THE BOREL HIERARCHY IN
INTUITIONISTIC MATHEMATICS

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The Borel Hierarchy in intuitionistic mathematics

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In memoriam magistri cari et amici mei
Johan J. de Iongh (1915–1999)

Abstract
In intuitionistic analysis one may prove, using Brouwer’s continuity principle and an axiom of countable choice, that the positively Borel sets form a really growing hierarchy. The continuity principle implies also that the Borel hierarchy has a remarkable fine structure.

Introduction
The paper is divided into six Sections. The main result is an intuitionistic Borel hierarchy theorem: it will be proved in Section 5. In Section 1 we explain intuitionistic analysis. We introduce positively Borel sets and demonstrate why the argument establishing the hierarchy that has been given by H. Lebesgue and E. Borel is of no help to us. In Section 2 we consider subsets of the set \( \mathbb{R} \) of real numbers and of Baire space \( \mathcal{N} \) that are \( F_\sigma \) but not \( G_\delta \), or \( G_\delta \) but not \( F_\sigma \). (We are using the terminology introduced by F. Hausdorff. An \( F_\sigma \) set is a countable union of closed sets, and a \( G_\delta \) set is a countable intersection of open sets). We compare our examples with examples given by L.E.J. Brouwer. Section 3 is a rather long intermezzo, exploring the wide variety of \( F_\sigma \) sets. In intuitionistic analysis, \( F_\sigma \) sets that are not \( G_\delta \) are very easy to find: we will see that the union of the two closed real segments \([0,1]\) and \([1,2]\) is an example. Clearly then, the union of two closed subsets of \( \mathbb{R} \) is not always closed. There are also unions of three closed subsets of \( \mathbb{R} \) (or, for that matter, \( \mathcal{N} \)) that do not coincide with any union of two closed sets, and so on. The intuitionistic hierarchy exhibits a fine structure that is absent from its non-intuitionistic counterpart.

We return to our main theme in Section 4 and establish the first countably many levels of the Borel hierarchy. We find that the fine structure that appeared in the class of the countable unions of closed sets also shows up at higher levels. The general hierarchy theorem requires a more subtle argument than the partial result.
of Section 4 and we give such an argument in Section 5.
The last Section collects some remarks on analytical and co-analytical sets and shows
the collapse of the intuitionistic projective hierarchy.

We now mention the titles of the Sections and of some of the subsections:

1. Intuitionistic analysis and positively Borel sets.
   1.1. Infinite sequences.
   1.2. Axioms of countable choice.
   1.3. The continuity principle.
   1.4. Cataloguing open subsets of Baire space $\mathcal{N}$.
   1.5. The finite levels of the Borel hierarchy in $\mathcal{N}$.
   1.6. The set of stumps.
   1.7. The transfinite levels of the Borel hierarchy in $\mathcal{N}$.
   1.8. The difficulty of complements.
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   2.1. $\mathbb{Q}$ belongs to $\Sigma^0_2 \setminus \Pi^0_2$.
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Acknowledgments

I dedicate this paper to the memory of Johan J. de Iongh. He introduced me to Brouwer’s intuitionistic mathematics. He also asked me the question that led to all the other ones. I have forgotten its exact formulation but it must have been on the second level of the Borel hierarchy. The seminar on intuitionistic mathematics he conducted in the seventies in Nijmegen was very important for shaping my thoughts. We studied Howard and Kreisel 1966 and Kleene and Vesley 1965 and of course Brouwer’s own papers. Wim Gielen made decisive and inspiring contributions to this Seminar, see Gielen, de Swart and Veldman 1981. The presentation of intuitionistic analysis in Section 1 of this paper owes much to him.

I want to express my thanks to him and to the other participants of this Seminar, among them Harrie de Swart and Jo Gielen. I feel also grateful towards my later students, in particular my Ph.D. students Tonny Hurkens and Frank Waaldijk. By their enthusiasm and sometimes critical interest they helped me to sustain my belief that intuitionistic mathematics is a worthwhile cause.
The notion of perhapsity, see section 3.3, grew out of a study of Frank Waaldijk’s notion of “weak stability”, see Waaldijk 1996.
1 Intuitionistic analysis and positively Borel sets

1.1 Infinite sequences

We are contributing to intuitionistic analysis. The logical constants have their constructive meaning and we follow the rules of intuitionistic logic. In particular, a disjunctive statement $A \lor B$ is considered proven only if either $A$ or $B$ is proven and a proof of an existential statement $\exists x \in V[A(x)]$ has to provide one with a particular element $x_0$ from the set $V$ and a proof of the corresponding statement $A(x_0)$.

Intuitionistic mathematics distinguishes itself from other varieties of constructive mathematics by its conception of the continuum.

An infinite sequence $\alpha$ of natural numbers $\alpha(0), \alpha(1), \alpha(2), \ldots$ may be the result of evaluating a finitely described algorithm. Sometimes, however, such an infinite sequence is constructed step by step by choosing its values one by one. The sequence then is never finished: at any point in time only finitely many values have been chosen. It may also happen that one starts building the sequence step by step and then, at some moment, decides to describe the whole of its continuation in finitely many words. Even so, we do not divide the infinite sequences of natural numbers into the ones that are given algorithmically and the (more or less freely) step-by-step created ones. We treat infinite sequences from the extensional point of view and disregard the manner of their construction. Every infinite sequence of natural numbers comes into being in many different ways, and always, even if it is given by an algorithm, may be imagined to be the result of a step-by-step-construction.

1.2 Axioms of countable choice

Let $\mathbb{N}$ be the set of natural numbers and $\mathcal{N}$ the set of all infinite sequences of natural numbers. We use $m, n, \ldots$ as variables over $\mathbb{N}$, and $\alpha, \beta, \ldots$ as variables over $\mathcal{N}$.

A first axiom of countable choice is the following one:

\[ \text{AC}_{0,0} \]

For every binary relation $R$ on $\mathbb{N}$, if for every $m$ there exists $n$ such that $mRn$, then there exists $\alpha$ such that, for every $m$, $mR\alpha(m)$.

One always may construct the promised $\alpha$ step-by-step; that is why we accept the axiom. We cannot, like non-intuitionistic mathematicians, define $\alpha$ by saying: let $\alpha(m)$ be the least $n$ such that $mRn$. One may be unable to find the least such $n$, for instance, if one knows $0R1$ but cannot decide if $0R0$ or not.

Before we introduce a second axiom of countable choice we agree on some notations. $\mathbb{N}^*$ is the set of all finite sequences of natural numbers. We let $\langle \, \rangle$ be a fixed bijective mapping from $\mathbb{N}^*$ onto $\mathbb{N}$. Such a function is called a coding of the set of finite sequences of natural numbers: $\langle a_0, a_1, \ldots, a_{k-1} \rangle$ is the code number of the finite sequence $(a_0, a_1, \ldots, a_{k-1})$. We assume that the empty sequence is coded by the number 0. We let $\ast$ denote concatenation: $\ast$ is a binary function on $\mathbb{N}$ such that, for all $m, n$, $m \ast n$ is the code number of the finite sequence obtained by putting the sequence coded by $n$ behind the sequence coded by $m$. 
We now define another function, called $J$, from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$: for all $m, n : J(m, n) := \langle m \rangle \ast n$. It is easy to see that $J$ is a bijective mapping from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}\{0\}$. We assume that $J$ is bi-monotone in the following sense: for all $m, n : J(m, n) \leq J(m + 1, n)$ and $J(m, n) \leq J(m, n + 1)$.

We let $K, L$ be the inverse functions of $J$, that is, $K$ and $L$ are functions from $\mathbb{N}\{0\}$ to $\mathbb{N}$ and for each $m, n \neq 0 : J(K(m), L(m)) = m$.

$J$ is a pairing function on $\mathbb{N}$.

We will see later that there are some advantages to the choice of this particular $J$.

Every $\alpha$ in $\mathcal{N}$ dissolves into an infinite sequence of elements $\alpha^0, \alpha^1, \alpha^2, \ldots$ of $\mathcal{N}$ if one defines, for all $m, n, \alpha^m(n) := \alpha(J(m, n))$.

Our second axiom of countable choice extends the first one and reads as follows:

<table>
<thead>
<tr>
<th>AC0,1</th>
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<tbody>
<tr>
<td>For every binary relation $R \subseteq \mathbb{N} \times \mathcal{N}$,</td>
</tr>
<tr>
<td>if for every $m$ there exists $\alpha$ such that $m R \alpha$,</td>
</tr>
<tr>
<td>then there exists $\alpha$ such that, for every $m, m R \alpha^m$.</td>
</tr>
</tbody>
</table>

We accept also this axiom for the reason that one may construct the promised $\alpha$ step-by-step. The difficulty of the construction is only slightly greater than in the case of the earlier axiom $\text{AC0,0}$. One has to start and keep going an infinite number of never finished constructions, one for $\alpha^0$, one for $\alpha^1$, and so on. At each stage exactly one of these constructions is brought one step further: at stage $n$ one defines $\alpha^{K(n)}(L(n))$.

### 1.3 The continuity principle

We define, give any $\alpha$ and any $n : \overline{\alpha}(n) := \langle \alpha(0), \alpha(1), \ldots, \alpha(n - 1) \rangle$.

If confusion is unlikely to arise, we sometimes write $\overline{\alpha}n$ for $\overline{\alpha}(n)$.

The continuity principle reads as follows:

<table>
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<tr>
<th>CP</th>
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<tbody>
<tr>
<td>For every binary relation $R \subseteq \mathcal{N} \times \mathcal{N}$,</td>
</tr>
<tr>
<td>if for every $\alpha$ there exists $m$ such that $\alpha R m$,</td>
</tr>
<tr>
<td>then for every $\alpha$ there exist $m, n$ such that for every $\beta$,</td>
</tr>
<tr>
<td>if $\overline{\alpha}(n) = \overline{\beta}(n)$, then $\beta R m$.</td>
</tr>
</tbody>
</table>

The continuity principle is classically false and it makes that intuitionistic analysis is not a subsystem of classical analysis. The axiom is adopted for the following reason. If one is able to determine for every $\alpha$ a suitable $m$, one will in particular be able to determine a suitable $m$ for any $\alpha$ that is given step-by-step. A number $m$, suitable for an $\alpha$ that is given step-by-step will be found at some moment of time, and at that moment only finitely many values of $\alpha$, say $\alpha(0), \alpha(1), \ldots, \alpha(n - 1)$ will be known. The number $m$ will therefore suit every $\beta$ that has its first $n$ values the same as $\alpha$.

We repeat the remark we made at the end of Section 1.1: every $\alpha$, even an algorithmically given one, can be thought of as resulting from a step-by-step-construction. Observe that the intuitionistic continuum is not a set in the sense of classical set theory. One does not create it by collecting all its previously created elements. The
continuum is better described as a kind of frame on which all kinds of projects for constructing possible elements may be executed.

1.4 Cataloguing open subsets of Baire space \( \mathcal{N} \)

Every binary relation \( R \) on Baire space \( \mathcal{N} \) may be viewed as a family of subsets of \( \mathcal{N} \): for every \( \beta \) in \( \mathcal{N} \), we let \( R\beta \) denote the set of all \( \alpha \) in \( \mathcal{N} \) such that \( \alpha R \beta \). One sometimes says that the relation \( R \) catalogues or parametrizes the family of sets \( R\beta, \beta \in \mathcal{N} \).

We now want to define the family of open subsets of \( \mathcal{N} \) by means of a suitably chosen binary relation on \( \mathcal{N} \). We start from the idea that a basic open set in Baire space \( \mathcal{N} \) is given by a finite sequence \( s \) of natural numbers and consists of all infinite sequences \( \alpha \) that have \( s \) as an initial part. An open set in general is a countable union of basic open sets. We therefore let \( G \) be the binary relation on \( \mathcal{N} \) given by:

\[
\text{for all } \alpha, \beta \text{ in } \mathcal{N} : \quad \alpha G \beta \text{ if and only if } \exists m \exists n [\beta(n) = \bar{\alpha}m + 1].
\]

A subset \( X \) of \( \mathcal{N} \) is open if and only if there exists \( \beta \) such that, for all \( \alpha, A \alpha \) if and only if \( \alpha G \beta \).

There are other binary relations on \( \mathcal{N} \) that produce the same class of subsets of \( \mathcal{N} \). For instance, let \( G' \) be the binary relation on \( \mathcal{N} \) given by:

\[
\text{for all } \alpha, \gamma \text{ in } \mathcal{N} : \quad \alpha G' \gamma \text{ if and only if } \exists m [\gamma(\bar{\alpha}m) = 0].
\]

(An open set is now thought of as being given by a decidable rather than an enumerable set of finite sequences.)

We show that \( G, G' \) parametrize the same class of subsets of \( \mathcal{N} \). We first define a binary relation \( \sqsubseteq \) on \( \mathbb{N} \) by: \( m \sqsubseteq n \) if and only if there exists \( e \) such that \( n = m * e \).

Given any \( \gamma \) one may define \( \beta \) such that, for all \( \alpha, \alpha G \beta \) if and only if \( \alpha G' \gamma \), as follows: for each \( n, \beta(n) = 0 \) if \( \gamma(n) \neq 0 \) and \( \beta(n) = n + 1 \) if \( \gamma(n) = 0 \).

Conversely, given any \( \beta \), one may define \( \gamma \) such that, for all \( \alpha, \alpha G \beta \) if and only if \( \alpha G' \gamma \), as follows: for each \( m, \gamma(m) = 0 \) if and only if there exists \( n < m \) such that \( \beta(n) \sqsubseteq m \).

1.5 The finite levels of the Borel hierarchy in \( \mathcal{N} \)

We introduce a sequence \( G_1, F_1, G_2, F_2, \ldots \) of binary relations on \( \mathcal{N} \), as follows:

(i) For all \( \alpha, \beta, \alpha G_1 \beta \) if and only if \( \exists m \exists n [\beta(n) = \bar{\alpha}m + 1] \) (so \( G_1 \) coincides with the relation \( G \) defined in Section 1.4) and \( \alpha F_1 \beta \) if and only if \( \forall m \forall n [\beta(n) \neq \bar{\alpha}m + 1] \).

(ii) For all \( n > 0 \), for all \( \alpha, \beta, \alpha G_{n+1} \beta \) if and only if \( \exists m [\alpha F_n \beta^m] \) and \( \alpha F_{n+1} \beta \) if and only if \( \forall m [\alpha G_n \beta^m] \).

We also introduce, for each \( n > 0 \), classes \( \Sigma_n^0, \Pi_n^0 \) of subsets of \( \mathcal{N} \).

A subset \( A \) of Baire space \( \mathcal{N} \) belongs to the class \( \Sigma_n^0, (\Pi_n^0, \text{ respectively}) \) if and only if there exists \( \beta \) such that, for every \( \alpha, A \alpha \) if and only if \( \alpha G_n \beta \) (\( A \alpha \) if and only if \( \alpha F_n \beta \), respectively). Sometimes we write \( A = G_n \beta \) (\( A = F_n \beta \), respectively) under these circumstances.
We define, for all $a$, for all $p, q$, $a^{p.q} := (a^p)^q$.

For every $\beta$ there exists $\gamma$ such that, for all $\alpha$, $\alpha G_{\beta}^\gamma$ if and only if $\exists n[\alpha G_{\beta}^n]$; we define, for all $n$, $\gamma^n := \beta^{K(n),L(n)}$. This might be taken as a proof of the fact that the class $\Sigma^0_1$ is closed under the operation of countable union. However, if we want to show: “for every sequence $X_0, X_1, X_2, \ldots$ of open subsets of $\mathcal{N}$, its union, $\bigcup_{n \in \mathbb{N}} X_n$, is also open”, we first use the axiom $\text{AC}_{0,1}$ in order to determine $\beta$ such that for each $n$, $X_n = G_{\beta}^n$.

Similarly, every class $\Sigma^0_1$ is closed under the operation of countable union, and every class $\Pi^0_1$ is closed under the operation of countable intersection.

1.6 The set of stumps

1.6.1

The description of the transfinite levels of the Borel hierarchy requires a further definition. We need something like countable ordinals and introduce *stumps*. The set $\text{St}$ of stumps is a subset of Baire space $\mathcal{N}$ and is given by the following inductive definition:

(i) $1$ is a stump. ($1$ is the element of $\mathcal{N}$ with the constant value 1). We sometimes call $1$ the *empty stump*.

(ii) For all $\beta$ in $\mathcal{N}$, if, for each $n$, $\beta^n$ is a stump, and $\beta(0) = 0$, then $\beta$ itself is a stump.

(iii) Clauses (i) and (ii) produce all stumps.

N. Lusin had some doubts about the legitimacy of accepting the set $\aleph_1$ of all countable ordinals and Brouwer sometimes shared these doubts. However, we accept the above definition, and, as a consequence of (iii), recognize the possibility of giving proofs and definitions by transfinite induction on $\text{St}$.

For every stump $\beta$, we define the successor of $\beta$, notation: $\beta^+$ or $S(\beta)$, by: $(S(\beta))(0) = 0$ and for every $n$, $(S(\beta))^n = \beta$.

1.6.2

With every stump $\beta$ we may associate the subset $B(\beta)$ of $\mathbb{N}$, consisting of all $m$ in $\mathbb{N}$ such that for every $n \subseteq m$, $\beta(n) = 0$. The set $\text{St}(\beta)$ of all finite sequences of natural numbers whose code number belongs to $B(\beta)$ is a “stump” in the sense given to this word by Brouwer. We mention three important properties of the set $\text{St}(\beta)$.

(i) We may decide, for every finite sequence of natural numbers, if it belongs to $\text{St}(\beta)$ or not.

(ii) Every initial part of a finite sequence in $\text{St}(\beta)$ belongs itself to $\text{St}(\beta)$.

(iii) Finally, to every $\gamma$ in $\mathcal{N}$ we may calculate $n$ such that $(\gamma(0), \ldots, \gamma(n-1))$ does not belong to $\text{St}(\beta)$, that is, there exists $n$ such that $\beta(n) \neq 0$.

Observe that $\text{St}(1) = \emptyset$; that is why $1$ is sometimes called the empty stump. We may decide, for every stump $\beta$, if $\beta = 1$ or not.
From now on we use $\sigma, \tau, \ldots$ as variables on the set $\text{Stp}$. We define binary relations $<, \leq$ on the set $\text{Stp}$ of stumps as follows: for every stump $\sigma$, $1 \leq \sigma$ and for no stump $\sigma$, $\sigma < 1$. Furthermore, all stumps $\sigma, \tau$ such that $\tau \neq 1$, $\tau \leq \sigma$ if and only if, for each $n$, $\tau^n < \sigma$, and $\sigma < \tau$ if and only if, for some $n$, $\sigma \leq \tau^n$.

A subset $P$ of $\text{Stp}$ is called hereditary if and only if for every stump $\sigma$, $\sigma$ belongs to $P$ if every $\tau < \sigma$ belongs to $P$. One may verify that every hereditary subset of $\text{Stp}$ coincides with $\text{Stp}$.

### 1.7 The transfinite levels of the Borel hierarchy in $\mathcal{N}$

#### 1.7.1

We introduce for every $\sigma$, binary relations $G_\sigma, F_\sigma$ on $\mathcal{N}$, as follows:

(i) $G_1 := G_1$ and $F_1 = F_1$.

$(G_1$ and $F_1$ were introduced in Section 1.5).

(ii) For every $\sigma \neq 1$, for all $\alpha, \beta$, $\alpha G_\sigma \beta$ if and only if $\exists m[\alpha F_{\sigma^m} \beta^m]$ and $\alpha F_\sigma \beta$ if and only if $\forall m[\alpha G_{\sigma^m} \beta^m]$.

We introduce, for each stump $\sigma$, classes $\Sigma^0_\sigma, \Pi^0_\sigma$ of subsets of $\mathcal{N}$.

A subset $A$ of $\mathcal{N}$ belongs to $\Sigma^0_\sigma$ ($\Pi^0_\sigma$, respectively) if and only if there exists $\beta$ such that, for all $\alpha, A \alpha$ if and only if $\alpha G_\sigma \beta$ ($A \alpha$ if and only if $\alpha F_\sigma \beta$, respectively).

We call $\beta$ in $\mathcal{N}$ repetitive if and only if for each $m$ there exists $n$ such that $n > m$ and $\beta^m = \beta^n$.

#### 1.7.2

We introduce a subclass $\text{Hrs}$ of $\text{Stp}$. The elements of $\text{Hrs}$ are called hereditarily repetitive stumps. The set $\text{Hrs}$ is given by the following inductive definition:

(i) $1$ is a hereditarily repetitive stump.

(ii) For all $\beta$, if, for each $n$, $\beta^n$ is a hereditarily repetitive stump, and $\beta$ is repetitive, and $\beta(0) = 0$ then $\beta$ is itself a hereditarily repetitive stump.

(iii) Clauses (i) and (ii) produce all hereditarily repetitive stumps.

For each hereditarily repetitive $\sigma$, the class $\Sigma^0_\sigma$, ($\Pi^0_\sigma$, respectively) is closed under the operation of countable union (countable intersection, respectively). These facts are formulated and proved like the corresponding facts for $\Sigma^0_n, (\Pi^0_n$, respectively) mentioned in Section 1.5.

### 1.8 The difficulty of complements

#### 1.8.1 Markov’s principle

The subsets of $\mathcal{N}$ that belong to one of the classes $\Sigma^0_\sigma$ are called the positively Borel subsets of $\mathcal{N}$. Observe that we did not use the operation of taking complements in
their construction.
This is because there are serious difficulties with the operation of taking complements. For every subset \( A \) of \( \mathcal{N} \), we let \( A^c \) denote the set of all \( \alpha \) in \( \mathcal{N} \) such that \( \neg \alpha \alpha \), that is, such that the assumption \( \alpha \alpha \) leads to a contradiction. One verifies easily that for every open \( A \), its complement \( A^c \) is closed. On the other hand, the statement that for every closed, its complement \( A^c \) is open cannot be proved intuitionistically. It is equivalent to

\[
\text{GMP} \quad \text{For every } \alpha, \quad \text{if } \neg \exists n[\alpha(n) = 0], \text{ then } \exists n[\alpha(n) = 0]
\]

a principle that might be called: the generalized Markov principle.
Markov enunciated this principle for algorithmically given sequences \( \alpha \), the generalization consists in its extension to arbitrary members of \( \mathcal{N} \).
Brouwer developed his famous creating subject arguments in order to show that a statement very similar to \( \text{GMP} \) is contradictory; he wanted this fact in order to conclude - unconvincingly - that the predicate \( \neg \exists n[\alpha(n) = 0] \) is inexpressible without negation and therefore that negation should not be removed from the language of intuitionistic mathematics.

1.8.2 Strong complements

We might try to remedy the situation by making the notion of the complement of a set more constructive.
We use the apartness relation. For all \( \alpha, \beta \) in \( \mathcal{N} \) we say \( \alpha \) is apart from \( \beta \) (notation: \( \alpha \neq \beta \)), if there exists \( n \) such that \( \alpha(n) \neq \beta(n) \).
This relation has the following well-known properties: for every \( \alpha, \beta, \gamma \): not \( \alpha \neq \alpha \) (\( \neq \) is irreflexive) and, if \( \alpha \neq \beta \), then \( \beta \neq \alpha \), \( \neq \) is symmetric and, if \( \alpha \neq \beta \), then either \( \alpha \neq \gamma \) or \( \gamma \neq \beta \) (\( \neq \) is co-transitive).
For every subset \( A \) of \( \mathcal{N} \), we let \( A^c \) (the strong complement of \( A \)) denote the set of all \( \alpha \) in \( \mathcal{N} \) such that, for each \( \beta \) in \( A \), \( \alpha \neq \beta \). One might hope that the strong complement of any closed set is open. In fact, we only succeeded in proving this for the special kind of closed subsets of \( \mathcal{N} \) that are commonly called spreads. A closed subset \( A \) of \( \mathcal{N} \) is a spread if and only if one may decide, for every \( n \), if there exists \( \alpha \) in \( A \) such that \( \exists m[\alpha m = n] \) or not. Given a spread \( A \), there exists \( \gamma \) in \( \mathcal{N} \) such that for each \( n \), \( \gamma(n) = 0 \) if and only if there exists \( \alpha \) in \( A \) such that \( \exists m[\alpha m = n] \). \( \gamma \) is sometimes called the spread-law governing the spread \( A \).
Let us prove that the strong complement of a spread is an open subset of \( \mathcal{N} \). If \( A \) is a spread and \( \gamma \) the spread-law governing \( A \) one may define a mapping \( r_A \) from \( \mathcal{N} \) to \( A \) such that for every \( \alpha \) in \( A \), \( r_A(\alpha) = \alpha \). \( r_A \) is called the canonical retraction of \( \mathcal{N} \) onto \( A \): in order to define \( r_A \), one first defines \( \delta \) as follows: \( \delta(0) = 0 \), and for all \( a, n \), if \( \gamma(\delta(a) * \langle n \rangle) = 0 \), then \( \delta(a * \langle n \rangle) = \delta(a) * \langle n \rangle \) and if \( \gamma(\delta(a) * \langle n \rangle) \neq 0 \), then \( \delta(a * \langle n \rangle) = \delta(a) * \langle n_0 \rangle \), where \( n_0 \) is the least \( p \) such that \( \gamma(\delta(a) * \langle p \rangle) = 0 \). One now prescribes, for all \( a, n \): \( r_A(\alpha) n = \delta(\alpha m) \).
Suppose now that \( A \) is a spread and that \( \beta \) is apart from every \( \alpha \) in \( A \). In particular, \( \beta \) is apart from \( r_A(\beta) \), and there exists \( n \) such that \( \exists n \neq r_A(\beta) n \). So there exists \( n \) such
that $\beta n$ is rejected by the spread-law governing $A$. Therefore the strong complement of $A$ is an open subset of $\mathcal{N}$.

The fact that we are unable to prove that the strong complement of every closed subset of $\mathcal{N}$ is open might be considered as a first weakness of the notion of “constructive complement”.

1.8.3 Strong complements are of no use at level 2

If we go up one level in the hierarchy the situation gets worse. Consider the sets $E_2 := \{a | \exists n \forall m [a^n(m) = 0] \}$ and $A_2 := \{a | \forall n \exists m [a^n(m) \neq 0] \}$. As we will see, the strong complement of the set $E_2$ is the set $A_2$, but the strong complement of $A_2$ does not coincide with $E_2$.

We first prove that the strong complement of the set $E_2$ is the set $A_2$.

We suppose $\beta$ is apart from every member of $E_2$. Given any $n$, we consider the sequence $\gamma$ defined by: $\gamma(0) := \beta(0)$ and, for each $p$, if $p \neq n$ then $\gamma^p := \beta^p$. Then $\gamma$ belongs to $E_2$, so $\beta \neq \gamma$, and therefore there exists $m$ such that $\beta^n(m) \neq 0$. We conclude that $\beta$ belongs to $A_2$. One also establishes easily that every member of $A_2$ is apart from every member of $E_2$.

We now verify that the strong complement of $A_2$ does not coincide with $E_2$. It contains $E_2$ but is much bigger. In order to see this we first remark that the strong complement of $A_2$ coincides with the set $B := \{a | a \in \mathcal{N} \exists n [a^n(\gamma(n)) = 0] \}$. Let us prove this. For every $\gamma, \alpha$ we define a sequence $f^*(\gamma, \alpha)$ as follows: $(f^*(\gamma, \alpha))(0) := \alpha(0)$ and, for every $n$, $(f^*(\gamma, \alpha))^n(\gamma(n)) := \max(1, \alpha^n(\gamma(n)))$ and, for every $n, p$, if $\gamma(n) \neq p$ then $(f^*(\gamma, \alpha))^n(p) = \alpha^n(p)$.

Observe that, for each $\gamma, \alpha$, $f^*(\gamma, \alpha)$ belongs to $A_2$. Suppose $\beta$ is apart from every member of $A_2$. Then $\beta$ is apart from every sequence $f^*(\gamma, \beta)$, therefore for every $\gamma$ there exists $n$ such that $\beta^n(\gamma(n)) = 0$, so $\beta$ belongs to $B$.

Conversely, suppose $\beta$ belongs to $B$ and $\alpha$ belongs to $A_2$. Using the axiom of countable choice $\text{AC}_{0,0}$ we find $\gamma$ in $\mathcal{N}$ such that for each $n$, $\alpha^n(\gamma(n)) \neq 0$. As there exists $n$ such that $\beta^n(\gamma(n)) = 0$, $\alpha$ is apart from $\beta$. So every element of $B$ belongs to the strong complement of $A_2$.

We now claim that the assumption that $B$ is a subset of $E_2$ leads to a contradiction. We have given a proof of this fact in Veldman 1982 and now offer a slightly different argument.

Let us assume that $B$ is a subset of $E_2$.

Considering sequences $\alpha$ with the property: for every $n$, $\alpha^{n+2} = \alpha^n$ we see that our assumption implies: for every $\alpha$ in $\mathcal{N}$, if $\forall p \forall q [a^0(p) = 0 \lor a^1(q) = 0]$, then either $\forall p [a^0(p) = 0]$ or $\forall q [a^1(q) = 0]$.

It is easy to see that the set $C := \{\alpha | \exists \gamma \in \mathcal{N} [\forall p \forall q [a^0(p) = 0 \lor a^1(q) = 0]] \}$ coincides with a spread, in fact, for every $\alpha, \alpha$ belongs to $C$ if and only if for each $m, \alpha^0m = \bar{0}$ or $\alpha^1m = \bar{0}$.

It is important now that the continuity principle $\text{CP}$ generalizes to spreads. Let $r_C : \mathcal{N} \rightarrow C$ be the canonical retraction of $\mathcal{N}$ onto $C$. $r_C$ is a continuous function and, for each $\alpha$ in $C$, $r_C(\alpha) = \alpha$. From our assumption we know that for every $\alpha$, if $\alpha$ belongs to $C$, then either $\alpha^0 = \bar{0}$ or $\alpha^1 = \bar{0}$, therefore, for every $\alpha$, either $(r_C(\alpha))^0 = \bar{0}$
or \((r_C(\alpha))^1 = 0\), and, using \(CP\), we calculate \(m\) such that, for every \(\alpha\), if \(\bar{\alpha}m = \bar{\gamma}m\), then \((r_C(\alpha))^0 = 0\) or, for every \(\alpha\), if \(\bar{\alpha}m = \bar{\gamma}m\), then \((r_C(\alpha))^1 = 0\). Therefore, for each \(\alpha\) in \(C\), if \(\bar{\alpha}m = \bar{\gamma}m\), then \(\alpha^1 = 0\), or, for each \(\alpha\) in \(C\), if \(\bar{\alpha}m = \bar{\gamma}m\), then \(\alpha^1 = 0\). This conclusion is false, as there exist \(\beta, \gamma\) in \(C\) such that \(\bar{\beta}m = \bar{\gamma}m = \bar{\delta}m\) and \(\beta^0 \neq 0\) and \(\gamma^1 \neq 0\).

Our statement that \(B\) is much bigger than \(E_2\) should be understood properly; we mean to say that the assumption that \(B\) is a subset of \(E_2\) leads to a contradiction, not that we are able to define members of \(B\) that are apart from every member of \(E_2\); we are not, and we are also unable to indicate a member of \(B\) such that the assumption that this member of \(B\) belongs to \(E_2\) leads to a contradiction (we do not know if such members of \(B\) exist, although the assumption of \(GMP\) excludes their existence).

One obtains similar conclusions from studying the sets

\[
\text{Fin} := \{\alpha \mid \exists n \forall m > n[\alpha(m) = 0]\} \quad \text{and} \quad \text{Inf} := \{\alpha \mid \forall n \exists m > n[\alpha(m) \neq 0]\}.
\]

\(\text{Fin}\) and \(\text{Inf}\) are (almost) the set of the characteristic functions of the finite and the infinite subsets of \(\mathbb{N}\), respectively. 

\(\text{Inf}\) is the strong complement of \(\text{Fin}\), but the strong complement of \(\text{Inf}\) is the set of all \(\alpha\) such that for every strictly increasing \(\gamma\) there exists \(n\) such that \(\alpha(\gamma(n)) = 0\). We call this set \textit{Almostfinite}. In Veldman 1999 it is shown that there is an uncountable variety of positively Borel sets \(X\) such that \(\text{Fin} \subseteq X \subseteq \text{Almostfinite}\), and \(X\) does not coincide with either \(\text{Fin}\) or \(\text{Almostfinite}\).

1.9 The diagonal argument

1.9.1

We want to find, for every (hereditarily repetitive) stump \(\sigma\) a subset of \(\mathcal{N}\) belonging to \(\Sigma^0_\sigma\) and not to \(\Pi^0_\sigma\), and also a subset of \(\mathcal{N}\) belonging to \(\Pi^0_\sigma\) and not to \(\Sigma^0_\sigma\). We try the famous diagonal argument due to G. Cantor, and first applied in this context by H. Lebesgue and E. Borel.

1.9.2

We first remark: for every stump \(\sigma\), for all \(\alpha, \beta, \gamma\), if \(\alpha F_\sigma \gamma\) and \(\beta G_\sigma (1 - \gamma)\), then \(\alpha \neq \beta\).

This is easily proved by transfinite induction on the set of stumps. It means that for every \(\sigma\) and every \(\gamma\), the sets \(F_\sigma \gamma\) and \(G_\sigma (1 - \gamma)\) are contained in each other's constructive complement.

The statement that the sets \(F_\sigma \gamma\) and \(G_\sigma (1 - \gamma)\) together make up the whole set \(\mathcal{N}\) is not true, and, speaking loosely, becomes more and more untrue with increasing \(\sigma\).

1.9.3

Let \(\sigma\) be a stump and let us try to find a set belonging to \(\Sigma^0_\sigma\) and not to \(\Pi^0_\sigma\). We observe that the diagonal set \(D_\sigma := \{\alpha \mid \alpha \in \mathcal{N} \land \alpha F_\sigma \alpha\}\) belongs to \(\Pi^0_\sigma\), and determine
\( \beta \) such that, for all \( \alpha, \alpha F_\sigma \beta \).

We now follow Cantor’s advice and define \( H_\sigma := \{ \alpha | \alpha \in \mathcal{N} \alpha G_\sigma (1 \supset \beta) \} \). We conjecture that \( H_\sigma \), while clearly belonging to \( \Sigma^0_\sigma \), does not belong to \( \Pi^0_\sigma \). So assume that \( H_\sigma \) does belong to \( \Pi^0_\sigma \) and determine \( \gamma \) such that, for all \( \alpha, H_\sigma \alpha \) if and only if \( \alpha F_\sigma \gamma \).

