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**ESTIMATION OF THE MEAN OF  
STATIONARY AND NONSTATIONARY  
ORNSTEIN-UHLENBECK PROCESSES**

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# Estimation of the mean of stationary and nonstationary Ornstein–Uhlenbeck processes

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## Abstract

The maximum likelihood estimator of the mean of stationary and nonstationary Ornstein–Uhlenbeck processes is determined based on observation in a rectangle.

**Key words:** Wiener sheet, Ornstein-Uhlenbeck sheet.

## 1 Introduction

The stationary Ornstein-Uhlenbeck process  $\{\tilde{X}(t) : t \geq 0\}$  is the stationary solution of the stochastic differential equation

$$d\tilde{X}(t) = -\alpha\tilde{X}(t)dt + \sigma dW(t), \quad (1)$$

where  $\alpha > 0$ ,  $\sigma > 0$  and  $\{W(t) : t \geq 0\}$  is a standard Wiener process. It is a zero mean Gaussian process with

$$\mathbb{E}\tilde{X}(t_1)\tilde{X}(t_2) = \frac{\sigma^2}{2\alpha}e^{-\alpha|t_2-t_1|},$$

and it can be also represented as follows

$$\tilde{X}(t) = e^{-\alpha t} \left( \tilde{X}(0) + \sigma \int_0^t e^{\alpha u} dW(u) \right),$$

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where  $\tilde{X}(0)$  is a zero mean normal random variable with  $\mathbb{E}\tilde{X}^2(0) = \sigma^2/(2\alpha)$ , independent of  $\{W(t) : t \geq 0\}$ . Consider the process  $\tilde{Y}(t) := \tilde{X}(t) + m$ ,  $t \geq 0$ . Denote by  $\mathbb{P}_{\tilde{Y}}$  and  $\mathbb{P}_{\tilde{X}}$  the measures generated on  $C([T_1, T_2] \rightarrow \mathbb{R})$  by the processes  $\tilde{Y}$  and  $\tilde{X}$ , respectively. Then  $\mathbb{P}_{\tilde{Y}}$  and  $\mathbb{P}_{\tilde{X}}$  are equivalent and the Radon–Nikodym derivative has the form

$$\frac{d\mathbb{P}_{\tilde{Y}}}{d\mathbb{P}_{\tilde{X}}}(\tilde{Y}) = \exp \left\{ -\frac{\alpha m^2}{2\sigma^2}(2 + \alpha(T_2 - T_1)) + \frac{\alpha m}{\sigma^2} \left( \tilde{Y}(T_1) + \tilde{Y}(T_2) + \alpha \int_{T_1}^{T_2} \tilde{Y}(v) dv \right) \right\},$$

hence the maximum likelihood estimator (MLE) of  $m$  based on the observation of  $\{\tilde{Y}(t) : t \in [T_1, T_2]\}$  is given by

$$\tilde{m} = \frac{\tilde{Y}(T_1) + \tilde{Y}(T_2) + \alpha \int_{T_1}^{T_2} \tilde{Y}(v) dv}{2 + \alpha(T_2 - T_1)},$$

and it has a normal distribution with mean  $m$  and variance  $\sigma^2\alpha^{-1}(2 + \alpha(T_2 - T_1))^{-1}$  (see Grenander [4], [5], Arató, M. [1]).

The process  $\{X(t) : t \geq 0\}$  given by

$$\begin{cases} dX(t) = -\alpha X(t)dt + \sigma dW(t), & t \geq 0, \\ X(0) = 0, \end{cases}$$

where  $\alpha \in \mathbb{R}$ ,  $\sigma > 0$ , can be considered as the Ornstein–Uhlenbeck process with initial condition  $X(0) = 0$ . It can be represented as

$$X(t) = \sigma \int_0^t e^{\alpha(u-t)} dW(u).$$

Let  $Y(t) := X(t) + m$ ,  $t \geq 0$ , and consider the measures  $\mathbb{P}_Y$  and  $\mathbb{P}_X$  generated on  $C([T_1, T_2] \rightarrow \mathbb{R})$  by the processes  $Y$  and  $X$ , respectively, where  $T_1 > 0$ . Then it can be shown that  $\mathbb{P}_Y$  and  $\mathbb{P}_X$  are equivalent and in case  $\alpha \neq 0$  the Radon–Nikodym derivative has the form

