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Abstract

Let $K = (K, | |)$ be a spherically (= maximally) complete non-archimedean
rank 1 valued field with valuation ring $B_K := \{ \lambda \in K : |\lambda| \leq 1 \}$. It is proved
(Theorem 3.8) that a $B_K$-module of finite rank is a direct sum of $B_{B_{K}}$-modules of
rank 1. The proof uses convexity techniques and seminorms. However to obtain
the announced result it is not sufficient to use only real-valued seminorms, (see
§2), so we are led to allow a more general range, a so-called $G$-module (see §3).

Introduction

Let $K$, $B_K$ be as above. A subset $A$ of a $K$-vector space $E$ is called absolutely convex
if $0 \in A$ and if $x, y \in A$, $\lambda, \mu \in B_K$ implies $\lambda x + \mu y \in A$ i.e. if $A$ is a $B_K$-submodule
of $E$. A $B_K$-module $B$ is said to be of finite rank if there is an $n \in \mathbb{N}$, an absolutely
convex $A \subset \mathbb{K}^n$ and a surjective $B_K$-module homomorphism $A \rightarrow B$. The smallest $n$
for which this is true is called the rank of $B$. (One can prove easily that it is the same
as the Fleischer rank introduced in [1].) The following natural question was stated in
[2], p. 35 as an open problem.

Q. Is every rank $n$ $B_K$-module a direct sum of $n$ rank 1 submodules?

For a non-spherically complete base field, a twodimensional indecomposable absolutely convex set is constructed in [3], p. 68 so the condition of spherical completeness of $K$ is necessary to obtain a positive answer.

In this note we prove that Q has a positive answer. During preparation of this note,
it was kindly pointed out by Prof. L. Fuchs that there is a direct purely algebraic
proof using the theory of [1], sketched as follows. Let $B$ be a finite rank $B_K$-module.
It is a surjective image of a finite rank torsion-free module $A$. As every rank one
submodule of $A$ is pure-injective, $A$ is completely decomposable. By [1], Th. 5.5, $B$ is
polyserial, by spherical completeness and [1], Th. 5.1 all uniserials are pure-injective
and therefore $A$ is a direct sum of uniserials.

Now we present an alternative proof, using techniques of convexity and seminorms.
To this end we write a $B_K$-module of rank $n$ as $T/S$ where $S \subset T$ are absolutely
convex sets in $K^n$ and study orthogonality properties of the Minkowski seminorms of
S and T. As we will see in §2 this method yields the result only for special, so-called edged sets, S and T. To obtain the full answer we extend the notion of Minkowski function by admitting a range set different from \([0, \infty)\), see §3.

1 Prelim inaries

Throughout \(K, B_K\) are as above. For a subset \(X\) of a \(K\)-vector space \(E\) we denote by \([X]\) the \(K\)-linear span of \(X\). An absolutely convex set \(A \subseteq E\) is called absorbing if \([A] = E\).

Let \(p\) be a (non-archimedean) seminorm on a \(K\)-vector space \(E\). Two subspaces \(D_1, D_2\) of \(E\) are called \(p\)-orthogonal if \(D_1 \cap D_2 = \{0\}\) and \(p(d_1 + d_2) = \max\{p(d_1), p(d_2)\}\) for all \(d_1 \in D_1, d_2 \in D_2\). If, in addition, \(E = D_1 \oplus D_2\) we call \(D_2 (D_1)\) a \(p\)-orthocomplement of \(D_1 (D_2)\).

A finite linearly independent sequence \(e_1, \ldots, e_n\) in \(E\) is called \(p\)-orthogonal if \(p(\sum_{i=1}^{n} \lambda_i e_i) = \max_{1 \leq i \leq n} p(\lambda_i e_i)\) for all \(\lambda_1, \ldots, \lambda_n \in K\) i.e. if \(Ke_i\) is \(p\)-orthogonal to \(\sum_{j \neq i} Ke_j\), for each \(i\).