Observe that \( \gamma F_\sigma \gamma \) if and only if \( \gamma G_\sigma (1 \supset \beta) \) but also \( \gamma F_\sigma \gamma \) if and only if \( \gamma F_\sigma \beta \), therefore \( \gamma F_\sigma \beta \) if and only if \( \gamma G_\sigma (1 \supset \beta) \).

If we assume \( \gamma F_\sigma \beta \) we find \( \gamma^{(1)} \) and therefore \( \gamma \neq \gamma \), which is absurd, therefore: not: \( \gamma F_\sigma \beta \) and not: \( \gamma G_\sigma (1 \supset \beta) \).

The difficulty now is that we are almost always unable to derive a contradiction from these two negative propositions. Let us consider some simple cases.

1.9.3.1. If \( \sigma = 1 \), we have a conclusion of the form: \( \forall n [A(n)] \) if and only if \( \exists n [\neg A(n)] \), where \( A \) is a decidable subset of the set \( \mathbb{N} \) of natural numbers. This conclusion is contradictory indeed: assume \( n \in \mathbb{N} \) and \( \neg A(n) \), then for every \( p, A(p) \), contradiction, therefore \( \neg \neg A(n) \), and, as \( A \) is decidable, \( A(n) \). So for each \( n, A(n) \). But then there exists \( p \) such that \( \neg A(p) \).

Contradiction.

1.9.3.2. If \( \sigma = S(1) \), we obtain a conclusion of the form:
\( \forall m \exists n [A(m, n)] \) if and only if \( \exists m \forall n [\neg A(m, n)] \), where \( A \) is a decidable subset of \( \mathbb{N} \times \mathbb{N} \).

We find \( \neg \exists m \exists n [A(m, n)] \) and \( \exists m \forall n [\neg A(m, n)] \), therefore \( \forall m \neg \neg \exists n [A(m, n)] \). We are led to a contradiction if we assume GMP, the generalized Markov-principle and replace \( \forall m \neg \neg \exists n [A(m, n)] \) by \( \forall m \exists n [A(m, n)] \). We also are led to a contradiction if we use Kuroda’s principle, which says the following: for every \( P \subseteq \mathbb{N} \), if \( \forall m [\neg P(m)] \), then \( \neg \forall m [P(m)] \). Without such non-intuitionistic assumptions we are helpless.

1.9.3.3. If \( \sigma = S(S(1)) \), we fail utterly. This time, also the generalized Markov principle and Kuroda’s principle do not save our attempt to reach a contradiction. In Veldman 1981 and Veldman 1990 it is stated wrongly, that the generalized Markov principle would enable one to mimick the argument due to Cantor and Lebesgue at all levels of the Borel hierarchy. The argument only proves that there are closed sets that are not open and open sets that are not closed, and the generalized Markov principle enables one only to extend this to the second level of the hierarchy.

1.9.3.4. Our failure at level 3 is explained by the fact, that in intuitionistic predicate logic, no contradiction follows from the three assumptions:

\[ \neg \forall x \exists y \forall z [P(x, y, z)], \neg \exists x \forall y \exists z [P(x, y, z)] \] and \( \forall x \forall y \forall z [P(x, y, z) \lor \neg P(x, y, z)] \).

The following example makes this clear.

Consider the statement: \( \neg \forall n \exists m [\alpha(n) = 0 \rightarrow \alpha(m) = 0] \).

Using the continuity principle, we may prove it, as follows:
Suppose \( \forall n \exists m [\alpha(n) = 0 \rightarrow \alpha(m) = 0] \). Determine, \( n, p \) such that \( \forall n [\alpha(n) = 0 \rightarrow \alpha(m) = 0] \). Let \( N := \max(p + 1, n + 1) \). Define \( \alpha := \{ N \times 1 \} \) and observe that we have a contradiction.

On the other hand: \( \neg \forall n \exists m [\alpha(n) = 0 \land \alpha(m) \neq 0] \).

And of course we may decide, for all \( \alpha, m, n \): either \( \alpha(n) = 0 \rightarrow \alpha(m) = 0 \) or \( \neg (\alpha(n) = 0 \rightarrow \alpha(m) = 0) \).
Although the diagonal argument does not solve our problem, the hierarchy theorem proven by it has an intuitionistic interpretation. We only have to use a translation of classical logic into intuitionistic logic as given by G. Gentzen or K. Gödel.

We first define a set $\text{Bor}$ of binary relations on $\mathcal{N}$.

The elements of $\text{Bor}$ might be called \textit{Borel-catalogues}.

$\text{Bor}$ is given by the following inductive definition.

(i) The set $G_1 := \{ (\alpha, \beta) | \exists m (\exists n [\beta(n) = \bar{\alpha}(m) + 1]) \}$ belongs to $\text{Bor}$.

$G_1$ is the binary relation on $\mathcal{N}$ introduced in Section 1.5 that catalogues the open subsets of $\mathcal{N}$.

(ii) For every $X$ in $\text{Bor}$, also $X^c := \mathcal{N} \setminus X$ belongs to $\text{Bor}$.

(iii) For every sequence $X_0, X_1, \ldots$ of elements of $\text{Bor}$:

the relations $\sum X_n$ and $\prod X_n$ belong to $\text{Bor}$, where, for all $\alpha, \beta$, $\alpha(\sum X_n)\beta$ if and only if, for some $n$, $\alpha(X_n)(\beta^n)$, and $\alpha(\prod X_n)\beta$ if and only if for every $n \in \mathbb{N}$ $\alpha(X_n)(\beta^n)$.

($\sum X_n$ and $\prod X_n$ might be called the \textit{sum} and the \textit{product} of the sequence $X_0, X_1, X_2, \ldots$ respectively.).

(iv) Clauses (i)-(iii) produce all elements of $\text{Bor}$.

A subset $A$ of $\mathcal{N}$ is \textit{catalogued} by $X$ in $\text{Bor}$ if and only if there exists $\alpha$ such that $A = X\alpha$. We now define a subclass $\text{Class}$ (from “classical”) of $\text{Bor}$, by almost the same inductive definition: we only remove the infinitary operation $S$, and replace (i) by (i)': The set $F_1 := \mathcal{N} \times \mathcal{N}\setminus G_1$ belongs to $\text{Class}$.

One verifies by transfinite induction: for all $X$ in $\text{Class}$, $X^{cc} = X$, that is, every element of $\text{Class}$ is a \textit{stable} subset of $\mathcal{N} \times \mathcal{N}$.

Following Cantor, one defines, given any $X$ in $\text{Class}$:

$D(X) := \alpha | \alpha(\mathcal{N}\setminus X)\alpha \beta$. $D(X)$ is catalogued by $X^c$ but does not occur among the sets catalogued by $X$. For suppose it does and determine $\beta$ such that for all $\alpha$, $\neg\alpha X\alpha$ if $\alpha X\beta$. Then, in particular, $\neg\beta X\beta$ if and only if $\beta X\beta$, so both $\neg\beta X\beta$ and $\beta X\beta$, contradiction.

\textbf{1.10 Changing the notion of complement?}

Sometimes, constructive mathematicians have been tempted to redefine the notion of complement. Following them, we say that $A, B$ are complementary positively Borel subset of $\mathcal{N}$, or more precisely, that $(A, B)$ is a \textit{complementary pair of positively Borel subsets of $\mathcal{N}$}, if and only if there exists $\alpha, \beta$ such that $A = F_\alpha B$ and $B = G_\alpha (1 - \beta)$, or, conversely, $A = G_\alpha B$ and $B = F_\alpha (1 - \beta)$. Brouwer gives a similar definition in Brouwer 1992, page 89, line 21-27, where he is studying sets from the second level of the positive Borel hierarchy. The same course is taken in Martin-Löf 1968, page 80, and in Bishop and Bridges 1985, pages 73-75. Brouwer goes so far as to introduce another name, “the counter-set of $A$”, for the set of all $\alpha$ such that the assumption: $\alpha \in A$ is contradictory.
We studied two complementary pairs of positively Borel subsets of $\mathcal{N}$ in Section 1.8.3, viz. the pairs $(A_2, E_2)$ and $(\text{Fin}, \text{Inf})$.

We proved that $E_2$ is part of the constructive complement of $A_2$, but does not coincide with it.

It is easy to see that for every complementary pair $(A, B)$ of Borel subsets of $\mathcal{N}$, for every $a \in A$, for every $\beta \in B$, $a \neq \beta$, that is $A$ is part of the constructive complement of $B$ and $B$ is part of the constructive complement of $A$.

It is important to remark that, given some positively Borel subset of $\mathcal{N}$, there may be many different positively Borel sets $\mathcal{B}$ such that $(\mathcal{B}, \text{Inf})$ is a complementary pair.

Example: Consider $\text{Fin} := \{ \alpha | \exists n \forall m > n [\alpha(m) = 0] \}$ and $\text{Inf} := \{ \alpha | \forall n \exists m > n [\alpha(m) \neq 0] \}$. Consider also $\text{Fin}^+: = \{ \alpha | \exists n [\alpha(n) = 0 \rightarrow \text{Fin}(\alpha)] \}$. As is shown in Veldman 1995, $\text{Fin}$ is a proper subset of $\text{Fin}^+$. It is easily seen, however, that $(\text{Fin}^+, \text{Inf})$ is a complementary pair. Uncountably many positively Borel sets $\mathcal{X}$ such that $\text{Fin} \subseteq \mathcal{X} \subseteq \text{Almostfinite}$ were introduced in Veldman 1995, and for each set $\mathcal{X}$ from that collection the pair $(\mathcal{X}, \text{Inf})$ is a complementary pair.

Even if $A$ is a closed subset of $\mathcal{N}$ we are unable to conclude: for all positively Borel sets $\mathcal{B}, \mathcal{C}$, if both $(A, B)$ and $(A, C)$ are complementary pairs, then $B = C$. This is because there are decidable subsets $P, Q$ of $\mathbb{N}$ such that we are sure that: $\forall n [P(n)]$ if and only if $\forall n [Q(n)]$ but we do not know: $\exists n [\neg P(n)]$ if and only if $\exists n [\neg Q(n)]$.

### 1.11 Borel subsets of the set $\mathbb{R}$ of real numbers

We spend a few words on one of the possible ways to introduce real numbers and Borel sets of real numbers in intuitionistic analysis.

#### 1.11.1

We first determine a fixed enumeration $\rho$ of the set $\mathbb{Q}$ of rational numbers. We then define, for each $n$, $L^*(n) := \rho(K(n + 1))$ and $R^*(n) := \rho(L(n + 1))$. We define binary relations $\sqsubseteq^*$ and $\sqsupseteq^*$ on $\mathbb{N}$, as follows: $m \sqsubseteq^* n$ if and only if $L^*(n) \leq L^*(m) \leq R^*(m) \leq R^*(n)$ and $m \sqsupseteq^* n$ if and only if $L^*(n) < L^*(m) \leq R^*(m) < R^*(n)$.

For all rational numbers $p, q$ we define $\bar{p}, q^* :=$ the least natural number $n$ such that $L^*(n) = p$ and $R^*(n) = q$.

$\alpha$ in $\mathcal{N}$ is called a real number if, for each $n, \alpha(n + 1) \sqsubseteq^* \alpha(n)$ and, for each $q, r \in \mathbb{Q}$, if $q < r$, then there exist $n$ such that either $q < L^*(\alpha(n))$ or $R^*(\alpha(n)) < r$. We denote the set of real numbers by $\mathbb{R}$. Observe that $\mathbb{R}$ is a $G_\delta$ subset of $\mathcal{N}$.

#### 1.11.2

For all $\alpha, \beta \in \mathbb{R}$, we define: $\alpha <^* \beta$ ($\alpha$ is really smaller than $\beta$) if and only if there exists $n$ such that $R^*(\alpha(n)) < L^*(\beta(n))$. We also define: $\alpha \neq^* \beta$ ($\alpha$ is really apart from $\beta$) if and only if either $\alpha <^* \beta$ or $\beta <^* \alpha$, and: $\alpha =^* \beta$ ($\alpha$ really coincides with $\beta$) if and only if the assumption $\alpha \neq^* \beta$ leads to a contradiction. Observe that
1.11 Borel subsets of the set \( \mathbb{R} \) of real numbers

\( \alpha \) really coincides with \( \beta \) if and only if for each \( n \), both \( L^*(\alpha(n)) \leq R^*(\beta(n)) \) and \( L^*(\beta(n)) \leq R^*(\alpha(n)) \).

1.11.3

The relation of real coincidence extends to subsets of \( \mathbb{R} \) as follows:
Let \( A, B \) be subsets of \( \mathbb{R} \). A really coincides with \( B \) if and only if every member of \( A \) really coincides with some member of \( B \), and conversely, every member of \( B \) really coincides with some member of \( A \).
It is very important that \( \mathbb{R} \) itself really coincides with a subspread of \( \mathcal{N} \) and that a bounded closed interval like \([0, 1]\) really coincides with a fan, or finitary spread, see Section 1.13.
Let \((a_0, b_0), (a_1, b_1), \ldots \) be a fixed enumeration of all pairs \((a, b)\) of rational numbers such that \( a < b \).
\( a \) is called a canonical real number if for each \( n \), \( a(n + 1) \leq^* a(n) \) and either \( a_n < L^*(a(n)) \) or \( R^*(a(n)) < b_n \).
Let \( \text{Crn} \) be the set of all canonical real numbers. \( \text{Crn} \) is a spread and really coincides with \( \mathbb{R} \).

1.11.4

Let \( a \) be a natural number. We define \( M(a) := \frac{1}{2}(L^*(a) + R^*(a)) \).
\( M(a) \) is the midpoint of the rational segment coded by \( a \). Let \( a, b \) be natural numbers.
We say that the rational segment coded by \( b \) is a regular subsegment of the rational segment coded by \( a \) if either: \( L^*(a) = L^*(b) \) and \( R^*(b) = M(a) \) or: \( L^*(b) = \frac{1}{2}(L^*(a) + M(a)) \) and \( R^*(b) = \frac{1}{2}(M(a) + R^*(a)) \) or: \( L^*(b) = M(a) \) and \( R^*(b) = R^*(a) \).
\( a \) in \( \mathbb{R} \) is called a regular real number if and only if for each \( n \), \( a(n + 1) \) is a regular subsegment of \( a(n) \).
Let \( \text{Rrn} \) be the set of all regular real numbers. \( \text{Rrn} \) is a spread and really coincides with \( \mathbb{R} \).
The set of all regular real numbers \( a \) such that \( L^*(a(0)) = 0 \) and \( R^*(a(0)) = 1 \) is a finitary spread or a fan really coinciding with the segment \([0, 1]\).

1.11.5

We now define binary relations \( F^*, G^* \) on \( \mathcal{N} \) as follows:
for all \( \alpha, \beta \), \( \alpha G^* \beta \) if and only if there exist \( m, n \) such that \( \alpha(m) \sqsubset^* \beta(n) \) and \( \alpha F^* \beta \) if and only if for all \( m, n \), not \( \alpha(m) \sqsubset^* \beta(n) \).
A subset \( X \) of \( \mathbb{R} \) is called open (closed, respectively) if and only if there exists \( \beta \) in \( \mathcal{N} \) such that for all \( \alpha \) in \( \mathbb{R} \), \( X \alpha \) if and only if \( \alpha G^* \beta \) (\( X \alpha \) if and only if \( \alpha F^* \beta \), respectively).
The relations \( G^*, F^* \) catalogue the classes \( \Sigma^0_1(\mathbb{R}) \), \( \Pi^0_1(\mathbb{R}) \) of the open and the closed subsets of \( \mathbb{R} \), respectively. By a definition similar to the one given in Section 1.7 one obtains, for each stump \( \sigma \), relations \( G^*_\sigma \) and \( F^*_\sigma \), cataloguing classes called \( \Sigma^0_\sigma(\mathbb{R}) \) and \( \Pi^0_\sigma(\mathbb{R}) \), respectively.
Any subset of \( \mathbb{R} \) belonging to some class \( \Sigma^0_\sigma(\mathbb{R}) \), \( \Pi^0_\sigma(\mathbb{R}) \) is called a positively Borel
1.11.6

The following observation will be useful in the sequel.
For every open subset $X$ of $\mathbb{R}$ there exists a decidable subset $D$ of $\mathbb{Q} \times \mathbb{Q}$ such that for every $\alpha$ in $\mathbb{R}$, $\alpha$ belongs to $X$ if and only if, for some $n$, the pair $(L^*(\alpha(n)), R^*(\alpha(n)))$ belongs to $D$.
We leave it to the reader to prove this observation.

1.12 Borel subsets of Polish spaces

We want to discuss arbitrary Polish spaces. In classical mathematics, a Polish space is defined as a completely metrizable separable topological space.

Intuitionistically, a Polish space is given by a function $d : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ satisfying the triangle inequality: for all $m,n,p$, $d(m,p) \leq d(m,n) + d(n,p)$.

Let $d : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ be such a function. We describe how to build the space determined by $d$ and how to define its Borel subsets.

We want to associate with every natural number $n$ the basic neighbourhood $B_n := B(K(n+1)), \rho(L(n+1))$, that is, the open ball with center $K(n+1)$ and radius $\rho(L(n+1))$.

We therefore define two binary relations on $\mathbb{N}$, $\sqsubset_d$ and $\not\sqsubset_d$, as follows. For all $m,n, m \sqsubset_d n$ if and only if $d(K(m+1), K(n+1)) + \rho(L(m+1)) < \rho(L(n+1))$ and $m \not\sqsubset_d n$ if and only if $d(K(m+1), K(n+1)) > \rho(L(m+1)) + \rho(L(n+1))$.

(Observe that, for all $m,n$, if $m \sqsubset_d n$, then $B_m$ is part of $B_n$, and, if $m \not\sqsubset_d n$, then $B_m$ lies outside $B_n$).

$\sqsubset_d$ and $\not\sqsubset_d$ are enumerable subsets of $\mathbb{N} \times \mathbb{N}$ but not necessarily decidable ones. We construct a function $\delta$ from $\mathbb{N}$ to $\mathbb{N}$ such that, for all $m,n$, $m \sqsubset_d n$ if and only if there exists $p$ such that $\delta(p) = J(m,n) - 1$.

We now let $X = X(d)$ be the set of all $\alpha$ in $\mathcal{N}$, such that for each $n$, $\alpha(n+1) \sqsubset_d \alpha(n)$ and also for every $m$ there exists $p$ such that $\rho(L(\alpha(p)+1)) < \frac{1}{m}$. Observe that $X$ is a $\Pi^0_4$-subset of $\mathcal{N}$. For all $\alpha, \beta$ in $X$ we define: $\alpha$ is $d$-apart from $\beta$, notation $\alpha \#_d \beta$, if and only if there exists $n$ such that $\alpha(n) \#_d \beta(n)$.

We also define, for all $\alpha, \beta$, $\alpha$ $d$-coincides with $\beta$ if the assumption $\alpha \not\#_d \beta$ leads to a contradiction.

Next we define binary relations $G(X)$ and $F(X)$ on $\mathcal{N}$ as follows: for all $\alpha, \beta$, $\alpha G(X) \beta$ if there exists $m, n$ such that $\alpha(m) \sqsubset_d \beta(n)$, and $\alpha F(X) \beta$ if for all $m, n$, not: $\alpha(m) \sqsubset_d \beta(n)$.

A subset $Y$ of $Z$ is called open (closed, respectively) if and only if there exists $\beta$ in $\mathcal{N}$ such that, for all $\alpha$ in $Z$, $\alpha G(X) \beta$ if and only if $\alpha G(X) \beta$ is open (closed, respectively).

The relations $G(X), F(X)$ catalogue the classes $\Sigma^0_4(X), \Pi^0_4(X)$ of the open and the closed subsets of $X$. By a definition similar to the one given in Section 1.7 one
obtains, for each stump $\sigma$, relations $G_\sigma(X), F_\sigma(X)$ cataloguing classes called $\Sigma^0_\sigma(X)$ and $\Pi^0_\sigma(X)$, respectively.
The sets belonging to some class $\Sigma^0_\sigma(X), \Pi^0_\sigma(X)$ are called the *positively Borel* subsets of $X$. Observe that each positively Borel subset of $X$ is closed under $=_d$, the relation of $d$-coincidence.

### 1.13 Further axioms of intuitionistic analysis: Brouwer’s Thesis

In Sections 1.1-3 we introduced everything we need for the proof of the intuitionistic Borel hierarchy theorem: an axiom of countable choice and the continuity principle. As Brouwer is using more for the examples we will cite in Section 2, we now mention these further principles of intuitionistic analysis. In Section 6, the last Section of the paper, we want to draw some other consequences from these further principles.

We called one of them *Brouwer’s Thesis* in Veldman 1981 and Veldman 1990 and we will also do so now.

**Brouwer’s Thesis:**

| BT | For every subset $P$ of $\mathbb{N}$:
|    | if for every $\alpha$ there exists $n$ such that $P(\alpha n)$, then there exists a stump $\beta$ such that for every $\alpha$ there exists $n$ such that $P(\alpha n)$ and $\forall m \leq n [\beta(\alpha m) = 0]$. |

(The notion of “stump” has been introduced in Section 1.6).

Brouwer thought that his Thesis could be seen to be true by reflection on the possible structure of a proof of the fact “for every $\alpha$ there exists $n$ such that $P(\alpha n)$''.

We refrain from discussing his argument.

Brouwer’s Thesis has important consequences and a particularly famous one is the *Fan Theorem.*

A *fan* or *finitary spread* is a subset $F$ of Baire space $\mathcal{N}$ such that there exist $\beta$ such that for every $\alpha, F(\alpha)$ if and only if, for each $n, \beta(\alpha n) = 0$, and for each $n$ such that $\beta(n) = 0$ there exists $m$ such that for all $k$, if $\beta(n * (k)) = 0$, then $k < m$.

**Fan Theorem:**

| FT | Let $F \subseteq \mathcal{N}$ be a fan.
|    | For every subset $P$ of $\mathbb{N}$:
|    | if for every $\alpha$ in $F$ there exists $n$ such that $P(\alpha n)$, then there exists $p$ such that for every $\alpha$ in $F$ there exists $n \leq p$ such that $P(\alpha n)$. |

Brouwer used the fan theorem for proving that every real function defined on $[0, 1]$ is uniformly continuous on $[0, 1]$. 
1.14 Further axioms of intuitionistic analysis: stronger continuity principles

In this Section we introduce two axioms that imply and are stronger than the continuity principle CP.

We first define a subset Fun of Baire space $\mathcal{N}$. If $\gamma$ belongs to Fun, we will use the words: “$\gamma$ codes a (continuous) function from $\mathcal{N}$ to $\mathbb{N}$”. We define, for each $\gamma$, $\gamma$ belongs to Fun if and only if $\forall x \exists n[\gamma(an) \neq 0]$.

If $\gamma$ belongs to Fun, we define, for each $\alpha$, the value of the function coded by $\gamma$ at $\alpha$, notation: $\gamma(\alpha)$, as follows:

$\gamma(\alpha) := \text{the least number } p \text{ such that } \exists m[\gamma(an) = p + 1 \land \forall n < m[\gamma(an) = 0]]$. We now call the following axiom $\text{AC}_{1,0}$ (where “AC” stands both for “axiom of choice” and “axiom of continuity”).

\[
\text{AC}_{1,0}
\]

For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$:

if for every $\alpha$ there exists $m$ such that $\alpha R m$, then there exists $\gamma$ in Fun such that for every $\alpha$, $\alpha R \gamma(\alpha)$.

This axiom needs an argument slightly more elaborate than the argument given for the continuity principle CP. We allow ourselves to construct the promised $\gamma$ step by step and consider the finite sequences of natural numbers one by one.

When considering such a finite sequence we imagine that it forms the beginning of an infinite sequence that is growing step by step, and we ask ourselves if, as such, it suffices for the determination of a natural number that suits this infinite sequence. If it does, we determine such a number, call it $p$, and let the value of $\gamma$ at (the code number of) this finite sequence be $p + 1$, if not, we let that value be 0. We may convince ourselves that for every $\alpha$, whether it is given by an algorithm or is constructed, more or less freely, step by step, there will exist $n$ such that $\gamma(an) \neq 0$, by reasons similar to the ones that made us accept the continuity principle CP.

Observe that, if $\gamma$ belongs to Fun and $\gamma(0) = 0$, then for each $n$, $\gamma^n$ belongs to Fun. For this reason, if $\gamma$ belongs to Fun and $\gamma(0) = 0$, $\gamma$ may be considered as the code of a continuous function from $\mathcal{N}$ to $\mathcal{N}$. If $\gamma$ belongs to Fun and $\gamma(0) = 0$, we define, for each $\alpha$, $\gamma|\alpha$ in $\mathcal{N}$ as follows: for each $n$, $(\gamma|\alpha)(n) := \gamma^n(\alpha)$.

If $\gamma$ belongs to Fun and $\gamma(0) = 0$ we define the range of $\gamma$, notation: $\text{Ran}(\gamma)$, as follows: $\text{Ran}(\gamma)$ is the set of all sequences $\gamma|\alpha$, where $\alpha$ belongs to $\mathcal{N}$.

We now introduce an axiom of choice/continuity that implies all earlier ones:

\[
\text{AC}_{1,1}
\]

For every binary relation $R \subseteq \mathcal{N} \times \mathcal{N}$:

if for every $\alpha$ there exists $\beta$ such that $\alpha R \beta$, then there exists $\gamma$ in Fun such that $\gamma(0) = 0$ and $\forall \alpha[\alpha R \gamma(\alpha)]$.

Once more, we construct the promised $\gamma$ step by step. When considering the code number $\alpha$ of a finite sequence of natural numbers we look for the least $n < \text{length}(\alpha)$ such that $\forall m < n \exists b \in a[\gamma^m(b) \neq 0]$. If there exists no such $n$ we define, for all $m$, $\gamma^m(\alpha) := 0$. If there exists such a number, we call it $n_0$. We imagine the finite sequence coded by $a$ to be the beginning of an infinite $\alpha$ that we are constructing
step by step. We were able already to determine the first $n_0$ values of a sequence $\beta$ such that $\alpha R \beta$ and now ask ourselves if $\alpha$ suffices to determine the next value. If so, we calculate this next value, call it $p$ and define $\gamma^{n_0}(\alpha) := p + 1$, if not, we define $\gamma^{n_0}(\alpha) := 0$. In both cases we define, for each $m \neq n_0$, $\gamma^m(\alpha) := 0$.

The argument that this procedure guarantees: $\gamma(0) = 0$, $\gamma$ belongs to $\text{Fun}$ and $\forall \alpha [\alpha R (\gamma|\alpha)]$ is similar to the argument given for the weaker axiom $\text{AC}_{1,0}$ and we do not spell it out.
2 Beginning the hierarchy

2.1 \( \mathbb{Q} \) belongs to \( \Sigma^0_2 \setminus \Pi^0_2 \)

We study Borel subsets of Baire space \( \mathcal{N} \), of the set \( \mathbb{R} \) of real numbers and of other Polish spaces. As we saw in Section 1, the positively Borel subsets of such spaces are obtained from their open and closed subsets by the repeated use of the operations of countable union and countable intersection. Let us first think of \( \mathbb{R} \).

One may ask: does there exist a countable union of closed subsets of \( \mathbb{R} \) different from every countable intersection of open subsets of \( \mathbb{R} \)?

(That is: an \( F_\sigma \) subset of \( \mathbb{R} \) that is not \( G_\delta \), or: a subset of \( \mathbb{R} \) belonging to \( \Sigma^0_2 \) but not to \( \Pi^0_2 \)).

Such sets exist and the set \( \mathbb{Q} \) of rational numbers is an example. Observe that \( \mathbb{Q} \) is a countable union of singletons and singletons are closed subsets of \( \mathbb{R} \). On the other hand, \( \mathbb{Q} \) is not a countable intersection of open subsets of \( \mathbb{R} \). For, suppose \( G_0, G_1, G_2, \ldots \) is a sequence of open subsets of \( \mathbb{R} \) such that \( \mathbb{Q} = \bigcap_{n \in \mathbb{N}} G_n \). Observe that each \( G_n \), containing \( \mathbb{Q} \), is dense in \( \mathbb{R} \). Let \( q_0, q_1, q_2, \ldots \) be an enumeration of \( \mathbb{Q} \) and determine \( \beta \) such that, for each \( n \), \( G_n = G^* \beta^n \). (See Section 1.11.5.)

We now consider \( \alpha \) in \( \mathbb{R} \) such that for each \( n \), \( L^* (\alpha(n)) < R^* (\alpha(n)) \) and either \( q_n < L^* (\alpha(n)) \) or \( R^* (\alpha(n)) < q_n \) and, for all \( n \) in \( \mathbb{N} \), for some \( p \), \( \alpha(n) \sqsupset \beta^n (p) \).

Then \( \alpha \) will belong to every \( G_n \) and be apart from every rational number. Contradiction, so \( \mathbb{Q} \) is not \( G_\delta \).

This argument is similar to the one used by R. Baire for his famous Category Theorem, and it is a constructive argument.

It is better to formulate its conclusion positively, that is without using negation, as follows:

Given any sequence \( G_0, G_1, G_2, \ldots \) of open subsets of \( \mathbb{R} \) such that \( \mathbb{Q} \subseteq \bigcap_{n \in \mathbb{N}} G_n \), one may construct a positively irrational number that belongs to each \( G_n \).

2.2 The dual question

Does there exist a countable intersection of open subsets of \( \mathbb{R} \) different from every countable union of closed subsets of \( \mathbb{R} \)?

There is no way to reduce this question to its dual, treated in Section 2.1. Let us see why.

We define \( \text{PosIrr} \) as the set of all real numbers \( \alpha \) that are apart from every rational number. \( \text{PosIrr} \) is obviously \( \Pi^0_2 \) and we guess that is not \( \Sigma^0_2 \). We try to prove this. Assume \( F_0, F_1, \ldots \) is a sequence of closed subsets of \( \mathbb{R} \) such that \( \text{PosIrr} = \bigcup_{n \in \mathbb{N}} F_n \).

Using \( \text{AC}_{0,1} \), determine \( \beta \) such that, for each \( n \), \( F_n = F^* \beta^n \) (see Section 1.11) and we define, for each \( n \), \( G_n := G^* (1 - \beta^n) \). Each \( G_n \) is an open subset of \( \mathbb{R} \) and \( F_n \) is the set of all real numbers that do not belong to \( G_n \). Is the set \( \mathbb{Q} \) of rational numbers a subset of \( \bigcap_{n \in \mathbb{N}} G_n \)? If \( \alpha \) in \( \mathbb{R} \) is rational, then \( \alpha \) does not belong to \( F_n \), so the
assumption that $\alpha$ does not belong to $G_n$ leads to a contradiction. Unfortunately, we cannot conclude from this that $\alpha$ does indeed belong to $G_n$, unless we are prepared to use Markov’s principle, see Section 1.8.1. But we are not, and we do not reach the conclusion that $Q$ is part of $\bigcap_{n \in \mathbb{N}} G_n$, and do not find a real number belonging to both $\text{PosIrr}$ and $\bigcap_{n \in \mathbb{N}} G_n$; we are unable to obtain the desired contradiction.

### 2.3 $\text{PosIrr}$ belongs to $\Pi_2^0 \setminus \Sigma_2^0$

We will now see how we are saved from our difficulty by the continuity principle. We define a subset $I$ of Baire space $\mathcal{N}$, as follows. For each $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $I$ if and only if for each $n$, $L^*(\alpha(n)) \leq L^*(\alpha(n + 1)) < R^*(\alpha(n + 1)) \leq R^*(\alpha(n))$ and $q_n < L^*(\alpha(n))$ or $R^*(\alpha(n)) < q_n$.

Observe that each member of $I$ is a real number that is really apart from every rational number, so $I$ is a subset of $\text{PosIrr}$. Moreover, each positively irrational real number really coincides with an element of $I$. And, most importantly, as a subset of Baire space $\mathcal{N}$, $I$ is a spread, in the sense defined in Section 1.8.2. We are going to use the fact that the continuity principle $\text{CP}$ generalizes to spreads, as we explained in Section 1.8.3.

Again assume $F_0, F_1, \ldots$ is a sequence of closed subsets of $\mathbb{R}$ such that $\text{PosIrr} = \bigcup_{n \in \mathbb{N}} F_n$. In particular then, for each $\alpha$ in $I$, there exists $n$ such that $\alpha$ belongs to $F_n$. Let $\alpha_0$ be some element of $I$. Applying the continuity principle $\text{CP}$, we find $m, n$ such that for each $\alpha$ in $I$, if $\bar{\alpha} = \overline{\alpha}_0 \alpha_m$, then $\alpha$ belongs to $F_n$. Now every positively irrational number $\alpha$ in the open interval $(L^*(\alpha(m)), R^*(\alpha(m)))$ will belong to $F_n$. As $F_n$ is closed, every number in $(L^*(\alpha(m)), R^*(\alpha(m)))$ will belong to $F_n$, in particular, some rational number will belong to $F_n$. Contradiction.

As in Section 2.1, we may give a positive turn to our conclusion:

> Given any sequence $F_0, F_1, F_2, \ldots$ of closed subsets of $\mathbb{R}$ such that $\text{PosIrr} \subseteq \bigcup_{n \in \mathbb{N}} F_n$, one may construct a rational number belonging to some $F_n$.

### 2.4 A less fruitful approach

Comparing the results of Sections 2.1 and 2.3, one might wonder if it is not possible to prove the following statement:

> For every sequence $F_0, F_1, \ldots$ of closed subsets of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} F_n \subseteq \text{PosIrr}$, there exists a positively irrational number that does not belong to any $F_n$.

We cannot prove this statement in full generality but are able to prove a special case. We assume that each set $F_n$ really coincides with a fan. This assumption is equivalent to the assumption that each set $F_n$ is both bounded
and located in \( \mathbb{R} \), that is, for every \( \alpha \in \mathbb{R} \) one may calculate the distance from \( \alpha \) to \( F_n \).

Now assume also that each set \( F_n \) consists of positively irrational numbers. Let \( q_0, q_1, \ldots \) be an enumeration of the rationals. Determine a fan \( G \) such that each element of \( F_0 \) really coincides with some element of \( G \). Observe that for each \( \alpha \) in \( G \) there exists \( n \) such that \( ||\alpha - q_0|| > \frac{1}{n} \). Using the continuity principle and the fan theorem, determine \( n_0 \) in \( \mathbb{N} \) such that, for each \( \alpha \) in \( G \), \( ||\alpha - q_0|| > \frac{1}{n_0} \). Now determine \( p \) such that \( q_0 < L^*(p) < R^*(p) < q_0 + \frac{1}{n_0} \) and define \( \alpha(0) := p \). By this choice of \( \alpha(0) \) we have ensured that \( \alpha \) will be really apart from \( q_0 \) and from every member of \( F_0 \). Continuing, and proceeding similarly, we define \( \alpha(1) \) in such a way that we are sure that \( \alpha \) will be really apart from \( q_1 \) and from every member of \( F_1 \). And so on.

