On Bass’ inverse degree approach to the Jacobian Conjecture and exponential automorphisms

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Abstract

In this paper we give a new equivalent description of the Jacobian Conjecture based on an idea of Bass in [1].

Introduction

In the beginning of the eighties Hyman Bass published a paper [1] in which he related the Jacobian Conjecture to the problem of estimating the degree of the inverse of a polynomial automorphism over an arbitrary \( \mathbb{Q} \)-algebra (see \( UB(n) \) below for a more accurate statement).

The aim of the paper is to give a new impulse to this approach, motivated by some recent discoveries.

The contents of this paper are arranged as follows.

In the first section we recall the Jacobian Conjecture and Bass’ result. In section two we discuss Derksen’s proof of Bass’ theorem and some recent results of Furter. In section three we study the nilpotency subgroup of the automorphism group of the polynomial ring over the algebras \( \mathbb{C}_m := \mathbb{C}[T]/(T^m) \), \( m \geq 1 \) and use these results to give a new formulation of the Jacobian Conjecture.

1 The Jacobian Conjecture and the universal coefficient method

In 1939 O. Keller in [7] investigated polynomial maps \( F = (F_1, \ldots , F_n) : \mathbb{C}^n \to \mathbb{C}^n \) i.e. each \( F_i \) belongs to \( \mathbb{C}[X] := \mathbb{C}[X_1, \ldots , X_n] \), the polynomial ring in \( n \) variables over \( \mathbb{C} \). He observed that if \( F \) is invertible, in the sense that there exists a polynomial map \( G = (G_1, \ldots , G_n) : \mathbb{C}^n \to \mathbb{C}^n \) such that \( G \circ F = 1_{\mathbb{C}^n} \), then \( \det JF \in \mathbb{C}^* \), where \( JF := \left( \frac{\partial F_i}{\partial X_j} \right) \) is the Jacobian matrix of \( F \): namely using the chain rule gives \( JG(F)JF = I_n \), so taking determinants one obtains \( \det JF \in \mathbb{C}^* \).

Conversely he wondered if the condition \( \det JF \in \mathbb{C}^* \) is sufficient to guarantee the
invertibility of \( F \). This problem became known as Keller’s problem but is more often referred to as

**Jacobian Conjecture.** If \( F : \mathbb{C}^n \to \mathbb{C}^n \) is a polynomial map with \( \det JF \in \mathbb{C}^* \), then \( F \) is invertible.

In spite of the efforts of many mathematicians over sixty years, this conjecture is still open for all \( n \geq 2 \). Many equivalent formulations are known (see [4]). In this paper we will investigate one such a formulation originating form Bass’s paper [1].

Starting point of his approach is to replace \( \mathbb{C} \) in the formulation of the Jacobian Conjecture by more general rings. So let \( R \) be a commutative ring containing 1 and \( R[X] := R[X_1, \ldots, X_n] \) the polynomial ring in \( n \) variables over \( R \). By a **polynomial map over \( R \)** we understand an element \( F = (F_1, \ldots, F_n) \in R[X]^n \) and we say that \( F \) is **invertible over \( R \)** if there exists \( G \in R[X]^n \) such that \( G \circ F = X (= (X_1, \ldots, X_n)) \).

Using the chain rule one gets: if \( F \) is invertible over \( R \), then \( \det JF \in R[X]^* \) (\( = \) the group of units of \( R[X] \)). Conversely, in general the condition \( \det JF \in R[X]^* \) does not imply that \( F \) is invertible over \( R \); just take \( n = 1 \) and \( F = X - X^p \in \mathbb{F}_p[X] \). Since \( F(x) = 0 \) for all \( x \in \mathbb{F}_p \), \( F : \mathbb{F}_p \rightarrow \mathbb{F}_p \) is not injective, so \( F \) is not invertible over \( \mathbb{F}_p \). Therefore we assume from now on: \( R \) is a \( \mathbb{Q} \)-**algebra**.

