}{1 - \gamma^2}.$$

As shown previously,

$$(1 - P_\nu(\alpha))^{-1} = \left( \frac{1 + \rho}{1 - \rho} \right)^2,$$

so the eigenvalues, and hence the condition number of the preconditioned matrix  $M^{(k)-1} A^{(k)}$  is bounded above,

$$\lambda_k \leq \frac{1}{\alpha}$$

when  $\alpha \in (0, 1)$  is sufficiently small to satisfy

$$f(\rho) \equiv \left( \frac{1 + \rho}{1 - \rho} \frac{1 - \rho^{\frac{1}{\nu}}}{1 + \rho^{\frac{1}{\nu}}} \right)^2 \leq 1 - \gamma^2 - \alpha b,$$

where  $\rho = \left( \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}} \right)^\nu$ . Due to the factor  $\alpha$  in  $\alpha b$ , the same bound for  $\nu$  holds as when  $b = 0$ , i.e., this inequality has a solution  $\alpha \in (0, 1)$  if  $\nu > (1 - \gamma^2)^{-\frac{1}{2}}$ . The above results are summarized in the next theorem.

**Theorem 2.1.** *Let  $B_1^{(k)}$  be spectrally equivalent approximations to  $A_1^{(k)}$  such that, uniformly in  $k$ , one has*

$$\mathbf{v}_1^T A_1^{(k)} \mathbf{v}_1 \leq \mathbf{v}_1^T B_1^{(k)} \mathbf{v}_1 \leq (1 + b) \mathbf{v}_1^T A_1^{(k)} \mathbf{v}_1, \text{ for all } \mathbf{v}_1.$$

Let  $\nu > (1 - \gamma^2)^{-\frac{1}{2}}$  and  $\rho \in (0, 1)$  be a solution of the inequality,

$$\left( \frac{1 + \rho}{1 - \rho} \right)^2 \alpha \leq 1 - \gamma^2 - \alpha b, \quad \alpha = \left( \frac{1 - \rho^{\frac{1}{\nu}}}{1 + \rho^{\frac{1}{\nu}}} \right)^2.$$

Consider the polynomials,

$$P_\nu(t) = \frac{T_\nu\left(\frac{1+\alpha-2t}{1-\alpha}\right) + 1}{T_\nu\left(\frac{1+\alpha}{1-\alpha}\right) + 1}.$$

Define  $M^{(0)} = A^{(0)}$  and for  $k = 1, 2, \dots, J_0$ , assuming that  $M^{(k-1)}$  has already been defined, construct  $\widetilde{M}^{(k)}$  as in Definition 2.1 version (ii) and then define

$$M^{(k)} = \begin{bmatrix} B_1^{(k)} & 0 \\ A_{21}^{(k)} + J_{12}^{(k)T}(A_1^{(k)} - B_1^{(k)}) & \widetilde{M}^{(k-1)} \end{bmatrix} \begin{bmatrix} I_1 & B_1^{(k)-1}(A_{12}^{(k)} + (A_1^{(k)} - B_1^{(k)})J_{12}^{(k)}) \\ 0 & I_2 \end{bmatrix}.$$

Then, the corresponding AMLI preconditioning matrix  $M^{(k)}$  (version ii) is spectrally equivalent to  $A^{(k)}$  with the following bounds:

$$\mathbf{v}^T A^{(k)} \mathbf{v} \leq \mathbf{v}^T M^{(k)} \mathbf{v} \leq \frac{1}{1-\gamma^2} \left( b + \left( \frac{1+\rho}{1-\rho} \right)^2 \right) \mathbf{v}^T A^{(k)} \mathbf{v} \leq \frac{1}{\alpha} \mathbf{v}^T A^{(k)} \mathbf{v}, \quad \text{for all } \mathbf{v}.$$

For  $\nu = 2$  and  $4\gamma^2 < 3$ , one has  $\alpha = \frac{3-4\gamma^2}{1+2b+\sqrt{3-4\gamma^2+(1+2b)^2}}$  and  $\lambda \leq \left( \frac{1}{\alpha} \left( \frac{1+\alpha}{2} \right)^2 + b \right) / (1-\gamma^2)$ . For  $\nu = 3$  and  $9\gamma^2 < 8$ ,  $\alpha \in (0, 1)$  is a root of the cubic equation  $\frac{(1-\alpha)^3}{(3+\alpha)^2} = (1-\gamma^2)[1-\alpha(1+b)]$ , or

$$b\alpha^3 + (1+6b+\gamma)\alpha^2 + (9b+6\gamma^2)\alpha + 9\gamma^2 - 8 = 0.$$

**Proof.** The general result has already been proved. The specific results for  $\nu = 2$  and  $\nu = 3$  follows by elementary computations.  $\diamond$

**Remark 2.3.** Somewhat sharper estimates for  $\lambda_k$  were derived in [9]. For version (i) the same type of bounds hold but without the factor  $(1-\gamma^2)^{-1}$  in the upper bound.

### 3. THE AMLI METHOD FOR GENERAL POSITIVE DEFINITE MATRICES

In this section we consider the construction of an AMLI method for general positive definite matrices, i.e., without assuming any underlying hierarchy of meshes and thus avoiding any (implicit or explicit) transformation to a corresponding HB block structure of the matrices. It will be shown that in order to construct an optimal order preconditioner the approximations  $B_1^{(k)}$  to  $A_1^{(k)}$  must be related to the Schur complements  $S_{A^{(k)}}$  in a certain way. It suffices to consider the two level form of the method as the multilevel extension can be done as shown in Section 2. For convenience, we delete the superscripts  $(k)$  whenever no confusion can arise.