**Proposition 1.1** Let \(E\) be an \(n\)-dimensional space over \(K\) \((n \in \mathbb{N})\), let \(p\) be a seminorm. Then each subspace of \(E\) has a \(p\)-orthocomplement. In particular, each \(p\)-orthogonal sequence can be extended to a \(p\)-orthogonal base of \(E\).

**Proof.** The statements are well-known for norms \(p\), ([3], 5.5, 5.15). We leave the extension to the case of seminorms \(p\) to the reader.

2 The edged case

Recall that for an absolutely convex subset \(A\) of a \(K\)-vector space, \(A^c := \bigcap_{r>1} \{\lambda a : \lambda \in K, |\lambda| \leq r, a \in A\}\) i.e., \(A^c = A\) if the valuation of \(K\) is discrete, \(A^c = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}\) if the valuation of \(K\) is dense. \(A\) is called edged if \(A^c = A\). The following is well-known.

**Proposition 2.1** For an absolutely convex subset \(A\) of a \(K\)-vector space the formula

\[
p_A(x) = \inf\{|\lambda| : \lambda \in K, x \in \lambda A\}
\]

defines a seminorm \(p_A\) on \([A]\). We have

\[
\{x \in [A] : p_A(x) < 1\} \subseteq A \subseteq \{x \in [A] : p_A(x) \leq 1\}.
\]

\(A\) is edged if and only if \(A = \{x \in [A] : p_A(x) \leq 1\}\).

**Proposition 2.2** Let \(n \in \mathbb{N}\), let \(p, q\) be seminorms on \(K^n\). Then there exists a base \(e_1, \ldots, e_n\) of \(K^n\) that is both \(p\)- and \(q\)-orthogonal.
Proof. (After [4], 1.10). It suffices to prove the existence of an \( e \in K^n \setminus \{0\} \) and a subspace \( D \) of \( K^n \) such that \( K^n = Ke \oplus D \), and \( Ke \) and \( D \) are both \( p \)- and \( q \)-orthogonal. If \( p(e) = 0 \) for some nonzero \( e \), let \( D \) be any \( q \)-orthocomplement of \( Ke \). Then trivially \( D \) and \( Ke \) are \( p \)-orthogonal. So, we may assume that \( p \) is a norm. Let \( e_1, \ldots, e_n \) be a \( p \)-orthogonal base of \( K^n \) (see 1.1). Set \( t := \max_i q(e_i)/p(e_i) = q(e_k)/p(e_k) \) for some \( k \in \{1, \ldots, n\} \). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then trivially \( D \) and \( Ke \) are \( p \)-orthogonal. So, we may assume that \( p \) is a norm.

Let \( e_1, \ldots, e_n \) be a \( p \)-orthogonal base of \( K^n \) (see 1.1). Set \( t := \max_i q(e_i)/p(e_i) = q(e_k)/p(e_k) \) for some \( k \in \{1, \ldots, n\} \). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \). Choose \( e := e_k \), let \( D \) be a \( q \)-orthocomplement of \( Ke \) (see 1.1). Then \( tp(x) \geq q(x) \) for all \( x \in K^n \).
3 The general case

From now on in §3, let \( G := \{ |\lambda| : \lambda \in K, \lambda \neq 0 \} \). It is a multiplicative subgroup of \((0, \infty)\). The following notion has been used successfully in Functional Analysis over infinite rank valued fields to define (semi)norms, see [6], [5] for a discussion.

**Definition 3.1** A **G-module** is a linearly ordered set \( X \) together with an action \( G \times X \to X \) (i.e. \( g_1 (g_2 x) = (g_1 g_2) x, 1 x = x \) for all \( g_1, g_2 \in G, x \in X \)) such that \( g_1 \geq g_2, x_1 \geq x_2 (g_1, g_2 \in G, x_1, x_2 \in X) \implies g_1 x_1 \geq g_2 x_2 \), and such that for each \( \varepsilon \in X \) and \( x \in X \) there exists a \( g \in G \) and that \( gx < \varepsilon \).