We formulate

\[ JC(R, n) \text{. If } F \in R[X]^n \text{ with } \det JF \in R[X]^*, \text{ then } F \text{ is invertible over } R. \]

Of course, replacing \( F \) by \( F - F(0) \) and then \( F \) by \( JF(0)^{-1} F \) we may assume that \( F \) is **normalised** i.e. \( F(0) = 0 \) and \( JF(0) = I \).

In other words, each \( F_i = X_i + \text{higher order terms} \).

It was shown in [2] that \( JC(\mathbb{C}, n) \) implies \( JC(R, n) \) for all \( \mathbb{Q} \)-algebras \( R \). Consequently we can investigate the Jacobian Conjecture by the method of universal coefficients i.e. we consider a “universal” polynomial map which Jacobian determinant is a unit in the polynomial ring. More precisely, let \( d, n \geq 1 \) be integers and consider a normalised \( n \)-tuple of universal polynomials \( F_1^u, \ldots, F_n^u \) of degree \( d \) i.e. each \( F_i^u \) is of the form

\[ F_i^u = X_i + \sum_{1<|\alpha|\leq d} A_{\alpha}^{(i)} X^\alpha \]

where all \( A_{\alpha}^{(i)} \) are variables. Let \( F^u := (F_1^u, \ldots, F_n^u) \). Then

\[ \det J_X F^u = 1 + \sum_{|\beta|>0} P_{\beta} X^\beta \]

and each \( P_{\beta} \) is a polynomial in the \( A_{\alpha}^{(i)} \) with coefficients in \( \mathbb{Z} \). To describe universally the condition that this determinant is a unit we fix an integer \( e \geq 1 \) and consider in \( \mathbb{Q}[A] \), the polynomial ring in the \( A_{\alpha}^{(i)} \) over \( \mathbb{Q} \), the ideal \( J \) generated by the polynomials \( P_{\beta}^e \). Put

\[ R_0 := \mathbb{Q}[A]/J. \]
Then the polynomial map $\overline{F}_u$ obtained by reducing all coefficients of all $F^i_j$ mod $J$ satisfies $\det J X \overline{F}_u$ is a unit in $R_0[X]$, since $\overline{F}_u = 0$ for all $|\beta| > 0$. So if $JC(\mathbb{C}, n)$ is true then as observed above in particular $JC(R_0, n)$ is true. Hence $\overline{F}_u$ has a polynomial inverse $g = (g_1, \ldots, g_n) \in R_0[X]^n$.

Let

$$C(n, d, e) := \deg_X g = \max_i \deg g_i$$

and choose $G_1, \ldots, G_n \in Q[A][X]$ such that $\overline{G}_i = g_i$ and $\deg_X G_i = \deg_X g_i$ for all $i$. So $\deg G_i \leq C(n, d, e)$ for all $i$. Furthermore for each $i$ we have

$$F^i_1(G_1, \ldots, G_n) - X_i \text{ and } G_i(F^1_1, \ldots, F^n_1) - X_i \in JQ[A][X]$$

From (1) one easily deduces that for any $R$-algebra $R$ and any normalised $F \in Aut_R R[X]$ with $\deg F \leq d$, say $F = X + \sum_{1 < |\alpha| \leq d} a^{(i)}_\alpha X^\alpha$ which satisfies $\det J F = 1 + \sum_{|\beta| > 0} p_\beta X^\beta$ and $p_0^\beta = 0$ for all $|\beta| > 0$, we have

$$\deg F^{-1} \leq C(n, d, e).$$

Namely the $a^{(i)}_\alpha$ are zeros of the ideal $J$. So making the substitutions $A^{(i)}_\alpha \to a^{(i)}_\alpha$ in (1) we deduce that $G(a) := (G_1(a), \ldots, G_n(a))$, obtained by substitution of $a = (a^{(i)}_\alpha)$ in the coefficients of $G = (G_1, \ldots, G_n)$, is the inverse of $F$. So $\deg F^{-1} \leq C(n, d, e)$.