#### 3.1. The condition number of the two level method.

**Lemma 3.1.** Let  $A$  and  $E$  be symmetric matrices and let  $A_1$  be positive definite.

Consider the transformation matrix  $K = \begin{bmatrix} I_1 & -A_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}$ . Then the transformed

matrices  $K^T A K$  and  $K^T E K$ , where  $E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$  take the form

$$(i) \begin{bmatrix} I_1 & 0 \\ -A_{21}A_1^{-1} & I_2 \end{bmatrix} \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} I_1 & -A_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & S_A \end{bmatrix}$$

$$(ii) \begin{bmatrix} I_1 & 0 \\ -A_{21}A_1^{-1} & I_2 \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} I_1 & -A_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} E_1 & -E_1A_1^{-1}A_{12} \\ -A_{21}A_1^{-1}E_1 & E_2 + A_{21}A_1^{-1}E_1A_1^{-1}A_{12} \end{bmatrix}$$

The corresponding quadratic forms satisfy

$$(iii) [\mathbf{v}_1^T, \mathbf{v}_2^T] K^T A K \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \mathbf{v}_1^T A_1 \mathbf{v}_1 + \mathbf{v}_2^T S_A \mathbf{v}_2$$

$$(iv) [\mathbf{v}_1^T, \mathbf{v}_2^T] K^T E K \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \leq (1+\zeta) \mathbf{v}_1^T E_1 \mathbf{v}_1 + (1+\zeta^{-1}) \mathbf{v}_2^T A_{21} A_1^{-1} E_1 A_1^{-1} A_{12} \mathbf{v}_2 + \mathbf{v}_2^T E_2 \mathbf{v}_2,$$

for any  $\zeta > 0$  and for all  $\mathbf{v}_1, \mathbf{v}_2$  of dimensions consistent with the block matrix partitioning of  $A$ , where  $E_1$  is assumed to be positive semidefinite.

**Proof.** Parts (i) - (iii) follow by straightforward computations. Part (iv) follows from

$$\begin{aligned} |\mathbf{v}_1^T E_1 A_1^{-1} A_{12} \mathbf{v}_2 + \mathbf{v}_2^T A_{21} A_1^{-1} E_1 \mathbf{v}_1| &= 2|\mathbf{v}_1^T E_1 A_1^{-1} A_{12} \mathbf{v}_2| = 2|(E_1^{\frac{1}{2}} \mathbf{v}_1)^T E_1^{\frac{1}{2}} A_1^{-1} A_{12} \mathbf{v}_2| \\ &\leq 2\{\mathbf{v}_1^T E_1 \mathbf{v}_1 \mathbf{v}_2^T A_{21} A_1^{-1} E_1 A_1^{-1} A_{12} \mathbf{v}_2\}^{\frac{1}{2}} \\ &\leq \zeta \mathbf{v}_1^T E_1 \mathbf{v}_1 + \zeta^{-1} \mathbf{v}_2^T A_{21} A_1^{-1} E_1 A_1^{-1} A_{12} \mathbf{v}_2 \end{aligned}$$

◇

Consider now the preconditioner in the block matrix factored form

$$(3.1) \quad B = \begin{bmatrix} B_1 & 0 \\ A_{21} & S_B \end{bmatrix} \begin{bmatrix} I_1 & B_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix},$$

where  $B_1$  is an approximation of  $A_1$  and  $S_B$  is an approximation of  $S_A$ . Note that

$$B = \begin{bmatrix} B_1 & A_{12} \\ A_{21} & S_B + A_{21}B_1^{-1}A_{12} \end{bmatrix},$$

so  $S_B$  is the Schur complement of  $B$ .

We assume that  $B_1$  is spectrally equivalent to  $A_1$  and  $S_B$  to  $S_A$  and that the following inequalities hold for some  $\beta \geq 1, \eta \geq 1$ .

$$(3.2,i) \quad \beta \mathbf{v}_1^T A_1 \mathbf{v}_1 \geq \mathbf{v}_1^T B_1 \mathbf{v}_1 \geq \mathbf{v}_1^T A_1 \mathbf{v}_1, \text{ for all } \mathbf{v}_1 \in \mathbb{R}^{n_k - n_{k-1}}$$

$$(3.2,ii) \quad \eta \mathbf{v}_2^T S_A \mathbf{v}_2 \geq \mathbf{v}_2^T S_B \mathbf{v}_2 \geq \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2, \text{ for all } \mathbf{v}_2 \in \mathbb{R}^{n_k - 1},$$

where  $\tilde{A} = \begin{bmatrix} B_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$  and  $S_A = A_2 - A_{21}A_1^{-1}A_{12}$ ,  $S_{\tilde{A}} = A_2 - A_{21}B_1^{-1}A_{12}$

We shall also assume that

$$(3.2,iii) \quad \alpha \mathbf{v}_2^T S_A \mathbf{v}_2 \geq \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2 \geq \mathbf{v}_2^T S_A \mathbf{v}_2, \text{ for all } \mathbf{v}_2,$$

where  $\alpha \geq 1$  and the left inequality is sharp, i.e., there exists a vector  $\hat{\mathbf{v}}_2$  such that

$$\alpha \hat{\mathbf{v}}_2^T S_A \hat{\mathbf{v}}_2 = \hat{\mathbf{v}}_2^T S_{\tilde{A}} \hat{\mathbf{v}}_2.$$

The right inequality in (iii) follows from the right inequality in (i), because (i) implies

$$\mathbf{v}^T \tilde{A} \mathbf{v} \geq \mathbf{v}^T A \mathbf{v} \text{ for all } \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}, \text{ and hence } \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2 \geq \mathbf{v}_2^T S_A \mathbf{v}_2.$$

As we shall see,  $\eta$  will be a lower bound for the estimate of the condition number of  $B^{-1}A$ . The value of  $\eta$  taken will depend on both the accuracy of the approximation of  $S_B$  to  $S_A$  and of  $B_1$  to  $A_1$ . More precisely, the latter dependence follows from the next theorem.

**Theorem 3.1.** *Let the assumptions (3.2,i-iii) hold. Then  $\eta \geq \alpha$  and*

$$\eta \geq 1 + \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T A_{21}(A_1^{-1} - B_1^{-1})A_{12}\mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2} \geq 1 + \frac{\mathbf{v}'_2{}^T A_{21}(A_1^{-1} - B_1^{-1})A_{12}\mathbf{v}'_2}{\mathbf{v}'_2{}^T S_A \mathbf{v}'_2}$$

where  $\mathbf{v}'_2$  is the eigenvector for the smallest eigenvalue of  $S_A$ .

**Proof.** By assumption (3.2,ii),  $\eta \mathbf{v}_2^T S_A \mathbf{v}_2 \geq \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2$  which shows that  $\eta \geq \alpha$ , where  $\alpha$  is defined in (3.2,iii). Further

$$\begin{aligned} (\eta-1)\mathbf{v}_2^T S_A \mathbf{v}_2 &\geq \mathbf{v}_2^T (S_{\tilde{A}} - S_A) \mathbf{v}_2 \\ &= \mathbf{v}_2^T A_{21}(A_1^{-1} - B_1^{-1})A_{12}\mathbf{v}_2, \quad \text{for all } \mathbf{v}_2 \end{aligned}$$

which shows the lower bounds in the second inequality.  $\diamond$

In general, unless  $S_B = S_{\tilde{A}}$ , a strict inequality  $\alpha < \eta$  holds. Further  $\alpha = 1$  if  $\beta = 1$  and  $\alpha$  is related to  $\beta$  in the following way.