We are unable to prove statement (\( * \)) in full generality, that is without assuming that each closed set \( F_n \) really coincides with a fan.

We did not obtain satisfying results in connection with statement:

\[
\text{For every sequence } G_0, G_1, \ldots \text{ of open subsets of } \mathbb{R} \text{ such that } \\
\bigcap_{n \in \mathbb{N}} G_n \subseteq \mathbb{Q}, \text{ there exists a rational number not belonging to any } \\
G_n.
\]

Let us see why. Assume \( G_0, G_1, \ldots \) is a sequence of open subsets of \( \mathbb{R} \) and \( \bigcap_{n \in \mathbb{N}} G_n \subseteq \mathbb{Q} \).

If we make the further assumption that \( \mathbb{Q} \) is contained in \( \bigcap_{n \in \mathbb{N}} G_n \), we are of course back in Section 2.1:

there will exist a positively irrational number in \( \bigcap_{n \in \mathbb{N}} G_n \). Contradiction.

So \( \mathbb{Q} \) is not contained in \( \bigcap_{n \in \mathbb{N}} G_n \).

This however does not enable us to indicate a rational number that will not belong to \( \bigcap_{n \in \mathbb{N}} G_n \).

The difficulty we encounter here in fact occurs already at the first level of the hierarchy. Consider the real open interval \( (0, 1) \).

We are unable to prove:

for every closed subset \( A \) of \( \mathbb{R} \): if \( A \subseteq (0, 1) \), there exist \( p \) in \( (0, 1) \), \( p \notin A \).

We can prove it only if we assume that \( A \) is located and therefore really coincides with a fan, and use the fan theorem.

The statements (\( * \)) and (\( ** \)) obviously are less good as a pattern for the intuitionistic Borel hierarchy theorem than the results proved in Sections 2.1 and 2.3.

### 2.5 Brouwer's own examples

We consider the examples given in Brouwer 1992.
2.5 Brouwer's own examples

2.5.1

We have to make some preparations.
Let $\text{Seq}_3$ be the set of all natural numbers $a$ such that, for each $i < \text{length}(a)$, $a(i) < 3$. We define a mapping $D$ from $\text{Seq}_3$ to $\mathbb{N}$, as follows. Choose a natural number $m$ such that $L^*(m) = 0$ and $R^*(m) = 1$ and define $D(\emptyset) := m$.

For each $a$ in $\text{Seq}_3$, $i < 3$, we define $D(a * (i))$ in such a way that $L^*(D(a * (i))) = L^*(D(a)) + i \cdot 3^{-n-1}$ and $R^*(D(a * (i))) = L^*(D(a) * (i)) + 3^{-n-1}$, where $n = \text{length}(a)$.

Let $S_3$ be the set of all $a$ in $\mathbb{N}$ such that, for each $n$, $a(n) < 3$. We define a function $\delta$ from $S_3$ to $\mathbb{R}$: for each $n$ in $\mathbb{N}$, for each $a$ in $S_3$, $(\delta(a))(n) := D(an)$. Observe that for every $a$ in $S_3$, $\delta(a)$ really coincides with $\sum a(n)3^{-n-1}$. The set of all $\delta(a)$, where $a$ belongs to $S_3$, coincides with the set of all real numbers $x$ in $[0,1]$ that have a ternary development, that is, we may decide, for every $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $x < m \cdot 3^{-n}$ or $m \cdot 3^{-n} \leq x$.

For each $n$, we let $K_n$ be the set of all $a$ in $S_3$ that assume the value 1 at most $n$ times, that is, for each $p$, $\# \{j \mid j < p |a(j) = 1\} < n + 1$.

It is easy to see that each $K_n$ is a fan, and that $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$.

For each $n$, we let $P_n$ be the set of all real numbers that coincide with some number $\delta(a)$ where $a$ belongs to $K_n$.

Each $P_n$ is a closed subset of $\mathbb{R}$ and $P_0 \subseteq P_1 \subseteq P_2 \subseteq \ldots$. We now let $A$ be $\bigcup_{n \in \mathbb{N}} P_n$.

$A$ is a countable union of an increasing sequence of closed subsets of $\mathbb{R}$. Brouwer called such sets outer limiting species. Not every countable union of closed sets is an outer limiting species; this is because the union of two closed sets is not necessarily closed, as we will see in Section 3.

(Among the outer limiting species Brouwer further distinguished the consolidated ones: there are countable unions of an increasing sequence of located closed sets).

The set $A$ does not coincide with any countable intersection of open subsets of $\mathbb{R}$. We paraphrase Brouwer's proof. Let $G_0, G_1, G_2, \ldots$ be a sequence of open subsets of $\mathbb{R}$, define $I := \bigcap_{n \in \mathbb{N}} G_n$ and suppose $A = I$. Using $\text{AC}_{0,1}$, determine $\beta$ such that for each $n$, $G_n = G^* \beta^n$. For every $n, m$, for every $a$ in $K_m$, there exists $p, q$ such that $(\delta(a))(q) \subseteq \beta^n(p)$. Brouwer now applies the fan theorem and calculates natural numbers $Q = Q(n,m)$ and $P = P(n,m)$ such that, for all $\alpha, \beta$ in $K_m$, there exists $q < Q$ and $p < P$ such that $(\delta(\alpha))(q) \subseteq \beta^n(p)$.

For every $\alpha$ in $S_3$, if $\alpha Q$ contains at most $m$ 1's, then there exists $\gamma$ in $K_m$ such that $\delta(\gamma)Q = \delta(\alpha Q)$ and $\delta \alpha$ will belong to $G_n$. Brouwer considers the sequence $Q_0 := Q(0,0), Q_1 := Q(1,1), \ldots$ and defines a special element $\alpha$ in $S_3$ as follows: $\alpha := 0(Q_0 + 1) * (1) * 0(Q_1 + 1) * (1) * \ldots$.

The real number $\delta(\alpha)$ will be apart from every member of $\bigcup_{n \in \mathbb{N}} P_n$, and still belong to $\bigcap_{n \in \mathbb{N}} G_n$.

Clearly, Brouwer proves a more constructive statement than the conclusion he is striving for, viz. that $A$ does not coincide with $I$. We did the same in Section 2.1. More important, Brouwer is using the fan theorem, and we used the continuity principle.
only. Brouwer might have given a proof of the correctness of his own example, avoiding the fan theorem, as follows. One may construct \( \beta \) belonging to \( \bigcap_{n \in \mathbb{N}} G_n \) but apart from every member of \( \bigcup_{n \in \mathbb{N}} F_n \), step by step. Define \( \alpha_0 := \emptyset \). \( \alpha_0 \) belongs to \( K_0 \), therefore \( \delta|\alpha_0 \) belongs to \( G_0 = G^* \beta^0 \), so determine \( M_0, p_0 \) such that \( (\delta|\alpha_0)(M_0) \sqsubseteq^* \beta^0(p_0) \).

Define \( \alpha_1 := \emptyset(M_0 + 1) \times (1) \times \emptyset \) and observe: \( \alpha_1 \in K_1 \), therefore \( \delta|\alpha_1 \in G_1 \), so determine \( M_1, p_1 \) such that \( (\delta|\alpha_1)(M_0 + 2 + M_1) \sqsubseteq^* \beta^1(p_1) \). Now define \( \alpha_2 := \emptyset(M_0 + 1) \times (1) \times \emptyset(M_1 + 1) \times (1) \times \emptyset \) and continue.

Let \( \alpha \) in \( S_3 \) be the limit of the sequence \( \alpha_0, \alpha_1, \alpha_2, \ldots \), and define \( \beta := \delta|\alpha \).

Brouwer observes that his result generalizes:

suppose \( F_0, F_1, \ldots \) is an increasing sequence of located and closed subsets of \( \mathbb{R} \) such that each \( F_n \) is nowhere dense, and \( \bigcup_{n \in \mathbb{N}} F_n \) is dense.

Then, given any sequence \( G_0, G_1, \ldots \) of open sets such that \( \bigcup_{n \in \mathbb{N}} F_n \subseteq \bigcap_{n \in \mathbb{N}} G_n \), there will exist \( \beta \) in \( \bigcap_{n \in \mathbb{N}} G_n \) apart from every member of \( \bigcup_{n \in \mathbb{N}} F_n \).

This generalization is correct (and our example \( Q \) may be seen to be a special case) but one does not need the condition that the sequence \( F_0, F_1, \ldots \) is increasing, nor the fact that each \( F_n \) is located.

2.5.2

Brouwer also gives an example of a countable intersection of open sets (an *inner limiting species*, in his terminology) that is no outer limiting species, and actually different from every countable union of closed sets.

Let \( L \) be the set of all \( \alpha \) in \( S_3 \) that assume the value 1 infinitely many times and let \( I \) be the set of all \( \beta \) in \( \mathbb{R} \) coinciding with some \( \delta|\alpha \), where \( \alpha \) belongs to \( L \).

Finally let \( B \) be the set of all \( \beta \) in \( \mathcal{N} \) such that for each \( n \), \( \alpha(n + 1) \sqsubseteq^* \alpha(n) \), and for each \( n \) there exists \( b \) such that \( D(b) = \alpha(n + 1) \) and \( b \) assumes the value 1 at least \( n \) times, that is, \( \#\{j|j < \text{length}(b)|b_j = 1\} \geq n \). It is easy to see that \( B \) is a subset of \( \mathbb{R} \) and that every member of \( I \) really coincides with some element of \( B \).

Moreover, as a subset of \( \mathcal{N} \), \( B \) is a spread.

Assume now that \( I \subseteq \bigcup_{n \in \mathbb{N}} F_n \), where each \( F_n \) is a closed subset of \( \mathbb{R} \). Then, given any \( \alpha \) in \( B \), one may calculate \( n \) such that \( \alpha \) belongs to \( F_n \). Let \( \alpha_0 \) be some element of \( B \).

Using the continuity principle we find \( m, n \) such that, for every \( \alpha \) in \( B \), if \( \bar{\alpha}m = \bar{\alpha}_0m \), then \( \alpha \) belongs to \( F_n \). As \( F_n \) is closed, and the set \( \{ \alpha|\alpha \in B|\bar{\alpha}m = \bar{\alpha}_0m \} \) is dense in the open interval \( \left( L^*(\alpha_0(m - 1)), R^*(\alpha_0(m - 1)) \right) \), we conclude that the interval \( \left( L^*(\alpha_0(m - 1)), R^*(\alpha_0(m - 1)) \right) \) is contained in \( F_n \). One easily finds \( \beta \) in \( F_n \) that does not belong to \( I \).

Brouwer’s application of the continuity principle in this proof is very similar to our use of it in Section 2.3.
2.6 Analytical and strictly analytical sets

Both in Section 2.3 and in Section 2.5.2 we did apply the continuity principle. We used the fact that both the set PosIrr of positively irrational numbers and the set \( I \) of all real numbers between 0 and 1 that admit of a ternary development in which the number 1 occurs infinitely many times really coincide with a spread.

We now introduce a more general but related notion.

A subset \( X \) of \( \mathbb{R} \) will be called strictly analytical if and only if there exists \( \gamma \) in \( \text{Fun} \) such that \( \gamma(0) = 0 \) and for each \( a, \alpha \) belongs to \( X \) and every element of \( X \) really coincides with some \( \gamma|\alpha \), where \( \alpha \) belongs to \( \mathcal{N} \).

(\( \text{Fun} \) is the set of codes of continuous functions from \( \mathcal{N} \) to \( \mathbb{N} \) as introduced in Section 1.14. If \( \gamma \) belongs to \( \text{Fun} \) and \( \gamma(0) = 0 \), then \( \gamma \) codes a function from \( \mathcal{N} \) to \( \mathcal{N} \), see Section 1.14.)

This notion extends to subsets of Baire space \( \mathcal{N} \) itself. A subset \( X \) of \( \mathcal{N} \) is strictly analytical if and only if there exists \( \gamma \) in \( \text{Fun} \) such that \( \gamma(0) = 0 \) and \( X \) coincides with \( \text{Ran}(\gamma) \).

The notion also extends in the obvious way to subsets of other Polish spaces.

A subset \( X \) of \( \mathbb{R} \) is called analytical if there exists a closed subset \( F \) of the product space \( \mathbb{R} \times \mathcal{N} \) such that \( X \) is the projection of \( F \), that is \( X = \{\alpha \in \mathbb{R} | \exists \beta [F(\alpha, \beta)]\} \).

Every strictly analytical set is analytical, but the converse is false, even if we restrict ourselves to inhabited analytical sets. The difference between strictly analytical and analytical sets is of the same kind as the difference between spreads and closed sets. Observe that every subset of \( \mathbb{R} \) that really coincides with a spread is strictly analytical. That is because for every subspread \( B \) of \( \mathcal{N} \) there exists a retraction \( r_B \) from \( \mathcal{N} \) onto \( B \), as we saw in Section 1.8.2.

2.7 A third proof

We want to establish the second level of the Borel hierarchy a third time. In Section 4, we intend to generalize this third proof to higher levels of the hierarchy. We first give another characterization of the classes \( \Pi^0_2(\mathcal{N}) \) and \( \Sigma^0_2(\mathcal{N}) \).

2.7.1

Given subsets \( X \) and \( Y \) of Baire space \( \mathcal{N} \) we define:

\( X \preceq Y \) ("\( X \) reduces to \( Y \)") if and only if there exists \( \gamma \) in \( \text{Fun} \) such that \( \gamma(0) = 0 \) and for all \( a, \alpha \) belongs to \( X \) if and only if \( \gamma|\alpha \) belongs to \( Y \).

So \( X \) reduces to \( Y \) if there exists an effective method to translate every question: "does \( \alpha \) belong to \( X \)" into a question "does \( \beta \) belong to \( Y \)?

This notion is called “Wadge-reducibility” in classical descriptive set theory. Its analogue in recursion theory is called “many-one-reducibility” or \( m \)-reducibility. We use the unadorned expression “reducible” as no other notion of reducibility figures in this paper.
2.7.2
Given a class \( K \) of subsets of Baire space \( \mathcal{N} \) and an element \( X \) of \( K \) we say that \( X \) is a complete element of \( K \) ("\( X \) is \( K \)-complete") if and only if for every subset \( Y \) of \( \mathcal{N} \), \( Y \) belongs to \( K \) if and only if \( Y \) reduces to \( X \).

2.7.3
We again consider the subsets \( A_2 \) and \( E_2 \) of \( \mathcal{N} \). We defined them in Section 1.8.3, as follows: \( A_2 := \{ \alpha | \forall m \exists n [\alpha^m(n) \neq 0] \} \) and \( E_2 := \{ \alpha | \exists m \forall n [\alpha^m(n) = 0] \} \). We claim that \( A_2, E_2 \) are \( \Pi_2^0 \)-complete, \( \Sigma_2^0 \)-complete, respectively, and we leave the simple proof of this fact to the reader.
Recall that \( (A_2, E_2) \) is a complementary pair of subsets of \( \mathcal{N} \) in the sense of Section 1.10, that is, every element of \( A_2 \) is apart from every element of \( E_2 \).

2.7.4 Theorem:
Every function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( E_2 \) into \( A_2 \) also maps some element of \( A_2 \) into \( A_2 \).

Proof: Let \( g \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( E_2 \) into \( A_2 \).
(We always assume that a (continuous) function from \( \mathcal{N} \) to \( \mathcal{N} \) is given to us by means of some code for that continuous function, in the sense of Section 1.14. We may as well say: \( g \) belongs to \( \text{Fun} \) and \( g(0) = 0 \).)
Define \( \alpha_0 := 0 \). \( \alpha_0 \) belongs to \( E_2 \), so \( g|\alpha_0 \) belongs to \( A_2 \) and we determine \( n_0 \) such that \( (g|\alpha_0)^0(n_0) \neq 0 \). We also determine \( m_0 \) such that, for all \( \alpha \), if \( \tilde{\alpha} m_0 = \tilde{\alpha}_0 m_0 \) then \( (g|\alpha)^0(m_0) = (g|\alpha_0)^0(m_0) \).
Define \( \alpha_1 := \tilde{\alpha} m_0 \ast (1) \ast 0 \). Then \( (\alpha_1)^0(m_0) = 1 \) and \( \alpha_1 \) belongs to \( E_2 \), so \( g|\alpha_1 \) belongs to \( A_2 \) and we determine \( n_1 \) such that \( (g|\alpha_1)^1(n_1) \neq 0 \). Determine \( m_1 \) such that \( m_1 > m_0 \) and for all \( \alpha \), if \( \tilde{\alpha} m_1 = \tilde{\alpha}_1 m_1 \), then \( (g|\alpha)^1(n_1) = (g|\alpha_1)^1(n_1) \).
Define \( \alpha_2 := \tilde{\alpha}_1 m_1 \ast (1) \ast 0 \).
Continue in this way.
Let \( \alpha \) be such that, for each \( k \), \( \tilde{\alpha} m_k = \tilde{\alpha}_k m_k \).
(\( \alpha \) is the limit of the sequence \( \alpha_0, \alpha_1, \ldots \)).
Then, for every \( k \), \( \alpha^k(m_k) = 1 \) and \( (g|\alpha)^k(n_k) \neq 0 \), so both \( \alpha \) and \( g|\alpha \) belong to \( A_2 \).
\( \Box \)

2.7.5
The set \( A_2 \) is strictly analytical. We define a function \( f_2 \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that \( A_2 \) coincides with the range of \( f_2 \). Let \( a \) belong to \( \mathcal{N} \). We define \( (f_2|a)(0) := 0 \) and for all \( m, n, (f_2|a)^m(n) := \alpha^m(n + 2) \) if \( n \neq a^m(0) \), and \( (f_2|a)^m(\alpha^m(0)) := \alpha^m(1) + 1 \).

2.7.6 Theorem:
Every function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( A_2 \) into \( E_2 \) also maps some element of \( E_2 \) into \( E_2 \).
Proof: Let $g$ be a function from $\mathcal{N}$ to $\mathcal{N}$ that maps $A_2$ into $E_2$. We now use the function $f_2$, defined in Section 2.7.5.

For all $\alpha$, $f_2|\alpha$ belongs to $A_2$, so $g \circ f|\alpha$ belongs to $E_2$, and there exists $n$ such that $(g \circ f_2|\alpha)^n = 0$.

We use the continuity principle and determine $m, n$ such that for all $\alpha$, if $\alpha m = \bar{y}_m$ then $(g \circ f_2|\alpha)^n = 0$. Now suppose $\alpha$ satisfies the following conditions: $\alpha(0) = 0$ for all $j < m : \alpha^j = (g \circ f_2|\alpha)|0$ and $\alpha^m = 0$.

Observe that for every $q$ there exist $\beta$ such that $\bar{\beta} m = \bar{y}_m$ and $(f_2|\beta) q = \bar{\alpha} q$, so for every $q$ there exists $\varepsilon$ such that $\varepsilon q = \bar{\alpha} q$ and $(g|\varepsilon)^n = 0$. Therefore both $\alpha^n = 0$ and $(g|\alpha)^n = 0$ that is, both $\alpha$ and $g|\alpha$ belong to $E_2$. 

We conclude this Section by showing how to extend Theorems 2.7.4 and 2.7.6 to other Polish spaces.

2.7.7 Corollary:

Let $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a (pseudo-)metric such that the corresponding Polish space $X(d)$ is perfect; that is, for every $m, n$ we may determine $p$ such that $0 < d(m, p) < \frac{1}{n}$. There exists a pair $A, E$ of subsets of $X(d)$ such that:

$A$ is $\Pi_0^0$, $E$ is $\Sigma_0^0$, $(A, E)$ is a complementary pair and, for every $\Sigma_0^0$ subset $C$ of $X(d)$, if $A \subseteq C$, there exists $\alpha$ in $C \cap E$, and for every $\Pi_2^1$ subset $D$ of $X(d)$, if $E \subseteq D$, there exists $\alpha$ in $D \cap A$.

Proof: We may conclude from theorems 2.7.4 and 2.7.6 that the corollary holds if $X(d)$ is a Baire space $\mathcal{N}$. We take $A = A_2$ and $E = E_2$. Suppose now that $X(d)$ is some perfect Polish space.

One may construct a strongly injective (continuous) mapping $i$ from $\mathcal{N}$ into $X(d)$. (So for all $\alpha, \beta$, if $\alpha \neq \beta$, then $i(\alpha) \neq i(\beta)$).

It is important now that for every open subset $C$ of $\mathcal{N}$ there exists an open subset $D$ of $X(d)$ such that $i(C) = i(\mathcal{N}) \cap D$. As $i$ is strongly injective it follows that for every complementary pair of subsets $C_0, C_1$ of $\mathcal{N}$, consisting of an open and a closed subset of $\mathcal{N}$, there exists a complementary pair $(D_0, D_1)$ of subsets of $X(d)$ consisting of open set $D_0$ and a closed set $D_1$ such that $i(C_0) = i(\mathcal{N}) \cap D_0$ and $i(C_1) = i(\mathcal{N}) \cap D_1$.

It is not difficult to see that there exist also a $\Pi_0^1$ subset $A$ of $X(d)$ and a $\Sigma_0^1$ subset $E$ of $X(d)$ such that $(A, E)$ is a complementary pair and $i(A_2) = i(\mathcal{N}) \cap A$ and $i(E_2) = i(\mathcal{N}) \cap E$.

Suppose now that $C$ is a $\Sigma_0^0$ subset of $X(d)$ and $A \subseteq C$ then $i^{-1}(A) \subseteq i^{-1}(C)$ and $i^{-1}(C)$ is $\Sigma_0^0$, and $A_2 \subseteq i^{-1}(C)$, so there exists $\alpha$ in $i^{-1}(C) \cap E_2$, so $i(\alpha)$ will belong to $C \cap E$.

In a similar way one proves if $D$ is a $\Pi_0^1$ subset of $X(d)$ and $E \subseteq D$, then there exists $\alpha$ in $D \cap A$. 

\[ \square \]
3 The many kinds of countable unions of closed sets

3.1 Some subsets of the set $\mathbb{R}$ of real numbers

3.1.1 The union of the two closed real segments $[0,1]$ and $[1,2]$ is not a closed subset of $\mathbb{R}$

Suppose $[0,1] \cup [1,2]$ is a closed subset of $\mathbb{R}$. It is easy to see that then $[0,1] \cup [1,2]$ really coincides with $[0,2]$. So for every $a$ from $[0,2]$ we may decide:
a belongs to $[0,1]$ or a belongs to $[1,2]$. There exist however numbers $a$ in $[0,2]$ for which we are unable to take this decision, numbers that, as one sometimes says, "waver around 1." We now construct such a number.

Let $p : \mathbb{N} \to \{0,1,\ldots,9\}$ be the decimal expansion of $\pi$. Define $a$ in $[0,2]$ as follows. For each $n$, if there is no $k < n$ such that for every $i < 99$, $p(k + i) = 9$, then $a(n) := r_1 - \frac{1}{n}, 1 + \frac{1}{n}$, and if $k_0 < n$ is the least $k$ such that for every $i < 99$, $p(k_0 + i) = 9$ then, if $k_0$ is even, $a(n) := r_1 - \frac{1}{n}, 1 - \frac{1}{n}$ and if $k_0$ is odd, $a(n) := r_1 + \frac{1}{n}, 1 + \frac{1}{n}$. We have no proof that $a$ belongs to $[0,1]$ and we have no proof that $a$ belongs to $[1,2]$.

Using the continuity principle we obtain a contradiction from the assumption that $[0,1] \cup [1,2]$ coincides with $[0,2]$, as follows. Consider the spread consisting of all regular real numbers $a$ such that $a(0) = [0,2]$. (Regular real numbers have been introduced in Section 1.11.4.) This spread is a fan that really coincides with the closed segment $[0,2]$.

The sequence $r_0, 2r_1, r_1, \frac{1}{2}, r_2, r_1, \frac{1}{3}, \ldots$ belongs to this fan. Applying the continuity principle we find $m$ such that either, for every regular $a$, if $a(m) = r_1 - \frac{1}{2m}, 1 + \frac{1}{2m}$, then $a$ belongs to $[0,1]$, or, for every regular real $a$, if $a(m) = r_1 - \frac{1}{2m}, 1 + \frac{1}{2m}$, then $a$ belongs to $[1,2]$. This is an obvious contradiction.

It is not difficult to see now that the set $[0,1] \cup [1,2]$ is not a $G_\delta$-set either. For if $G$ is an open set containing $[0,1] \cup [1,2]$, also $[0,2]$ forms part of $G$. Consequently, if $[0,1] \cup [1,2]$ forms part of a $G_\delta$ set $X$, also $[0,2]$ forms part of $X$, and $[0,1] \cup [1,2]$ does not coincide with $X$.

3.1.2 The set $A := \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not a closed subset of $\mathbb{R}$

Observe that we may decide, for each rational interval, if it contains an element of $A$ or not. Therefore, the set of all real numbers $a$ such that for each $n$, the rational interval coded by $a(n)$ contains at least one member of $A$, is a closed subset of $\mathbb{R}$. We call this set the closure of $A$, notation $\overline{A}$. The set of all regular real numbers belonging to $\overline{A}$ is a spread. Suppose that $A$ is a closed subset of $\mathbb{R}$, then every member of $\overline{A}$ coincides with a member of $A$. Using the continuity principle we determine $m,n$ such that either every element of $[0, \frac{1}{2n}] \cap \overline{A}$ coincides with $\frac{1}{2n}$, or every element of $[0, \frac{1}{2n}] \cap \overline{A}$ coincides with 0. Both alternatives are false. Observe that, for every real number $a$, if $a$ is really apart from 0, then $a$ belongs to $A$ if and only if $a$ belongs to $\overline{A}$. As, for every real $a$, $\neg(\neg(\alpha \# 0 \lor a \approx 0)$ we find that $\overline{A}$ coincides with the set of all real numbers $\alpha$ such that $\neg(\alpha \in A)$, that is, $\overline{A}$ coincides with the double complement $A^{cc}$ of $A$. 


Now let, for every subset \( X \) of \( \mathbb{R} \), \( \text{Perhaps}(X) \) be the set of all real numbers \( y \) such that there exists \( i \) in \( 1 \) with the property: if \( y \neq x \), then \( y \) belongs to \( X \). It is easily seen that \( \text{Perhaps}(X) \) forms part of \( X^c \) but the converse is not generally true. There are subsets \( X \) of \( \mathbb{R} \) such that the gap between \( \text{Perhaps}(X) \) and \( X^c \) is very wide, for instance the set \( \text{Rat} \) of all real numbers coinciding with a rational. Starting from \( \text{Rat} \), iterating the operation \( \text{Perhaps} \), and taking countable unions at limit stages, one obtains uncountably many subsets \( Y \) such that \( \text{Rat} \subseteq Y \subseteq \text{Rat}^c \). (see Veldman 1999).

Observe that \( A \) behaves tamely in this respect. \( \overline{A} \) coincides with \( \text{Perhaps}(A) \), and \( \text{Perhaps}(\overline{A}) \) coincides with \( A \), so \( A \) has “perhapsity 1”, in the sense of Veldman 1999.

### 3.1.3 The set \( A := \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \} \) does not coincide with a finite union of closed sets

We have seen that \( A \) is not closed and proceed by induction. Observe that no infinite and decidable subset of \( A \) is closed. Let \( n \) be a positive integer and suppose we have verified that no decidable infinite subset of \( A \) coincides with a union of \( n \) closed sets.

Now assume \( C_0, C_1, \ldots, C_n \) are closed sets such that \( A = C_0 \cup C_1 \cup \cdots \cup C_n \). Using \( AC_{0,0} \) we determine \( a \) such that for each \( p, \) the number \( \frac{1}{p+1} \) belongs to \( C_{a(p)} \).

We claim the following: for every \( m \) there exists \( q > m \) such that \( a(q) \neq 0 \).

Let us prove this claim.

Assume that, for some natural number \( m \), \( a(m + 1) = 0 \). We define a real number \( \beta \) as follows.

\[
\beta(0) := \overline{0}, \frac{1}{m+2}, \quad \text{and for every } q, \text{if there exists } j \leq q+1 \text{ such that } a(m+j+1) = 0, \\
\beta(q+1) := \overline{R*(\beta(q)), R*(\beta(q))} \quad \text{and if, for all } j \leq q+1, a(m+j+1) \neq 0 \\
\beta(q+1) := \overline{0}, \frac{1}{m+q+1}.
\]

Observe that the real number \( \beta \) belongs to \( C_0 \), and also to \( A \). So either \( \beta \) coincides with 0 or, for some \( p, \beta \) coincides with \( \frac{1}{p} \).

If however \( \beta \) coincides with 0, then each number \( \frac{1}{p+1} \), where \( q > m \), belongs to \( C_0 \). So \( A \) coincides with \( C_0 \cup \{1, \frac{1}{2}, \ldots, \frac{1}{m+1} \} \) and is a closed set. But, as we saw in the previous Section, \( A \) is not a closed subset of \( \mathbb{R} \). We conclude that for some \( p, \beta \) coincides with \( \frac{1}{p} \), so there exists \( q > m \) such that \( a(q) \neq 0 \).

This ends the proof of our claim.

We now construct a strictly increasing \( \gamma \) such that, for each \( p, a(\gamma(p)) \neq 0 \) and consider \( B := \{ \frac{1}{\gamma(p)+1} : p \in \mathbb{N} \} \cup \{0\} \).

Observe that \( B \) coincides with \( (B \cap C_1) \cup (B \cap C_2) \cup \cdots \cup (B \cap C_n) \). For each \( i, 1 \leq i \leq n \), we let \( D_i \) be a decidable subset of \( \mathbb{Q} \times \mathbb{Q} \) such that, for every \( \alpha \) in \( \mathbb{R} \), \( \alpha \) belongs to \( C_i \) if and only if, for each \( n, \) the pair \( (L*(\alpha(n)), R*(\alpha(n))) \) belongs to \( D_i \) (cf. Section 1.11.6). Observe also that each set \( B \cap C_i \) contains every real number \( \alpha \) such that, for each \( n, \) the rational interval \( (L*(\alpha(n)), R*(\alpha(n))) \) both contains a member of \( B \) and belongs to \( D_i \). For such a number belongs itself to \( C_i \) and therefore
to \( A \), and we may decide if it belongs to \( B \). The assumption that it does not belong to \( B \) easily leads to a contradiction. Observe also that we may decide, for every rational interval, if it belongs to \( D_i \) and contains a member of \( B \). Therefore each set \( B \cap C_i \) is a closed subset of \( \mathbb{R} \), and this contradicts the induction hypothesis. We conclude that \( A \) is not a union of \( n \) closed sets and that the same is true for every infinite decidable subset of \( A \).

### 3.1.4 The set \( A := \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \} \) is not a countable intersection of open subsets of \( \mathbb{R} \)

It suffices to observe the following: if \( G \) is an open set containing \( A \), there exists \( n \) such that for every \( p > n \) the number \( \frac{1}{p} \) belongs to \( G \), and therefore \( G \) contains \( \overline{A} \).

So, if \( G_0, G_1, \ldots \) is a sequence of open sets and \( A \subseteq \bigcap_{n \in \mathbb{N}} G_n \), also \( \overline{A} \subseteq \bigcap_{n \in \mathbb{N}} G_n \), and, as \( \overline{A} \) does not coincide with \( A \), neither does \( \bigcap_{n \in \mathbb{N}} G_n \).

Like the set \([0,1] \cup [1,2] \), the set \( A \), although in classical eyes a closed subset of \( \mathbb{R} \), is intuitionistically an example of an \( F_\sigma \)-set different from every \( G_\delta \)-set.

### 3.1.5 The open real interval \((0,1)\) differs from every finite union of closed subsets of \( \mathbb{R} \)

The proof of this fact is similar to the proof of the fact that \( A := \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \} \) is not a finite union of closed sets. \((0,1)\) itself is not closed, and we proceed by induction. Suppose that, for some positive integer \( n \) we have verified that \((0,1)\) is not a union of \( n \) closed sets. Assume now that \( C_0, C_1, \ldots, C_n \) are closed sets such that \((0,1) = C_0 \cup C_1 \cup \cdots \cup C_n \).

Applying an axiom of countable choice, we determine \( \alpha \) such that for each \( p \), the number \( \frac{1}{\alpha(p)+1} \) belongs to \( C_{\alpha(p)} \). Using the induction hypothesis we prove that there are infinitely many \( p \) such that \( \alpha(p) = 0 \) and we construct a strictly increasing \( \gamma \) such that, for each \( p \), \( \alpha(\gamma(p)) = 0 \). Each number \( \frac{1}{\gamma(p)+1} \) belongs to \( C_0 \), and \( C_0 \) is closed, so \( 0 = \lim_{p \to \infty} \frac{1}{\gamma(p)+1} \) belongs to \( C_0 \) and therefore to \((0,1)\).

Contradiction.

### 3.1.6

We have seen, in Section 3.1, that there exist non-closed subsets of \( \mathbb{R} \) that coincide with a union of two closed sets. Do there also exist unions of three closed sets that do not coincide with any union of two closed sets? (Observe that the set \([0,1] \cup [1,2] \cup [2,3] \) is the union of \([0,1] \cup [2,3] \) and \([1,2] \) and each one of the latter two sets is closed).

#### 3.1.6.1

We again consider the set \( A := \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \} \).

Suppose \( B, C \) are decidable, infinite and mutually disjoint subsets of \( A \) such that \( A = B \cup C \).

We claim that the set \( \overline{B} \cup \overline{C} \) is not closed. For suppose it is. Then \( \overline{B} \cup \overline{C} \) will coincide with \( \overline{A} \), so every member of \( \overline{A} \) will belong to either \( \overline{B} \) or \( \overline{C} \), and using the continuity
principle we find \( m \) such that either \([0, \frac{1}{2m}] \cap \overline{A}\) forms part of \( \overline{B} \), or \([0, \frac{1}{2m}] \cap \overline{A}\) forms part of \( \overline{C} \), and both alternatives are false.

We claim that for every positive \( n \), if \( B_0, B_1, \ldots, B_n \) are decidable, infinite and mutually disjoint subsets of \( A \) and \( A = B_0 \cup B_1 \cup \cdots \cup B_n \), then \( \overline{B_0} \cup \overline{B_1} \cup \cdots \cup \overline{B_n} \) is a union of \( n + 1 \) closed subsets of \( \mathbb{R} \) not coinciding with any union of \( n \) closed subsets of \( \mathbb{R} \). We verify this claim as follows.