In particular if we take $e = 1$ i.e. look at automorphisms with $\det J F = 1$ we get: if $JC(\mathbb{C}, n)$ is true, then $UB(n)$ is true, where $UB(n)$ is the following statement.

$UB(n)$. For every $d > 1$ there exists a positive integer $C(n, d)$ such that for any $R$-algebra $R$ and any $F \in Aut_R R[X]$ with $\det J F = 1$ and $\deg F \leq d$, the degree of $F^{-1}$ is bounded by $C(n, d)$.

Remark 1.1 By $C(n, d)$ we denote from now on the smallest positive integer with the property described in $UB(n)$ (in case it exists).

Now the remarkable point, observed by Bass in [1] is that the converse is true as well i.e. $UB(n)$ implies $JC(\mathbb{C}, n)$.

In fact a stronger converse with a very elegant proof was obtained by Harm Derksen in [3]. This result will be described in the next section.

2 Derksen’s proof and Furtür’s example

To describe Derksen’s result we introduce a new statement $UB(n)$. For every $d > 1$ there exists a positive integer $C(n, d)$ such that for any $\mathbb{C}$-algebra of the form

$$\mathbb{C}_m := \mathbb{C}[T]/(T^m), \quad m \geq 1$$

and every $F \in \text{Aut}_{\mathbb{C}_m} \mathbb{C}_m[X]$ with $\det J F = 1$ and $\deg F \leq d$, the degree of $F^{-1}$ is bounded by $C(n, d)$ (so independent of $m$).
Theorem 2.1 (Derksen, [3]) \( \overline{UB}(n) \) implies \( JC(\mathbb{C}, n) \).

Proof. Let \( F \in \mathbb{C}[X]^n \) be a normalised polynomial map with \( \det JF = 1 \) and \( \deg F = d \). Let \( G \in \mathbb{C}[[X]]^n \) be its formal inverse and let for each \( i \geq 1 \) \( G_{(i)} \) denote its homogeneous component of degree \( i \). We will show that \( G_{(i)} = 0 \) for all \( \ell > C(n, d) \) (so \( G \) is a polynomial map and hence \( F \) is invertible). Therefore introduce one new variable \( T \) and define

\[
F^T := T^{-1}F(TX) = X + TF_2 + \cdots + T^{d-1}F_{(d)}
\]

and

\[
G^T = T^{-1}G(TX) \in \mathbb{C}[T][[X]]^n.
\]

One easily verifies that \( \det J_F F^T = \det J_F(TX) = 1 \). Furthermore

\[
F^T \circ G^T = X = G^T \circ F^T,
\]

the composition considered as formal power series in \( X \) with coefficients in \( \mathbb{C}[T] \). Now let \( \ell > C(n, d) \). Reducing (2) mod \( T^\ell \) we see that \( F^T \in \text{Aut}_{\mathbb{C}[T]} \mathbb{C}[X] \) with inverse \( G^T \). Also \( \det J_F F^T = 1 \) and \( \deg F^T \leq d \). So by \( \overline{UB}(n) \) we get

\[
\deg G^T \leq C(n, d).
\]

However \( G^T = X + G_{(2)}T + \cdots + G_{(\ell)}T^{\ell-1} \). If \( G_{(\ell)} \neq 0 \), since \( T^\ell \neq 0 \) in \( \mathbb{C}_\ell \), we get \( \deg G^T = \ell > C(n, d) \), contradicting (3). So \( G_{(\ell)} = 0 \) for all \( \ell > C(n, d) \), as desired. \( \square \)

Corollary 2.2 The statements \( UB(n), \overline{UB}(n) \) and \( JC(\mathbb{C}, n) \) are equivalent.