By (3.2,iii),

$$\begin{aligned} (\alpha-1)\hat{\mathbf{v}}_2^T S_A \hat{\mathbf{v}}_2 &= \hat{\mathbf{v}}_2^T (S_{\tilde{A}} - S_A) \hat{\mathbf{v}}_2 = \hat{\mathbf{v}}_2^T A_{21}(A_1^{-1} - B_1^{-1})A_{12}\hat{\mathbf{v}}_2 \\ &= \hat{\mathbf{v}}_2^T \tilde{A}_{21}(I - \tilde{B}_1^{-1})\tilde{A}_{12}\hat{\mathbf{v}}_2 \\ &\leq (1 - \beta^{-1})\hat{\mathbf{v}}_2^T \tilde{A}_{21}A_1^{-1}A_{12}\hat{\mathbf{v}}_2 \end{aligned}$$

where  $\tilde{A}_{21} = A_{21}A_1^{-1}$ ,  $\tilde{B}_1 = A_1^{-\frac{1}{2}}B_1A_1^{-\frac{1}{2}}$ . Hence

$$(3.3) \quad \alpha - 1 \leq q(\hat{\mathbf{v}}_2)(1 - \beta^{-1})$$

where

$$q(\hat{\mathbf{v}}_2) = \frac{\hat{\mathbf{v}}_2^T A_{21}A_1^{-1}A_{12}\hat{\mathbf{v}}_2}{\hat{\mathbf{v}}_2^T S_A \hat{\mathbf{v}}_2}.$$

Here  $q$  can take large values for some vector, typically for a ‘‘smooth’’ vector close to the lowest harmonic vector. However, for the particular vector  $\hat{\mathbf{v}}_2$  it can be expected that  $q$  takes moderate values. At any rate, (3.3) shows how  $\alpha - 1$  is related to  $1 - \beta^{-1}$ . It is further seen from Theorem 3.1 that for (ii) to hold for a not too large constant  $\eta$ ,  $B_1$  must be related to  $A_1$  so that

$$A_1^{-1}A_{12}\mathbf{v}_2 = B_1^{-1}A_{12}\mathbf{v}_2$$

or

$$A_1 B_1^{-1} A_{12} \mathbf{v}_2 = A_{12} \mathbf{v}_2$$

for some ‘‘smooth’’ vector  $\mathbf{v}_2$  such as the eigenvector to the smallest eigenvalue of  $S_A$ . Otherwise

$$\frac{\mathbf{v}_2^T (S_{\tilde{A}} - S_A) \mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2} \simeq q(\mathbf{v}_2)(1 - \beta^{-1})$$

where  $q(\mathbf{v}_2)$  is large, because  $\mathbf{v}_2^T S_A \mathbf{v}_2 / \mathbf{v}_2^T \mathbf{v}_2$  is small. In this case  $\eta$  would be much larger than  $1 - \beta^{-1}$ . The conclusion is that in order for  $B$  in (3.1) to be an efficient preconditioner for a general spd matrix  $A$ , the approximation  $B_1$  of  $A_1$  must be related

to the Schur complement  $S_A$ , and in particular, the action of  $B_1^{-1}$  must be close to the action of  $A_1^{-1}$  for “smooth” vectors.

The inequalities (3.2,i-iii) imply the following spectral relation between  $A$  and  $B$ .

**Theorem 3.2.** *Let (3.2,i-iii) hold. Then*

$$\mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T B \mathbf{v} \leq \kappa \mathbf{v}^T A \mathbf{v}, \quad \text{for all } \mathbf{v}$$

where

$$\kappa \leq \beta + \frac{1}{2}[\eta - 1 + (\alpha - 2)(\beta - 1)] + \frac{1}{2}\sqrt{[\eta - 1 + (\alpha - 2)(\beta - 1)]^2 + 4\beta(\alpha - 1)(\beta - 1)}.$$

In particular, if  $\beta = 1$  then  $\kappa \leq \eta$ , if  $\alpha = \eta$  then  $\kappa \leq \beta\eta$  and when  $\alpha < \eta$  then  $\kappa < \beta\eta$ .

**Proof.** From

$$\begin{aligned} B - A &= (B - \tilde{A}) + (\tilde{A} - A), \\ B - \tilde{A} &= \begin{bmatrix} 0 & 0 \\ 0 & (S_B - S_{\tilde{A}}) \end{bmatrix}, \quad \tilde{A} - A = \begin{bmatrix} (B_1 - A_1) & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and the right inequalities in (i) and (ii) it follows that  $\mathbf{v}^T (B - \tilde{A}) \mathbf{v} \geq 0$ ,  $\mathbf{v}^T (\tilde{A} - A) \mathbf{v} \geq 0$  so  $\mathbf{v}^T B \mathbf{v} \geq \mathbf{v}^T A \mathbf{v}$ , for all  $\mathbf{v}$ .

To prove the upper bound we use Lemma 3.1 with  $E_1 = B_1 - A_1$ ,  $E_2 = S_B - S_{\tilde{A}}$  and the inequality  $\frac{a+b}{c+d} \leq \max\left(\frac{a}{c}, \frac{b}{d}\right)$ ,  $a, b \geq 0$ ,  $c, d > 0$ . This shows that

$$(3.4) \quad \frac{\mathbf{v}^T (B - A) \mathbf{v}}{\mathbf{v}^T A \mathbf{v}} \leq \max \left\{ (1 + \zeta) \sup_{\mathbf{v}_1} \frac{\mathbf{v}_1^T (B_1 - A_1) \mathbf{v}_1}{\mathbf{v}_1^T A_1 \mathbf{v}_1}, \sup_{\mathbf{v}_2} \frac{(1 + \zeta^{-1}) \mathbf{v}_2^T A_{21} A_1^{-1} (B_1 - A_1) A_1^{-1} A_{12} \mathbf{v}_2 + \mathbf{v}_2^T (S_B - S_{\tilde{A}}) \mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2} \right\}$$

for any  $\zeta > 0$ .