Suppose that \( C_0, C_1, \ldots, C_{n-1} \) are closed sets such that \( \overline{B_0} \cup \overline{B_1} \cup \cdots \cup \overline{B_n} \) coincides with \( \overline{C_0} \cup \overline{C_1} \cup \cdots \cup \overline{C_{n-1}} \). As each set \( B_i \) coincides with a spread, we find, using the continuity principle, for each \( i < n + 1 \) numbers \( p_i \), \( \overline{B_i} \) forms part of \( C_p \). Without loss of generality we assume \( p_0 = p_1 = 0 \) and define \( m = \max(m_0, m_1) \). So \([0, \frac{1}{2m}] \cap (\overline{B_0} \cup \overline{B_1})\) forms part of \( C_0 \). As \( C_0 \) is closed, also \([0, \frac{1}{2m}] \cap (\overline{B_0} \cup \overline{B_1})\) forms part of \( C_0 \), and therefore also of \( \overline{B_0} \cup \overline{B_1} \cup \cdots \cup \overline{B_n} \).

Reasoning now as in the first paragraph of this Section, we find \( p, i \) such that \([0, \frac{1}{2p}] \cap (\overline{B_0} \cup \overline{B_1})\) forms part of \( B_t \) and thus we obtain a contradiction.

**3.1.6.2** Suppose that \( B, C \) are decidable and infinite subsets of \( A \). The continuity principle implies: if \( B \subseteq C \) then there exists \( m \) such that \([0, \frac{1}{2m}] \cap \overline{B} \subseteq \overline{C} \), so there exists \( n \) such that for every \( p > n \), if \( \frac{1}{p} \) belongs to \( B \), then \( \frac{1}{p} \) belongs to \( C \).

**3.1.6.3** Suppose that \( B_0, B_1, \ldots \) is an infinite sequence of mutually disjoint infinite decidable subsets of \( A \). One may verify that the set \( \bigcup_{n \in \mathbb{N}} \overline{B_n} \) is a countable union of closed sets, congruent with \( A \), but not coinciding with either \( A \) or \( \overline{A} \) or with any finite union of closed subsets of \( \mathbb{R} \).

**3.1.6.4** Given any infinite and decidable subset \( B \) of \( A \setminus \{0\} := \{1, \frac{1}{2}, \ldots \} \) we determine a strictly increasing \( \gamma \) such that \( B := \{\frac{1}{\gamma(n)+1} | n \in \mathbb{N}\} \) and define \( B' := \{\frac{1}{\gamma(2n)+1} | n \in \mathbb{N}\} \).

We now construct a function associating to each \( a \) from the set Seq of all (code numbers of) finite sequences with values 0,1 a subset \( A_a \) of \( A \) as follows:

\[
A(1) := A \setminus \{0\} \quad \text{and, for each } a \in \text{Seq, } A_{a+(0)} = (A_a)^' \quad \text{and} \quad A_{a+(1)} = (A_a)^'' .
\]

Suppose \( B, C \) are subsets of \( A \). We say that \( A \) is an almost-subset of \( C \) if all but finitely many members of \( B \) belong to \( C \).

Suppose \( B_0, B_1, \ldots \) is a sequence of infinite, decidable subsets of \( A \), and for each \( n \), \( B_{n+1} \) is an almost-subset of \( B_n \).

We define a sequence \( b_0, b_1, \ldots \) of natural numbers:

for each \( n \), \( b_n \) is the least element of \( B_n \) that belongs to every \( B_k \), \( k < n \). We call the set \{\( b_k | k \in \mathbb{N} \)\} the almost-limit of the sequence \( B_0, B_1, \ldots \), notation \( \text{AL}(B_n) \).

Observe that \( \text{AL}(B_n) \) is an almost-subset of each set \( B_n \).

We now define a map associating to each \( a \) in Cantor space \( C \) a subset \( A_a \) of \( A \) as follows: \( A_a := \text{AL}(A_{a(0)}, \ldots, a(n-1)) \). Observe that for all \( a, \beta \) in \( C \), if \( a \not= \beta \), then \( A_a \cap A_\beta \) is a finite set. On the other hand each \( A_a \) is infinite and \( A \subseteq \bigcup_{a \in C} A_a \subseteq \overline{A} \).
Observe also that for all \( \alpha, \beta \) in \( \mathcal{C} \), if \( \alpha \neq \beta \), then \( \overline{A_\alpha} \cup \overline{A_\beta} \neq \overline{A_\alpha \setminus A_\beta} \) whereas for each \( \alpha \), \( \overline{A_\alpha} \cup \overline{A_\alpha \setminus A_\alpha} \subseteq \overline{A} \).

We may also associate to every stump \( \sigma \) and every infinite decidable subset \( B \) of \( A \setminus \{0\} \) an infinite subset \( B[\sigma] \) of \( B \), as follows:

(i) \( B[0] := B \)

(ii) for each stump \( \sigma \neq 0 \) and each \( B \) we first build a sequence \( B^{(0)}, B^{(1)}, \ldots \) as follows: \( B^{(0)} := B \) and, for each \( n \), \( B^{(n+1)} := B^{(n)}[\sigma^n] \).

We then define: \( B[\sigma] := (\bigcup_{n \in \mathbb{N}} (B^{(n)})^\prime) \).

One may verify that for all stumps \( \sigma, \tau \), if \( \sigma < \tau \), then \( A[\tau] \) is an almost-subset of \( A[\sigma] \), but \( A[\sigma] \setminus A[\tau] \) is infinite, so \( A[\sigma] \cup A[\tau] \) is different from \( A[\tau] \cup A[\tau] \). On the other hand, each set \( A[\sigma] \cup A[\tau] \) contains \( A \) and is part of \( \overline{A} \).

### 3.2 Some subsets of Baire space \( \mathcal{N} \)

#### 3.2.1

Given subsets \( P, Q \) of \( \mathcal{N} \) we let \( D(P, Q) \) be the set of all \( \alpha \) such that either \( P(\alpha^0) \) or \( Q(\alpha^1) \). We call \( D(P, Q) \) the disjunction of \( P \) and \( Q \). A subset \( X \) of \( \mathcal{N} \) reduces to \( D(P, Q) \) if and only if there exist subsets \( X_0, X_1 \) of \( \mathcal{N} \) such that \( X = X_0 \cup X_1 \) and \( X_0, X_1 \) reduce to \( P, Q \), respectively. If there exists \( \alpha \) such that \( \neg Q(\alpha) \), then \( P \) reduces to \( D(P, Q) \).

#### 3.2.2

Given a subset \( P \) of \( \mathcal{N} \) and a positive integer \( n \), we let \( D^n(P) \) be the set of all \( \alpha \) such that \( \exists k < n[P(\alpha^k)] \).

A subset \( X \) of \( \mathcal{N} \) reduces to \( D^n(P) \) if and only if there exist subsets \( X_0, X_1, \ldots, X_{n-1} \) of \( \mathcal{N} \), each one of them reducing to \( P \), such that \( X = \bigcup_{k<n} X_k \).

Consider \( A_1 := \{ \alpha | \forall n[\alpha(n) = 0] \} = \{ 0 \} \).

\( A_1 \) is a complete element of the class of closed subsets of \( \mathcal{N} \), that is, a subset \( X \) of \( \mathcal{N} \) is closed if and only if \( X \) reduces to \( A_1 \).

It is not difficult to see now that \( D^n(A_1) \) is a complete element of the class of unions of \( n \) closed sets.

#### 3.2.3 Theorem:

(i) \( D^2(A_1) \) is not closed, and

(ii) for each \( n > 1 \), \( D^{n+1}(A_1) \) is not a union of \( n \) closed sets.

**Proof:**
3.2 Some subsets of Baire space $\mathcal{N}$

(i) First observe that for each finite sequence $s$ one may decide if there exists $\alpha$ in $D^2(A_1)$ such that $s$ is an initial part of $\alpha$. So, if $D^2(A_1)$ is closed, $D^2(A_1)$ coincides with the set of all $\alpha$ such that for every $m$, $\alpha^0m = 0m$ or $\alpha^1m = 0m$.

Let us call the latter set $\overline{D^2(A_1)}$, the closure of $D^2(A_1)$.

Observe that $\overline{D^2(A_1)}$ is a spread and that $0$ belongs to $\overline{D^2(A_1)}$.

Assuming that $\overline{D^2(A_1)}$ coincides with $D^2(A_1)$, we apply the continuity principle and find $m$ such that for every $\alpha$ in $\overline{D^2(A_1)}$, if $\alpha m = 0m$, then $\alpha^0 = 0$, or for every $\alpha$ in $D^2(A_1)$, if $\alpha m = 0m$, then $\alpha^1 = 0$. Both alternatives are false.

(ii) Suppose, for some $n > 1$, that $D^{n+1}(A_1)$ coincides with $C_0 \cup \cdots \cup C_{n-1}$, where each $C_i$ is a closed subset of $\mathcal{N}$.

Define for each $j < n + 1$, $S_j = \{\alpha|\alpha^j = 0\}$ and observe that each $S_j$ is a spread containing $0$ and that $D^{n+1}(A_1)$ coincides with $\bigcup_{j<n+1} S_j$.

Applying the continuity principle $n+1$ times, we find for every $j < n + 1$ numbers $r_{ij}$, $i_j$ such that every $\alpha$ in $S_j$ such that $\alpha r_{ij} = 0$ belongs to $C_{i_j}$. Without loss of generality we assume $i_0 = i_1 = 0$ and we define $m := \max(m_0, m_1)$.

So every $\alpha$ in $D^2(A_1)$ such that $\alpha m = 0m$ will belong to $C_0$, and therefore to $D^{n+1}(A_1)$, and reasoning as in (i) we find a contradiction.

3.2.4

Given a subset $P$ of $\mathcal{N}$ we let $Un(P)$ the set of all $\alpha$ in $\mathcal{N}$ such that for each $n$, $P(\alpha^n)$. Using an axiom of countable choice one proves that a subset $X$ of $\mathcal{N}$ reduces to $Un(P)$ if and only if there exists a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ such that $X = \bigcap_{n \in \mathbb{N}} X_n$ and each $X_n$ reduces to $P$.

Theorem: For each $n$, $D^{n+1}(A_1)$ does not reduce to $Un(D^n(A_1))$, that is, $D^{n+1}(A_1)$ does not coincide with any countable intersection of unions of $n$ closed sets.

Proof: We consider the case $n = 2$, leaving the other cases to the reader.

Consider the set $AM_1$ of all $\alpha$ in $\mathcal{N}$ that assume at most one time a value different from $0$, so $AM_1 := \{\alpha|\text{for all } m, n, \text{ if } \alpha(m) \neq 0 \text{ and } \alpha(n) \neq 0, \text{ then } m = n\}$. Observe that $AM_1$ is a spread and that $AM_1$ forms part of the spread $D^3(A_1)$. Remark that for every $\alpha$ in $AM_1$, if $\alpha \neq 0$, then $\alpha$ belongs to $D^3(A_1)$.

Let $C_{0,0}, C_{0,1}, C_{1,0}, C_{1,1}, \ldots$ be a sequence of closed subsets of $\mathcal{N}$ such that $D^3(A_1) = \bigcap_{n \in \mathbb{N}} C_{n,0} \cup C_{n,1}$. Arguing as in Section 3.2.3, we find for each $n$ some $m, i$ such that for all $\alpha$ in $D^3(A_1)$, if $\alpha m = 0m$, then $\alpha$ belongs to $C_{n, i}$. So for every $\alpha$ in $AM_1$, either $\alpha m = 0m$ and $\alpha$ belongs to $C_{n,0} \cup C_{n,1}$, or $\alpha m \neq 0m$, and $\alpha$ belongs to $D^3(A_1)$ and therefore to $C_{n,0} \cup C_{n,1}$. We conclude that $AM_1$ is contained in $\bigcap_{n \in \mathbb{N}} C_{n,0} \cup C_{n,1}$ and therefore in $D^3(A_1)$. We now use the continuity principle and find $m, i$ such that for all $\alpha$, if $\alpha$ belongs to $AM_1$ and $\alpha m = 0m$, then $\alpha^1 = 0$. Contradiction.
3.2.5

For every subset $X$ of $N$ we let $\text{Perhaps}(X)$ be the set of all $\alpha$ such that there exists $\beta$ in $X$ with the property if $\alpha \neq \beta$, then $X(\alpha)$.

We now prove: $\text{Perhaps}(D^2(A_1))$ coincides with $D^2(A_1)$.

For suppose $\alpha$ belongs to $D^2(A_1)$. Let $\beta$ be the sequence such that $\beta(0) = \alpha(0)$, and $\beta^0 = 0$ and for each $n > 0$, $\beta^n = \alpha^n$. One verifies easily that $\beta$ belongs to $D^2(A_1)$, and if $\alpha \neq \beta$, then $\alpha^1 = 0$, and $\alpha$ belongs to $D^2(A_1)$. Therefore, $D^2(A_1)$ is part of $\text{Perhaps}(D^2(A_1))$.

The converse is obvious.

We now define, for every subset $X$ of $N$ and every positive integer $n$, $\text{Perhaps}_n(X)$: it is the set of all $\alpha$ such that there exists a finite sequence $\beta_0, \beta_1, \ldots, \beta_{n-1}$ of elements of $X$ with the property: if for every $j < n$, $\alpha \neq \beta_j$, then $X(\alpha)$. We prove: for each $n$, $\text{Perhaps}_n(D^{n+1}(A_1))$ coincides with $D^{n+1}(A_1)$.

For suppose $\alpha$ belongs to $D^{n+1}(A_1)$. Let $\beta_0, \beta_1, \ldots, \beta_{n-1}$ be a finite sequence of elements of $D^{n+1}(A_1)$ such that, for every $j < n$, $(\beta_j)^j = 0$ and, for every $k \neq j$, $(\beta_j)^k = \alpha^k$ and $\beta_j(0) = \alpha(0)$. If for every $j < n$, $\alpha \neq \beta_j$, then $\alpha^n = 0$ and $\alpha$ belongs to $D^{n+1}(A_1)$.

**Theorem:** $D^3(A_1)$ is not a subset of $\text{Perhaps}(D^3(A_1))$.

**Proof:** Consider the set $AM_2$ of all $\alpha$ that assume at most two times a value different from 0.

Observe that $AM_2$ is part of $D^3(A_1)$, and is a spread containing 0. Now assume $D^3(A_1)$, and therefore also $AM_2$, is a subset of $\text{Perhaps}(D^3(A_1))$.

Observe that for every $\alpha$ in $AM_2$ one may determine $i < 3$ and $\beta$ such that $\beta^i = 0$ and: if $\alpha \neq \beta$, then $\alpha$ belongs to $D^3(A_1)$.

Apply the continuity principle and determine $m, i$ such that for every $\alpha$ in $AM_2$ such that $\overline{\alpha}^m = \overline{\alpha}^m$ there exists $\beta$ such that $\beta^i = 0$, and if $\alpha \neq \beta$, then $\alpha$ belongs to $D^3(A_1)$. We claim that now $AM_1$ is a subset of $D^3(A_1)$.

Without loss of generality we may assume $i = 0$. In order to prove our claim we construct a function $f$ from $N$ to $N$ such that for all $\alpha$, $(f(\alpha)m = \overline{\alpha}^m$ and, for every $j \neq m + 1$, $(f(\alpha)^0(j) = 0$ and $(f(\alpha)^0(m + 1) = 1$ and $(f(\alpha)^1 = \overline{\alpha}^m * \alpha^1$ and $(f(\alpha)^2 = \overline{\alpha}^m * \alpha^2$, and $(f(\alpha)(0) = 0$.

Observe that for every $\alpha$ in $AM_1$, $f(\alpha)$ belongs to $AM_2$ and $(f(\alpha)^0 \neq 0$, and $(f(\alpha)m = \overline{\alpha}^m$, therefore $f(\alpha)$ belongs to $D^3(A_1)$ and so $\alpha^1 = 0$ or $\alpha^2 = 0$.

So for every $\alpha$ in $AM_1$ either $\alpha^1 = 0$ or $\alpha^2 = 0$.

By a reasoning similar to the one used in Section 3.2.4 we obtain a contradiction.

Similarly one proves, for each $n > 1$, that $D^{n+2}(A_1)$ is not a part of $\text{Perhaps}_n(D^{n+2}(A_1))$ but coincides with $\text{Perhaps}_{n+1}(D^{n+2}(A_1))$.

Using the terminology introduced in Veldman 1999 one might say that $D^{n+2}(A_1)$ has perhaposity $n + 1$ but not perhaposity $n$.

Let us take a closer look at these perhaposities.
3.3 An interlude on perhaps

3.3.1 Perhapsive subsets of $\mathcal{N}$

We now introduce a binary operation Perhaps on the class of subsets of $\mathcal{N}$. Given subsets $X, Y$ of $\mathcal{N}$ such that $X \subseteq Y$ we let Perhaps$(X, Y)$ be the set of all $\alpha$ such that there exists $\beta$ in $X$ with the property: if $\alpha \neq \beta$, then $\alpha$ belongs to $Y$ ("$\alpha$ belongs to $X$, perhaps to $Y$.").

$Y$ is a subset of Perhaps$(X, Y)$, but the converse may be false.

There is a connection with the unary operation Perhaps that we considered in Section 3.2.5 (and earlier, when we were studying subsets of $\mathbb{R}$, in Section 3.1.2). For every subset $X$ of $\mathcal{N}$, Perhaps$(X) = $ Perhaps$(X, X)$. A subset $X$ of $\mathcal{N}$ is called perhapsive if and only if $X$ coincides with Perhaps$(X)$. (In Veldman 1999, following Waaldijk 1996, we used the term "weakly stable" for "perhapsive"). Every $\Pi_1^0$ subset of $\mathcal{N}$ is perhapsive. Let us prove this. Let $G_0, G_1, G_2, \ldots$ be a sequence of open subsets and consider $B = \bigcap_n G_n$. Suppose $\beta$ belongs to $B$ and $\alpha$ satisfies: if $\alpha \neq \beta$, then $\alpha$ belongs to $B$. We claim that $\alpha$ itself belongs to $B$. For, given any $n$, calculate $m$ such that every $\gamma$ such that $\gamma m = \beta m$ belongs to $G_n$. Now consider $\bar{\alpha}m$ and $\bar{\beta}m$. If $\bar{\alpha}m = \bar{\beta}m$, then $\alpha$ belongs to $G_n$. If $\bar{\alpha}m \neq \bar{\beta}m$, then $\alpha \neq \beta$, and $\alpha$ belongs to $B$ and so to $G_n$. So $\alpha$ belongs to every $G_n$ and therefore to $B$.

A similar argument shows: Every $\Pi_1^1$ subset of $\mathcal{N}$ is perhapsive.

(A subset $X$ of $\mathcal{N}$ is called $\Pi_1^1$ or co-analytical if and only if there exists a closed subset $F$ of $X$ such that $X = \{ \alpha^0 | \alpha \in F \}$, see Section 6.5.)

We may conclude from Section 3.2.5 that the set $D^2(A_1)$ is not perhapsive. Therefore, this set is not co-analytical. (Observe that every set that reduces to a perhapsive set is itself perhapsive.)

Remark that $D^2(A_1)$ is a fairly simple Borel set. (In classical descriptive set theory every Borel set is co-analytical).

3.3.2 The perhapsive closure of a countable and dense subset of $\mathcal{N}$

3.3.2.1 We want to study iterations of the operation Perhaps.

We introduce, for every subset $X$ of $\mathcal{N}$ and every stump $\sigma$, a subset $P(\sigma, X)$ of $\mathcal{N}$, as follows: $P(1, X) = X$, and, for every non-empty stump $\sigma$, $P(\sigma, X) = $ Perhaps$(X, \bigcup_{n \in \mathbb{N}} P(\sigma^n, X))$.

(This definition is a bit different from the one used in Veldman 1999. Let us compare the two definitions.

We define for every subset $X$ of $\mathcal{N}$ and every stump $\sigma$ a set $P^*(\sigma, X)$ as follows. $P^*(X) = X$ and for every non-empty stump $\sigma$, $P^*(\sigma, X) = \bigcup_{n \in \mathbb{N}} $ Perhaps$(X, P^*(\sigma^n, X))$.

$P^*$ coincides with the operation called $P$ defined in Veldman 1999.

One may verify that for each $\sigma$, $P^*(\sigma, X) \subseteq P(\sigma, X)$ and also that, for each $\sigma$, $P(\sigma, X) \subseteq P^*(S(\sigma), X)$. Let us prove the latter fact. We use induction and assume that for some non-empty stump $\sigma$, for each $n$, $P(\sigma^n, X) \subseteq P^*(S(\sigma^n), X)$. Then
\[ \mathbb{P}(\sigma, X) = \text{Perhaps}(X, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X)) \subseteq \text{Perhaps}(X, \bigcup_{n \in \mathbb{N}} \mathbb{P}^*(S(\sigma^n), X)) \subseteq \text{Perhaps}(X, \bigcup_{n \in \mathbb{N}} \mathbb{P}^*(\sigma, X)). \]

One may prove that for all \( \sigma, \tau \), if \( \sigma \leq \tau \), then \( \mathbb{P}(\sigma, X) \subseteq \mathbb{P}(\tau, X) \).

It may occur that for all \( \sigma, \tau \), if \( \sigma < \tau \), then \( \mathbb{P}(\sigma, X) \) is a proper subset of \( \mathbb{P}(\tau, X) \).

Examples have been given in Veldman 1999.

One may prove that for all \( a, t \), \( a \leq t \), then \( \mathbb{P}(a, X) \subseteq \mathbb{P}(t, X) \).

It may occur that for all \( u, t \), if \( a < t \), then \( \mathbb{P}(a, X) \) is a proper subset of \( \mathbb{P}(t, X) \).

Examples have been given in Veldman 1999.

One may take \( X = \text{Fin} = \{ a \in X | \exists m \forall n > m[a(n) = 0] \} \), the set of all \( a \) that assume only finitely many times a value different from 0. Generalizing this example, one may consider any countable and dense subset \( D = \{ d_0, d_1, d_2, \ldots \} \) of Baire space \( \mathcal{N} \). Using Brouwer’s thesis one may show that for every \( \alpha \), \( \alpha \) belongs to some \( \mathbb{P}(\sigma, D) \) if and only if for every \( \gamma \) there exists \( n \) such that \( \alpha(\gamma(n)) = d_n(\gamma(n)) \) (that is, if we express ourselves metrically, the distance between \( \alpha \) and \( d_n \) is less than \( 1/10^{\alpha(\gamma(n))} \), where we are using a widely used metric on \( \mathcal{N} \), so, one might also say: every attempt to give evidence that \( \alpha \) is apart from every \( d_n \) will fail).

We do not prove this fact as the proof is similar to the proof of its particular case given in Veldman 1999.

**3.3.2.2** Let us introduce some notation. Given a countable and dense subset \( D = \{ d_0, d_1, d_2, \ldots \} \) of \( \mathcal{N} \), let \( D^* \) be the set we considered a moment ago, consisting of all \( a \) such that for every \( \gamma \) there exists \( n \) such that \( a(\gamma(n)) = d_n(\gamma(n)) \).

\( D^* \) might be called the **perhapsive closure** of the set \( D \), for the following reasons:

(i) \( D \subseteq D^* \) and \( \text{Perhaps}(D, D^*) = D^* \).

(ii) For every subset \( Y \) of \( \mathcal{N} \), if \( D \subseteq Y \) and \( \text{Perhaps}(D, Y) = Y \), then \( D^* \subseteq Y \).

(As to (ii), if \( Y \subseteq \mathcal{N} \) satisfies the requirements, one proves by induction, for every \( \sigma \), \( \mathbb{P}(\sigma, D) \subseteq Y \). Using Brouwer’s Thesis, one concludes \( D^* \subseteq Y \).

We claim that the set \( D^* \) is perhapsive, that is: \( \text{Perhaps}(D^*, D^*) = D^* \).

In order to prove this, we need the following remark:

for all subsets \( X, Y, Z \) of \( \mathcal{N} \): if \( X \subseteq Y \) and \( \text{Perhaps}(X, Y) \subseteq Z \), then \( \text{Perhaps}(X, Z) \subseteq \text{Perhaps}(Y, Z) \).

Let us prove this remark. Suppose \( \alpha \) belongs to \( \text{Perhaps}(X, Y, Z) \) and determine \( \beta \) in \( \text{Perhaps}(X, Y) \) such that if \( \alpha \neq \beta \), then \( \alpha \) belongs to \( Z \). Also determine \( \gamma \) in \( X \) such that, if \( \beta \neq \gamma \), then \( \beta \) belongs to \( Y \). Suppose now: \( \alpha \neq \gamma \); then either \( \alpha \neq \beta \) or \( \beta \neq \gamma \), so either \( \alpha \) belongs to \( Z \) and therefore to \( \text{Perhaps}(Y, Z) \), or \( \beta \) belongs to \( Y \), so \( \alpha \) belongs to \( \text{Perhaps}(Y, Z) \).

Now observe: \( \text{Perhaps}(D, D^*) \subseteq D^* \), and assume, for some \( \sigma \), for each \( n \), \( \text{Perhaps}(D, D^*) \subseteq D^* \). By the above remark, for each \( n \):

\( \text{Perhaps}(D, D^*) \subseteq D^* \) and also \( \text{Perhaps}(D^*, D^*) \subseteq D^* \).

We conclude that for every \( \sigma \), \( \text{Perhaps}(\sigma, D^*) \subseteq D^* \), and using Brouwer’s The-
sis again, Perhaps\((D^*, D^*) \subseteq D^*\), that is: \(D^*\) is perhapsive. Obviously, \(D^*\) is the least perhapsive set \(X\) such that \(D \subseteq X\).

### 3.3.3 A remark on subsets of \(\mathcal{N}\) of finite perhapsity

In Section 3.2.5 we considered, given a subset \(X\) of \(\mathcal{N}\) and a natural number \(n\), the set \(\text{Perhaps}_n(X)\), consisting of all \(\alpha\) for which there exist elements \(\beta_0, \beta_1, \ldots, \beta_{n-1}\) of \(X\) with the property: if for all \(j < n\), \(\alpha \neq \beta_j\), then \(\alpha\) belongs to \(X\). It is easy to see that \(\text{Perhaps}_n(X)\) is part of \(\mathcal{P}(n, X)\) but we have no reason to assert the converse.

### 3.3.4 Subsets of \(\mathcal{N}\) of bounded perhapsity

Given a subset \(X\) of \(\mathcal{N}\) it may happen that, for some stump \(\sigma\), \(\mathcal{P}(S(\sigma), X)\) coincides with \(\mathcal{P}(\sigma, X)\). We then say that \(X\) has perhapsity \(\sigma\), and also, disregarding \(\sigma\), that \(X\) has bounded perhapsity. (We are using the latter expression in a more natural and slightly less strict sense than in Veldman 1999).

Suppose that \(\mathcal{P}(\sigma, X)\) coincides with \(\mathcal{P}(S(\sigma), X)\) and consider \(Y = \mathcal{P}(\sigma, X)\). Observe that \(X \subseteq Y\) and \(\text{Perhaps}(X, Y) = Y\).

One proves by induction that for every \(\tau\): \(\mathcal{P}(\tau, Y) \subseteq Y\).

It is also true, that for every \(Z \subseteq \mathcal{N}\) if \(X \subseteq Z\) and \(\text{Perhaps}(X, Z) \subseteq Z\), then \(Y \subseteq Z\).

So \(Y\) deserves to be called the perhapsive closure of \(X\). As in Section 3.3.2 we may prove that \(Y\) is perhapsive.

If \(X\) is not of bounded perhapsity we do not see how to define a perhapsive closure of \(X\), except of course in some special cases see for instance Section 3.3.2.

If \(X\) has perhapsity \(\sigma\), and for every \(\tau\), if \(X\) has perhapsity \(\tau\), then \(\sigma \leq \tau\), we say that \(X\) has perhapsity exactly \(\tau\).

In general, we may not expect to be able to find for any given \(X\) of bounded perhapsity, a stump \(\sigma\) such that \(X\) has perhapsity exactly \(\sigma\), as not every inhabited set of stumps has (in this sense) a least element.

If both \(X, Y\) are subsets of \(\mathcal{N}\) of bounded perhapsity we say that \(X\) has lower perhapsity than \(Y\) if for each \(\tau\), if \(\mathcal{P}(S(\tau), Y) = \mathcal{P}(\tau, Y)\) there exists \(\sigma < \tau\) such that \(\mathcal{P}(S(\sigma), X) = \mathcal{P}(\sigma, X)\).

### 3.4 Sets of larger and larger perhapsity and larger and larger complexity

#### 3.4.1

In Veldman 1999 we have shown how to obtain sets of arbitrarily large but bounded perhapsity. The sets we found were analytical sets and (probably) not Borel. We want to show now that we may find such sets within the class \(\Sigma^0_2\).
3.4.2
We define, for every stump $\sigma$, a subset $C_\sigma$ of $\mathcal{N}$, as follows: $C_1 := A_1 = \{0\}$ and for every non-empty stump $\sigma$, and every $\alpha$, $\alpha$ belongs to $C_\sigma$ if and only if either $\alpha^0 = 0$ or there exists $n$ such that $n+1$ is the least $k > 0$ such that $\alpha^0(k) \neq 0$ and $\alpha^1$ belongs to $C_\sigma \cap (n+1)$. It is easy to see that each set $C_\sigma$ belongs to $\Sigma^0_2$.
Observe that a classical mathematician would believe every $C_\sigma$ to be closed.

3.4.3
A subset $X$ of Baire space $\mathcal{N}$ is called closable if and only if for every finite sequence $s$ of natural numbers one may decide if there exists $\alpha$ in $X$ starting with $s$, that is, such that $\exists n[\bar{\alpha}n = s]$.
If $X$ is closable we let $\overline{X}$ denote the set of all $\alpha$ such that for every $n$ there exists $\beta$ in $X$ such that $\bar{\alpha}n = \bar{\beta}n$. $\overline{X}$ is a spread and is called the closure of $X$.
If $X$ is closable, then for every $\sigma$, $\mathbb{P}(\sigma, X) \subseteq \overline{X}$, and $\mathbb{P}(\sigma, X)$ itself is also closable and $\mathbb{P}(\sigma, X) = \overline{X}$.
It is easy to see that each one of the sets $C_\sigma$ introduced in 3.4.2 is closable.

3.4.4
We claim that for each $\sigma$, $\overline{C_\sigma}$ is contained in $\mathbb{P}(\sigma, C_\sigma)$.
(We may conclude then that $\mathbb{P}(S(\sigma), C_\sigma)$ coincides with $\mathbb{P}(\sigma, C_\sigma)$, so $C_\sigma$ has perhapsity $\sigma$.)
We prove this claim by induction.
Observe that $C_0$ is closed and coincides with $\mathbb{P}(1, C_0)$.
Now assume that $\sigma$ is a non-empty stump and for each $n$, $\overline{C_\sigma^n}$ coincides with $\mathbb{P}(\sigma^n, C_{\sigma^n})$.
It follows that, for each $n$, the collection of all $\alpha$ in $\overline{C_\sigma}$ such that $n+1 = \mu(k[\alpha^0(k) \neq 0]$ is contained in $\overline{\mathbb{P}(\sigma^{K(n+1)}, C_\alpha)}$.
We now prove that $\overline{C_\sigma}$ coincides with $\mathbb{P}(\sigma, C_\sigma)$.
Assume that $\alpha$ belongs to $\overline{C_\sigma}$ and define $\beta$ such that $\beta(0) := \alpha(0)$ and $\beta^0 := 0$ and for each $n > 0$, $\beta^n := \alpha^n$.
Observe that $\beta$ belongs to $C_\sigma$, and, if $\alpha \neq \beta$, then $\alpha^0 \neq 0$, and $\alpha$ belongs to $\mathbb{P}(\sigma^{K(n+1)}, C_\sigma)$, where $n+1$ is the least $k > 0$ such that $\alpha^0(k) \neq 0$. So $\alpha$ belongs to $\mathbb{P}(\sigma, C_\sigma)$.
We may conclude that, for each $\sigma$, $\overline{C_\sigma}$ is the perhapsive closure of $C_\sigma$, that is the least perhapsive set containing $C_\sigma$.

3.4.5
We now prove, for every non-empty stump $\sigma$, for every $\tau$, if $C_\sigma$ has perhapsity $\tau$, then $\tau$ is non-empty, and for every $m$ there exists $n$ such that $C_{\sigma^m}$ has perhapsity $\tau^n$. So assume that $\sigma$ is a non-empty stump and $C_\sigma$ has perhapsity $\tau$.
Let us first verify that $\tau$ is non-empty.
So suppose $C_\sigma$ has perhapsity 1. Then Perhaps($C_\sigma$) coincides with $C_\sigma$, that is, $C_\sigma$ itself is perhapsive. Therefore every set $\mathbb{P}(\tau, C_\sigma)$ is part of $C_\sigma$, in particular $\mathbb{P}(\sigma, C_\sigma)$ is part of $C_\sigma$, so $\overline{C_\sigma}$ is part of $C_\sigma$ and $C_\sigma$ is closed.
Now consider $E = \{ \alpha | AM_1(\alpha^0) \land \alpha^1 = 0 \}$ and observe that $E \subseteq \overline{C}_\sigma$. Therefore $E$ is part of $C_\sigma$ and for every $\alpha$ in $E$ we may decide $\alpha^0 = 0$ or $\exists n[\alpha^0(n) \neq 0]$. Using the continuity principle (and the fact that $E$ is a spread) one obtains a contradiction.

We conclude that $\tau$ is non-empty. As $C_\sigma$ has perhapsity $\tau$, $\mathbb{P}(\tau, C_\sigma)$ is the perhapsive closure of $C_\sigma$ and coincides with $\overline{C}_\sigma$. So for every $\alpha$ in $\overline{C}_\sigma$ we may determine $\beta$ in $C_\sigma$ such that, if $\alpha \neq \beta$, then $\alpha$ belongs to some set $\mathbb{P}(\tau^n, C_\sigma)$.

Observe that then, for every $\alpha$ in $\overline{C}_\sigma$ we may determine $i$ such that either $i = 0$ and there exists $\beta$ such that $\beta^0 = 0$ and if $\alpha \neq \beta$, then $\alpha$ belongs to some set $\mathbb{P}(\tau^n, C_\sigma)$, or $i > 0$ and there exists $\beta$ such that $\beta^0(i) > 0$ and if $\alpha \neq \beta$ then $\alpha$ belongs to some set $\mathbb{P}(\tau^n, C_\sigma)$.

Now we use the fact that $\overline{C}_\sigma$ is a spread and 0 belongs to $\overline{C}_\sigma$. We apply the continuity principle and first assume that we find $p, i > 0$ such that for all $\alpha$ in $\overline{C}_\sigma$, if $\bar{\alpha}p = \bar{0}p$, there exists $\beta$ such that $\beta^0(i) > 0$ and if $\alpha \neq \beta$, then $\alpha$ belongs to some set $\mathbb{P}(\tau^n, C_\sigma)$. Applying the continuity principle once more we find $q, n$ ($q$ will be greater than $J(0, i - 1)$) such that, for all $\alpha$ in $\overline{C}_\sigma$, if $\bar{\alpha}q = \bar{0}q$, then $\alpha$ belongs to $\mathbb{P}(\tau^n, C_\sigma)$. It follows that $\overline{C}_\sigma$ itself is contained in $\mathbb{P}(\tau^n, C_\sigma)$ and, for every $m$, $\overline{C}_{\sigma^m}$ is contained in $\mathbb{P}(\tau^n, C_\sigma)$.