Summarizing: instead of polynomial maps over \( \mathbb{C} \) with \( \det JF = 1 \) we may as well study the degree of the inverse of polynomial automorphisms with Jacobian determinant 1 over arbitrary \( \mathbb{Q} \)-algebras or over the \( \mathbb{C} \)-algebras \( \mathbb{C}_m \), \( m \geq 1 \).

The question naturally arises: if \( JC(\mathbb{C}, n) \) is true, what is a reasonable candidate for \( C(n, d) \)?

If \( k \) is a field it is well-known (see [2], [10], [9], [11] and [4]) that if \( F \in \text{Aut}_k k[X] \) with \( \deg F = d \), then

\[
\deg F^{-1} \leq d^{n-1}.
\]

This bound is sharp: take for example the triangular map

\[
F = (X_1 + X_2^d, X_2 + X_3^d, \ldots, X_{n-1} + X_n^d, X_n).
\]

Then \( \deg F^{-1} = d^{n-1} \). Using standard arguments one deduces from (4) that the same inequality holds if \( k \) is a reduced ring (i.e. no nilpotent elements except zero). So one is tempted to ask: is \( C(n, d) = d^{n-1} \)?
A negative answer to this question was given by Furter in January 1996 (see [6]). He found the following counterexample in case $n = 2$:

Let $R = \mathbb{C}_2$ and $\varepsilon := \overline{T} \in R$. Put

$$F = (X + \varepsilon X^3, (1 - 3\varepsilon X^2)Y + X^2).$$

Then $F \in \text{Aut}_R R[X, Y]$ with $\det JF = 1$, $\deg F = 3$. However $\deg F^{-1} = 4 > 2^2 - 1 = 3$.

Before I proceed let me make

A personal note on history of mathematics

When I read Furter’s fax send to Adjamagbo containing the counterexample mentioned above I realised that it was of a very special structure, namely the first component contains only $X$, i.e. $F = (f(X), F_2)$. Then using $\det JF = 1$ one easily gets that $F_2 = (f'(X)^{-1}Y + c(X))$ for some $c(X) \in \mathbb{C}_2[X]$. To get the simplest example where $\deg F^{-1} > \deg F$ one has to take $c(X) = X^2$ and $f(X) = X + \varepsilon X^3$. So it was clear to me how Furter must have found his example.

Now the point I want to make is the following: suppose Furter would have lived some two thousand years ago and I would have been a historian of mathematics, then I would have been sure how Furter had found his example.

However a few weeks after his discovery I met him in person and asked him how he found the example. He explained me that he heavily used a computer and solved a huge system of equations, set special coefficients zero to simplify the computations and finally produced one solution: exactly the one I described above as the simplest example in my family with $\deg F^{-1} > \deg F$. How wrong would I have been if I were that historian!

After the above example was found, Furter went on to study $C(2, 3)$. Together with Fournie and Pinchon in [5] they were able to show that $C(2, 3) = 9$. This result again heavily used a computer!

Looking at the equations in their paper I was able to produce the following explicit example where the bound 9 is obtained.

Example 2.3 Let $R = \mathbb{C}_7$, $\varepsilon = \overline{T}$ and $F = (F_1, F_2)$ defined by

$$F_1 = X - \frac{1}{27}\varepsilon^3 X^2 - 2\varepsilon XY + \frac{64}{27}\varepsilon^6 X^3 + \frac{8}{3}\varepsilon^4 X^2 Y + 4\varepsilon^2 XY^2 + Y^3$$

$$F_2 = Y + \frac{8}{3}\varepsilon^3 XY + \varepsilon Y^2$$

Then $F \in \text{Aut}_R R[X, Y]$, $\det JF = 1$, $\deg F = 3$ and $\deg F^{-1} = 9$.

In order to “simplify” this example we make the following general observation: Let $F \in \text{Aut}_{C_m} C_m[X]$. Put $\varepsilon := \overline{T}$ and let $\bar{F}$ be obtained by reducing the components of $F \mod \varepsilon$. So $\bar{F} \in \text{Aut}_{C_m} C_m[X]$ and define $F_* := \bar{F} \circ \bar{F}^{-1}$. Then $F_* = X$.