Here

$$\begin{aligned} \mathbf{v}_2^T A_{21} A_1^{-1} (B_1 - A_1) A_1^{-1} A_{12} \mathbf{v}_2 &= \mathbf{v}_2^T \tilde{A}_{21} (\tilde{B}_1 - I) \tilde{A}_{12} \mathbf{v}_2 \\ &\leq \mathbf{v}_2^T \tilde{A}_{21} \tilde{B}_1 (I - \tilde{B}_1^{-1}) \tilde{A}_{12} \mathbf{v}_2 \leq \beta \mathbf{v}_2^T \tilde{A}_{21} (I - \tilde{B}_1^{-1}) \tilde{A}_{12} \mathbf{v}_2 \\ &= \beta \mathbf{v}_2^T A_{21} (A_1^{-1} - B_1^{-1}) A_{12} \mathbf{v}_2 = \beta \mathbf{v}_2^T (S_{\tilde{A}} - S_A) \mathbf{v}_2. \end{aligned}$$

Hence, using this and

$$S_B - S_{\tilde{A}} = S_B - S_A - (S_{\tilde{A}} - S_A)$$

it follows from (3.4),

$$\begin{aligned} \frac{\mathbf{v}^T (B - A) \mathbf{v}}{\mathbf{v}^T A \mathbf{v}} &\leq \max \left\{ (1 + \zeta)(\beta - 1), ((1 + \zeta^{-1})\beta - 1) \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T (S_{\tilde{A}} - S_A) \mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2} \right. \\ &\quad \left. + \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T (S_B - S_A) \mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2} \right\} \\ &= \max \{ (1 + \zeta)(\beta - 1), ((1 + \zeta^{-1})\beta - 1)(\alpha - 1) + \eta - 1 \} \end{aligned}$$

Taking here  $\zeta$  such that the upper bound is minimized, i.e. letting

$$(1 + \zeta)(\beta - 1) = ((1 + \zeta^{-1})\beta - 1)(\alpha - 1) + \eta - 1,$$

it follows (for  $\beta > 1$ )

$$\zeta = \widehat{\zeta} = \frac{1}{2} \left( \frac{\eta-1}{\beta-1} + \alpha - 2 \right) + \frac{1}{2} \sqrt{\left( \frac{\eta-1}{\beta-1} + \alpha - 2 \right)^2 + 4\beta \frac{\alpha-1}{\beta-1}}$$

and

$$\begin{aligned} \kappa &= 1 + (1 + \widehat{\zeta})(\beta-1) \\ &= \beta + \frac{1}{2}[\eta-1 + (\alpha-2)(\beta-1)] + \frac{1}{2} \sqrt{[\eta-1 + (\alpha-2)(\beta-1)]^2 + 4\beta(\alpha-1)(\beta-1)}. \end{aligned}$$

When  $\beta = 1$  it follows  $\kappa \leq \eta$  and when  $\alpha = \eta$  it follows

$$\kappa \leq \beta + \frac{1}{2}[(\eta-1)\beta - (\beta-1)] + \frac{1}{2} \sqrt{[(\eta-1)\beta + (\beta-1)]^2} = \eta\beta.$$

Further  $\kappa$  is a monotonically increasing function of  $\alpha$ .  $\diamond$

**Remark 3.1.** When  $\beta, \alpha \ll \eta$ , say  $\beta - 1 = \xi_0(\eta-1)$ ,  $\alpha - 1 = \xi_1(\eta-1)$ ,  $\xi_0, \xi_1 \ll 1$ , then the bound in Theorem 3.2 shows  $\kappa \leq (1 + O(\max\{\xi_0, \xi_1\}))\eta$  while the general bound  $\beta\eta$  shows  $\kappa \leq (1 + \xi_0\eta)\eta$ . Hence, the more accurate bound grows much slower for increasing  $\beta$  near the value 1, when  $\eta$  is large. In fact, a computation shows that  $\kappa \leq \eta + (\beta-1)(1 + \beta(\alpha-1)) \simeq \eta + \beta - 1$  when  $\beta, \alpha \ll \eta$ , while the bound  $\beta\eta = \eta + \eta(\beta-1)$ .

Consider now the case where the opposite inequalities hold, i.e.

$$(3.5.i) \quad \beta^{-1} \mathbf{v}_1^T A_1 \mathbf{v}_1 \leq \mathbf{v}_1^T B_1 \mathbf{v}_1 \leq \mathbf{v}_1^T A_1 \mathbf{v}_1, \text{ for all } \mathbf{v}_1 \in \mathbb{R}^{n_k - n_{k-1}}$$

$$(3.5.ii) \quad \eta^{-1} \mathbf{v}_2^T S_A \mathbf{v}_2 \leq \mathbf{v}_2^T S_B \mathbf{v}_2 \leq \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2, \text{ for all } \mathbf{v}_2 \in \mathbb{R}^{n_{k-1}}$$

$$(3.5.iii) \quad \alpha^{-1} \mathbf{v}_2^T S_A \mathbf{v}_2 \leq \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2 \leq \mathbf{v}_2^T S_A \mathbf{v}_2, \text{ for all } \mathbf{v}_2 \in \mathbb{R}^{n_{k-1}},$$

where  $\beta \geq 1$ ,  $\eta \geq 1$  and  $\alpha \geq 1$ . Here the latter inequality is sharp for a vector  $\widehat{\mathbf{v}}_2$ , i.e.

$$\alpha^{-1} \widehat{\mathbf{v}}_2^T S_A \widehat{\mathbf{v}}_2 = \widehat{\mathbf{v}}_2^T S_{\tilde{A}} \widehat{\mathbf{v}}_2.$$

That  $\mathbf{v}_2^T (S_A - S_{\tilde{A}}) \mathbf{v}_2 \geq 0$  follows from  $\mathbf{v}_1^T (A_1 - B_1) \mathbf{v}_1 > 0$ , which implies  $\mathbf{v}^T (A - \tilde{A}) \mathbf{v} \geq 0$  and, in particular,  $\mathbf{v}_2^T (S_A - S_{\tilde{A}}) \mathbf{v}_2 \geq 0$ . Further,  $\eta^{-1} \mathbf{v}_2^T S_A \mathbf{v}_2 \leq \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2$  shows that  $\alpha \leq \eta$ . Similar to Theorem 3.1, the following lower bound holds,

$$(\eta-1) \mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2 \geq \mathbf{v}_2^T (S_A - S_{\tilde{A}}) \mathbf{v}_2 = \mathbf{v}_2^T A_{21} (B_1^{-1} - A_1) A_{12} \mathbf{v}_2,$$

or

$$\eta \geq 1 + \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T A_{21} (B_1^{-1} - A_1) A_{12} \mathbf{v}_2}{\mathbf{v}_2^T S_{\tilde{A}} \mathbf{v}_2} \geq 1 + \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T A_{21} (B_1^{-1} - A_1) A_{12} \mathbf{v}_2}{\mathbf{v}_2^T S_A \mathbf{v}_2}$$

Also the following relation between  $1 - \alpha^{-1}$  and  $\beta - 1$  holds.