Next, we assume that we find $p$ such that for all $\alpha$ in $\overline{C}_\sigma$, if $\bar{\alpha}p = \bar{0}p$, there exists $\beta$ such that $\beta^0 = 0$ and, if $\alpha \neq \beta$, then $\alpha$ belongs to some set $\mathbb{P}(\tau^n, C_\sigma)$. Let $m$ be a natural number. Consider the set of all $\alpha$ in $\overline{C}_\sigma$ such that $J(m, p)$ is the least $k$ such that $\alpha^0(k) \neq 0$ and $\bar{\alpha}p = \bar{0}p$. Observe that this set is a spread and that every element of it belongs to some set $\mathbb{P}(\tau^n, C_\sigma)$. Consider the sequence $\beta$ in $AM_1$ such that $\beta^0(J(m, p)) = 1$ and determine $q, n$ such that $q > J(m, p)$ and for every $\alpha$ in $\overline{C}_\sigma$, if $\bar{\beta}q = \bar{\alpha}q$ then $\alpha$ belongs to $\mathbb{P}(\tau^n, C_\sigma)$.

Observe that now, for every $\alpha$, if $\bar{\beta}q = \bar{\alpha}q$ and $\alpha^1$ belongs to $\overline{C}_{\sigma^m}$, then $\alpha$ belongs to $\mathbb{P}(\tau^n, C_\sigma)$, and therefore $\overline{C}_{\sigma^m}$ will be part of $\mathbb{P}(\tau^n, C_\sigma)$, so $C_{\sigma^m}$ has perhapsity $\tau^n$.

3.4.6

We conclude from the result in Section 3.4.5:

For every $\sigma, \tau$, if $C_\sigma$ has perhapsity $\tau$, then $\sigma \leq \tau$.

(We use induction. If $C_\sigma$ has perhapsity $\tau$, and $\sigma$ is non-empty, then $\tau$ is non-empty and for each $m$ there exists $n$ such that $C_{\sigma^n}$ has perhapsity $\tau^n$, and therefore, according to the induction hypothesis, $\sigma^n \leq \tau^n$; so $\sigma \leq \tau$).

Therefore, every $C_\sigma$ has perhapsity exactly $\sigma$.

Also, if $\sigma < \tau$, then $C_\sigma$ has lower perhapsity then $C_\tau$.

3.4.7 Not only growing perhapsity but also growing complexity

We now want to study the question for which $\sigma, \tau$ the set $C_\sigma$ reduces to the set $C_\tau$.

We intend to show for all stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $C_\sigma$ reduces to $C_\tau$, and if $\sigma < \tau$, then $C_\tau$ does not reduce to $C_\sigma$.

3.4.7.1 We make a preliminary remark. Let, for each $n$, $f_n$ be the function from Baire space $\mathcal{N}$ to itself such that, for every $\alpha$, $f_n[\alpha] = \bar{0}n \ast \alpha$. 
Observe that \( f_n \) reduces \( C_1 \) to \( C_1 \), that is, for every \( \alpha, \alpha = 0 \) if and only if \( f_n|\alpha = 0 \). We want to construct, for all \( \alpha, \alpha \) a function \( f_{\alpha,n} \) from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( C_\sigma \) to \( \mathcal{C}_\tau \) such that for all \( \alpha f_{\alpha,n} = 0 \). We do so by induction. Assume \( \sigma \) is a non-empty stump and for all \( n, k \) a function \( f_{\sigma,n,k} \) from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( C_{\sigma,n} \) to \( C_{\tau,n} \) such that for all \( \alpha f_{\sigma,n,k} = 0 \).

We want to construct for all \( \alpha \) such that \( \sigma \) is a non-empty stump and for all \( n, k \) a function \( f_{\sigma,n,k} \) from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( C_{\sigma,n} \) to \( C_{\tau,n} \) such that for all \( \alpha f_{\sigma,n,k} = 0 \).

We have to build such functions for \( C_\sigma \) itself.

Let \( k \) be a natural number and let us construct a function \( f = f_{\sigma,k} \). We take care that for every \( \alpha, m, n \), if \( n < k \), then \( (f|\alpha)^0(J(m,n)) = 0 \), and if \( n > k \), then \( (f|\alpha)^0(J(m,n)) : = c\gamma(J(m,n - k)) \). Further, for each \( m \), if, for every \( n \leq m \), \( (f|\alpha)^0(n) = 0 \), then \( (f|\alpha)^1(m) = 0 \). Now assume \( m, n \) are natural numbers and \( \alpha^0 (J(m,n)) \neq 0 \) and for each \( p < J(m,n) \), \( \alpha^0(p) = 0 \). Then \( f(\alpha)^0(J(m,n + k)) \neq 0 \) and for each \( p < J(m,n + k) \), \( (f|\alpha)^0(p) = 0 \). We define \( (f|\alpha)^1 = f_{\alpha,m,J(m,n+k)}(\alpha^1) \).

We also prescribe, for every \( n > 1 \), \( (f|\alpha)^n = 0 \).

It is not difficult to verify that \( f \) satisfies the requirements.

3.4.7.2 We prove, by induction, for every \( \tau \), for every \( \sigma \), if \( \sigma \leq \tau \), then \( C_\sigma \) reduces to \( C_\tau \). If \( \tau \) is empty there is nothing to prove. Assume \( \tau \) is non-empty and \( \sigma \leq \tau \), that is, for every \( \alpha \), there exists \( n \) such that \( \sigma^m \leq \tau^n \) and \( C_{\sigma^m} \) reduces to \( C_{\tau^n} \).

Using an axiom of countable choice we determine \( \gamma \) in \( \mathcal{N} \) and a sequence \( \gamma_0, \gamma_1, \ldots \) of functions from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( m, g_m \) reduces \( C_{\sigma^m} \) to \( C_{\tau^m} \). We build a function \( g \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for each \( \alpha, (g|\alpha)^0(0) = 0 \) and for all \( m, n \), \( (g|\alpha)^0(J(\gamma(m), m + n)) = \alpha^0(J(m,n)) \), and, \( (g|\alpha)^0(p) = 0 \) if there are no \( m, n \) such that \( p = J(\gamma(m), m + n) \).

We further take care that for every \( p \), for all \( m, n \), if \( \alpha^0(J(m,n)) \neq 0 \) and for all \( p < J(m,n) \), \( \alpha^0(p) = 0 \), then \( (g|\alpha)^1 = f_{\gamma(m),\gamma(m+n+1)}(\alpha^1) \).

We leave it to the reader to verify that \( g \) reduces \( C_\sigma \) to \( C_\tau \).

3.4.7.3 We now prove, again by induction, for every \( \tau \), for every \( \sigma \), if \( \tau < \sigma \), then \( C_\sigma \) does not reduce to \( C_\tau \).

Observe that \( A_1 = C_1 \) is a closed and a perhapsive set. As for every non-empty stump \( \sigma \), the set \( C_\sigma \) is not a perhapsive subset of \( \mathcal{N} \), \( C_\sigma \) does not reduce to \( C_1 \).

(We once again use the fact that every set reducing to a perhapsive set is itself perhapsive).

3.4.7.4 We now prove the induction step.

Let \( \tau \) be a stump such that for every \( \tau' < \tau \), for every \( \sigma \), if \( \tau' < \sigma \), then \( C_\sigma \) does not reduce to \( C_{\tau'} \).

We want to prove that for every \( \sigma \), if \( \tau < \sigma \), then \( C_\sigma \) does not reduce to \( C_\tau \).

So let \( \sigma \) be a stump such that \( \tau < \sigma \), and assume that \( g \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( C_\sigma \) to \( C_\tau \).

We will derive a contradiction.

We first show: for every \( \alpha \), if \( \alpha^0 = 0 \), then \( (g|\alpha)^0 = 0 \).

Assume \( \alpha^0 = 0 \), therefore \( \alpha \) belongs to \( C_\sigma \), therefore either \( (g|\alpha)^0 = 0 \) or there
exists $n$ such that $(g|\alpha)(n+1) \neq 0$. Assume the latter and calculate $m,p$ such that $m$ is the least $n$ such that $(g|\alpha)^0(n+1) \neq 0$ and for all $\beta$, if $\beta p = \overline{\alpha} p$, then $(g|\beta)^0(m+2) = (g|\alpha)^0(m+2)$.

Observe that for each $\beta$, if $\beta p = \overline{\alpha} p$, then $g|\beta$ belongs to $C_{\tau}$ if and only if $(g|\beta)^1$ belongs to $C_{\tau K(m+1)}$. Let $f$ be a function reducing $C_{\sigma}$ to $C_{\sigma}$ such that for all $\beta$, $(f|\beta)p = \overline{\alpha} p$. (see Section 3.4.7.1).

The function $\beta \mapsto (g \circ f|\beta)^1$ reduces $C_{\sigma}$ to $C_{\tau K(m+1)}$ and $\tau := \tau K(m+1) < \tau < \tau$, so we have a contradiction.

We now calculate $q$ such that $\tau \leq \sigma q$.

We define a sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of elements of $\mathcal{N}$ such that for every $m$, $J(q,m)$ is the least $n$ such that $(\alpha_m)^0(n) \neq 0$ and $(\alpha_m)^1 = 0$.

Observe that, for each $m$, $\alpha_m$ belongs to $C_{\sigma}$, therefore $g|\alpha_m$ belongs to $C_{\tau}$ therefore either $(g|\alpha_m)^0 = 0$ or for some $n$, $(g|\alpha_m)^0(n+1) \neq 0$.

Assume that $m$ is a natural number and for some $n$, $(g|\alpha_m)^0(n+1) \neq 0$. Calculate $t,p$ such that $t$ is the least $n$ such that $(g|\alpha_m)^0(n+1) = 0$ and for each $\beta$, if $\beta p = \overline{\alpha} m p$, then $(g|\beta)^0(t+2) = (g|\alpha_m)^0(t+2)$.

Observe that for each $\beta$, if $\beta p = \overline{\alpha} m p$, then $g|\beta$ belongs to $C_{\tau}$ if and only if $(g|\beta)^1$ belongs to $C_{\tau K(m+1)}$. Let $f$ be a function reducing $C_{\sigma}$ to $C_{\sigma}$ such that for all $\beta$, $(f|\beta)p = \overline{\alpha} p$.

Now construct a function $h$, such that for each $\beta$, $J(q,m)$ is the least $n$ such that $(h|\beta)^0(n) \neq 0$ and $(h|\beta)p = \overline{\alpha} m p$ and $(h|\beta)^1 = f|\beta$. It is easy to see that the function $\beta \mapsto (g \circ h|\beta)^1$ reduces $C_{\sigma}$ to $C_{\tau K(m+1)}$. Observe $\tau' = \tau K(m+1) < \tau$ and $\tau' < \sigma^q$. We have a contradiction.

We conclude that for each $m$, $(g|\alpha_m)^0 = 0$.

We have seen earlier that $(g|0)^0 = 0$.

We conclude that for every $\alpha$, if $AM_1(\alpha)$, that is, $\alpha$ assumes at most one time a value different from 0, then $(g|\alpha)^0 = 0$, therefore: $C_{\tau}(g|\alpha)$, therefore: $C_{\tau}(\alpha)$, therefore: $\alpha^0 = 0$ or $\alpha^0 \neq 0$.

This again leads to a contradiction, as follows.

$AM$ is a spread, and using the continuity principle, we determine $m$ such that either for all $\alpha$, if $AM_1(\alpha)$ and $\overline{m} = \overline{0} m$, then $\alpha^0 = 0$ or, for all $\alpha$, if $AM_1(\alpha)$ and $\overline{m} = \overline{0} m$, then $\alpha^0 \neq 0$. Both alternatives are false.

### 3.5 Comparing $C_2$ and $D^2(A_1)$

Let us consider the set $C_2 := C_{S(2)}$ that we defined in Section 3.4.2.

Observe that, for each $\alpha$, $\alpha$ belongs to $C_2$ if and only if either $\alpha^0 = 0$ or both there exist $k > 0$ such that $\alpha^k(k) \neq 0$ and $\alpha^1 = \overline{0}$.

Observe also that, for each $\alpha$, $\alpha$ belongs to $D^2(A_1)$ if and only if either $\alpha^0 = 0$ or $\alpha^1 = 0$.

We want to show that the sets $C_2$ and $D^2(A_1)$ do not reduce to each other.

Let us first assume that $D^2(A_1)$ reduces to $C_2$ and let $g$ be a (continuous) function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, $\alpha$ belongs to $D^2(A_1)$ if and only if $g|\alpha$ belongs to $C_2$. Observe that $D^2(A_1)$ is the union of the two spreads $\{\alpha|\alpha^0 = 0\}$ and $\{\alpha|\alpha^1 = 0\}$ and that $\overline{0}$ is a common member of these two spreads. Applying the continuity
principle we find $m$ and for each $i < 2$, $p_i < 2$ and $n_i > 0$ such that either $p_i = 0$ and for all $\alpha$, if $\alpha^i = 0$ and $\bar{\alpha} m = \bar{\alpha} n_i$, then $(g|\alpha)^0 = 0$, or $p_i = 1$ and for all $\alpha$, if $\alpha^i = 0$ and $\bar{\alpha} m = \bar{\alpha} n_i$, then $(g|\alpha)^0(n_i) \neq 0$.

Let us first assume $p_0 = p_1 = 0$. We now may reduce $D^2(A_1)$ to $A_1$, as follows. Let $h$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all $\alpha$, $(h|\alpha)^0 = \bar{\alpha} m \cdot \alpha^0$ and $(h|\alpha)^1 = \bar{\alpha} m \cdot \alpha^1$. Then for all $\alpha$, $\alpha$ belongs to $D^2(A_1)$ if and only if $(g|(h|\alpha))^0 = 0$. As we know, $D^2(A_1)$ does not reduce to $A_1$, so we have a contradiction.

We now assume that $p_0 = 1$. We calculate $n$ such that for all $\alpha$, if $\bar{\alpha} m = \bar{\alpha} n$, then $(g|\alpha)^0(n_0) = (g|\alpha)^0(n_0) \neq 0$.

Observe that for every $\alpha$, if $\bar{\alpha} m = \bar{\alpha} n$, then $g|\alpha$ belongs to $C_2$ if and only if $(g|\alpha)^1 = 0$, therefore $\alpha$ belongs to $D^2(A_1)$ if and only if $(g|\alpha)^1 = 0$, that is $D^2(A_1)$ reduces to $A_1$; contradiction.

The assumption $p_1 = 1$ leads also to a contradiction. We conclude that $D^2(A_1)$ does not reduce to $C_2$.

Let us now assume that $C_2$ reduces to $D^2(A_1)$ and let $h$ be a (continuous) function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, $\alpha$ belongs to $C_2$ if and only if $h|\alpha$ belongs to $D^2(A_1)$. We now consider a sequence $\gamma_n, \gamma_1, \ldots$ of elements of $AM_1$, (that is each $\gamma_n$ assumes at most one time a value different from 0), such that for each $n$, $(\gamma_n)^0(n + 1) = 1$. Observe that for each $n$, $\gamma_n$ belongs to $C_2$ and determine $p_n < 2$ such that $(h|\gamma_n)^{p_n} = 0$.

Also calculate $p < 2$ such that $(h|0)^p = 0$.

Consider $B := \{\gamma|AM_1(\gamma)|\forall n[\gamma(n + 1) \neq 0 \rightarrow p_n = p]\}$, and observe: $B$ is a spread and for each $\gamma$ in $B$, $(h|\gamma)^p = 0$, therefore $h|\gamma$ belongs to $D^2(A_1)$ and $\gamma$ belongs to $C_2$, in particular, either $\gamma = 0$ or $\gamma \# 0$.

Applying the continuity principle we find $n$ such that for all $\gamma$ in $B$, if $\bar{\gamma} n = \bar{\gamma} n$, then $\gamma = 0$, so for every $m > n$, $p_m \neq p$. We then conclude: for every $\gamma$ in $AM_1$, if $\bar{\gamma} n = \bar{\gamma} n$, and $\gamma \# 0$, then $(h|\gamma)^1 - p = 0$, therefore also $(h|0)^1 - p = 0$, so for every $\gamma$ in $AM_1$, one has $(h|\gamma)^1 - p = 0$, therefore $h|\gamma$ belongs to $D^2(A_1)$ and $\gamma$ belongs to $C_2$, therefore $\gamma = 0$ or $\gamma \# 0$.

So, applying once more the continuity principle we find $p$ such that for every $\gamma$ in $AM_1$, if $\bar{\gamma} p = \bar{\gamma} p$, then $\gamma = 0$. Contradiction.

### 3.6 Productive upper bounds

#### 3.6.1

We want to study the operation $\mathcal{E}$ associating to every sequence $P_0, P_1, P_2, \ldots$ of subsets of $\mathcal{N}$ another subset of $\mathcal{N}$, called $\mathcal{E} P_n$, and defined by: for all $\alpha$, $\alpha$ belongs to $\mathcal{E} P_n$ if and only if either $\alpha^0 = \emptyset$ or $\alpha \# \emptyset$ and $\alpha^{n+1}$ belongs to $P_n$, where $n$ is the least $k$ such that $\alpha^0(k) \neq 0$.

It is easy to see that each set $P_n$ reduces to the set $\mathcal{E} P_n$. (Given $n$, we construct $g$ from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha$, $n$ is the least $k$ such that $(g|\alpha)^n(k) \neq 0$ and $(g|\alpha)^n = \alpha$. Then $g$ reduces $P_n$ to $\mathcal{E} P_n$).
So the set $\mathcal{E} P_n$ is, in the sense of our reducibility relation, an upper bound for the sequence $P_0, P_1, P_2, \ldots$.

It is not its least upper bound. We may form another set, notation: $\bigcup_{n \in \mathbb{N}} P_n$, as follows:

for every $\alpha$, $\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} P_n$ if and only if $\alpha \circ S$, that is, the composition of $\alpha$ and the successor function, or "$\alpha$ without its first element", or $\lambda n \cdot \alpha(n + 1)$, belongs to $P_\alpha(0)$.

We might call $\bigcup_{n \in \mathbb{N}} P_n$ the disjoint union or the sum of the sequence $P_0, P_1, P_2, \ldots$.

One may verify that each set $P_n$ reduces to $\bigcup_{n \in \mathbb{N}} P_n$, and that $\bigcup_{n \in \mathbb{N}} P_n$ reduces to every $\mathcal{E} Q_n$ of $\mathcal{N}$ such that each $P_n$ reduces to $Q$. The class $\Sigma_2^0$ is closed both under the operation of countable disjoint union and under the operation $\mathcal{E}$.

The operation $\mathcal{E}$ may remind the reader of the sets introduced in Section 3.4.2. For each non-empty stump $\sigma$, the set $C_\sigma$ introduced in that section is obtained from the sequence of sets $C_\sigma^0, C_\sigma^1, \ldots$ by an operation similar to $\mathcal{E}$.

A subset $X$ of $\mathcal{N}$ is called a dense subset of $\mathcal{N}$ if and only if for each $a$ there exist $\alpha, m$ such that $\bar{a}m = a$ and $\alpha$ belongs to $X$. A subset of $\mathcal{N}$ is called a co-dense subset of $\mathcal{N}$ if and only if $\mathcal{N} \setminus X$ is a dense subset of $X$.

### 3.6.2

Let $P_0, P_1, P_2, \ldots$ be a sequence of co-dense subsets of $\mathcal{N}$ such that for each $n, p$ there exists $m$ such that $m > p$ and $P_m$ does not reduce to $P_n$.

We claim that the set $D(A_1, \mathcal{E} P_n)$ does not reduce to the set $\mathcal{E} P_n$. Let us prove this claim.

Suppose $g$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D(A_1, \mathcal{E} P_n)$ to $\mathcal{E} P_n$. We make three observations and obtain a contradiction.

**Observation (i):** For all $\alpha$, if $\alpha^{1,0} = 0$, then $(g|\alpha)^0 = 0$.

For suppose $\alpha^{1,0} = 0$ and $(g|\alpha)^0 \neq 0$. Let $n$ be the least $k$ such that $(g|\alpha)^0(k) \neq 0$. As $g$ is continuous there exists $m$ such that for all $\beta$, if and only if $\bar{m}m = \bar{a}m$, then $n$ is the least $k$ such that $(g|\beta)^0(k) \neq 0$.

We determine $p > m$ such that $P_p$ does not reduce to $P_n$.

We define a mapping $h$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\beta$, $h|\beta m = \bar{a}m$ and $p$ is the least $k$ such that $(g|\beta)^1(0)(k) \neq 0$ and $(h|\beta)^1(p+1) = \beta$ and $(h|\beta)^0 \neq 0$.

Observe that for each $\beta$, $\beta$ belongs to $P_p$ if and only if $h|\beta$ belongs to $D(A_1, \mathcal{E} P_n)$ if and only if $g(h|\beta)$ belongs to $\mathcal{E} P_n$ if and only if $(g(h|\beta))^{n+1}$ belongs to $P_n$.

So $P$ reduces to $P_n$. Contradiction.

Clearly, for all $\alpha$, if $\alpha^{1,0} = 0$, then also $(g|\alpha)^0 = 0$ and our first observation is proved.

**Observation (ii):** There exists $m$ such that for all $\alpha$, if $\bar{a}m = \bar{b}m$ and $\alpha^0 = 0$, then $(g|\alpha)^0 = 0$. In order to see this, observe that the set $\{\alpha|\alpha^0 = 0\}$ is a spread that
contains 0 and is contained in \( D(A_1, \mathcal{C} P_n) \).

Observe that for every \( \alpha \), if \( \alpha^0 = 0 \), then \( \alpha \) belongs to \( \mathcal{C} P_n \), and apply the continuity principle. Either there exists \( m \) such that for all \( \alpha \), if \( \overline{\alpha m} = \overline{\alpha m} \) and \( \alpha^0 = 0 \), then \( (g(\alpha))^0 = 0 \), (that is: the desired conclusion) or there exists \( m \), such that for all \( \alpha \), if \( \overline{\alpha m} = \overline{\alpha m} \) then \( (g(\alpha))^0 \neq 0 \). But the latter of these two alternatives contradicts observation (i) as there exists \( \alpha \) such that \( \overline{\alpha m} = \overline{\alpha m} \) and \( \alpha^{1,0} = 0 \).

Observation (iii): From observations (i), (ii) it follows that \( D^2(A_1) \) reduces to \( A_1 \): let us prove this.

Let \( m \) be a number with the property mentioned in Observation (ii).

We first determine for each \( n \) a sequence \( \beta_n \) such that \( \beta_n \) does not belong to \( P_n \) and \( \overline{\beta_n m} = \overline{\alpha m} \). We then construct a mapping \( h \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for each \( \alpha \), \( (h(\alpha))^0 = \overline{\alpha m} \), \( (h(\alpha))^{1,0} = \overline{\alpha m} \), and for each \( n \), \( (h(\alpha))^{1,n+1} = \beta_n \). Observe that, for each \( \alpha \) belongs to \( D^2(A_1) \) if and only if \( h(\alpha) \) belongs to \( D(A_1, \mathcal{C} P_n) \) if and only if \( g(h(\alpha)) \) belongs to \( \mathcal{C} P_n \) if and only if \( (g(h(\alpha)))^0 = 0 \). So \( D^2(A_1) \) reduces to \( A_1 \).

But, as we know, \( D^2(A_1) \) does not reduce to \( A_1 \). Contradiction.

3.6.3

Let \( X, Y \) be subsets of Baire space \( \mathcal{N} \). We say that \( X \) strictly reduces to \( Y \), notation: \( X \prec Y \), if and only if \( X \) reduces to \( Y \) but \( Y \) does not reduce to \( X \).

Let \( P_0, P_1, P_2, \ldots \) be a sequence of subsets of \( \mathcal{N} \) such that, for all \( n \), \( P_n \prec D(A_1, P_n) \) and for all \( n \) there exists \( m > n \) such that \( P_n \leq P_m \), and for all \( n, p \), there exists \( \alpha \) such that \( \alpha p = \overline{\alpha p} \) and \( \alpha \) does not belong to \( P_n \).

One may establish by a slight variation of the proof given in 3.6.1 that \( \mathcal{C} P_n \prec D(A_1, \mathcal{C} P_n) \). This statement also follows from our more general Theorem 3.6.5.

3.6.4

Let \( P_0, P_1, \ldots, P_{n-1} \) be a finite sequence of subsets of \( \mathcal{N} \).

We define subsets \( D(P_0, P_1, \ldots, P_{n-1}) \) and \( C(P_0, P_1, \ldots, P_{n-1}) \) of \( \mathcal{N} \) as follows. For each \( \alpha \), \( \alpha \) belongs to \( D(P_0, P_1, \ldots, P_{n-1}) \) if and only if there exists \( j < n \) such that \( \alpha^j \) belongs to \( P_j \) and \( \alpha \) belongs to \( C(P_0, P_1, \ldots, P_{n-1}) \) if and only if for all \( j < n \), \( \alpha^j \) belongs to \( P_j \).

If for all \( j < n \), \( P = P_j \) we write \( D^n(P) \) for \( D(P_0, P_1, \ldots, P_{n-1}) \) and \( C^n(P) \) for \( C(P_0, P_1, \ldots, P_{n-1}) \).

3.6.5

Let \( P_0, P_1, P_2, \ldots \) be a sequence of co-dense subsets of \( \mathcal{N} \) such that for each \( n \), \( A_1 \) reduces to \( P_n \) and there exists \( m \) such that \( P_m \) does not reduce to \( D(P_0, P_1, \ldots, P_{n-1}) \).

Define \( Q := \mathcal{C} P_n \): we assert that \( Q \) is a co-dense subset of \( \mathcal{N} \) and that for all \( k, \ell \), the set \( D(D^k(A_1), D^\ell(Q)) \) reduces strictly to the set \( D(D^{k+1}(A_1), D^\ell(Q)) \).
The proof of this assertion takes several steps. It is easy to see that \( \mathcal{N} \setminus \mathcal{Q} \) is dense in \( \mathcal{N} \) and that \( D(D^k(A_1), D^l(Q)) \) reduces to \( D(D^{k+1}(A_1), D^l(Q)) \). We only have to show that, conversely, \( D(D^{k+1}(A_1), D^l(Q)) \) does not reduce to \( D(D^k(A_1), D^l(Q)) \).

3.6.5.1 To this end, we assume that \( g \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( D(D^{k+1}(A_1), D^l(Q)) \) to \( D(D^k(A_1), D^l(Q)) \).

For every \( a, p \) we define a number \( c(a, p) \) as follows:

\[
c(a, p) := \# \{ j \mid j < \ell | a^{1,0,0} p = \bar{0} \}.
\]

Observe that, for each \( a, p, \ell \geq c(a, p) \geq c(a, p + 1) \).

We claim that for every \( j < \ell \), for every \( a \), if for every \( p, c(a, p) > j \) then for every \( p, c(g|a, p) > j \). We prove this by induction.

3.6.5.2 We first prove: if for every \( p, c(a(p)) > 0 \), then for every \( p, c(g|a, p) > 0 \).

Suppose that we find some \( a \) such that for every \( p, c(a, p) > 0 \), and that we also find some \( r \) such that \( c(a, r) = 0 \).

We calculate \( n_0, n_1, \ldots, n_{\ell-1} \) such that for each \( j < \ell \), \( n_j \) is the least \( p \) such that \((g|a)^{1,0,0}(p) \neq 0 \). We calculate \( m \) such that for every \( \beta \), if \( \bar{\beta}m = \bar{a}m \), then for each \( j < \ell \), \( n_j \) is the least \( p \) such that \((g|\beta)^{1,0,0}(p) \neq 0 \).

Observe that for each \( \beta \), if \( \bar{\beta}m = \bar{a}m \), then \( g|\beta \) belongs to \( D(D^k(A_1), D^l(Q)) \) if and only if either \((g|\beta)^{0,0}(p) \neq 0 \) and for every \( j < \ell \), \((g|\beta)^{1,0,0}(p) \neq 0 \) for every \( j < \ell \), \((h|\beta)^{1,0,0}(p) \neq 0 \).

Observe that for all \( \beta \), if \( \bar{\beta}m = \bar{a}m \), then \( \beta \) belongs to \( D(D^{k+1}(A_1), D^l(Q)) \) if and only if \( h|\beta \) belongs to \( D(D^k(A_1), D^l(Q)) \) if and only if \( h|\beta \) belongs to \( D(A_1, P_{n_0}, P_{n_1}, \ldots, P_{n_{\ell-1}}) \). We determine \( N > m \) such that \( P_N \) does not reduce to \( D(D^k(A_1), D^l(Q)) \).

We also determine a function \( f \) such that for every \( \beta \), \( f|\beta m = \bar{a}m \) and for every \( \beta \), \( \beta \) belongs to \( P_N \) if and only if \( f|\beta \) belongs to \( D(D^{k+1}(A_1), D^l(Q)) \).

(We obtain this function \( f \) as follows. As \( c_a(m) > 0 \), we may assume that \( a^{1,0,0}m = \bar{0}m \). We take care that for each \( \beta \), \( f|\beta m = \bar{a}m \), and \( N \) is the least \( p \) such that \((f|\beta)^{1,0,0}(p) \neq 0 \) and \((f|\beta)^{0,0,N+1} = \beta \) and \((f|\beta)^{0,0,N+1} \neq 0 \) does not belong to \( D^k(A_1) \) and for each \( j \), if \( 0 < j < \ell \), then \((f|\beta)^{1,0,0,J} \neq 0 \) does not belong to \( Q \).

Observe that \( h \circ f \) reduces \( P_N \) to \( D(D^k(A_1), P_{n_0}, P_{n_1}, \ldots, P_{n_{\ell-1}}) \).

Contradiction.

We conclude that if for every \( p, c(a, p) > 0 \), then for every \( p, c(g|a, p) > 0 \).

3.6.5.3 Assume now that for some \( j \) such that \( j + 1 < \ell \) we have seen that for all \( a \), if for every \( p, c(a, p) > j \), then for every \( p, c(g|a, p) > j + 1 \).

We want to show for all \( a \), if for every \( p, c(a, p) > j + 1 \), then for every \( p, c(g|a, p) > j + 1 \).

Suppose that we find some \( a \) such that for every \( p, c(a, p) > j + 1 \) and that we also find some \( q \) such that \( c(g|a, q) \leq j \).

We will obtain a contradiction.
We calculate \( m \) such that for every \( \beta \), if \( 3m = \alpha m \), then \( c(\beta; q) = c(\beta; q) \).

We consider the set \( B := \{ \beta | 3m = \alpha m \} \) and for every \( p \), \( c(\beta; p) \geq j \).

Observe that \( B \) is a spread and that the induction hypothesis implies that for every \( \beta \) in \( B \), and every \( p \geq q \), \( c(\beta; p) > j \) for every \( \beta \) in \( B \), then \( \beta \) belongs to \( D(D^k(A_1), D^\ell(Q)) \) and \( \beta \) itself belongs to \( D(D^{k+1}(A_1), D^\ell(Q)) \).

So the spread \( B \) is a subset of the set \( D(D^{k+1}(A_1), D^\ell(Q)) \) and \( \alpha \) belongs to \( B \).

Using the continuity principle we find \( q, i \) such that either for every \( \beta \) in \( B \), if \( \beta q = \alpha q \) and for every \( p \), \( c(\beta; p) > j \), then \( \beta \) is \( 0 \) or for every \( \beta \), if \( \beta q = \alpha q \) and for every \( p \), \( c(\beta; p) \geq j \), then \( \beta \) belongs to \( Q \). The first alternative is obviously false; and the second one is seen to be false by another application of the continuity principle, as follows.

We determine \( r, n \) such that either for every \( \beta \), if \( \beta r = \alpha r \) and for every \( p \), \( c(\beta; p) > j \), then \( \beta \) is \( 0 \), or for every \( \beta \), if \( \beta r = \alpha r \) and for every \( p \), \( c(\beta; p) \geq j \), then \( \beta \) belongs to \( Q \). The first of these two sub-alternatives is false as \( c(\alpha, r) > j + 1 \) so we may choose \( \beta \) such that \( \beta \) is \( 0 \), \( \beta r = \alpha r \) and for every \( p \), \( c(\beta; p) > j \). The second one is also false, as \( A_i \setminus P_n \) is dense in \( A_i \). Contradiction.

Clearly, for all \( \alpha \), if for every \( p \), \( c(\alpha; p) > j + 1 \), then for every \( p \), \( c(\alpha; p) > j + 1 \).

We now consider, for each \( i < k + 1 \), the set \( B_i = \{ \alpha_0'^i = 0 \} \). Observe that each set \( B_i \) is a spread contained in \( D(D^{k+1}(A_1), D^\ell(Q)) \) and that the sequence \( 0 \) belongs to every \( B_i \).

We observe that for each \( i < k + 1 \), for each \( \beta \) in \( B_i \), \( \beta \) belongs to \( D(D^k(A_1), D^\ell(Q)) \) and apply the continuity principle. For each \( i < k + 1 \) we may determine \( r_i, q_i, m_i \) such that either (i) for each \( \beta \) in \( B_i \), if \( \beta r_i = \alpha r_i \), then \( (\beta; r_i) \geq 0 \) or (ii) for each \( \beta \) in \( B_i \), if \( \beta r_i = \alpha r_i \), then \( (\beta; r_i) \geq 0 \) or (iii) for each \( \beta \) in \( B_i \), if \( \beta r_i = \alpha r_i \), then \( (\beta; r_i) \geq 0 \).

The alternative (iii) is excluded by the conclusion formulated in the first sentence of this Section, as for every \( p \), \( c(\alpha, p) = 1 \).

We also consider, for each \( j < \ell \), the set \( C_j = \{ \alpha_1'^j, 0 = 0 \} \). Again, each set \( C_j \) is a spread contained in \( D(D^{k+1}(A_1), D^\ell(Q)) \) and the sequence \( 0 \) is a member of each \( C_j \).