We claim that it suffices to find a uniform bound for the degrees of the automorphisms $F_*^{-1}$, for all $F_*$ with bounded degree. More precisely define
For every $d \geq 1$ there exists a positive integer $C_*(n,d)$ such that for any $\mathbb{C}$-algebra $\mathbb{C}_m$, $m \geq 1$ and any $F \in \text{Aut}_{\mathbb{C}_m}[X]$ satisfying $\det JF = 1$, $\deg F \leq d$ and $\overline{F} = X$, the degree of $F^{-1}$ is bounded by $C_*(n,d)$ i.e. independent of $m$.

**Proposition 2.4** $\overline{UB}_*(n)$ implies $\overline{UB}(n)$.

**Proof.** Let $F \in \text{Aut}_{\mathbb{C}_m}[X]$ with $\det JF = 1$ and $\deg F \leq d$. Put $F_* := F \circ \overline{F}^{-1}$. So $F^{-1} = \overline{F}^{-1} \circ F_*^{-1}$, whence

$$\deg F^{-1} \leq \deg \overline{F}^{-1} \cdot \deg F_*^{-1} \leq (\deg F)^{n-1} C_*(n, \deg F_*).$$

(5)

Since $F_* = F \circ \overline{F}^{-1}$ we get

$$\deg F_* \leq \deg F \cdot (\deg F)^{n-1} = (\deg F)^n.$$  

(6)

From (5) and (6) we get

$$\deg F^{-1} \leq (\deg F)^{n-1} C_*(n, (\deg F)^n) \leq d^{n-1} C_*(n, d^n)$$

which implies $\overline{UB}(n)$.

Now let's return to example 2.3. In spite of the difference between $\deg F^{-1} (= 9)$ and $\deg F (= 3)$ I found to my surprise that $\deg F_*^{-1} = \deg F_*$. Also for Furter's example I found equality of these degrees! These and several other examples led me to the following question:

**Question 2.5** Is $C_*(2,d) = d$?

To study this question we are going to investigate the Nilpotency subgroup of $\text{Aut}_{\mathbb{C}_m}[X]$ i.e. the set of all $F \in \text{Aut}_{\mathbb{C}_m}[X]$ satisfying $\overline{F} = X$. The results are given in the next section.
3 The Nilpotency subgroup and exponential automorphisms

To study $\text{Aut}_{C_m}[X]$ we recall some results of [4]. Let $A$ be a $\mathbb{Q}$-algebra and $\ell : A \to A$ or $\mathbb{Q}$-linear map. Then $\ell$ is called locally nilpotent if for every $a \in A$ there exists a positive integer $q$ such that $\ell^q(a) = 0$. Now suppose that $D$ is a derivation on $A$ which is locally nilpotent. To such a derivation define $\exp D : A \to A$ by the formula

$$\exp D(a) := \sum_{i \geq 0} \frac{1}{i!} D^i(a).$$

Then $\exp D$ is a $\mathbb{Q}$-automorphism of $A$ with inverse $\exp(-D)$ (see for example proposition 2.1.1 in [4]). Such an automorphism of $A$ is called an exponential automorphism. To decide if a given ring homomorphism $f : A \to A$ is an exponential automorphism, define $E : A \to A$ by $E := f - 1_A$. The following criterion was obtained in [4], proposition 2.1.3.

**Proposition 3.1** Let $f : A \to A$ be a ring homomorphism. Then $f$ is an exponential automorphism of $A$ if and only if $E$ is locally nilpotent. Furthermore, if $E$ is locally nilpotent then the map $D : A \to A$ defined by

$$D(a) = \sum_{i \geq 1} (-1)^{i+1} \frac{E^i(a)}{i},$$

for all $a \in A$

is a locally nilpotent derivation on $A$ and $f = \exp D$.