We have

$$\begin{aligned} (1 - \alpha^{-1}) \widehat{\mathbf{v}}_2^T S_A \widehat{\mathbf{v}}_2 &= \widehat{\mathbf{v}}_2^T (S_A - S_{\tilde{A}}) \widehat{\mathbf{v}}_2 = \widehat{\mathbf{v}}_2^T A_{21} (B_1^{-1} - A_1) A_{12} \widehat{\mathbf{v}}_2 \\ &\leq (\beta - 1) \widehat{\mathbf{v}}_2^T A_{21} A_1^{-1} A_{12} \widehat{\mathbf{v}}_2 \end{aligned}$$

or

$$1 - \alpha^{-1} \leq (\beta - 1) q(\widehat{\mathbf{v}}_2).$$

As in Theorem 3.1, this shows that  $\eta$  takes large values unless  $B_1$  is properly related to  $A_1$  for ‘smooth’ vectors  $\mathbf{v}_2$ .

The following spectral relation between  $A$  and  $B$  holds.

**Theorem 3.3.** *Assume that (3.5,i-iii) hold. Then*

$$\kappa^{-1} \mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T B \mathbf{v} \leq \mathbf{v}^T A \mathbf{v}, \text{ for all } \mathbf{v}$$

where

$$\kappa \leq \beta + \frac{1}{2}[\eta - 1 + ((1 - \alpha^{-1})\eta - 1)(\beta - 1)] + \frac{1}{2}\sqrt{[\eta - 1 + ((1 - \alpha^{-1})\eta - 1)(\beta - 1)]^2 + 4\beta(1 - \alpha^{-1})\eta(\beta - 1)}.$$

Further  $\kappa \leq \eta$  if  $\beta = 1$  and  $\kappa \leq \eta\beta$  if  $\alpha = \eta$ . If  $\alpha < \eta$  then  $\kappa < \eta\beta$ .

**Proof.** The lower bound

$$\frac{\mathbf{v}^T A \mathbf{v}}{\mathbf{v}^T B \mathbf{v}} \geq 1 \text{ for all } \mathbf{v},$$

follows from

$$A - B = (A - \tilde{A}) + (\tilde{A} - B),$$

where the right hand sides of the inequalities (3.5,i,ii) imply that  $\mathbf{v}^T(A - \tilde{A})\mathbf{v} \geq 0$  and  $\mathbf{v}^T(\tilde{A} - B)\mathbf{v} \geq 0$ .

To find an upper bound we use

$$\sup_{\mathbf{v}} \frac{\mathbf{v}^T(A - B)\mathbf{v}}{\mathbf{v}^T B \mathbf{v}} = \sup_{\mathbf{v}} \frac{\mathbf{v}^T K^T(A - B)K \mathbf{v}}{\mathbf{v}^T K^T B K \mathbf{v}},$$

where  $K = \begin{bmatrix} I_1 & -B_1^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}$ , to find

$$\mathbf{v}^T K^T B K \mathbf{v} = \mathbf{v}_1^T B_1 \mathbf{v}_1 + \mathbf{v}_2^T S_B \mathbf{v}_2$$

and

$$\begin{aligned} \mathbf{v}^T K^T(A - B)K \mathbf{v} &= \mathbf{v}_1^T(A_1 - B_1)\mathbf{v}_1 + 2\mathbf{v}_1^T(I - A_1 B_1^{-1})A_{12}\mathbf{v}_2 \\ &\quad + \mathbf{v}_2^T[(S_{\tilde{A}} - S_B) + A_{21}B_1^{-1}(A_1 - B_1)B_1^{-1}A_{12}]\mathbf{v}_2. \end{aligned}$$

Here for any  $\zeta > 0$ ,

$$2|\mathbf{v}_1^T(I - A_1 B_1^{-1})A_{12}\mathbf{v}_2| \leq \zeta \mathbf{v}_1^T(A_1 - B_1)\mathbf{v}_1 + \zeta^{-1} \mathbf{v}_2^T A_{21} B_1^{-1}(A_1 - B_1) B_1^{-1} A_{12} \mathbf{v}_2.$$

Further, we use  $A_{\tilde{A}} - S_B = (S_{\tilde{A}} - S_A) + (S_A - S_B)$ , where

$$\begin{aligned} S_{\tilde{A}} - S_A &= A_{21}(A_1^{-1} - B_1^{-1})A_{12} = \tilde{A}_{21}(I - \tilde{B}_1^{-1})\tilde{A}_{12}, \\ \tilde{A}_{12} &= A_1^{-\frac{1}{2}}A_{12}, \tilde{B}_1 = A_1^{-\frac{1}{2}}B_1A_1^{-\frac{1}{2}}, \end{aligned}$$

and

$$C \equiv A_{21}B_1^{-1}(A_1 - B_1)B_1^{-1}A_{12} = \tilde{A}_{21}(\tilde{B}_1^{-2} - \tilde{B}_1^{-1})\tilde{A}_{12}.$$

Hence

$$\begin{aligned} \mathbf{v}_2^T C \mathbf{v}_2 &\leq \beta \mathbf{v}_2^T \tilde{A}_{21}(\tilde{B}_1^{-1} - I)\tilde{A}_{12}\mathbf{v}_2 = \beta \mathbf{v}_2^T(S_A - S_{\tilde{A}})\mathbf{v}_2 \\ &\leq \beta(1 - \alpha^{-1})\mathbf{v}_2^T S_A \mathbf{v}_2. \end{aligned}$$