Using the continuity principle and reasoning as above we may determine, for each \( j < \ell \), numbers \( p_j, r_j, s_j \) such that either (i) for each \( \beta \) in \( C_j \), if \( \beta r_j = \alpha r_j \), then \( (\beta; r_j) \geq 0 \) or (ii) for each \( \beta \) in \( C_j \), if \( \beta r_j = \alpha r_j \), then \( (\beta; r_j) \geq 0 \). We now may reduce \( D^{k+1}(A_1) \) to \( D^{k+1}(A_1) \). We first build a function \( h \) from \( N \) to \( N \) as follows. Let \( N \) be the greatest of the numbers \( n_0, n_1, \ldots, n_k, p_0, \ldots, p_\ell - 1 \). For each \( \alpha \) in \( N \), for each \( i < k + 1 \), we define \( (h|\alpha)^0, i := 0 \) if \( \alpha \) and for each \( j < \ell \), we define \( (h|\alpha)^1, 0 := 0 \). Furthermore, we define \( h|\alpha \) in such a way that for each \( j < \ell \), for each \( n \), \( (h|\alpha)^1, n+1 \) does not belong to \( P_n \).

Observe that for every \( \alpha \), \( \alpha \) belongs to \( D^{k+1}(A_1) \) if and only if \( h|\alpha \) belongs to
Productive upper bounds

\( D(D^{k+1}(A_1), D^l(Q)) \) if and only if there exists \( i < k \) such that \( (g \circ h^i|_0)_0 = 0 \) or there exists \( j < l \) such that \( (g \circ h^j|_0)_0 = 0 \), that is, \( D^{k+l+1}(A_1) \) reduces to \( D^{k+l}(A_1) \).

But, as we saw in Section 3.2.4, \( D^{k+l+1}A_1 \) does not reduce to \( D^{k+l}A_1 \). Contradiction. This concludes the proof of the assertion in Section 3.6.5.

3.6.6

The following remark is a consequence of the result of section 3.6.5.

Let \( P_0, P_1, P_2, \ldots \) be a sequence of co-dense subsets of \( \mathcal{N} \) and assume that for each \( n \) there exists \( m \) such that the set \( P_m \) does not reduce to the set \( D(P_0, P_1, \ldots, P_{n-1}) \).

Let \( Q := \bigwedge_{n \in \mathbb{N}} P_n \). Then, for all \( l \), \( D^l(Q) \alpha D^l+1(Q) \), that is, \( D^l(Q) \) reduces to \( D^{l+1}(Q) \), but \( D^{l+1}(Q) \) does not reduce to \( D^l(Q) \).

(Observe that \( A_1 \) reduces to \( Q \). So according to 3.6.5, for each \( l \), \( D^lQ \alpha D(A_1, D^lQ) \alpha D^{l+1}(Q) \).

3.6.7

It will be clear that the result 3.6.6 may be applied repeatedly. Starting from the sequence \( A_1 \alpha D^2(A_1) \alpha D^3(A_1) \ldots \) we first form \( Q_0 := \bigwedge_{n \in \mathbb{N}} D^n(A_1) \) and observe \( Q_0 \alpha D^2(Q_0) \alpha D^3(Q_0) \ldots \), we then form \( Q_1 := \bigwedge_{n \in \mathbb{N}} D^n(Q_0) \) and observe \( Q_1 \alpha D^2(Q_1) \alpha \ldots \). We now define a sequence \( Q_0, Q_1, \ldots \) of subsets of \( \mathcal{N} \), as follows: for each \( m \), \( Q_{m+1} := \bigwedge_{n \in \mathbb{N}} D^n(Q_m) \). Then we “diagonalize” and form \( Q_\omega := \bigwedge_{m \in \mathbb{N}} Q_m \). We can go further and further but we never leave the class \( \Sigma_2^0 \) of countable unions of closed sets.

We now want to prove a counterpart to Corollary 3.6.6. Let \( X \) be a subset of \( \mathcal{N} \). \( X \) is inhabited if and only if there exists \( \alpha \) such that \( \alpha \) belongs to \( X \) and \( X \) is co-inhabited if and only if there exists \( \alpha \) such that \( \alpha \) does not belong to \( X \).

3.6.8

Let \( P_0, P_1, P_2, \ldots \) be a sequence of inhabited subsets of \( \mathcal{N} \) and assume that for each \( n \) there exists \( m \) such that \( P_m \) does not reduce to \( C(P_0, P_1, \ldots, P_{n-1}) \).

Let \( Q := \bigwedge_{n \in \mathbb{N}} P_n \). We assert that \( Q \) is an inhabited subset of \( \mathcal{N} \), and that for all \( l \), \( C^l(Q) \alpha C^{l+1}(Q) \), that is, the set \( C^l(Q) \) reduces to the set \( C^{l+1}(Q) \) but the set \( C^{l+1}(Q) \) does not reduce to the set \( C^l(Q) \).

The proof of this assertion takes several steps.

Let \( P_0, P_1, P_2 \ldots \) be a sequence of subsets of \( \mathcal{N} \) satisfying the conditions of the theorem. Define \( Q := \bigwedge_{n \in \mathbb{N}} P_n \)
3.6.8.1 Our first observation is that $Q$ is not a closed set. We prove this as follows. Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be a sequence of elements of $N$ such that for each $n$, $\alpha_n$ belongs to $P_n$. Let $f$ be a function from $N$ to $N$ such that for each $\alpha$, $(f(\alpha))^0 = \alpha$ and, for each $n$, $(f(\alpha))^{n+1} = \alpha_n$. Then, for each $\alpha$, $\alpha = 0$ or $\alpha \neq 0$ if and only if $Q(f(\alpha))$.

Suppose $Q$ is closed. Observe that for each $\alpha, n$, there exists $\beta$ in $Q$ such that $(f(\alpha)n) = \beta n$, so $f(\alpha)$ itself belongs to $Q$ and we may decide $\alpha = 0$ or $\alpha \neq 0$. Using the continuity principle we find an $m$ such that for all $\alpha$, if $\alpha m = 0$ then $\alpha = 0$. Contradiction.

3.6.8.2 We have to prove that for each $n$, $C_n(Q) < C_{n+1}(Q)$.

It is easy to see that $Q$ is an inhabited subset of $N$ and that, for each $n$, $C_n(Q)$ reduces to $C_{n+1}(Q)$.

We have to show that, for each $n$, $C_{n+1}(Q)$ does not reduce to $C_n(Q)$.

To this end, assume that $n$ is a natural number and that $g$ is a function from $N$ to $N$ reducing $C_{n+1}(Q)$ to $C_n(Q)$. For every $\alpha, p, n$ we define a natural number $c_n(\alpha, p)$ as follows:

$$c_n(\alpha, p) := \# \{ j \mid j < n, (g(\alpha)^j)^0 p = 0 \}.$$

3.6.8.3 We claim that for every $\alpha$, if $\alpha$ belongs to $C_{n+1}(Q)$ and, for every $p$, $c_{n+1}(\alpha, p) > 0$ then, for every $p$, $c_n(g(\alpha), p) > 0$.

In order to prove this, let us assume that we find some $\alpha$ such that $\alpha$ belongs to $C_{n+1}(Q)$ and for every $p$, $c_{n+1}(\alpha, p) > 0$ and that we also find some $q$ such that $c_n(g(\alpha), q) = 0$. We then determine numbers $\ell_0, \ell_1, \ldots, \ell_{n-1}$ such that, for each $j < n$, $\ell_j$ is the least $p$ such that $(g(\alpha)^j)^0 p \neq 0$. We also determine $m$, such that for every $\beta$, if $\beta m = \alpha m$, then for each $j < n$, $\ell_j$ is the least $p$ such that $(g(\alpha)^j)^0 p \neq 0$. There exists $j < n + 1$ such that $\alpha^j m = 0$ and without loss of generality we assume $\alpha^j m = 0$. We now calculate $N$ such that $N > m$ and $P_N$ does not reduce to $C(P_{\ell_0}, P_{\ell_1}, \ldots, P_{\ell_{n-1}})$.

We build a function $h$ from $N$ to $N$ such that, for each $\beta$, $h(\beta) = \alpha m$, and $N$ is the least $p$ such that $(h(\beta)^0 p) \neq 0$ and $(h(\beta)^0 m) = 0$ and for all $j > 0$, $(h(\beta)^j) = \alpha^j$. Observe that for each $\beta$, $\beta$ belongs to $P_N$ if and only if $(h(\beta)^0 m) = 0$ and only if $h(\beta)$ belongs to $C_{n+1}(Q)$ if and only if for each $j < n$, $(h(\beta)^j)^{\ell_{j+1}}$ belongs to $P_{\ell_j}$, that is: $P_N$ reduces to $C(P_{\ell_0}, P_{\ell_1}, \ldots, P_{\ell_{n-1}})$. Contradiction.

3.6.8.4 Suppose that we have shown, for some $j < n$, for every $\alpha$ in $C_{n+1}(Q)$, if for every $p$, $c_{n+1}(\alpha, p) > j$ then for every $p$, $c_n(g(\alpha), p) > j$.

We now prove: for every $\alpha$ in $C_{n+1}(Q)$, if for every $p$, $c_{n+1}(\alpha, p) > j + 1$ then for every $p$, $c_n(g(\alpha), p) > j + 1$.

Suppose we find $\alpha$ in $C_{n+1}(Q)$, such that for every $p$, $c_{n+1}(\alpha, p) > j + 1$ and that we also find some $q$ such that $c_n(g(\alpha), q) = j$. Without loss of generality we may further assume $c_n(g(\alpha), q) = j$ and for every $i < n$, $(g(\alpha)^i)^0 q = 0$ if and only if $i < j$.

We determine $m$ such that for every $\beta$, if $\beta m = \alpha m$, then for every $i < n$, $(g(\beta)^i)^0 q = (g(\alpha)^i)^0 q$. As $c_{n+1}(\alpha, m) > j + 1$ we assume: for every $i \leq j$, $\alpha^i m = 0$. 

We determine \( \ell_1, \ldots, \ell_{n-1} \) such that for every \( i \), if \( j \leq i \leq n - 1 \), then \( \ell_i < q \) and \( \ell_i \) is the least \( p \) such that \( (g|\alpha)^{i,0}(p) \neq 0 \).

We calculate \( N \) such that \( N > m \) and \( P_N \) does not reduce to \( C(P_{i_1}, \ldots, P_{n-1}) \).

We now define a function \( h \) from \( \mathcal{N} \) to \( \mathcal{N} \), in such a way that for every \( \beta \), \( h|\beta m = \alpha m \) and for every \( i \leq j \), \( h|\beta^{i,0} = 0 \) and \( N \) is the least \( p \) such that \( (h|\beta)^{j,0}(p) = 0 \) and \( (h|\beta)^{j,N+1} = \beta \) and for all \( i \), if \( j < i < n + 1 \) then \( (h|\beta)^{i} = \alpha^{i} \).

Observe that for every \( \beta \), for every \( a \), \( c_{n+1}(h|\beta, p) \geq j \), therefore, by the induction hypothesis, for every \( p \), \( c_{n+1}(g(h|\beta), p) \geq j \), and consequently, for every \( i < j \), \( (g(h|\beta))^{i,0} = 0 \) and thus: \( (g(h|\beta))^{j} \) belongs to \( Q \).

Therefore, for every \( \beta \), \( \beta \) belongs to \( P_N \) if and only if \( (h|\beta)^{j} \) belongs to \( Q \) if and only if \( h|\beta \) belongs to \( C^{n+1}(Q) \) if and only if \( g \circ h|\beta \) belongs to \( C^n(Q) \) if and only if for every \( i \), if \( j < i < n - 1 \), then \( (g \circ h|\beta)^{i,j} = 0 \).

So \( P_N \) reduces to \( C(P_1, \ldots, P_{n-1}) \). Contradiction.

### 3.6.8.5
We thus obtain the following conclusion:
for every \( j \leq n \), for every \( \alpha \) in \( C^{n+1}(Q) \), if, for every \( p \), \( c_{n+1}(\alpha, p) > j \), then for every \( p \), \( c_{n}(g(\alpha), p) > j \).

In particular, for every \( \alpha \) in \( C^{n+1}(Q) \), if for every \( p \), \( c_{n+1}(\alpha, p) \geq n \), then for every \( p \), \( c_{n}(g(\alpha), p) = n \), that is, for every \( j < n \), \( (g(\alpha))^{j,0} = 0 \).

We now define a function \( h \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \beta \), for every \( j < n \), \( (h|\beta)^{j} = 0 \) and \( (h|\beta)^{n} = \beta \). Then for every \( \beta \), \( \beta \) belongs to \( Q \) if and only if \( h|\beta \) belongs to \( C^{n+1}(Q) \) if and only if, for every \( j < n \), \( (g \circ h|\beta)^{j,0} = 0 \).

Therefore \( Q \) reduces to \( A_1 \) and \( Q \) is a closed subset of \( \mathcal{N} \).

However, in Section 3.6.8.1 we saw that \( Q \) is not a closed subset of \( \mathcal{N} \).
Therefore \( C^{n+1}(Q) \) does not reduce to \( C^n(Q) \).

### 3.6.9
Theorem 3.6.8 may be used to build hierarchies within the class \( \Sigma_2^{0} \) of countable unions of closed sets, very much like its counterpart, Theorem 3.6.6, see Remark 3.6.7.

### 3.7 Finite intersections of finite unions of closed sets

#### 3.7.1
The set \( C^2(D^2(A_1)) \) does not reduce to the set \( D^2(A_1) \).

In order to see this, observe that \( C^2(D^2(A_1)) \) coincides with the union of four spreads, \( B_{00}, B_{01}, B_{10} \) and \( B_{11} \) where for all \( i, j < 2 \), \( B_{ij} := \{ a | (\alpha^i)^j = 0 \text{ and } (\alpha^j)^i = 0 \} \) and that \( 0 \) belongs to each of these spreads.

Assume that \( f \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( C^2(D^2(A_1)) \) to \( D^2(A_1) \). Using the continuity principle four times we find \( N \) in \( N \) and for all \( i, j < 2 \) a number \( m_{i,j} < 2 \) such that for all \( \alpha \) in \( B_{ij} \), if \( (\alpha|\alpha)^{m_{i,j}} = 0 \), then \( (f|\alpha)^{m_{i,j}} = 0 \).

Without loss of generality we assume \( m_{00} = m_{01} = 0 \).

It is not difficult to see that for every \( \alpha \), if \( \alpha \) assumes at most one time a value different
from 0, then \((f|\alpha)^0 = 0\), therefore \(f|\alpha\) belongs to \(C^2(D^2(A_1))\) and \(\alpha\) itself belongs to \(C^2(D^2(A_1))\).

Consider \(AM_1 := \{\alpha\} \) For all \(m, n \) if \(\alpha(m) \neq 0 \) and \(\alpha(n) \neq 0\), then \(m = n\) and observe that this set is a spread.

We leave it to the reader to derive, using the continuity principle, a contradiction from the supposition that each \(\alpha\) in \(AM_1\) belongs to \(D^2(D^2(A_1))\) and \(\alpha\) itself belongs to \(C^2(D^2(A_1))\).

A slight extension of the above argument shows that \(C^2(D^2(A_1))\) does not reduce to \(D^3(A_1)\). More generally let \(n_0, n_1, \ldots, n_{k-1}\) be a finite sequence of nonzero natural numbers and consider the set \(P := \bigcap_{j<k} C(D^{n_j}(A_1), D^{n_{j+1}}(A_1), \ldots, D^{n_{k-1}}(A_1))\). Let \(n := \prod_{j<k} n_j\). One may prove that \(P\) reduces to \(D^n(A_1)\) but not to \(D^{n-1}(A_1)\).

Observe that \(D^3(A_1)\) does not reduce to \(C^2(D^2(A_1))\). We have seen, in Theorem 3.2.4 that \(D^3(A_1)\) does not reduce to \(\bigcap_{j<k} C(D^{n_j}(A_1), D^{n_{j+1}}(A_1), \ldots, D^{n_{k-1}}(A_1))\), and obviously, \(C^2(D^2(A_1))\) reduces to \(\bigcap_{j<k} C(D^{n_j}(A_1), D^{n_{j+1}}(A_1), \ldots, D^{n_{k-1}}(A_1))\).

3.7.2

We wish to show how one may decide, given any two finite sequences \((m_0, m_1, \ldots, m_{s-1})\) and \((n_0, n_1, \ldots, n_{t-1})\) of nonzero natural numbers, if the set \(C(D^{m_0}(A_1), D^{m_1}(A_1), \ldots, D^{m_{s-1}}(A_1))\) reduces to the set \(C(D^{n_0}(A_1), D^{n_1}(A_1), \ldots, D^{n_{t-1}}(A_1))\).

We first introduce some notation. Given natural numbers \(m, n\) we say \(m < n\) if and only if \(m, n\) are coding finite sequences of natural numbers of the same length, and for each \(i < \text{length}(m)\), \(m_i < n_i\).

We also define, for every \(m\), subsets \((CD)^m A_1\) and \((CD)^{m+1} A_1\), as follows. For every \(\alpha\) in \(N\), \(\alpha\) belongs to \((CD)^{m+1} A_1\) if and only if for every \(i < \text{length}(m)\) there exists \(p < (m)_i\) such that \(\alpha^{i,p} = 0\) and \(\alpha\) belongs to \((CD)^{m} A_1\) if and only if for every \(i < \text{length}(m)\), \(\alpha^{i,m}_i = 0\).

Observe that each set \((CD)^m A_1\) is a spread, and that for every \(m\), the set \((CD)^{m+1} A_1\) coincides with \(\bigcap_{n < m} (CD)^n A_1\).

If there exists \(j < \text{length}(m)\) such that \((m)_j = 0\), then \((CD)^m A_1 = \emptyset\).

3.7.3

Observe that for each \(m\), if \((m)_0 = 0\), then \((CD)^m A_1 = \emptyset\). Observe also that for each \(m\), \((CD)^{m+1} A_1\) is of the same degree of reducibility as \((CD)^m A_1\). By way of example, we show that the set \(C(D^3(A_1), D^2(A_1))\) reduces to the set \(D^2(A_1)\).

For every \(\alpha\), \(\alpha\) belongs to \(C(D^3(A_1), D^2(A_1))\) if and only if \(\alpha^{0,0} = 0\) and \(\alpha^{0,1} = 0\) or \(\alpha^{1,0} = 0\) and \(\alpha^{1,1} = 0\). We therefore define a function \(f\) from \(N\) to \(N\) such that for every \(\alpha, n\), \((f|\alpha)^0(2n) = (f|\alpha)^1(2n) = \alpha^{0,0}(n)\) and \((f|\alpha)^0(2n+1) = \alpha^{1,0}(n)\) and \((f|\alpha)^1(2n+1) = \alpha^{1,1}(n)\), then \(f\) will reduce the set \(C(D^3(A_1), D^2(A_1))\) to the set \(D^2(A_1)\).

For every nonegative rational number \(q\), we let \([q]\) be the greatest natural number \(n\) such that \(n \leq q\). For all \(n, j, p\) we define a natural number \(c = c(n, j, p)\) as follows:
length(c) = length(n) and for all $i < \text{length}(n)$, if $i \neq j$, then $(c)_i = (n)_i$ and if $j < \text{length}(n)$ and $p \leq (n)_j$ then $(c)_j = \left\lceil \frac{(n)_j}{p} \right\rceil$ and if $j < \text{length}(n)$ and $p > (n)_j$ then $(c)_j = 0$.

3.7.4 Theorem:

(i) For all $p, n$, the set $D^p(A_i)$ reduces to the set $(CD)^n A_1$ if and only if there exists $j < \text{length}(n)$ such that $p \leq (n)_j$.

(ii) For all $p, m, n$, the set $(CD)^{\ast}m A_i$ reduces to the set $(CD)^n A_1$ if and only if there exists $j < \text{length}(n)$ such that $p < (n)_j$ and $(CD)^{\ast}m A_1$ reduces to $(CD)^{\ast}(n, j, p) A_1$.

Proof: We only prove (ii) as it is not difficult to see that (ii) implies (i).

(Let 0 encode the empty sequence of natural numbers. Then $(CD)^0 A_1 = \mathcal{N}$ reduces to every set $(CD)^n A_1$.)

Let us prove that the given condition is necessary. So assume $f$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $(CD)^{\ast}m A_i$ to the set $(CD)^n A_1$. Using the continuity principle a finite number of times we calculate a number $s$ and construct a function $F$ associating to every number $q < (p)^{\ast}m$ a number $F(q) < n$ such that for every $a$, if $a$ belongs to $(CD)_q A_1$ and $\alpha = \emptyset s$, then $f|a$ belongs to $(CD)_{F(q)} A_1$. We claim that there must exist $j < \text{length}(n)$ such that for all $q, q' < (p)^{\ast}m$, if $(q)_0 \neq (q')_0$, then $(F(q))_j \neq (F(q'))_j$. As $F$ is a finite function we may decide in finitely many steps if there exists such a number $j$.

Suppose that we find no such number $j$. We then consider the set $B := \{a|a^0$ has at most one value different from 0 and for each $i > 0$, $\alpha^i = \emptyset\}$. We determine for every $j < \text{length}(n)$ numbers $q_j, q'_j < (p)^{\ast}m$ such that $(q_j)_0 \neq (q'_j)_0$ and $t_j := (F(q_j))_j = (F(q'_j))_j$.

Observe that for every $j < \text{length}(n)$, for every $a$ in $B$, for every $p$, there exists $\beta$ in $(CD)_q A_1 \cup (CD)_{q'} A_1$ such that $\beta p = \emptyset p$, and therefore: $(f|a)^{j, t_j} = 0$.

Therefore, for every $a$ in $B$, $f|a$ belongs to $(CD)^n A_1$, so $a$ itself belongs to $(CD)^{\ast}m A_1$ and $a^0$ belongs to $D^p(A_1)$.

We thus see that $AM_i := \{a|a$ assumes at most one time a value different from 0$\}$ is contained in $D^p(A_1)$. But $AM_i$ is a spread, and we calculate $m, i$ such that for every $a$ in $AM_i$, if $\alpha m = \emptyset m$, then $\alpha^i = \emptyset$. Contradiction.

So we are sure to find some $j < \text{length}(n)$ such that for all $q, q' < (p)^{\ast}m$, if $(q)_0 \neq (q')_0$, then $(F(q))_j \neq (F(q'))_j$. Let $j_0$ be such a number. Observe that $p$ cannot be greater than $(n)_j$. We now consider, for every $i < p$ the set

$$C_i := \{t|t < (n)_{j_0} \text{ there exists } q < (p)^{\ast}m \text{ such that } (q)_0 = i \text{ and } (F(q))_{j_0} = t\}.$$ 

Observe that for all $i, i' < p$, if $i < i'$, then $C_i \cap C_{i'} = \emptyset$.

We calculate $i_0$ such that the number of elements of $C_{i_0}$ is at most $\left\lceil \frac{(n)_{j_0}}{p} \right\rceil$. Observe that for all $\alpha$, if $\alpha^{i_0} = \emptyset$, and $\alpha s = \emptyset s$, then for every $j < \text{length}(m)$, $\alpha^{j+1}$ belongs to $D^{(\ast)} A_1$ if and only if for every $j < \text{length}(n)$, if $j \neq j_0$, then $(f|a)^{j}$ belongs to
$D^{(n)}_r(A_1)$ and there exists $t$ in $C_{l_0}$ such that $(f|\alpha)^{j_0,t} = 0$.

We conclude that the set $(CD)^mA_1$ reduces to the set $(CD)^c_{(n,j_0,p)}A_1$.

We now verify that the given condition is sufficient.

Suppose that the set $(CD)^mA_1$ reduces to the set $(CD)^c_{(n,j,p)}A_1$.

Remark that for all $i,j$, the set $C(D^i(A_1), D^j(A_1))$ reduces to the set $D^{i+j}(A_1)$. Using this remark we conclude that the set $(CD)^{m+p}A_1$ reduces to the set $CD^{q}A_1$ where 

$$\text{length}(q) = \text{length}(n), \text{ and for each } i < \text{length}(n), \text{ if } i \neq j, \text{ then } (q)_i = (n)_i \text{ and if } j < n, \text{ then } (q)_j = \left\lfloor \frac{n_j}{p} \right\rfloor \cdot p \leq (n)_j.$$ 

Therefore the set $(CD)^{m+p}A_1$ reduces to the set $(CD)^nA_1$.

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3.7.5

Given natural numbers $m,n$ we may decide in finitely many steps if $(CD)^mA_1$ reduces to $(CD)^nA_1$. We apply Theorem 3.7.4 and use induction on length$(m)$.
4 The finite Borel hierarchy theorem

In this Section, we study the first countably many levels of the Borel hierarchy. We restrict our attention to Baire space \( \mathcal{N} \).

4.1

We define a sequence \( A_1, E_1, A_2, E_2, \ldots \) of subsets of \( \mathcal{N} \), as follows:

(i) for each \( \alpha \) in \( \mathcal{N} \), \( \alpha \) belongs to \( A_1 \) if and only if for every \( m \), \( \alpha(m) = 0 \) and \( \alpha \) belongs to \( E_1 \) if and only if for some \( m \), \( \alpha(m) \neq 0 \).
(ii) for each \( n \), for each \( \alpha \), \( \alpha \) belongs to \( A_{n+1} \) if and only if, for every \( m \), \( \alpha^m \) belongs to \( E_n \), and \( \alpha \) belongs to \( E_{n+1} \) if and only if, for some \( m \), \( \alpha^m \) belongs to \( A_n \).

The proof of the following remarks is left to the reader:

(iii) for each \( n > 0 \), the pair \( (A_n, E_n) \) is a complementary pair of subsets of \( \mathcal{N} \) in the sense of Section 1.10, and therefore, for every \( \alpha \) in \( A_n \), for every \( \beta \) in \( E_n \), \( \alpha \) is apart from \( \beta \).
(iv) for each \( n > 0 \), \( A_n \) is a complete element of the class \( \Pi^0_n \), that is, for every subset \( X \) of \( \mathcal{N} \), \( X \) belongs to \( \Pi^0_n \) if and only if \( X \) reduces to \( A_n \). Similarly, \( E_n \) is a complete element of the class \( \Sigma^0_n \).

4.2

Let \( (P, Q) \) be a pair of subsets of \( \mathcal{N} \).
We call the pair \( (P, Q) \) incomparable if and only if \( P \) does not reduce to \( Q \) and \( Q \) does not reduce to \( P \).
We intend to show that, for each \( n \), the pair \( (A_n, E_n) \) is incomparable. But, as we saw in Section 2, we probably may prove more.
We therefore introduce the following notion:
Let \( (P, Q) \) be a pair of subsets of \( \mathcal{N} \).
We say that \( (P, Q) \) is strongly incomparable if and only if
(i) for every (continuous) function \( f \) from \( \mathcal{N} \) to \( \mathcal{N} \), if \( f \) maps \( P \) into \( Q \), then \( f \) maps some member of \( Q \) into \( Q \),
and (ii) for every (continuous) function \( f \) from \( \mathcal{N} \) to \( \mathcal{N} \), if \( f \) maps \( Q \) into \( P \), then \( f \) maps some member of \( P \) into \( P \).
Observe that every strongly incomparable pair is incomparable.
We have seen, in Section 2, that the pair \( (A_2, E_2) \) is strongly incomparable.
We will now prove that every pair \( (A_n, E_n) \) is strongly incomparable.

4.3

We need, for each \( n \), functions \( f_n \) and \( g_n \) such that \( A_n = \text{Ran}(f_n) \) and \( E_n = \text{Ran}(g_n) \).
We obtain them by induction.
We first define \( f_1 \) and \( g_1 \) as follows:
For every \( \alpha \), \( f_1|\alpha := 0 \) and for every \( \alpha \), \( (g_1(\alpha))(\alpha(0)) = \alpha(1) + 1 \) and for every \( m \) if \( m \neq \alpha(0) \), then \( (g_1|\alpha)(m) = \alpha(m + 2) \).
For each \( n, m, \alpha \) we define \((f_{n+1}|\alpha)^m := g_n((\alpha^m))\), and \((f_{n+1}|\alpha)(0) := \alpha(0)\).
For each \( n, \alpha \), we define \((g_{n+1}|\alpha)(0) := \alpha(0)\) and \((g_{n+1}|\alpha)^{(1)} := f_n((\alpha \circ S)^{(1)})\) and for every \( m \), if \( m \neq 1 \), then \((g_{n+1}|\alpha)^m = (\alpha \circ S)^m\). We leave it to the reader to verify that, for each \( n \), \( A_n = \text{Ran}(f_n)\) and \( E_n = \text{Ran}(g_n)\).
We thus see that each one of the sets \( A_1, E_1, \ldots \) is strictly analytical.

It is not true that every inhabited Borel set is strictly analytical. It is not even true that every inhabited and closed subset of \( \mathcal{N} \) is strictly analytical.

We will make use of the following property of the functions \( f_n \).

For every \( \gamma, \alpha, n, m \), if \((f_n|\gamma)^m = \tilde{\alpha}^m\) and \( \alpha \) belongs to \( A_n \), then there exists \( \delta \) such that \( \delta m = \tilde{\gamma}m \) and \( \alpha = f_n|\delta \).

4.4 Lemma:
The pair \((A_1, E_1)\) is strongly incomparable.

Proof: (i) Let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( A_1 \) into \( E_1 \). Determine \( n \) such that \( (f|\alpha)(0) ) \neq 0 \). Determine \( m \) such that for every \( \alpha \), if \( \tilde{\alpha}^m = \tilde{0}^m \), then \((f|\alpha)(n) = (f|\alpha)(0) \). Observe that the sequences \( \tilde{0}^m \ast 1 \) and \( f|\tilde{0}^m \ast 1 \) both belong to \( E_1 \).
(ii) Let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( E_1 \) into \( A_1 \). We claim that \( f|\alpha = 0 \). For suppose \( f|\alpha \neq 0 \) and determine \( n \) such that \((f|\alpha)(n) \neq 0 \). Determine \( m \) such that for every \( \alpha \), if \( \tilde{\alpha}^m = \tilde{0}^m \), then \((f|\alpha)(n) = (f|\alpha)(0) \). Observe that the sequences \( \tilde{0}^m \ast 1 \) and \( f|\tilde{0}^m \ast 1 \) both belong to \( E_1 \).
Contradiction. Therefore \( f|\alpha = 0 \) and both \( 0 \) and \( f|\alpha \) belong to \( A_1 \).

4.5 Lemma:
Let \( n > 0 \) and suppose that every function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( E_n \) into \( A_n \) also takes some member of \( A_n \) into \( A_n \).
Then every function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( A_{n+1} \) into \( E_{n+1} \) will take some member of \( E_{n+1} \) into \( E_{n+1} \).

Proof: Let \( g \) maps \( A_{n+1} \) into \( E_{n+1} \). For every \( \alpha \), \( f_{n+1}|\alpha \) belongs to \( A_{n+1} \), so there exists \( m \) such that \((g \circ f_{n+1}|\alpha)^m \) belongs to \( A_n \). Using the continuity principle we find \( p, m \) such that for every \( \alpha \), if \( \tilde{\alpha}p = \tilde{0}p \), then \((g \circ f_{n+1}|\alpha)^m \) belongs to \( A_n \). We now define a function \( h \) from \( \mathcal{N} \) to \( \mathcal{N} \), as follows.
For every \( \alpha \), \( (h|\alpha)(0) := 0 \) and for every \( i \neq p \), \( (h|\alpha)^i := (f_{n+1}|\alpha)^i \), but \( (h|\alpha)^p := \alpha \).
Observe that for every \( \alpha \), if \( \alpha \) belongs to \( E_n \), then \( h|\alpha \) belongs to \( A_{n+1} \) and there exists \( \gamma \) such that \( \tilde{\gamma}p = \tilde{0}p \) and \( h|\alpha = f_{n+1}|\gamma \), and therefore \((g \circ h|\alpha)^m \) belongs to \( A_n \).
We now determine \( \beta \) such that both \( \beta \) and \((g \circ h|\beta)^m \) belong to \( A_n \), and observe that both \( h|\beta \) and \( g \circ h|\beta \) belong to \( E_{n+1} \).
4.6 Lemma:

Let \( n > 1 \) and suppose that every function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( E_n \) into \( A_n \) also takes some member of \( A_n \) into \( A_n \).

Then every function from \( \mathcal{N} \) to \( \mathcal{N} \) that maps \( E_{n+2} \) into \( A_{n+2} \) will take some member of \( A_{n+2} \) into \( A_{n+2} \).

**Proof:** Suppose \( g \) maps \( E_{n+2} \) into \( A_{n+2} \).

We now build a mapping \( F \) from \( \mathbb{N} \times \mathcal{N} \) to \( \mathcal{N} \). For all \( m, \gamma, \alpha \), we define \( F(m, \gamma, \alpha)(0) = \alpha(0) \) and \( F(m, \gamma, \alpha))^{n+1} := f_{n+1}^\gamma \) and for each \( i \neq m \), \( F(m, \gamma, \alpha))^{i} := \alpha^i \). Observe that for all \( m, \gamma, \alpha \) the sequence \( F(m, \gamma, \alpha) \) belongs to \( E_{n+2} \), therefore \( g \circ F(m, \gamma, \alpha) \) belongs to \( A_{n+2} \), in particular \( (g \circ F(m, \gamma, \alpha))^{m} \) belongs to \( E_{n+1} \).

Using the axiom AC, we build functions \( c \) and \( d \) from \( \mathbb{N} \times \mathcal{N} \) to \( \mathcal{N} \) such that for all \( m, \gamma, \alpha, \gamma', \alpha' \), \( \gamma(c(m, \gamma, \alpha)) = \gamma'(c(m, \gamma, \alpha)) \) and \( \alpha(c(m, \gamma, \alpha)) = \alpha'(c(m, \gamma, \alpha)) \) then \( g \circ F(m, \gamma, \alpha))^{m} \) belongs to \( A_{n} \).

We now construct a sequence \( h_0, h_1, h_2, \ldots \) of functions from \( \mathcal{N} \) to \( \mathcal{N} \), and two sequences \( r_0, r_1, r_2, \ldots \) and \( t_0, t_1, t_2, \ldots \) of functions from \( \mathcal{N} \) to \( \mathbb{N} \), as follows.

We use induction and first define \( h_0, t_0 \) and \( r_0 \). For each \( \beta \), \( (h_0)(\beta)(0) := 0 \) and for each \( i \), \( (h_0)(\beta)^i := f_{n+1}^\beta(0) \). Observe that \( h_0 \beta = F(0,0,h_0 \beta) \) and define \( r_0(\beta) := c(0,0,h_0 \beta) \) and \( t_0(\beta) := d(0,0,h_0 \beta) \).

We now define \( h_1 \). For each \( \beta \), \( (h_1)(\beta)(0) := 0 \), and \( (h_1)(\beta)^i(0) := (h_0)(\beta)^i(0) \) and for each \( i \neq r_0(\beta), (h_1)(\beta)^{i+1} := (h_0)(\beta)^{i+1} \) but \( (h_1)(\beta)^i,r_0(\beta) := \beta \). Also, for each \( j > 0 \), \( (h_1)(\beta)^i := (h_0)(\beta)^i \).