**Lemma 3.2** Let \( X \) be a G-module, let \( x \in X \). If \( g \in G \), \( gx = x \) then \( g = 1 \).

**Proof.** The set \( \{ g \in G : gx = x \} \) is easily seen to be a proper subgroup \( H \) of \( G \). If \( h \in H, h > 1 \) and \( g \in G, g \geq 1 \) then \( 1 \leq g \leq h^n \) for some \( n \). It follows that \( H = G \), a contradiction.

Obvious examples of G-modules are G itself, the group \((0, \infty)\) or any union of multiplicative cosets of G in \((0, \infty)\). For a more interesting example, let \( X \) be a G-module, let \( Y \) be a totally ordered set. Then \( X \times Y \) becomes a G-module under the lexicographic ordering and the action

\[
g(x, y) = (gx, y) \quad (g \in G, x \in X, y \in Y).
\]

We adjoin an element \( 0_X \) to \( X \) for which \( 0_X < x \), \( 0 x = 0_X = 0.0 \) for every \( x \in X \) but from now on we will write \( 0 \) instead of \( 0_X \).

**Definition 3.3** Let \( E \) be a K-vector space, let \( X \) be a G-module. An **X-seminorm** is a map \( p : E \to X \cup \{0\} \) such that \( p(0) = 0 \), \( p(\lambda x) = |\lambda| p(x) \), \( p(x+y) \leq \max(p(x), p(y)) \) for all \( \lambda \in K, x, y \in E \).

**Remark.** It is not hard to see that Proposition 1.1 remains valid if we replace \( p \) by an X-seminorm. (For a formal proof for norms, see [6], 3.3.)

To define the kind of seminorms we are interested in, let \( X := (0, \infty) \times \{0, 1\} \) with the lexicographic ordering. Then for each \( r \in (0, \infty) \) the element \((r, 1)\) is an immediate successor of \((r, 0)\) which suggests the notation \( r^n \) for \((r, 0)\) and \( r^+ \) for \((r, 1)\). The action defined above now reads as \( |\lambda|r^+ = (|\lambda|r)^+ \) \((\lambda \in K, \lambda \neq 0)\). Thus, we have ‘doubled’ every positive real number \( r \) by giving it a successor \( r^+ \), and we write \( X = (0, \infty) \cup (0, \infty)^+ \) where \( (0, \infty)^+ := \{ r^+ : r \in (0, \infty) \} \).

From now on in this note we assume that the valuation of \( K \) is dense and let \( X_K := G \cup (0, \infty)^+ \) (which is a G-submodule of \((0, \infty) \times (0, \infty)^+\) we have just introduced).

**Theorem 3.4** Let \( A \) be an absolutely convex subset of a K-vector space. Then the formula

\[
q_A(x) = \begin{cases} \ p_A(x) & \text{if } p_A(x) = \min\{|\lambda| : x \in \lambda A\} \\ \ p_A(x)^+ & \text{otherwise} \end{cases}
\]
defines an $X_K$-seminorm $q_A \geq p_A$ on $[A]$ for which $A = \{x \in [A] : q_A(x) \leq 1\}$.

**Proof.** We first prove

(*) \quad q_A(x) \leq |\lambda| \iff x \in \lambda A \quad (x \in [A], \lambda \in K, \lambda \neq 0)

yielding the desired identity $A = \{x \in [A] : q_A(x) \leq 1\}$.

Let $q_A(x) \leq |\lambda|$. If $q_A(x) = |\mu|$ for some $\mu \in K$ then $x \in \mu A \subset \lambda A$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $p_A(x) \leq q_A(x) < |\lambda|$ so $p_A(\lambda^{-1}x) < 1$ hence $\lambda^{-1}x \in A$ by 2.2.