It follows that

$$\begin{aligned} \sup_v \frac{(Kv)^T(A-B)(Kv)}{(Kv)^T B (Kv)} &\leq \max \left\{ (1 + \zeta) \sup_{\mathbf{v}_1} \frac{\mathbf{v}_1^T(A_1 - B_1)\mathbf{v}_1}{\mathbf{v}_1^T B_1 \mathbf{v}_1}, \right. \\ &\quad \left. [(1 + \zeta^{-1})\beta - 1](1 - \alpha^{-1}) \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T S_A \mathbf{v}_2}{\mathbf{v}_2^T S_B \mathbf{v}_2} + \sup_{\mathbf{v}_2} \frac{\mathbf{v}_2^T(S_A - S_B)\mathbf{v}_2}{\mathbf{v}_2^T S_B \mathbf{v}_2} \right\} \\ &\leq \max\{(1 + \zeta)(\beta - 1), [(1 + \zeta^{-1})\beta - 1](1 - \alpha^{-1})\eta + \eta - 1\}. \end{aligned}$$



Letting now  $\zeta$  satisfy

$$(1+\zeta)(\beta-1) = [(1+\zeta^{-1})\beta-1](1-\alpha^{-1})\eta + \eta - 1$$

we find as in the proof of Theorem 3.2,

$$\zeta = \frac{1}{2} \left( \frac{\eta-1}{\beta-1} + (1-\alpha^{-1})\eta-1 \right) + \frac{1}{2} \sqrt{\left( \frac{\eta-1}{\beta-1} + (1-\alpha^{-1})\eta-1 \right)^2 + 4\beta \frac{(1-\alpha^{-1})\eta}{\beta-1}}$$

and an elementary computation completes the proof.  $\diamond$

**3.2. Multilevel extension.** The multilevel extension of the above two level methods can be done as in Section 2, by defining the Schur complement matrix  $S_B$  by

$$S_B = A^{(k-1)}[I - P_{\nu_k}(M^{(k-1)^{-1}}A^{(k-1)})]^{-1}$$

where  $P_\nu$  is a matrix of degree  $\nu_k$  and  $M^{(k)}$  is the preconditioner on level  $k$ .

For a general matrix, for instance one which is not defined by finite element matrices for a sequence of mesh levels, we must construct or define the next matrix  $A^{(k-1)}$  for each coarser level. A possible way is to let  $A^{(k-1)}$  be the Schur complement of the matrix

$$\tilde{B}^{(k)} \equiv \begin{bmatrix} D_1^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_2^{(k)} \end{bmatrix}$$

where  $D_1^{(k)}$  is a diagonal matrix spectrally equivalent to  $A_1^{(k)}$  such that  $\epsilon(D_1^{(k)} - A_1^{(k)})$  is positive semi-definite for  $\epsilon = -1$  or  $\epsilon = 1$ . Here the Schur complement  $A_2^{(k)} - A_{21}^{(k)} D_1^{(k)^{-1}} A_{12}^{(k)}$  can be formed explicitly. The arising new connections in  $A_{21}^{(k)} D_1^{(k)^{-1}} A_{12}^{(k)}$  correspond frequently to an approximate difference type matrix. For instance, this is the case when  $\tilde{B}^{(k)}$  is a five point (2D) or seven point (3D) difference matrix in which case the Schur complement is a nine point or 27 point difference matrix, respectively.

The matrix  $B_1^{(k)}$  on the other hand, can be a more accurate approximation to  $A_1^{(k)}$  than  $D_1^{(k)}$ . As we have seen from Theorem 3.2 and 3.3, we can come arbitrarily close to the condition number of the two level matrix with exact blocks  $B_1^{(k)} = A_1^{(k)}$ , by increasing the accuracy of this approximation.

#### 4. SOME EFFICIENT MODIFICATIONS AND IMPLEMENTATIONAL ASPECTS OF THE AMLI METHOD

In this section we comment shortly on some modifications of the method to improve its efficiency and implementation.

To avoid the recursive overhead associated with the AMLI method when the polynomial degree  $\nu_k \geq 2$ , one can use a fixed polynomial  $P_1(t) = 1 - t$  of first degree on most levels and a higher degree polynomial only at certain stabilization levels. It can readily be seen that in general the condition number then grows geometrically but the permitted maximal value of  $\nu$  at stabilization points to keep the computational expense of optimal order grows correspondingly also if a nearly uniform mesh refinement is used. Therefore, the polynomial degree at stabilization can be chosen as

$\nu = (1 - \gamma^2)^{-(\mu+1)/2}$  where  $\mu \geq 0$  is the number of intermediate levels where  $\nu_k = 1$ . This value suffices to stabilize the condition number. For further details, see [17], [7] and [2].

At any rate, in the method as described in Section 2 and 3, one must estimate the spectral parameters  $\beta, \eta$  (or, equivalently,  $b$  and  $\eta = (1 - \gamma^2)^{-1}$  in Section 2) to enable the computation of the lower eigenvalue bound  $\alpha$  used in the Chebyshev polynomial.

**4.1. Estimates of spectral parameters.** As has been shown in [8] and [9], for instance, in the context of finite element methods the parameter  $\gamma$  can be computed locally for each finite element by letting  $\gamma$  take the maximum of its local values. More generally, for frequently used approximations  $B_1$  and  $S_B$  of  $A_1$  and  $S_A$ , respectively, the computation of  $\beta$  and  $\eta$  can be done locally on small sized matrices, in the following way. This will be illustrated for the common case where some off-diagonal entries in the matrix  $A_1$  are deleted and each entry is compensated for by adding it to the diagonal entry in the same row. To preserve symmetry, these operations take place for pairs of symmetrically located entries.

Consider first the case where some of the positive off-diagonal entries are deleted and compensated for, for instance to form an  $M$ -matrix if all positive entries have been deleted. Let the resulting matrix be  $M$ . For the analysis of the condition number of  $M^{-1}A$  we let  $A^{(i,j)}$  be a submatrix which corresponds to moved entries with indices  $i, j$  such that

$$A = \sum_{(i,j)} A^{(i,j)}.$$

The matrix  $A^{(i,j)}$  contains zero entries except at rows  $i, j$  and at rows  $k, k \neq i, j$  which contain entries  $a_{ki} \neq 0$  or  $a_{kj} \neq 0$ . Let  $K$  be the set of the latter. It is further assumed that the diagonal entries in the latter rows have been modified in such a way that all matrices  $A^{(i,j)}$  are positive semidefinite and, if singular, its nullspaces are spanned by the vector  $\mathbf{e} = (1, 1, \dots, 1)$  and by no other vector.

In the context of finite element matrices for second order elliptic differential equations each such matrix corresponds to a subdomain. If the zero order term in the operator is absent, then  $A^{(i,j)}$  is singular if it corresponds to an interior subdomain or, if it contains edge on the boundary, the boundary conditions are of Neumann type. Domains for different matrices  $A^{(i,j)}$  can be partly overlapping.