Observe that for every \( \beta \), \( h_1(\beta) = F(1,0,h_1(\beta)) \).

Suppose that \( n \) is a positive natural number and that we constructed \( h_n \) and ensured that for every \( \beta \), \( h_n(\beta) = F(n,0,h_n(\beta)) \). We define \( r_n(\beta) := \max(r_{n-1}(\beta),c(n,0,h_n(\beta))) \) and \( t_n(\beta) := d(n,0,h_n(\beta)) \). We define \( h_{n+1} \) as follows. For each \( \beta \), \( (h_{n+1})(\beta)(0) := 0 \), and for each \( i \neq r_n(\beta), (h_{n+1})(\beta)^i := (h_n)(\beta)^i \), and \( (h_{n+1})(\beta)^n(0) := (h_n)(\beta)^n(0) \) and for each \( i \neq r_n(\beta), (h_{n+1})(\beta)^{n+1} := (h_n)(\beta)^{n+1} \) but \( (h_{n+1})(\beta)^n,r_n(\beta) := \beta \).

Finally, we build a function \( h \) from \( \mathcal{N} \) to \( \mathcal{N} \).

For each \( \beta \), we define \( (h)(\beta)(0) := 0 \) and for each \( i \), \( (h)(\beta)^i := (h_{n+1})(\beta)^i \). Observe that for all \( i, j \) if \( j > i \), then \( (h)(\beta)^i := (h)(\beta)^j \). So \( h(\beta) \) is the limit of the converging sequence \( h_0(\beta), h_1(\beta), \ldots \).

We claim that for each \( \beta \), if \( \beta \) belongs to \( E_n \), then for each \( m, (g \circ h)(\beta)^m \) belongs to \( A_n \). We prove this claim as follows. Assume that \( \beta \) belongs to \( E_n \) and let \( m \) be a natural number. Then \( (h)(\beta)^m \) coincides with \( (h_{n+1})(\beta)^m \) and there exists \( \gamma \) such that \( \gamma r_m(\beta) = \gamma r_m(\beta) \) and \( (h_{n+1})(\beta)^m = f_{n+1}^\gamma \). Observe that \( h(\beta) = F(m,\gamma,h(\beta)) \) and that \( (h)(\beta)^m(\beta) = (h_{n+1})(\beta)^m \).

(The latter equality is ensured by the fact that for each \( m, \beta \), \( r_m(\beta) \leq r_m(\beta) \).)

We may conclude that \( (g \circ h)(\beta)^m \) belongs to \( A_n \). Therefore, for each \( \beta \), if \( \beta \) belongs to \( E_n \), then for all \( m, i \), \( (g \circ h)(\beta)^m,i \) belongs to \( E_{n-1} \).

We define a function \( k \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that for all \( m, i \), \( (k)(\beta)^m,i = (g \circ h)(\beta)^m,i \).

Observe that, for all \( \beta \), if \( \beta \) belongs to \( E_n \), then \( k(\beta) \) belongs to \( A_n \). So we may determine \( \beta^* \) such that both \( \beta^* \) and \( k(\beta) \) belong to \( A_n \). Remark that both \( h(\beta^*) \) and \( g \circ h)(\beta^*) \) belong to \( A_{n+2} \).
Observe that we use the axiom $\text{AC}_{1,0}$ in the proof of Lemma 4.6.
Theorem 4.7 may be proven by means of the axiom $\text{CP}$ rather than the stronger axiom $\text{AC}_{1,0}$. This will follow from Section 5. Theorem 5.2 is stronger than Theorem 4.7 and is proven from $\text{CP}$ alone.

4.7 Theorem: (Finite Borel hierarchy theorem).

For each positive $n$, every function from $\mathcal{N}$ to $\mathcal{N}$ that maps $E_n$ into $A_n$ also takes some member of $A_n$ into $A_n$, and every function from $\mathcal{N}$ to $\mathcal{N}$ that maps $A_n$ into $E_n$, also takes some member of $E_n$ into $E_n$.

**Proof:** The case $n = 1$ has been treated in Lemma 4.4.
The case $n = 2$ has been treated in Theorems 2.7.4 and 2.7.6.
The general statement now follows by induction from Lemmas 4.5 and 4.6.

4.8 For every $n$, the class $\Pi^0_n$ is not closed under the operation of finite union

We have seen, in Section 3.2, that the set $D^2(A_1)$ does not reduce to the set $A_1$.
We want to prove now that, for each positive $n$, the set $D^2(A_n)$ does not reduce to the set $A_n$.
Actually, we prove that, for each positive $n$, the set $D(A_1, A_n)$ does not reduce to the set $A_n$.
The proof is not easy and we have to make some preparations.

4.8.1

For every $\alpha$ in $\mathcal{N}$, $\alpha$ in $\mathbb{N}$ we define an infinite sequence $^a\alpha$, as follows.
Determine the finite sequence coded by $a$, say $a = (a_0, a_1, \ldots, a_{k-1})$ and define: $^a\alpha := \alpha^{a_0, a_1, \ldots, a_{k-1}}$.
If $a = 0$, $a$ codes the empty sequence and $^a\alpha := \alpha$.

4.8.2

For all $\alpha$ in $\mathcal{N}$, $\alpha$ in $\mathbb{N}$, we define:
$\alpha$ passes through $a$ (or: $\alpha$ contains $\alpha$, or: $\alpha$ belongs to $a$), notation: $\alpha \in a$, if and only if $a \in a$.
So $\alpha \in a$ if and only if the finite sequence coded by $a$ forms an initial part of the infinite sequence $\alpha$. 
4.8.3 Lemma:

Let \( n \) be a positive natural number. For every natural number \( k \), for every infinite sequence \( \gamma \) there exist a natural number \( l \) and a finite set \( B \) of natural numbers, containing 0, with the following properties:

(i) For each \( a \) in \( B \), \( \text{length}(a) \) is even and \( \text{length}(a) < n \).

(ii) For all \( \alpha \) in \( \mathcal{N} \), if \( \alpha \) passes through \( (f_n|\gamma)l \) and for all \( a \) in \( B \), the sequence \( \alpha^a \) belongs to \( A_{n-\text{length}(a)} \), then there exists \( \delta \) passing through \( \gamma l \) such that \( \alpha = f_n|\delta \).

(iii) For all \( \alpha \), if there exists \( \delta \) passing through \( \gamma l \) such that \( \alpha = f_n|\delta \), then for all \( a \) in \( B \), the sequence \( \alpha^a \) belongs to \( A_{n-\text{length}(a)} \).

Proof: We use induction.

First suppose \( n = 1 \). Recall that for each \( \gamma \), \( f_1|\gamma = 0 \). Regardless of which \( k \), \( \gamma \) are given, we define \( l = 0 \) and \( B = \{0\} \).

Next suppose \( n = 2 \). Recall that for each \( \gamma \), \( (f_2|\gamma)(0) = \gamma(0) \) and for each \( m \), \( (f_2|\gamma)^m(\gamma^m(0)) = \gamma^m(1)+1 \) and for each \( i \), if \( i \neq \gamma^m(0) \), then \( (f_2|\gamma)^m(i) = \gamma^m(i+2) \).

Suppose we are given a number \( k \) and an infinite sequence \( \gamma \). We define \( l := \max(\gamma^m(0), l) + 1 \) and \( B := \{0\} \).

We now assume that \( n \) is a positive natural number and that we verified the first \( n+1 \) cases of the lemma. We are going to prove the case \( n+2 \). Recall that for every \( \gamma \), for every \( m \), \( (f_{n+2}|\gamma)^m(\gamma^m(0)) = f_n|(\gamma^m \circ S)^m(1) \) and for every \( m \), if \( i \neq \gamma^m(1) \), then \( (f_{n+2}|\gamma)^m(i) = (\gamma^m \circ S)^m(1) \).

Suppose we are given a number \( k \) and an infinite sequence \( \gamma \). For each \( \gamma^m(0) \) we define \( m := \max(\gamma^m(0), l)+1 \) and \( B := \{0\} \).

We apply the just proved case \( n \) of the Lemma \( k \) times and find for each \( m < k \) a number \( l_m \) and a finite set \( B_m \) of natural numbers such that for every \( a \), if \( a \) passes through \( f_n|B_m l_m \), and for every \( a \) in \( B_m \), the sequence \( \alpha^a \) belongs to \( A_{n-\text{length}(a)} \), then there exists \( \delta \) passing through \( B_m l_m \) such that \( f_n|\delta = \alpha \), and also, if there exists \( \delta \) passing through \( B_m l_m \) such that \( f_n|\delta = \alpha \), then for every \( a \) in \( B_m \), the sequence \( \alpha^a \) belongs to \( A_{n-\text{length}(a)} \). We now define \( B := \bigcup_{m<k} \{\langle m, \gamma^m(1)\rangle \ast a \mid a \in B_m\} \cup \{0\} \) and \( l := \max_{m<k}(\gamma^m(1), l_m) \).

Observe that for every \( a \), if \( f_{n+2}|\gamma l = \alpha l \), then for each \( m < k \), \( f_{n+2}|(\gamma^m \circ S)^m(1) l_m = \alpha^m(\gamma^m(1)) l_m \).

Observe also that for every \( a \), if for each \( a \) in \( B \), the sequence \( \alpha \) belongs to \( A_{n+2-\text{length}(a)} \), then for each \( m < k \), for each \( a \) in \( B \), the sequence \( \alpha^m \gamma^m(1) \) belongs to \( A_{n-\text{length}(a)} \). Observe finally that for all \( \delta \), if \( \delta \) passes through \( \gamma l \), then for each \( m < k \), the sequence \( (\delta^m \circ S)^m(1) \) passes through \( (\gamma^m \circ S)^m(1) l_m \). Suppose \( \alpha \) belongs to \( \mathcal{N} \) and \( f_{n+2}|\gamma l \) is \( \alpha l \) and for each \( a \) in \( B \), the sequence \( \alpha^a \) belongs to \( A_{n-\text{length}(a)} \).

We determine \( \delta_0, \delta_1, \ldots, \delta_k \) such that, for each \( m < k \), \( f_n|\delta_m = \alpha^m \gamma^m(1) \) and \( \delta_m \) passes through \( B_m l_m = (\gamma^m \circ S)^m(1) l_m \).

Observe that \( 0 \) belongs to \( B \), and therefore \( \alpha \) belongs to \( A_{n+2} \). So for every \( m \geq k \) we may determine a number \( p_m \) and a sequence \( \delta_m \) such that \( f_n|\delta_m = \alpha^m \gamma^m(1) \).
We now define $\zeta$ such that $f_{n+2}[\zeta] = \alpha$, as follows:

We define $\zeta(0) := f_0(0)$ and, for each $m < k$, $\zeta^m(0) = \alpha^m(0)$ and $\zeta^m(1) := \gamma^m(1)$ and $(\zeta^m \circ S)^\gamma(1) := \delta_m$ and for each $m < k$, for each $i \neq \gamma^m(1)$, $(\zeta^m \circ S)^i := \alpha^m_i$. We also define for each $m \geq k$, $\zeta^m(0) := \alpha^m(0)$ and $\zeta^m(1) = p_m$ and $(\zeta^m \circ S)^p_m = \delta_m$ and for each $m \geq k$, for each $i \neq p_m$, $(\zeta^m \circ S)^i = \alpha^m_i$.

We then determine $\zeta$ as follows: for all $\zeta$, there exist for every $\delta$, $\zeta \circ \delta$ coincides with $\alpha$. Conversely, suppose that $\zeta$ passes through $\gamma_l$ and $f_{n+2}[\zeta]$ coincides with $\alpha$. Then for each $m < k$, $\zeta^m(0) = \gamma^m(0)$ and $(\zeta^m \circ S)^{\gamma^m(1)}l_m = (\gamma^m \circ S)^{\gamma^m(1)}l_m$, and $f_{n+1}[\zeta \circ S]^\gamma(l_m) coincides with $\alpha^{m,\gamma^m(1)}$. It is now easily seen that for each $a$, $\zeta \circ S$ belongs to $A_{n-\text{length}(\alpha)}$. 

4.8.4 Theorem:

For each positive $n$, the set $D(A_n, A_1)$ does not reduce to $A_n$.

Proof: Suppose $n$ is a positive natural number and $g$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D(A_n, A_1)$ to $A_n$. We define a function $F$ from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$, as follows: for all $\gamma, \alpha$, $(F(\gamma, \alpha))(0) := \alpha(0)$ and $(F(\gamma, \alpha))^i := \alpha^i$ and for each $i > 0$, $(F(\gamma, \alpha))^i = \alpha^i$.

Observe that for each $\gamma, \alpha$, $(F(\gamma, \alpha))^0$ belongs to $A_n$, therefore $F(\gamma, \alpha)$ belongs to $D(A_n, A_1)$, therefore $g \circ F(\gamma, \alpha)$ belongs to $A_n$ and there exists $\delta$ such that $g \circ F(\gamma, \alpha) = f_{n}[\delta]$.

We apply the axiom AC$_{1,1}$ and determine a function $G$ from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$ such that for all $\gamma, \alpha$, $g \circ F(\gamma, \alpha) = f_0 \circ G(\gamma, \alpha)$. We intend to build a sequence $\beta_0, \beta_1, \ldots$ of elements of $\mathcal{N}$ with the following properties.

(i) For each $k$, $\beta_{2k} = F(0, 0)k$ and $\beta_{2k+1}$ does not belong to $A_n$ but $(\beta_{2k})^1 = 0$ and there exists $\delta$ such that $g \circ F(\gamma, \alpha) = f_{n}[\delta]$ and $\delta k = G(0, 0)k$. 

(ii) For each $k$, $\beta_{2k+1} = F(0, 0)k$ and $\beta_{2k+1}$ does not belong to $A_n$ and there exists $\delta$ such that $g \circ F(\gamma, \alpha) = f_{n}[\delta]$ and $\delta k = G(0, 0)k$.

Suppose $k$ is a natural number. We will construct $\beta_{2k}$ and $\beta_{2k+1}$.

We first determine $m$ such that, for all $\gamma, \alpha$, if $\gamma m = \alpha m = 0m$, then $G(\gamma, \alpha)k = G(0, 0)k$.

We now apply Lemma 4.8.3 and we determine a natural number $l$, and a finite set $B$ of natural numbers such that, for all $\alpha$, if $\alpha l = g \circ F(0, 0)l = f_{n}[\delta] = G(0, 0)l$ and for all $a$ in $B$, $a \alpha$ belongs to $A_{n-\text{length}(\alpha)}$, then there exists $\delta$ such that $\delta k = G(0, 0)k$ and $f_{n}[\delta] = \alpha$, and, if there exist $\delta$ passing through $G(0, 0)l$ such that $f_{n}[\delta] = \alpha$, then, for all $a$ in $B$, $a \alpha$ belongs to $A_{n-\text{length}(\alpha)}$. We calculate $m$ such that for all $\gamma, \alpha$, if $\gamma m = \alpha m = 0m$, then $g \circ F(\gamma, \alpha)l = f_{n}[\delta] = G(0, 0)l$ and $G(\gamma, \alpha)$ passes through $G(0, 0)l$.

We now consider the set $C := \{\zeta\}$ for all $i < m$, $\zeta^{0,i} := (F(0, 0))^{0,i}$. Observe that for every $\zeta$ in $C$ if, for every $i \geq m$, $\zeta^{0,i}$ belongs to $E_{n-1}$, then $\zeta^{0,i}$ belongs to $A_n$, and there exist $\gamma, \alpha$ such that $\zeta = F(\gamma, \alpha)$ and $\gamma m = \alpha m = 0m$.

We now build a function $H$ from $\mathcal{N}$ to $\mathcal{N}$.

For each $\beta$ in $\mathcal{N}$ we define: $(H(\beta))(0) := 0$ and $(H(\beta))^1 := 0$ and, for all $i < m$, $(H(\beta))^{0,i} := F(0, 0)^{0,i}$, and, for all $i \geq m$, $(H(\beta))^{0,i} := \beta$ and, for all $j > 1$, $(H(\beta))^j := 0$. 


Observe that for every $\beta$, if $\beta$ belongs to $E_{n-1}$, then $(H(\beta))^0$ belongs to $A_n$, and there exist $\gamma, \alpha$ such that $\overline{\gamma}m = \overline{\alpha}m = \overline{0}m$ and $H(\beta) = F(\gamma, \alpha)$ so $g \circ H(\beta)$ belongs to $A_n$ and there exists $\delta$ passing through $G(\overline{0}, \overline{0})l$ such that $g \circ H(\beta) = f_\alpha[l]$, therefore for each $a$ in $B$ the sequence $a(g \circ H(\beta))$ belongs to $A_{n-\text{length}(a)}$.

Now observe that $B$ is a finite set and that the set $\{a | \text{For every } a \in B, \text{the sequence } a(g \circ H(\beta)) \text{ belongs to } A_{n-\text{length}(a)} \}$ reduces to $A_{n-2}$.

Applying the hierarchy theorem, we may determine a sequence $\beta^*$ in $\mathcal{N}$ such that both $\beta^*$ belongs to $A_{n-1}$ and for every $a$ in $B \setminus \{0\}$ the sequence $a(g \circ H(\beta^*))$ belongs to $A_{n-\text{length}(a)}$.

Observe that (i) $(H(\beta^*))^0$ does not belong to $A_n$, and (ii) $(H(\beta^*))^1 = 0$, so for all $a$ in $B$, the sequence $a(g \circ H(\beta^*))$ belongs to $A_{n-\text{length}(a)}$, and also $g \circ H(\beta^*)$ passes through $g \circ F(\overline{0}, \overline{0})l$, therefore (iii) there exists $\delta$ passing through $G(\overline{0}, \overline{0})k$ such that $g \circ H(\beta^*)$ equals $f_\alpha[l]$. We thus define: $\beta^* := H(\beta^*)$.

We now describe the construction of $\beta_{2k+1}$.

We determine $m$ such that for all $\gamma, \alpha$, if $\overline{\gamma}m = \overline{\alpha}m = \overline{0}m$, then $G(\gamma, \alpha) k = G(\overline{0}, \overline{0}) k$.

We define $\alpha^*$ in $\mathcal{N}$ as follows: $\alpha^*(0) = 0$ and $(\alpha^*)^i := \overline{0}m * (1) * 0$ and for each $i \neq 1$, $(\alpha^*)^i := 0$.

We define: $\beta_{2k+1} := F(\overline{0}, \alpha^*)$.

Observe that $(\beta_{2k+1})^1 = (\alpha^*)^1 \neq 0$, but $g \mid \beta_{2k+1} = f_n [G(\overline{0}, \alpha^*)]$ and $G(\overline{0}, \alpha^*) k = G(\overline{0}, \overline{0}) k$.

So $\beta_{2k+1}$ fulfils the requirements.

We now define a function $T$ from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha, k$, if $k$ is the least $n$ such that $\alpha(n) \neq 0$, then $T(\alpha) = \beta_k$. Observe that $T(\overline{0}) = F(\overline{0}, \overline{0})$.

Observe also that for every $\alpha$ there exists $\delta$ such that $g \mid T(\alpha) = f_n [\delta]$, therefore $g \mid T(\alpha)$ belongs to $A_n$ and thus either $(T(\alpha))^0$ belongs to $A_n$ or $(T(\alpha))^1 = 0$, so either for every $k$, if $k$ is the least $n$ such that $\alpha(n) \neq 0$, then $k$ is odd, or for every $k$, if $k$ is the least $n$ such that $\alpha(n) \neq 0$, then $k$ is even.

We now obtain a contradiction by the continuity principle: We determine $m$ such that either for all $\alpha$, if $\overline{\alpha}m = \overline{0}m$, and $\alpha \neq 0$, then the least $n$ such that $\alpha(n) \neq 0$ is odd, or for all $\alpha$, if $\overline{\alpha}m = \overline{0}m$ and $\alpha \neq 0$ then the least $n$ such that $\alpha(n) \neq 0$ is even.

Both alternatives are obviously false.

4.8.5

Theorem 4.8.4 implies that $\mathcal{D}^2(A_n)$ does not reduce to $A_n$.

One may prove, for all $n, p$, that $\mathcal{D}(\mathcal{D}^p(A_n), A_1)$ does not reduce to $\mathcal{D}^p(A_n)$.

We then may apply the result from Section 3.6.5 and, as in Sections 3.6.6-9 obtain a lot of classes of subsets of $\mathcal{N}$, each of them containing $\Pi_n^0$ and contained in $\Sigma_{n+1}^0$. 
5 The Borel hierarchy theorem

5.1
For each stump $\sigma$, we introduce subsets $A_\sigma$ and $E_\sigma$ of $\mathcal{N}$ as follows.

(i) For each $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $A_1 := A_1$ if and only if for every $m$, $\alpha(m) = 0$ and $\alpha$ belongs to $E_1 := E_1$ if and only if for some $m$, $\alpha(m) \neq 0$.

(ii) For every stump $\sigma \neq 1$, for every $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $A_\sigma$ if and only if for every $m$, $\alpha^m$ belongs to $E_{\sigma^m}$, and $\alpha$ belongs to $E_\sigma$ if and only if for some $m$, $\alpha^m$ belongs to $A_{\sigma^m}$.

One verifies easily that, for each stump $\sigma$, $\Pi^0_0$ is the class of all subsets of $\mathcal{N}$ that reduce to $A_\sigma$, and $\Sigma^0_0$ is the class of all subsets of $\mathcal{N}$ that reduce to $E_\sigma$. Moreover, if $\alpha$ belongs to $A_\sigma$ and $\beta$ to $E_\sigma$, then there exists $n$ such that $\alpha(n) \neq \beta(n)$. We now formulate the theorem that we want to prove.

5.2 Theorem: (Borel hierarchy theorem).

Let $\sigma$ be a hereditarily repetitive stump. Every function from $\mathcal{N}$ to $\mathcal{N}$ that maps $E_\sigma$ into $A_\sigma$ also takes some member of $A_\sigma$ into $A_\sigma$, and every function from $\mathcal{N}$ to $\mathcal{N}$ that maps $A_\sigma$ into $E_\sigma$, also takes some member of $E_\sigma$ into $E_\sigma$.

The proof of this theorem will occupy us during the rest of the Section. The structure of the proof of this theorem is different from the structure of the proof of the finite Borel Hierarchy Theorem, Theorem 4.7. There is no straightforward extension of Lemma 4.6 to transfinite levels.

5.3
We need the fact that the sets $A_\sigma$, $E_\sigma$ are, all of them, strictly analytical. For every stump $\sigma$ we define functions $f_\sigma$ and $g_\sigma$ from $\mathcal{N}$ to $\mathcal{N}$ such that $A_\sigma = \operatorname{Ran}(f_\sigma)$ and $E_\sigma = \operatorname{Ran}(g_\sigma)$. We let $f_1$ and $g_1$ coincide with the functions $f_\sigma$, $g_\sigma$, respectively that we defined in Section 4.3.

For every stump $\sigma \neq 1$, for every $\alpha$, we define: $(f_\sigma|\alpha)(0) := \alpha(0)$ and for every $m$, $(f_\sigma|\alpha)^m := g_{\sigma^m}|\alpha^m$ and $(g_\sigma|\alpha)(0) := \alpha(0)$ and $(g_\sigma|\alpha)^{\alpha(1)} := f_{\sigma^m}|(\alpha \circ S)^{\alpha(1)}$ and for every $m \neq \alpha(1)$, $(g_\sigma|\alpha)^m := (\alpha \circ S)^m$.

We leave it to the reader to verify that the functions $f_\sigma$, $g_\sigma$ satisfy the requirements. We now prove a Lemma reminiscent of Lemma 4.8.3 but more simple.

5.4 Lemma:
Let $\sigma$ be a stump.

For every natural number $k$, for every infinite sequence $\gamma$ there exists a natural number $l$, such that, for every $\alpha$, if $\alpha$ belongs to $A_\sigma$ and $\gamma(0) = \alpha(0)$ and, for each $i < l$, $\alpha^i = (f_\sigma|\gamma)^i$, then there exists $\delta$ such that $\delta k = \tau k$ and $\alpha = f_\sigma|\delta$. 


5.5 Lemma:

Let $C$ be a spread and $a$ a natural number.
We consider $a$ as the code-number of a finite sequence of natural numbers. We define:

- $a \sqsubseteq b := \text{there exist } m \text{ such that } a \ast m = b$
  (that is, the finite sequence coded by $a$ is an initial part of the finite sequence coded by $b$).
- $b \text{ extends } a.$

- $a \perp b := \text{not: } a \sqsubseteq b \text{ and not: } a \sqsubseteq b$
  ($a, b$ are incompatible.)

5.4.2

Let $C$ be a spread and $a$ a natural number.
We consider $a$ as the code-number of a finite sequence of natural numbers. We define:

- $a$ is free in $C$ if and only if for each $a$ in $C$, for each $(i, (i)\beta)$ such that not $a \sqsubseteq b$, then $\beta$ belongs to $C$.
  (So if we take a member $\alpha$ of $C$ and replace its subsequence $^\alpha a$ by some other sequence, then the resulting sequence will belong to $C$.)

We also define: $a$ is almost-free in $C$ if and only if there exists $n$ such that for each $a$ in $C$, for each $(i, (i)\beta)$ such that not $a \sqsubseteq b$, $\beta (b)$, then $\beta$ belongs to $C$. So, in this case, if we take a member $\alpha$ of $C$ and replace its subsequence $^\alpha a$ by some sequence of the form $\beta n \ast \delta$, then the resulting sequence will belong to $C$.

Proof: We leave the proof to the reader.

5.5 Lemma:

Let $C$ be a spread and $a$ a natural number that is almost-free in $C$.
Let $\sigma$ be a hereditarily repetitive stump and $f$ a function from $C$ to $N$ such that for every $\alpha$ in $C$: if $^\alpha a$ belongs to $A_\sigma$, then $f|\alpha$ belongs to $E_\sigma$.
We may determine natural numbers $m, n$ and a subspread $D$ of $C$ with the following properties:

(i) For every $b \perp a$, if $b$ is almost-free in $C$, then $b$ is almost-free in $D$.
(ii) If $\sigma = 1$, then for each $\alpha$ in $D$, both $^\alpha a$ and $f|\alpha$ belong to $E_1 = E_\sigma$.
(iii) If $\sigma \neq 1$, then $\sigma^m = \sigma^n$

(iv) If $\sigma \neq 1$ and $\sigma^m = 1$, then for each $\alpha$ in $D$ both $^\alpha a$ and $(f|\alpha)^n$ belong to $A_1$ and therefore both $^\alpha a$ and $f|\alpha$ belong to $E_2$.
(v) If $\sigma \neq 1$ and $\sigma^m \neq 1$, then $a \ast (m)$ is free in $D$ and for all $\alpha$ in $D$:

($\ast$) $^\alpha a$ belongs to $A_\sigma$ if and only if $^\alpha a \ast (m)$ belongs to $E_{\sigma^m}$
and ($\ast \ast$) if $^\alpha a \ast (m)$ belongs to $E_{\sigma^m}$, then $(f|\alpha)^n$ belongs to $A_{\sigma^m}$.

Proof: We first consider the case $\sigma = 1$.
So we assume: for every $\alpha$ in $C$, if $^\alpha a = 0$, then $f|\alpha \neq 0$. We now consider the minimal element $\alpha_0$ of $C$. $\alpha_0$ is defined recursively. For each $n$, $\alpha_0(n) :=$ the least $p$ such that $\alpha_0 \ast n \ast (p)$ is admitted by $C$. 

Observe that $\alpha_0 = 0$, as $a$ is almost free in $C$.

Determine $m$ such that $(f|\alpha_0)(m) \neq 0$.

Determine $p$ such that for all $\alpha$ in $C$, if $\overline{\alpha}p = \overline{\alpha_0}p$, then $(f|\alpha)(m) = (f|\alpha_0)(m)$.

Determine $q$ such that for all $\alpha$ in $C$, for all $\beta$, if $\alpha_0$ is replaced by $\overline{\beta}q \ast \beta$, then the resulting sequence still belongs to $C$.

Let $p$ be the greatest of the two numbers $p, q$.

Let $D$ be the set of all $\alpha$ in $C$ such that $\overline{\alpha}p = \overline{\alpha_0}p$ and $\alpha(a \ast (n)) = 1$. It will be clear that $D$ satisfies the requirements.

We now consider the case $\sigma \neq 1$. We first determine $q$ such that for all $\alpha$ in $C$, for all $\beta$, if $\alpha_0$ is replaced by $\overline{\beta}q \ast \beta$, then the resulting sequence still belongs to $C$. We determine $\delta_0, \ldots, \delta_{q-1}$ such that, for each $i < q$, $\delta_i = \overline{\beta}q$ and $\delta_i$ belongs to $E_{\sigma^j}$.

Let $\gamma$ be an element of $K$.

Let $B(\gamma)$ be the set of all $\alpha$ in $C$ such that for every $i < q$, $\alpha^i = \delta_i$, and for every $i \geq q$, $(\alpha^i)^j = (f_{\sigma}|\gamma)^j$.

Observe that $B(\gamma)$ is a subsprad of $C$.

Observe that for every $\alpha$ in $B(\gamma)$, the sequence $\alpha_0$ belongs to $A_{\sigma}$. So for every $\alpha$ in $B(\gamma)$ there exists $n$ such that $(f|\alpha)^{\langle n \rangle}$ belongs to $A_{\sigma^\langle n \rangle}$. Let $\alpha_0$ be the minimal element of the spread $B(\gamma)$. We apply the continuity principle and determine $p, n$ such that for every $\gamma, \alpha$, if $\overline{\gamma}p = \overline{\beta}p$ and $\overline{\alpha}p = \overline{\alpha_0}p$ and $\alpha$ belongs to $B(\gamma)$ then $(f|\alpha)^{\langle n \rangle}$ belongs to $A_{\sigma^\langle n \rangle}$.

We now distinguish two subcases.

**Subcase (i):**

$\sigma^n = 1$. Determine $m > p$ such that $\sigma^m = \sigma^n = 1$. We let $F$ be the set of all $\alpha$ in $C$ such that $\overline{\alpha}p = \overline{\alpha_0}p$ and for each $i \neq m$, $(\alpha^i)^j = (\alpha_0^i)^j$. Because of the minimality of $\alpha_0$, every $b \perp a$ that is almost-free in $C$ is also almost-free in $F$. Observe that $a \ast (m)$ is free in $F$, and for all $\alpha$ in $F$ if $(\alpha^m)^{\langle n \rangle}$, then $(f|\alpha)^{\langle n \rangle} = 0$.

Let $D$ be the set of all $\alpha$ in $F$ such that $(\alpha^m)^{\langle n \rangle} = 0$. Observe that for all $\alpha$ in $D$, both $(\alpha^m)^{\langle n \rangle}$ and $(f|\alpha)^{\langle n \rangle}$ belong to $A_1$.

Observe that every $b \perp a$ that is almost-free in $C$ is also almost-free in $D$.

**Subcase (ii):**

$\sigma^n \neq 1$. Determine $m > p$ such that $\sigma^m = \sigma^n$. We let $D$ be the set of all $\alpha$ in $C$ such that $\overline{\alpha}p = \overline{\alpha_0}p$ and for each $i \neq m$, $(\alpha^i)^j = (\alpha_0^i)^j$.

Observe that $a \ast (m)$ is free in $D$.

Observe also that for all $\alpha$ in $D$, if $(\alpha^m)^{\langle n \rangle}$ belongs to $E_{\sigma^m}$, then there exists $\gamma$ such that $\overline{\gamma}p = \overline{\beta}p$ and $\alpha$ belongs to $B(\gamma)$, so and therefore $(f|\alpha)^{\langle n \rangle}$ belongs to $A_{\sigma^\langle n \rangle}$.

Observe finally that every $b \perp a$ that is almost-free in $C$ is also almost-free in $D$.

$\Diamond$

### 5.6

We now prove Theorem 5.2.
5.6.1

Let \( \sigma \) be a hereditarily repetitive stump and let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \alpha \), if \( \alpha \) belongs to \( A_\sigma \) then \( f|\alpha \) belongs to \( E_\sigma \).
We want to construct \( \alpha \) in \( \mathcal{N} \) such that both \( \alpha \) and \( f|\alpha \) belong to \( E_\sigma \).

5.6.2

We shall construct a sequence \( C_0, C_1, \ldots \) of spreads and a sequence \( a_0, a_1, \ldots \) of decidable subsets of \( \mathbb{N} \times \mathbb{N} \) and a sequence \( a_0, a_1, \ldots \) of natural numbers such that the following are true:

(i) For each \( n \), \( F_n \) is a function, and \( F_n \subseteq F_{n+1} \). For each \( a, n \) we may decide whether \( a \) belongs to the domain of \( F_n \) or not, and also if there exists \( m \) such that \( a * \langle m \rangle \) belongs to the domain of \( F_n \) or not.
If \( a \) belongs to the domain of \( F_n \), and there does not exist \( m \) such that \( a * \langle m \rangle \) belongs to the domain of \( F_n \), we say that \( a \) is a maximal element of the domain of \( F_n \). For all \( n, a, b \), if \( (a, b) \) belongs to \( F_n \), then \( \text{length}(a) = \text{length}(b) \) and \( a^\sigma = b^\sigma \). Moreover, for all \( a \), if \( a \) is a maximal element of the domain of \( F_n \) and \( \text{length}(a) \) is even and \( a^\sigma \neq 1 \), then \( a \) is almost-free in \( C_n \) and for all \( a \) in \( C_n \), if \( a^\sigma \) belongs to \( A_{\sigma} \), then \( b(f|\alpha) \) belongs to \( E_{\sigma} \). Also, for all \( a \), if \( \text{length}(a) \) is odd and \( a^\sigma \neq 1 \), then \( a \) is not a maximal element of the domain of \( F_n \) and for all \( a \) in \( C_n \), both \( a^\sigma \) and \( b(f|\alpha) \) belong to \( A_1 \).

(ii) For each \( n \), \( \text{length}(a_n) = n \) and there exists \( m \) such that \( a_{n+1} = a_n * \langle m \rangle \) and for every \( a \) in \( C_{n+1} \), \( \alpha n = a_n \).

(iii) Define \( F := \bigcup_{n \in \mathbb{N}} F_n \).

\[ (0, 0) \] belongs to \( F \) and for all \( (a, b) \) in \( F \), if \( \text{length}(a) \) is even, and \( a^\sigma \neq 1 \), then there exist \( m, n \) such that \( (a * \langle m \rangle, b * \langle n \rangle) \) belongs to \( F \), and if \( \text{length}(a) \) is odd and \( a^\sigma \neq 1 \), then for all \( n, (a * \langle n \rangle, b * \langle n \rangle) \) belongs to \( F \).