If, conversely, $x \in \lambda A$ and $q_A(x) = |\mu|$ for some $\mu \in K$ then $|\mu| = \min\{|\nu| : x \in \nu A\} \leq |\lambda|$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $r < |\nu|$ for all $\nu$ for which $x \in \nu A$, so $r < |\lambda|$, hence $q_A(x) = r^+ < |\lambda|$. To show that $q_A$ is a seminorm, let $x \in [A], \lambda \in K$. If $q_A(x) = |\mu|$ for some $\mu \in K$ then $x \in \mu A$ so that $\lambda x \in \lambda \mu A$ so that by (*) $q_A(\lambda x) = |\lambda\mu| = |\lambda|q_A(x)$. If $q_A(x) = r^+$ for some $r \in (0, \infty)$ then $x \in \mu A$ for all $|\mu| > r$ so $\lambda x \in \nu A$ for all $|\nu| > r|\lambda|$, hence $q_A(\lambda x) \leq |\lambda|$ for all $|\nu| > r|\lambda|$ i.e. $q_A(\lambda x) \leq (r|\lambda|)^+ = |\lambda|r^+ = |\lambda|q_A(x)$. So we have proved $q_A(\lambda x) \leq |\lambda|q_A(x)$.

To prove the converse inequality (which is only needed for $\lambda \neq 0$) we observe that $|\lambda|q_A(x) = |\lambda|q_A(\lambda^{-1}\lambda x) \leq |\lambda||\lambda^{-1}q_A(\lambda x) = q_A(\lambda x)$. Finally we prove the strong triangle inequality $q_A(x+y) \leq \max(q_A(x),q_A(y))$. Suppose $q_A(x) \leq q_A(y)$. If $q_A(y) = |\lambda|$ for some $\lambda \in K$ then by (*) $y \in \lambda A$ and also $x \in \lambda A$ so $x+y \in \lambda A$, implying $q_A(x+y) \leq |\lambda|$. If $q_A(y) = r^+$ for some $r \in (0, \infty)$ then for all $\lambda \in K$ with $|\lambda| > r$ we have $y \in \lambda A$ and also $x \in \lambda A$ so $x+y \in \lambda A$. We see that $q_A(x+y) \leq |\lambda|$ for all $|\lambda| > r$ i.e. $q_A(x+y) \leq r^+$.

**Lemma 3.5** Let $p,q$ be $X_K$-seminorms on a $K$-vector space $E$. If \( \{x \in E : p(x) \leq 1\} \subset \{x \in E : q(x) \leq 1\} \) then $p \geq q$.

**Proof.** By obvious scalar multiplication we have

\[ \{x \in E : p(x) \leq |\lambda|\} \subset \{x \in E : q(x) \leq |\lambda|\} \]

for each $\lambda \in K^\times$. Then the above inclusion is also true for $\lambda = 0$. Now let $r^+ \in (0, \infty)^+$. From

\[ \{x \in E : p(x) \leq r^+\} = \bigcap_{\lambda \in K, |\lambda| > r} \{x \in E : p(x) < |\lambda|\} \]

and a similar formula for $q$ we obtain

\[ \{x \in E : p(x) \leq s\} \subset \{x \in E : q(x) \leq s\} \]

for every $s \in X_K \cup \{0\}$. It follows that $q \leq p$.

**Corollary 3.6** Let $E$ be a $K$-vector space, let $p$ be an $X_K$-seminorm.

(i) If $A := \{x \in E : p(x) \leq 1\}$ then $p = q_A$.

(ii) Let $B : E \to E$ be a linear map. If $p(x) \leq 1$ implies $p(Bx) \leq 1$ for all $x \in E$ then $p(Bx) \leq p(x)$ for all $x \in E$. 