For the analysis of the condition number of  $\tilde{M}$  to  $\tilde{A}$  we reorder each matrix  $A^{(i,j)}$  and consider only the submatrix containing the nonzero entries, which are found in rows  $i, j$  and  $k \in K$ . We denote this submatrix  $\tilde{A}$ , i.e.

$$\tilde{A} = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \quad \text{where} \quad A_1 = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

and  $A_{12}, A_{21}, A_2$  contains the remaining entries, where  $\text{diag}(A_2)$  has been modified as described above.

The deletion of entries  $a_{ij}, a_{ji}$  results in the matrix

$$\tilde{A} = \tilde{M} - \tilde{N}, \quad \tilde{M} = M^{(i,j)}$$

where

$$\widetilde{M} = \begin{bmatrix} M_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} (a_{ii} + a) & 0 \\ 0 & (a_{jj} + a) \end{bmatrix},$$

$$N_1 = a \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad a = a_{ij} = a_{ji}.$$

We assume that  $A$  is sparse, which implies that there are few local couplings in  $A$  so  $\widetilde{A}$  and  $\widetilde{M}$  have small order. We consider now the case where  $\widetilde{A}$ , and hence  $\widetilde{M}$ , are singular. (It is readily seen that the estimate (4.2) below holds also for the general case.) It holds

$$\mathbf{v}^T \widetilde{M} \mathbf{v} \geq \mathbf{v}_1^T S^{(1)} \mathbf{v}_1, \quad \text{for all } \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix},$$

where  $\mathbf{v}_1 \in \mathbb{R}^2$  and  $S^{(1)} = M_1 - A_{12} A_2^{-1} A_{21}$ .

Here  $S^{(1)}$  is positive semidefinite. Now  $\widetilde{M}$  is singular for the vector  $\mathbf{e}$ , and therefore  $S^{(1)}$  is singular for the vector  $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Therefore  $S^{(1)}$ , being symmetric, must have

the form  $b \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$  for some positive constant  $b$ . Since  $S^{(1)} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , it follows that we can compute the coefficient  $b$  by computing this action of  $S^{(1)}$ . Since, by assumption,  $\widetilde{M}$  has small order, this computation is not costly.

It follows from the above that

$$(4.1) \quad 0 \leq \mathbf{v}^T \widetilde{N} \mathbf{v} = \mathbf{v}_1^T N_1 \mathbf{v}_1 = \frac{a}{b} \mathbf{v}_1^T S_1 \mathbf{v}_1 \leq \frac{a}{b} \mathbf{v}^T \widetilde{M} \mathbf{v} \quad \text{for all } \mathbf{v}$$

By summation, using  $M = \sum M^{(ij0)}$  and taking the maximum of the quotient  $\frac{a}{b}$  for all such deleted and moved entries  $a_{ij}$ , we find

$$\mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T M \mathbf{v} = \mathbf{v}^T (A + N) \mathbf{v} \leq \mathbf{v}^T A \mathbf{v} + \max\left(\frac{a}{b}\right) \mathbf{v}^T M \mathbf{v}$$

or

$$(4.2) \quad \mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T M \mathbf{v} \leq \frac{1}{1 - \max\left(\frac{a}{b}\right)} \mathbf{v}^T A \mathbf{v}$$

assuming that  $\max\left(\frac{a}{b}\right) < 1$ . However, the inequalities in (4.1) are sharp so  $\max\left(\frac{a}{b}\right) \geq 1$  would mean that  $\widetilde{A} = \widetilde{M} - \widetilde{N}$  is negative definite, or at least singular for some vector  $\mathbf{v} \neq \mathbf{e}$ , which contradicts the assumption made. Therefore (4.2) holds.

Similarly, if we delete and diagonally compensate for some of the negative off-diagonal entries we find for  $A = M + N$ , that

$$\left(1 - \max\left(\frac{c}{d}\right)\right) \mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T M \mathbf{v} \leq \mathbf{v}^T A \mathbf{v}, \quad \text{for all } \mathbf{v},$$

where  $-c < 0$  is an entry moved and  $d$  is the corresponding entry in the local Schur complement matrix.

If  $A$  is an  $M$ -matrix it is also diagonally dominant as it is positive semidefinite and singular for the vector  $\mathbf{e}$ . The same holds for  $M$  as well. Here we assume that no entry in the diagonal of  $M$  becomes zero. This implies that  $\max\left(\frac{c}{d}\right) < 1$ . The above shows the next Lemma.

**Lemma 4.1.** *Let  $A$  be symmetric and positive semidefinite and assume that some positive  $(a_{ij})$  and some negative off-diagonal entries  $(c_{ij})$  of  $A$  have been deleted to form a matrix  $M$  after diagonal compensation of all deleted entries. Then the following spectral relation holds,*

$$\left(1 - \max_{(i,j)} \frac{c_{ij}}{d_{ij}}\right) \mathbf{v}^T A \mathbf{v} \leq \mathbf{v}^T M \mathbf{v} \leq \frac{1}{1 - \max_{(i,j)} \frac{a_{ij}}{b_{ij}}} \mathbf{v}^T A \mathbf{v}$$

for all  $\mathbf{v}$ , where  $b_{ij}$  and  $d_{ij}$  are the entries in the corresponding local Schur complement matrices.

**4.2. An example.** In this subsection we illustrate an application of Lemma 4.1 for finite element matrices that arises in  $P_1$ –nonconforming elements applied to model second order elliptic operator of the form  $\mathcal{L}u = -\operatorname{div}(a\nabla u)$  discretized on polygonal domain  $\Omega$  partitioned into triangles. We assume that the resulting element matrices have non–positive off–diagonal elements. This is the case if the coefficient  $a$  is piecewise constant with respect to the triangles of the triangulation  $\mathcal{T}_h$  used and if all angles of the triangles are less or equal to  $\pi/2$ .

The idea explained below applies for unstructured meshes as well, i.e., we do not necessarily assume that the triangles of  $\mathcal{T}_h$  are obtained by uniform refinement of an initial coarse triangulation of  $\Omega$ .