Suppose for a moment that we succeeded in constructing \( C_0, C_1, \ldots \) and \( F_0, F_1, \ldots \) and \( a_0, a_1, \ldots \) such that all our requirements are fulfilled. We let \( \alpha \) be the element of \( \mathcal{N} \) such that, for each \( n \), \( \alpha n = a_n \). Observe that \( \alpha \) belongs to every \( C_n \).
We may prove that both \( \alpha \) and \( f|\alpha \) belong to \( E_\sigma \).

In the proof of this fact we use “induction on \( \sigma \”).
This principle may be formulated as follows:

Let \( \sigma \) be a stump and \( P \) a subset of \( \mathbb{N} \).
Suppose: for every \( a \) in \( \sigma \):
if for every \( n \) such that \( a * \langle n \rangle \) is in \( \sigma \), \( a * \langle n \rangle \) belongs to \( P \), then \( a \) belongs to \( P \).
Then 0 (the code number of the empty sequence) belongs to \( P \).

We prove, by induction on \( \sigma \): For all \( a \) in \( \sigma \), for all \( b \) in \( \sigma \), if \( (a, b) \) belongs to \( F \) then if \( \text{length}(a) \) is even, both \( a^\alpha \) and \( b(f|\alpha) \) belong to \( E_{(a^\sigma)} \), and if \( \text{length}(a) \) is odd, both
\[ ^a \alpha \text{ and } b(f|\alpha) \text{ belong to } A^{(*_\sigma)}. \]

5.6.3

We now explain how to construct the objects \( C_0, C_1, \ldots \) and \( F_0, F_1, \ldots \) and \( a_0, a_1, \ldots \), as announced in Section 5.6.2.

We define: \( C_0 := \mathcal{N} \) and \( F_0 := \{0,0\} \) and \( a_0 := 0 \).

Now suppose \( n \) is a natural number and we constructed \( C_n, F_n, a_n \). Suppose also that \( C_n, F_n, a_n \) satisfy the requirements formulated in 5.6.2 (i) and (ii). We indicate how to find \( C_{n+1}, F_{n+1}, a_{n+1} \).

We first determine \( a, q \) such that \( n + 1 = J(a,q) \). We then decide if \( a \) is a maximal element of the domain of \( F_n \) of even length, and if so, if \( a \) is almost-free in \( C_n \). If one of these questions is answered negatively, we do “nothing”, that is, we define \( C_{n+1} := C_n \) and \( F_{n+1} := F_n \) and \( a_{n+1} := a_n * \langle m_0 \rangle \) where \( m_0 \) is the least \( m \) such that \( a_n * \langle m \rangle \) is admitted by \( C_n \).

If \( a \) is indeed a maximal element of the domain of \( F_n \), and almost-free in \( C_n \), we apply Lemma 5.5. We may distinguish two cases.

Case (i): \( ^a \sigma = 1 \).

We determine \( b \) such that \( (a, b) \) belongs to \( F_n \). We know: for every \( \alpha \) in \( C_n \), if \( ^a \alpha \) belongs to \( A_1 \), then \( b(f|\alpha) \) belongs to \( A_1 \). We now define: \( C'_n := \{ \alpha | \alpha \in C_n \wedge \text{clos} a_n = a_n \} \). Observe that \( C'_n \) is a subspace of \( C_n \) and that every \( e \) that is almost free in \( C_n \) is still almost free in \( C'_n \).

Applying Lemma 5.5, we find a subspace \( D \) of \( C'_n \) such that every \( e \) that is incompatible with \( a \) and almost free in \( C'_n \) is still almost free in \( D \) and, for all \( \alpha \) in \( D \), both \( ^a \alpha \) and \( b(f|\alpha) \) belong to \( E_1 \). We define \( C_{n+1} := D \) and \( F_{n+1} := F_n \) and \( a_{n+1} := a_n * \langle m_0 \rangle \), where \( m_0 \) is the least \( m \) such that \( a_n * \langle m \rangle \) is admitted by \( C_{n+1} \).

Case (ii): \( ^a \sigma \neq 1 \).

We determine \( b \) such that \( (a, b) \) belongs to \( F_n \). We let \( C'_n \) be the spread consisting of all \( \alpha \) in \( C_n \) such that \( \text{clos} \alpha = a_n \).

Observe that for all \( \alpha \) in \( C'_n \), if \( ^a \alpha \) belongs to \( A^{(*_\sigma)} \), then \( b(f|\alpha) \) belongs to \( E^{(*_\sigma)} \), and observe that \( a \) is almost-free in \( C'_n \).

Applying Lemma 5.5 we determine natural numbers \( m, p \) and a subspace \( D \) of \( C'_n \) such that either

- Case (ii)a: \( ^a (m) \sigma \neq 1 \) and \( a * \langle m \rangle \) is free in \( D \), and for all \( \alpha \) in \( D \), if \( ^a (m) \alpha \) belongs to \( E^{(*_\sigma)} \), then \( b^{(p)} (f|\alpha) \) belongs to \( A^{(*_\sigma)} \). We define: \( C_{n+1} := D \) and \( F_{n+1} := F_n \cup \{ (a * \langle m \rangle, b * \langle p \rangle) \} \cup \{ (a * \langle m_q \rangle, b * \langle p_q \rangle) | q \in \mathbb{N} \} \) and \( a_{n+1} := a_n * \langle m_0 \rangle \), where \( m_0 \) is the least \( r \) such that \( a_n * \langle r \rangle \) is admitted by \( C_{n+1} \).

- Case (ii)b: \( ^a (m) \sigma = 1 \) and for all \( \alpha \) in \( D \) both \( ^a (m) \alpha \) and \( b^{(p)} (f|\alpha) \) belong to \( A_1 \).

We define: \( C_{n+1} := D \) and \( F_{n+1} := F_n \cup \{ (a * \langle m \rangle, b * \langle p \rangle) \} \) and \( a_{n+1} := a_n * \langle m_0 \rangle \) where \( m_0 \) is the least \( r \) such that \( a_n * \langle r \rangle \) is admitted by \( C_{n+1} \).

Our proof of the first statement of Theorem 5.2 is now complete.
5.6.4

Now suppose \( \sigma \) is a hereditarily repetitive stump and \( f \) is a function from \( N \) to \( N \) mapping \( E_\sigma \) into \( A_\sigma \).

Observe that, for each \( n \), for every \( \alpha \), if \( \alpha^n \) belongs to \( A_\sigma \), then \( (f|\alpha)^n \) belongs to \( E_\sigma^n \).

So we may argue as in Sections 5.6.2-3 in order to obtain \( \alpha \) such that both \( \alpha \) and \( f|\alpha \) belong to \( A_\sigma \).

We also reach this conclusion by using the result obtained in Section 5.6.3, as follows.

We define a function \( g \) from \( N \) to \( N \) that is closely related to \( f \). For each \( \alpha \) we define \( (g|\alpha)(0) = \alpha(0) \) and for each \( n, (g|\alpha)^n := f((\alpha^n)) \). We let \( \sigma^+ := S(\sigma) \) (see Section 1.6.1) be the nonempty stump \( \tau \) such that for each \( n, \tau^n = \sigma \). Observe that, for every \( \alpha \), if \( \alpha \) belongs to \( A_{\sigma^+} \), then for each \( n \) \( \alpha^n \) belongs to \( E_\sigma \), and \( (g|\alpha)^n \) belongs to \( A_\sigma \).

As \( \sigma \) is hereditarily repetitive, there exists a function \( h \) from \( N \) to \( N \) such that for every \( \alpha, \forall n [(\alpha^n) \in A_\sigma] \) if and only if \( h|\alpha \in A_\sigma \).

We define a function \( k \) from \( N \) to \( N \) in such a way that for each \( \alpha, (k|\alpha)^0 := \alpha \) and for each \( n > 0, (k|\alpha)^n \in E_\sigma \).

Observe that for every \( \alpha \), \( \alpha \in A_\sigma \) if and only if \( k|\alpha \in E_{\sigma^+} \).

Observe that the function \( k \circ h \circ g \) maps \( A_{\sigma^+} \) into \( E_{\sigma^+} \).

Using the result of Section 5.6.3 we find \( \alpha \) such that both \( \alpha \), itself and \( (k \circ h \circ g)|\alpha \), belong to \( E_{\sigma^+} \). So for some \( m \), \( \alpha^m \) belongs to \( A_{\sigma^+} \) and also \( h \circ g|\alpha \) belongs to \( A_\sigma \), therefore also \( (g|\alpha)^m = f|(\alpha^m) \) belongs to \( A_\sigma \).

The proof of Theorem 5.2, the intuitionistic Borel Hierarchy Theorem, is now complete.

5.7 Theorem:

For each hereditarily repetitive stump \( \sigma \), the set \( D(A_\sigma, A_1) \) does not reduce to the set \( A_\sigma \).

Proof: One may obtain the proof by extending the reasoning followed in the proof of Theorem 4.8.4.

It is an easy corollary of Theorem 5.7 that for each hereditarily repetitive stump \( \sigma \), the set \( D^2(A_\sigma) \) does not reduce to the set \( A_\sigma \). One may also prove that for every hereditarily repetitive stump \( \sigma \), for every \( p \), the set \( D(D^pA_\sigma, A_1) \) does not reduce to the set \( D^pA_\sigma \).

Applying the results of Section 3.6.5-9 one obtains a lot of classes of subsets of \( N \), each of them containing \( \Pi^p_\sigma \) and contained in \( \Sigma^3_\sigma^+ \).
6 Some remarks on analytical, strictly analytical and co-analytical sets

6.1

For every $\beta$ in $\mathcal{N}$ we let $C_\beta$ be the set of all $\alpha$ in $\mathcal{N}$ such that for some $\gamma$ in $\mathcal{N}$, for every $n$, $\beta(\langle \alpha n, \gamma n \rangle) = 0$. A subset $X$ of $\mathcal{N}$ is called analytical if and only if there exists $\beta$ in $\mathcal{N}$ such that $X$ coincides with $C_\beta$.

The class $\Sigma^1_1$ of all analytical subsets of $\mathcal{N}$ is closed under the operations of countable union and countable intersection, and it contains all Borel subsets of $\mathcal{N}$.

We define a special subset $E_1$ of $\mathcal{N}$, as follows: for every $\alpha$, $\alpha$ belongs to $E_1$ if and only if for some $\gamma$, for all $n$, $\alpha(\gamma n) = 0$. The set $E_1$ is analytical, and it is easily seen that for every subset $X$ of $\mathcal{N}$, $X$ is analytical if and only if $X$ reduces to $E_1$. In particular every Borel subset of $\mathcal{N}$ reduces to $E_1$.

It follows from the Borel Hierarchy Theorem that the set $E_1$ itself is not a Borel subset of $\mathcal{N}$. We may prove a slightly stronger statement: for every positively Borel subset $C$ of $\mathcal{N}$, if $E_1 \subseteq C$, then there exist $\alpha$ in $C$ such that $\alpha$ is apart from every member of $E_1$.

6.2

We call a subset $X$ of $\mathcal{N}$ strictly analytical if and only if either $X = \emptyset$ or there exists a function $f$ from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with $\text{Ran}(f)$. Every strictly analytical subset of $\mathcal{N}$ is an analytical subset of $\mathcal{N}$ but it is not true that every inhabited and closed subset of $\mathcal{N}$ is strictly analytical, see Veldman 1990.

Let $X, Y$ be subsets of $\mathcal{N}$. We call $(X, Y)$ a separate pair of subsets of $\mathcal{N}$ if and only if every member of $X$ is apart from every member of $Y$.

6.3 Theorem: (Lusin Separation Theorem, Intuitionistic Version).

Let $f, g$ be functions from $\mathcal{N}$ to $\mathcal{N}$ such that for all $\alpha, \beta$, if $\alpha \neq \beta$, then $f(\alpha) \neq g(\beta)$. There exists Borel subsets $C, D$ of $\mathcal{N}$ such that $\text{Ran}(f) \subseteq C$ and $\text{Ran}(g) \subseteq D$, and $(C, D)$ is a separate pair of sets.

Proof: We will use a principle of double bar induction:

Let $P$ be a subset of $\mathbb{N} \times \mathbb{N}$ such that for all $\alpha, \beta$ there exists $n$ such that $\langle \alpha n, \beta n \rangle$ belongs to $P$. Let $Q$ be a subset of $\mathbb{N} \times \mathbb{N}$ such that $P \subseteq Q$ and for all $a, b$ in $\mathbb{N}$, if, for all $m, n$, the pair $\langle a \ast \langle m \rangle, b \ast \langle n \rangle \rangle$ belongs to $Q$, then $\langle a, b \rangle$ belongs to $Q$. Then $\langle \emptyset, \emptyset \rangle$ belongs to $Q$. 
This principle follows from Brouwer’s Thesis, as formulated in Section 1.13. We leave
the proof as an exercise to the reader.
We apply the principle by defining $P, Q$, as follows:

\[ P(a, b) := \text{for every } \alpha \text{ passing through } a, \text{ for every } \beta \text{ passing through } b, \]

\[ f|\alpha \# f|\beta \]

\[ Q(a, b) := \text{there exist Borel sets } C, D \text{ such that } (C, D) \text{ is a separate pair of } \]

sets and for every $\alpha$, if $\alpha$ passes through $a$, then $f|\alpha$ belongs to $C$
and if $\alpha$ passes through $b$, then $g|\alpha$ belongs to $D$.

Suppose $a, b$ belong to $\mathbb{N}$ and for all $m, n$ there exist Borel sets $C, D$ such that $(C, D)$
is a separate pair of subsets of $\mathcal{N}$, and for all $\alpha$, if $\alpha$ passes through $a \ast \langle m \rangle$, then $f|\alpha$
belongs to $C$ and if $\alpha$ passes through $b \ast \langle n \rangle$, then $g|\alpha$ belongs to $D$.

Using the countable axiom of choice $\text{AC}_{0,1}$ we find for each $m, n$ a suitable pair of
sets $(C_{m,n}, D_{m,n})$.

Define: $C := \bigcup_{m} C_{m,n}$ and $D := \bigcup_{n} D_{m,n}$. Observe that $(C, D)$ is a separate pair
of Borel subsets of $\mathcal{N}$ and that for all $\alpha$, if $\alpha$ passes through $a$, then $f|\alpha$ belongs to
$C$, and if $\alpha$ passes through $b$, then $g|\alpha$ belongs to $D$.

\[ \Box \]

\textbf{6.4}

Let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$. $f$ is called strongly injective if and only if, for all
$\alpha, \beta$, if $\alpha \# \beta$, then $f|\alpha \# f|\beta$.

Let $X$ be a subset of $\mathcal{N}$, and $n$ a natural number. We say that $X$ has diameter $\leq 2^{-n}$
if and only if for all $\alpha, \beta$ in $X$, $\bar{\alpha} n = \bar{\beta} n$.

\textbf{6.4.1 Theorem:}

Let $f$ be a strongly injective function from $\mathcal{N}$ to $\mathcal{N}$. Ran$(f)$ is a Borel subset of $\mathcal{N}$.

\textbf{Proof:} Assume that $f$ is a strongly injective function from $\mathcal{N}$ to $\mathcal{N}$. Let $n$ be a
natural number.

We determine a decidable subset $B$ of $\mathbb{N}$ such that:

(i) for every $\alpha$ in $\mathcal{N}$ there exists exactly one $n$ such that $\bar{\alpha} n$ belongs to $B$, and

(ii) for every $\alpha$ in $B$, for all $\alpha, \beta$ passing through $a$, $\bar{f}|\alpha = \bar{f}|\beta$.

Observe that for all $a, b$ in $B$, if $a \neq b$, then for all $\alpha$ passing through $a$, for all $\beta$
passing through $b$, $f|\alpha \neq f|\beta$. Applying Theorem 6.3 we find a separate pair $(C, D)$
of Borel sets such that for all $\alpha$, if $\alpha$ passes through $a$, then $f|\alpha$ belongs to $C$ and if $\alpha$
passes through $b$, then $f|\alpha$ belongs to $D$.

Applying the axiom $\text{AC}_{0,1}$ of countable choice we find for every pair $a, b$ of
elements of $B$ such that $a \neq b$ a suitable separate pair $(C_{a,b}, D_{a,b})$ of Borel sets.

We now define, for every $a$ in $B$,

\[ C_a := \bigcap \{ C_{a,b} \cap D_{b,a} \mid b \in B \mid b \neq a \} \cap \{ \beta | \beta \in \mathcal{N} \mid \bar{\beta} n = \bar{f}|a * \bar{a} n \}. \]

Observe that each $C_a$ is a Borel set of diameter $\leq 2^{-n}$, and for all $a, b$ in $B$, if $a \neq b$, then $(C_a, C_b)$ is a separate pair, and for every $\alpha$ there exists $m$ such that $f|\alpha$ belongs
to $C_{3n}$.

We now carry out this construction for every $n$, applying again the axiom $\text{AC}_{0,1}$ of countable choice. We obtain for each $n$ a decidable subset $B_n$ of $\mathbb{N}$ and for each $a$ in $B_n$ a Borel set $C^n_a$ of diameter $\leq 2^{-n}$ such that for all $a, b$ in $B_n$, if $a \neq b$, then $(C^n_a, C^n_b)$ is a separate pair of subsets of $\mathcal{N}$, and for every $\alpha$ there exists $m$ such that $f|\alpha$ belongs to $C^n_{3m}$.

We further assume, obviously without loss of generality, that for each $n$, for each $a$ in $B_{n+1}$ there exists $b \subseteq a$ such that $b$ belongs to $B_n$, and length$(a) \geq n + 1$.

We define, for each $n$, for each $a$ in $B_n$:

$$E^n_a = \bigcap \{C^j_b \mid j \leq n, b \subseteq a, b \text{ belong to } B_j\}.$$ 

Assume that for some $n, a, b$, we find $\beta$ belonging to both $E^n_a$ and $E^n_{b+1}$. It is easy to see that $a$ must be an initial part of $b$. Observe that we still have: for each $a$ in $B_n$, $E^n_a$ is a Borel set of diameter $\leq 2^{-n}$, for all $a, b$ in $B_n$, if $a \neq b$, then $(E^n_a, E^n_b)$ is a separate pair of subsets of $\mathcal{N}$, and for every $\alpha$ there exists $m$ such that $f|\alpha$ belongs to $E^n_{3m}$.

We now define $H := \bigcap \bigcup_{n \in \mathbb{N}} E^n_a$.

We claim that $H$ is a Borel set that coincides with Ran($f$).

It is clear that Ran($f$) is included in $H$.

Now assume that $\beta$ belongs to $H$. We determine, for each $n$ an element $b_n$ of $B_n$ such that $\beta$ belongs to $E^n_{b_n}$. Observe that $b_0 \subseteq b_1 \subseteq b_2 \subseteq \ldots$ and for each $n$ length$(b_n) \geq n$, so there exists exactly one $a$ passing through every $b_n$. Therefore $f|\alpha$ belongs to $\bigcap_{n \in \mathbb{N}} E^n_{b_n}$. As for each $n$, $E^n_{b_n}$ has diameter $\leq 2^{-n}$, $\beta$ must coincide with $f|\alpha$.

\section*{6.5}

For every $\beta$ in $\mathcal{N}$ we let $D_\beta$ be the set of all $\alpha$ in $\mathcal{N}$ such that for every $\gamma$ in $\mathcal{N}$ there exists $n$ such that $\beta((\alpha n, \gamma n)) \neq 0$. A subset $X$ of $\mathcal{N}$ is called co-analytical if and only if there exists $\beta$ in $\mathcal{N}$ such that $X$ coincides with $D_\beta$.

We define a special subset $A^1_\text{Fun}$ of $\mathcal{N}$, as follows:

For every $\alpha$, $\alpha$ belongs to $A^1_\text{Fun}$ if and only if for every $\gamma$ there exists $n$ such that $\alpha(\gamma n) \neq 0$. The set $A^1_\text{Fun}$ is co-analytical and it is easily seen that for every subset $X$ of $\mathcal{N}$, $X$ is co-analytical if and only if $X$ reduces to $A^1_\text{Fun}$.

Observe that the set $A^1_\text{Fun}$ is the same as the set $\text{Fun}$, that we defined in Section 1.14. We leave it to the reader to verify that the class $\Pi^1_2$ of co-analytical subsets of $\mathcal{N}$ is closed under the operation of countable intersection. It is not closed, however, under the operation of finite union, as follows from the next Theorem.

\subsection*{6.5.1 Theorem:}

The set $D^2(A_1)$ is not co-analytical.

\textbf{Proof:} Assume that the set $D^2(A_1)$ is co-analytical and let $f$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^2(A_1)$ to $A^1_\text{Fun}$. So, for every $\alpha$, if either $\alpha^0 = 0$ or $\alpha^1 = 0$, then for every
there exists \( n \) such that \((f|\alpha)(\overline{\gamma}n) \neq 0\). We claim that for every \( \alpha \), if \( \alpha \) belongs to \( D^2A_1 \) then for every \( \gamma \) there exists \( n \) such that \((f|\alpha)(\overline{\gamma}n) \neq 0\).

For suppose \( \alpha \) belongs to \( D^2A_1 \). Define \( \beta_0 \) such that \((\beta_0)^0 = 0\) and \( \beta_0(0) = \alpha(0) \) and for all \( n > 0 \), \((\beta_0)^n = \alpha^n\). Observe that \( \beta_0 \) belongs to \( D^2(A_1) \).

Let \( \gamma \) be an element of \( \mathcal{N} \). Determine \( n, m \) such that \((f|\beta_0)(\overline{\gamma}n) \neq 0\) and for each \( \delta \), if \( \overline{\delta}m = \overline{\beta_0}m \), then \((f|\delta)(\overline{\gamma}n) = (f|\beta_0)(\overline{\gamma}n)\).

Now consider \( \overline{\alpha}m \). If \( \overline{\alpha}m = \overline{\beta_0}m \), then \((f|\alpha)(\overline{\gamma}n) \neq 0\). If \( \overline{\alpha}m \neq \overline{\beta_0}m \), then \( \alpha^1 = 0\), so \( \alpha \) belongs to \( D^2A_1 \) and there exists \( k \) such that \((f|\alpha)(\overline{\gamma}k) \neq 0\).

We conclude that \( D^2(A_1) \) coincides with \( D^2(A_1) \) and obtain a contradiction, see Theorem 3.2.3.

\[ \square \]

6.5.2 Theorem:

Let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \alpha \), \( f|\alpha \) belongs to \( A_1^1 \). There exists \( \beta \) in \( A_1^1 \) such that for every \( \alpha \), \( f|\alpha \neq \beta \).

Proof: This is an elementary result.

Let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that \( \text{Ran}(f) \subseteq A_1^1 \). We show how to find \( \gamma \) in \( A_1^1 \) such that for every \( \alpha \), \( f|\alpha \neq \gamma \). We apply Cantor’s diagonal argument and define a function \( h \) from \( \mathcal{N} \) to \( \mathbb{N} \) by: for every \( \alpha \), \( h(\alpha) := (f|\alpha)(\alpha) + 1 \). Determine \( \beta \) in \( A_1^1 \) such that, for every \( \alpha \), \( h(\alpha) = \beta(\alpha) \). Observe that for every \( \alpha \), \( (f|\alpha)(\alpha) \neq \beta(\alpha) \) and \( f|\alpha \neq \beta \).

\[ \square \]

Observe that the set \( E_1^1 \) is strictly analytical and that \((A_1^1, E_1^1)\) is a separate pair of subsets of \( \mathcal{N} \). Therefore, we may conclude, from Theorem 6.3 as well as from Theorem 6.5.2, that \( A_1^1 \) is not strictly analytical.

Brouwer’s Thesis implies that the set \( A_1^1 \) coincides with the set \( \text{Stp} \).

We define, for all \( \alpha, \beta \) in \( \mathcal{N} \): \( \alpha \leq^* \beta \) if and only if there exists \( \gamma \) in \( \mathcal{N} \) such that (i) for every \( n \), if for every \( m \subseteq n \), \( \alpha(m) = 0 \), then for every \( m \subseteq \gamma(n) \), \( \beta(m) = 0 \) and (ii) for all \( m, n \) if \( m \subseteq n \) and \( m \neq n \), then \( f(m) \subseteq f(n) \) and \( f(m) \neq f(n) \). One may prove that for all strumps \( \sigma, \tau \), \( \sigma \leq^* \tau \) if and only if \( \sigma \leq \tau \). (The relation \( \leq \) has been defined in Section 1.6.3).

6.5.3 Theorem: (Boundedness Theorem)

Let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \alpha \), \( f|\alpha \) belongs to \( A_1^1 \).

There exists \( \beta \) in \( A_1^1 \) such that, for every \( \alpha \), \( f(\alpha) \leq^* \beta \).

Proof: Let \( f \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \alpha \), \( f|\alpha \) is a “bar”, that is, an element of \( A_1^1 \).

We know that \( f \) belongs to \( \mathcal{N} \) and that for every \( \alpha, n \) there exists \( m \) such that \( f^m(\overline{\alpha}n) > 0 \). We also know that for every \( \alpha, \gamma \) there exists \( n \) such that \((f|\alpha)(\overline{\gamma}n) > 0\), and that for every \( \alpha, \gamma \) there exist \( m, n \) such that for all \( \delta \), if \( \overline{\delta}m = \overline{\alpha}m \), then \((f|\delta)(\overline{\gamma}n) = (f|\alpha)(\overline{\gamma}n) > 0\). In some sense, therefore, we have a “bar” in \( \mathcal{N} \times \mathcal{N} \), that
“contains” every bar $f|\alpha$. The proof consists in making this easy observation more precise.

We need a preliminary remark.

Let $P,Q$ be functions from $\mathbb{N}$ to $\mathbb{N}$ such that, for each $b$ in $\mathbb{N}$, $b = J(P(b),Q(b)) - 1$. (We are paying for our decision to introduce a non-surjective pairing function $J : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$.)

We now define a function $\beta$ from $\mathbb{N}$ to $\{0,1\}$, as follows:

Let $\alpha$ be a natural number. We determine $n = \text{length}(\alpha)$ and natural numbers $a(0),\ldots,a(n-1)$ such that $\alpha = \langle a(0),\ldots,a(n-1) \rangle$. We consider $a' := \langle P(a(0)),\ldots,P(a(n-1)) \rangle$ and $a'' := \langle Q(a(0)),\ldots,Q(a(n-1)) \rangle$. We define: $\beta(\alpha) = 1$ if and only if there exist $k,\ell \leq n$ such that $f^P(a(0))\ldots,P(a(k-1)))$ ($\langle Q(a(0)),\ldots,Q(a(\ell-1)) \rangle > 0$ and, for each $j < \ell$, $f^P(a(0))\ldots,P(a(k-1)))$ ($\langle Q(a(0)),\ldots,Q(a(\ell-1)) \rangle = 0$.

Observe that for each $\alpha$ there exists $n$ such that $\beta(\alpha n) \neq 0$.

Let $\alpha$ be an element of $\mathcal{N}$. Define $\alpha' := P \circ \alpha$ and $\alpha'' := Q \circ \alpha$.

Determine $p$ such that $(f|\alpha')(\alpha''p) \neq 0$. Determine $q$ such that $f^\alpha p(\alpha''q) = 1 + (f|\alpha')(\alpha''p)$ and, for all $j < q$, $f^\alpha p(\alpha''q) = 0$. Let $n := \text{max}(p,q)$.

Observe that for each $\alpha$, $f|\alpha \leq^* \beta$. Let $\alpha$ be an element of $\mathcal{N}$. Define $\gamma$ in $\mathcal{N}$ as follows: for each $\alpha : \gamma(\alpha) = \langle J(\alpha(0),\alpha(0)) - 1,\ldots,J(\alpha(n-1),\alpha(n-1)) - 1 \rangle$ where $n = \text{length}(\alpha)$.

Observe that, for every $n$, if, for every $m \subseteq n$, $(f|\alpha)(m) = 0$, then, for every $m \subseteq \gamma(n)$, $\beta(m) = 0$. Observe also that for all $m, n$ if $m \subseteq n$ and $m \neq n$, then $\gamma(m) \subseteq \gamma(n)$ and $\gamma(m) \neq \gamma(n)$.

\[\text{6.5.4 Lemma:}\]

For every stump $\sigma$, the set $\{\alpha | \alpha \in \mathcal{N} | \alpha \leq^* \sigma\}$ is a Borel subset of $\mathcal{N}$.

\textbf{Proof:} Exercise.

\[\boxdot\]

\[\text{6.5.5 Theorem:}\]

Every subset of $\mathcal{N}$ that is both strictly analytical and co-analytical, is a Borel subset of $\mathcal{N}$.

\textbf{Proof:} Let $X$ be a subset of $\mathcal{N}$ that is both strictly analytical and co-analytical. Determine a function $f$ from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with $\text{Ran}(f)$. Determine a function $g$ from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha$, $\alpha$ belongs to $X$ if and only if $g|\alpha$ belongs to $A^1_1$.

Observe that $g \circ f$ maps $\mathcal{N}$ into $A^1_1$.

Determine $\beta$ in $A^1_1$ such that for every $\alpha$, $g \circ f|\alpha \leq^* \beta$.

Applying Brouwer’s Thesis, we may assume that $\beta$ is a stump. Observe that for every $\alpha$, $\alpha$ belongs to $X$ if and only if $g|\alpha \leq^* \beta$.

So $X$ reduces to a Borel subset of $\mathcal{N}$ and is itself a Borel subset of $\mathcal{N}$.

\[\boxdot\]
6.5.6 Theorem:

$A^1_1$ is not a Borel subset of $\mathcal{N}$ and for every Borel set $X$, every function from $\mathcal{N}$ to $\mathcal{N}$ that maps $A^1_1$ into $X$ also takes some member of $E^1_1$ into $X$.

**Proof:** We prove: for every hereditarily repetitive stump $\sigma$ there exists a $\Pi^1_1$-subset of $\mathcal{N}$ that does not reduce to the set $E_\sigma$. Let $\sigma$ be a hereditarily repetitive stump. We let $P_\sigma$ be the set of all $\alpha$ such that for all $\beta$ in $E_\sigma$: $\alpha \not= \beta$.

Observe that $P_\sigma$ is part of $P_\sigma$.

We have seen in Section 5.3 that there exists a function $g_\sigma$ from $\mathcal{N}$ to $\mathcal{N}$ such that $E_\sigma$ coincides with $\text{Ran}(g_\sigma)$. Therefore, for every $\alpha$, $\alpha$ belongs to $P_\sigma$ if and only if for every $\beta$, $g_\sigma(\beta) \not\in \alpha$.

We thus see that the set $P_\sigma$ is co-analytical, and reduces to $A^1_1$.

Now assume that $\sigma$ is a (continuous) function from $\mathcal{N}$ to $\mathcal{N}$ mapping $P_\sigma$ into $E_\sigma$.

Then $\sigma$ maps also $A_\sigma$ into $E_\sigma$, and by the Borel Hierarchy Theorem there exists $\alpha$ such that both $\alpha$ and $\sigma(\alpha)$ belong to $E_\sigma$.

So $\sigma$ does not reduce $P_\sigma$ to $E_\sigma$.

We leave it to the reader to prove the second, sharper conclusion of the Theorem.

$\Box$

The question how to prove that $A^1_1$ is not positively Borel, was asked but not answered in Veldman 1981.

6.6 The collapse of the projective hierarchy

6.6.1

We introduce two more classes of subsets of $\mathcal{N}$, $\Pi^2_2$ and $\Sigma^2_2$.

Let $X$ be a subset of $\mathcal{N}$.

We define: $X$ belongs to $\Pi^2_2$ if and only if there exists a decidable subset $C$ of $\mathbb{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\forall \gamma \exists \beta \forall n[\langle \overline{\alpha} n, \overline{\beta} n, \overline{\gamma} n \rangle \in C]$.

We define: $X$ belongs to $\Sigma^2_2$ if and only if there exists a decidable subset $C$ of $\mathbb{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\exists \gamma \forall \beta \exists n[\langle \overline{\alpha} n, \overline{\beta} n, \overline{\gamma} n \rangle \not\in C]$.

Every subset of $\mathcal{N}$ that is either analytical or co-analytical belongs to both $\Pi^2_2$ and $\Sigma^2_2$.

In classical descriptive set theory the elements of $\Sigma^2_2$ are called PCA-sets (projections of complements of analytical sets), and the elements of $\Pi^2_2$ are called CPCA-sets (complements of PCA-sets).

The class $\Sigma^1_1$ is closed under the operations countable union, countable intersection and “existential” projections, the class $\Pi^1_1$ is closed under the operations countable intersection and “universal” projection. We conjecture that it is not closed under the operation of finite union, but we did not succeed in proving this.

6.6.2 Theorem:

Every subset of $\mathcal{N}$ that belongs to $\Pi^1_1$, also belongs to $\Sigma^1_2$.
Proof: Suppose that $X$ is a subset of $\mathcal{N}$ belonging to $\Pi_3^1$. Determine a decidable subset $C$ of $\mathbb{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\forall \gamma \exists \gamma n \left[ (\alpha n, \beta n, \gamma n) \in C \right]$. We now apply the axiom $\text{AC}_{1,1}$ and remark that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\exists \delta \left[ A_1^1 (\delta) \land \delta (0) = 0 \land \forall \gamma \forall n \left[ (\alpha n, \delta [\gamma n, \gamma n]) \in C \right] \right]$. We may remove the symbol $|$ as follows. Observe that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\exists \delta \left[ A_1^1 (\delta) \land \delta (0) = 0 \land \forall \gamma \forall n \left[ (\alpha n, \delta [\gamma n, \gamma n]) \in C \right] \right]$. It will now be clear that $X$ belongs to $\Sigma_2^1$. We conjecture that $\Pi_3^1$ is a proper subclass of $\Sigma_2^1$, but do not know how to prove it.

6.6.3

We define two more classes of subsets of $\mathcal{N}$, $\Pi_3^1$ and $\Sigma_3^1$. Let $X$ be a subset of $\mathcal{N}$. We define: $X$ belongs to $\Pi_3^1$ if and only if there exists a decidable subset $C$ of $\mathbb{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\forall \gamma \exists \gamma n \left[ (\alpha n, \beta n, \gamma n) \in C \right]$. We define: $X$ belongs to $\Sigma_3^1$ if and only if there exists a decidable subset $C$ of $\mathbb{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\forall \gamma \exists \gamma n \left[ (\alpha n, \beta n, \gamma n) \in C \right]$. It is useful to add a final warning. There is no reason to assume that the universal projection of a Borel set of $\mathcal{N}$ is co-analytical. For instance, the set consisting of all $\alpha$ such that $\forall \beta \exists n \forall m [a(\beta n, m) = 0]$, does not seem to be co-analytical, although we do not have a proof that it is not. Observe that $\Pi_3^1 \subseteq \Pi_2^1$, therefore $\Sigma_2^1 \subseteq \Sigma_2^1$.
REFERENCES