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Proof. (i) is a direct consequence of \( \{ x \in E : p(x) \leq 1 \} = \{ x \in E : p_A(x) \leq 1 \} \) and Lemma 3.5. For (ii) apply 3.5 to the seminorms \( p \) and \( p \circ B \).

**Proposition 3.7** Let \( n \in \mathbb{N} \), let \( p \) and \( q \) be \( X_K \)-seminorms on \( K^n \). Then there is a base of \( E \) that is both \( p \)- and \( q \)-orthogonal.

**Proof.** Like in the proof of 2.2 we prove the existence of an \( e \in K^n \setminus \{ 0 \} \) and an \((n - 1)\)-dimensional subspace \( D \) such that \( K^n = Ke \oplus D \) where \( Ke \) and \( D \) are both \( p \)- and \( q \)-orthogonal, and we may assume that \( p \) is a norm. Let \( e_1, e_2, \ldots, e_n \) be a \( p \)-orthogonal base of \( K^n \) (see Remark following 3.3). For each \( i \in \{1, \ldots, n\} \) let \( C_i := \{ \lambda \in K : p(\lambda e_i) \leq 1 \} \) and \( A_i := C_i e_i \). Then by \( p \)-orthogonality

\[
\{ x \in K^n : p(x) \leq 1 \} = A_1 + \cdots + A_n.
\]

Now set \( l(A_i) := \{ t \in X_K \cup \{ 0 \} : \text{there is an} \ a \in A_i \ \text{with} \ t \leq q(a) \} \). Then \( l(A_i) \) is an initial part of \( X_K \cup \{ 0 \} \), so \( l(A_1), \ldots, l(A_n) \) are linearly ordered by inclusion; let \( l(A_1) \) be the largest one. Set \( e := e_1 \). If \( l(A_1) = \{ 0 \} \) then \( q = 0 \) and we can take \( D = [e_2, \ldots, e_n] \), so assume \( q \not= 0 \) on \( A_1 \). Now let \( D \) be a \( q \)-orthogonal complement of \( Ke \) (Remark following 3.3) and let \( P : D + Ke \to D \) be the natural projection. We finish the proof by showing that \( Ke \) and \( D \) are \( p \)-orthogonal, i.e. that \( p(x) \leq 1 \) implies \( p(Px) \leq 1 \) (3.6 (ii)). Let \( x \in K^n, p(x) \leq 1 \). Then \( x = a_1 + \cdots + a_n \) where \( a_i \in A_i \) for each \( i \). We have, for each \( i \), \( q(a_i) \in l(A_i) \subseteq l(A_1) \), so \( q(a_i) \leq q(b) \) for some \( b \in A_1 \) and \( q(b) \not= 0 \). Then \( q(Pa_i) \leq q(a_i) \leq q(b) \). Now \( Pa_i \in [b] \) so \( Pa_i = \lambda b \) for some \( \lambda \in K \). We see that \( |\lambda| q(b) \leq q(b) \) implying \( |\lambda| \leq 1 \) by 3.2, so \( Pa_i \in A_i \). Then \( Px = \sum Pa_i E A_i \) i.e., \( p(Px) \leq 1 \), and we are done.

**Remark.** The above proof is valid for an \( X_K \)-seminorm \( p \) and an \( X \)-seminorm \( q \) for any \( G \)-module \( X \). I do not know whether the conclusion of 3.7 holds for an \( X \)-seminorm \( p \) and a \( Y \) seminorm \( q \) where \( X \) and \( Y \) are arbitrary \( G \)-modules.

The following corollary obtains.

**Theorem 3.8** (Let \( K \) be spherically complete and) let \( B \) be a \( B_K \)-module of finite rank. Then \( B \) is a direct sum of submodules of rank \( \leq 1 \).

**Proof.** The proofs of Proposition 2.3 and Corollary 2.4 can formally be taken over, where \( p_S \) and \( p_T \) are replaced by the \( X_K \)-seminorms \( q_S \) and \( q_T \) respectively.
References