Now let $R$ be a commutative $\mathbb{Q}$-algebra. The nilpotency subgroup of $\text{Aut}_R R[X]$, denoted by $N(R, n)$, consists of all $F$ such that $F(X_i) = X_i + g_i$ for all $i$, where each $g_i$ is a nilpotent element of $R[X]$ or equivalently belongs to $\eta R[X]$, where $\eta$ is the nilradical of $R$.

**Proposition 3.2** If $F \in N(R, n)$, then $F = \exp D$ for some $D \in \text{Der}_R R[X]$ with $D = 0$ ($D$ is obtained from $D$ by reducing the coefficients of $D$ modulo $\eta$).

**Proof.** Put $A := R[X]$ and $E := F - 1_A$.

i) Using proposition 3.1 we need to show that $E$ is locally nilpotent. So let $a \in A$.

We must prove that $E^p(a) = 0$ for some $p \geq 1$. Therefore replacing $R$ by the subalgebra of $R$ generated by all coefficients appearing in $a$ and $F$ we may assume that $R$ is noetherian and hence that $\eta^m = 0$ for some $m \geq 1$.

ii) Now let $h \in R[X]$. Since each $g_i \in \eta R[X]$ the same holds for $E(h) = h(X_1 + g_1, \ldots, X_n + g_n) - h(X_1, \ldots, X_n)$. So

$$E(R[X]) \subseteq \eta R[X].$$

(7)
Since $E$ is $R$-linear, applying $E$ to (7) gives $E^2(R[X]) \subset \eta^2 R[X]$. Repeating this argument we finally arrive at $E^m(R[X]) \subset \eta^m R[X] = 0$ as desired. Finally the formula of $D$ given in proposition 3.1 together with (7) gives that $\overline{D} = 0$. □

Now we return to the study of $\overline{UB}_*(n)$ i.e. we apply the previous results to the $\mathbb{C}$-algebras $\mathbb{C}_m$. So each $F \in \text{Aut}_{\mathbb{C}_m}\mathbb{C}_m[X]$ is of the form $\exp D$ with $\overline{D} = 0$ (observe that the nilradical of $\mathbb{C}_m$ equals $(\varepsilon)$, where $\varepsilon = T$ satisfies $\varepsilon^m = 0$).

The next question is: how to characterize amongst the $F \in \text{Aut}_{\mathbb{C}_m}\mathbb{C}_m[X]$ those who satisfy the additional property $\det JF = 1$.

The answer to this question is given by the following result.

**Theorem 3.3** Let $F = \exp D$ where $D$ is a $\mathbb{C}_m$-derivation on $\mathbb{C}_m[X]$ satisfying $\overline{D} = 0$. Then $\det JF = 1$ if and only if $\text{div} D = 0$, where $\text{div} D = \Sigma \partial_i(DX_i)$.

The proof of this result is based on the following result due to Nowicki in [8]. Let $R$ be a $\mathbb{Q}$-algebra and $D$ an $R$-derivation on $R[X]$. Define $\exp TD : R[X] \to R[X][[T]]$ by the formula

$$\exp TD(g) = \sum_{i \geq 0} \frac{T^i}{i!} D^i(g), \text{ for all } g \in R[X].$$

Then (see [4], proposition 1.2.14) $\exp TD$ is a ring homomorphism. To simplify the notations we write $J_X(\exp TD)$ instead of $\left( \frac{\partial \exp TD(X_1)}{\partial X_j} \right)_{1 \leq i,j \leq n}$.

**Theorem 3.4** (Nowicki, [8]). Define $B_0, B_1, \ldots$ in $R[X]$ by

$$\det J_X(\exp TD) = \sum_{p \geq 0} \frac{1}{p!} B_p T^p.$$ 

Then $B_0 = 1$ and $B_{p+1} = dB_p + D(B_p)$ for all $p \geq 0$, where $d := \text{div} D$.