We first describe a coarsening strategy based on forming macroelements. Assume that we have partitioned  $\Omega$  into a number of macroelements  $E$ , where each  $E$  consists of at most four triangles  $e_1, \dots, e_m$  of  $\mathcal{T}_h$  (i.e.,  $m \leq 4$ ). Let  $A_E$  be the assembled matrix composed from the element matrices corresponding to the fine elements  $e_i$ . Based on Lemma 4.1 we will delete a number of connections in  $A_E$  to create  $M_E$  which after assembly will provide the desired approximation to  $A$  and also a coarse sparse matrix  $A_c$  will be generated. Moreover, coarse element matrices will be constructed as proper Schur complements of  $M_E$ . Since we will compensate for negative off–diagonal entries, by Lemma 4.1 the following inequality will hold:

$$(4.3) \quad \mathbf{v}^T M_E \mathbf{v} \leq \mathbf{v}^T A_E \mathbf{v}, \quad \text{for all } \mathbf{v} = \mathbf{v}_E.$$

We now select coarse degrees of freedom according to the following rule. If two macroelements  $E_1$  and  $E_2$  share an edge of a fine–grid element we select only one midpoint (which is a fine–grid degree of freedom) per intersection  $E_1 \cap E_2$  to be a coarse degree of freedom. Also, the rule is to have at most three coarse degrees of freedom per macroelement  $E$ . Finally, for each fine grid element that contains a coarse degree of freedom, we break the connections between the remaining two fine grid degrees of freedom (which are not coarse) and compensate for them on the diagonal, i.e., each  $A_{e_i}$  creates  $M_{e_i}$  and after assembling  $M_{e_i}$  on each macroelement  $E$  one ends up with  $M_E$  that satisfies (4.3). By construction we will also have the following two–level form of  $M_E$ ,

$$(4.4) \quad M_E = \begin{bmatrix} M_f & M_{fc} \\ M_{cf} & M_c \end{bmatrix}.$$

where  $M_f = \begin{bmatrix} M_b^f & 0 \\ 0 & M_i^f \end{bmatrix}$ . Here the first block corresponds to the fine degrees of freedom that are on the boundary edges of  $E$  and the rest is in the interior of  $E$ . The coarse element matrix  $A_E^c \equiv M_c - M_{cf}M_f^{-1}M_{fc}$ , i.e., is the Schur complement of  $M_E$ . One has,

$$\mathbf{v}_c^T A_E^c \mathbf{v}_c \leq \mathbf{v}^T M_E \mathbf{v} \leq \mathbf{v}^T A_E \mathbf{v}.$$

Hence,

$$\mathbf{v}_c^T A_E^c \mathbf{v}_c \leq \mathbf{v}_c^T S_E \mathbf{v}_c = \inf_{\mathbf{v}_f} \mathbf{v}^T A_E \mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \mathbf{v}_f \\ \mathbf{v}_c \end{bmatrix}.$$

Note that  $A_E^c$  is a three-by-three matrix and its off-diagonal entries are non-positive. Hence after assembling of all  $A_E^c$  the resulting coarse matrix  $A^c$  will be an  $M$ -matrix and the following inequality will hold

$$\mathbf{v}_c^T A^c \mathbf{v}_c \leq \mathbf{v}_c^T S_A \mathbf{v}_c,$$

where  $S_A$  is the two-level Schur complement of the fine-grid matrix  $A$ . We will also have that  $A_c$  will be spectrally equivalent to  $S_A$  (because this holds on element level due to Lemma 4.1). Our matrix  $B$  is then defined as follows:

$$B = \begin{bmatrix} M_f & M_{fc} \\ M_{cf} & A_c + M_{cf}M_f^{-1}M_{fc} \end{bmatrix}.$$

Note that after assembling of  $M_E$  the resulting block of  $M$ ,  $M_f$  (corresponding to the fine degrees of freedom that are not coarse), is block diagonal (note the form of the element matrices  $M_E$ ). Hence  $M_f^{-1}$  is easily computable (and sparse). One can easily see that by construction  $B$  is spectrally equivalent to  $M$  and hence to  $A$ . Note that  $A_c$  is assembled from the Schur complements of the element matrices  $M_E$ . One has,

$$\mathbf{v}^T B \mathbf{v} \leq \mathbf{v}^T M \mathbf{v},$$

since

$$\mathbf{v}^T (B - M) \mathbf{v} = \mathbf{v}_c^T (A_c - S_M) \mathbf{v}_c \leq 0.$$

One also has that  $S_M$  and  $A_c$  are spectrally equivalent, hence

$$\mathbf{v}^T (M - B) \mathbf{v} = \mathbf{v}_c^T S_M \mathbf{v}_c \leq \gamma_2 \mathbf{v}_c^T A_c \mathbf{v}_c \leq \gamma_2 \mathbf{v}^T B \mathbf{v}.$$

Note that the above procedure can be recursively applied, now to  $A_c$  or rather to its element matrices  $A_E^c$  by further agglomerating at most four adjacent  $E_i$  and so on. Some additional details on agglomerating elements can be found in [13]. Of course, the spectral equivalence constants, in the unstructured case, can depend on the shape and size of the generated macroelements, and hence a relevant AMLI stabilization of the multilevel iteration may be in order.

**4.3. Concluding remarks.** It follows from Lemma 4.1 that in some cases one can compute the required eigenvalue bounds cheaply. However, for some general problems and approximation methods, it is of interest to modify the AMLI method so that such estimates of method parameters are not required. As has been shown, e.g., in [17], [7], this is possible if one uses the Lanczos method on each stabilization level to compute the eigenvalue information needed in choosing the Chebyshev polynomials.

These estimates must start with the second last coarsest mesh and proceeds in recursion for each higher stabilization level, when the previous lower level eigenvalue information has been computed to sufficient accuracy and the corresponding Chebyshev polynomial has been constructed. This process is not costly as few Lanczos steps will be needed since the arising condition numbers are small. For details, see [17] and [7].

There is an alternative parameter free implementation of the AMLI method (see [10] or the survey [18]). It uses inner iterations. For instance, the arising Schur complement systems which have been replaced by an approximate Chebyshev iteration (with a fixed number,  $\nu_k$  of iterations) in Section 2, can be solved by a parameter free inner iteration method, such as the conjugate gradient method.

Since the matrix polynomial generated in a conjugate gradient method depend on the initial residual vector, such a method corresponds to using variable preconditioners, i.e. a preconditioner which may change from one iteration to the next. This corresponds to some nonsymmetry in the preconditioned matrix and there is no Krylov subspace generated by a fixed matrix. Therefore some form of a generalized conjugate gradient method must be used, see [1], for further details. In practice, the perturbations of the preconditioner are small, and it suffices using few additional vectors in the vector updates.

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