**Proof of Theorem 3.3**

(i) Suppose $d := \text{div} D = 0$. Then by Nowicki’s theorem $B_p = 0$ for all $p \geq 1$, whence $\det J_X(\exp TD) = 1$, so $\det J_X F = 1$.

(ii) Now assume that $F = \exp D$, where $\overline{D} = 0$ and $\det JF = 1$. Put $d = \text{div} D$.

Suppose $d \neq 0$. Write $d = \varepsilon^r d_r + \cdots + \varepsilon^{m-1} d_{m-1}$, with $d_r \neq 0$ and $d_i \in \mathbb{C}[X]$ for all $i$. Observe that $r \geq 1$, since $\overline{D} = 0$. Again by Nowicki’s theorem

$$\det J_X(\exp TD) = \sum_{p \geq 0} \frac{1}{p!} B_p T^p,$$
with $B_0 = 1$ and $B_{p+1} = dB_p + D(B_p)$ for all $p \geq 0$. By induction on $p$ it follows that $B_p \in \epsilon^{r+p-1}\mathbb{C}_m[X]$ for all $p \geq 1$.

Consequently

$$1 = \det J_X(\exp D) = \sum_{p \geq 0} \frac{1}{p!} B_p = 1 + \epsilon^r d_r + \epsilon^{r+1}(\cdots),$$

which implies that $d_r = 0$ (since $r \geq 1$), a contradiction. □

As a consequence of theorem 3.3, proposition 2.4 and Corollary 2.2 we get

**Corollary 3.5** $JC(\mathbb{C}, n)$ is equivalent to the following statement.

For every $d \geq 1$ there exists a positive integer $C_*(n, d)$ such that for every $m \geq 1$ and every $D \in \text{Der}_{\mathbb{C}_m} \mathbb{C}_m[X]$ with $\overline{D} = 0$ and $\text{div} D = 0$ we have:

- if $\deg \exp D \leq d$, then $\deg \exp(-D) \leq C_*(n, d)$.

Now let’s return to question 2.5. One easily verifies that the $\mathbb{C}_m$-derivations on $\mathbb{C}_m[X, Y]$ with $\overline{D} = 0$ and $\text{div} D = 0$ are of the form

$$D = f_Y \partial_X - f_X \partial_Y$$

for some $f \in \mathbb{C}_m[X, Y]$ with $\overline{f} = 0$ and $f(0,0) = 0$. This $f$ is unique.

So by theorem 3.3 question 2.5 is equivalent to

**Question 3.6** Let $f \in \mathbb{C}_m[X, Y]$ with $\overline{f} = 0$ and $f(0,0) = 0$. Put $D = f_Y \partial_X - f_X \partial_Y$. Is $\deg \exp D = \deg \exp(-D)$?

As observed above, a positive answer would imply $JC(\mathbb{C}, 2)$.

However in March 1998 the first counterexample was found by Stefan Maubach. A little later the following example was given by Charles Cheng.

Let $f = Y^3\epsilon + X Y^3\epsilon^2 + \frac{3}{10} Y^5\epsilon^3$ with $\epsilon^4 = 0$. Then

$$\exp D = (X + 3Y^2\epsilon + 3XY^2\epsilon^2, Y - \epsilon^2 Y^3)$$

and

$$\exp(-D) = (X - 3Y^2\epsilon - 3XY^2\epsilon^2 + 3Y^4\epsilon^3, Y + \epsilon^2 Y^3).$$

So $\deg \exp D = 3$, and $\deg \exp(-D) = 4$.

However this example is of the form $F = \exp D$ where $D = f_Y \partial_X - f_X \partial_Y$ with $\deg_X f \leq 1$ and $f \in \mathbb{C}_m[X, Y]$. For these maps it was shown by Furter in [6] that $\deg F^{-1} \leq 4(\deg F)^4$. So the next case to study is

**Question 3.7** What can be said about $\deg \exp(-D)$ is case $\deg_X f \leq 2$?
References


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