$$\begin{aligned} \frac{d\mathbb{P}_Y}{d\mathbb{P}_X}(Y) = \exp \left\{ -\frac{\alpha m^2}{2\sigma^2} \left( \frac{2}{1 - e^{-2\alpha T_1}} + \alpha(T_2 - T_1) \right) \right. \\ \left. + \frac{\alpha m}{\sigma^2} \left( \coth(\alpha T_1)Y(T_1) + Y(T_2) + \alpha \int_{T_1}^{T_2} Y(v) dv \right) \right\}, \end{aligned}$$

hence the MLE of  $m$  based on the observation of  $\{Y(t) : t \in [T_1, T_2]\}$  is

$$\hat{m} = \frac{\coth(\alpha T_1)Y(T_1) + Y(T_2) + \alpha \int_{T_1}^{T_2} Y(v) dv}{2/(1 - e^{-2\alpha T_1}) + \alpha(T_2 - T_1)},$$

and it has a normal distribution with mean  $m$  and variance  $\sigma^2\alpha^{-1}(2/(1 - e^{-2\alpha T_1}) + \alpha(T_2 - T_1))^{-1}$  (this can be proved similarly as Theorem 1 in this paper).

Arató, M. [1] studied also the complex-valued stationary Ornstein-Uhlenbeck process, that is, the stationary solution of the equation (1), where now  $\alpha \in \mathbb{C}$  with  $\operatorname{Re}\alpha > 0$ ,  $\sigma > 0$  and  $\{W(t) : t \geq 0\}$  is a complex-valued standard Wiener process. Consider again the process  $\tilde{Y}(t) := \tilde{X}(t) + m$ ,  $t \geq 0$ , where  $m \in \mathbb{C}$  is the unknown parameter. The complex-valued processes  $\{\tilde{Y}(t) : t \geq 0\}$  and  $\{\tilde{X}(t) : t \geq 0\}$  can be considered as processes with values in  $\mathbb{R}^2$  as well. Let  $\mathbb{P}_{\tilde{Y}}$  and  $\mathbb{P}_{\tilde{X}}$  the measures generated on  $C([T_1, T_2] \rightarrow \mathbb{R}^2)$  by these processes, respectively. Then  $\mathbb{P}_{\tilde{Y}}$  and  $\mathbb{P}_{\tilde{X}}$  are equivalent and the Radon-Nikodym derivative has the form

$$\frac{d\mathbb{P}_{\tilde{Y}}}{d\mathbb{P}_{\tilde{X}}}(\tilde{Y}) = \exp \left\{ -\frac{|m|^2}{2\sigma^2}(2\operatorname{Re}\alpha + |\alpha|^2(T_2 - T_1)) + \frac{m}{\sigma^2} \left( \alpha \operatorname{Im}\tilde{Y}(T_1) + \bar{\alpha} \operatorname{Re}\tilde{Y}(T_1) + \bar{\alpha} \operatorname{Im}\tilde{Y}(T_2) + \alpha \operatorname{Re}\tilde{Y}(T_2) + |\alpha|^2 \int_{T_1}^{T_2} \tilde{Y}(v) dv \right) \right\}.$$

The MLE of  $m$  based on the observation of  $\{\tilde{Y}(t) : t \in [T_1, T_2]\}$  is given by

$$\tilde{m} = \frac{\alpha \operatorname{Im}\tilde{Y}(T_1) + \bar{\alpha} \operatorname{Re}\tilde{Y}(T_1) + \bar{\alpha} \operatorname{Im}\tilde{Y}(T_2) + \alpha \operatorname{Re}\tilde{Y}(T_2) + |\alpha|^2 \int_{T_1}^{T_2} \tilde{Y}(u) du}{2\operatorname{Re}\alpha + |\alpha|^2(T_2 - T_1)},$$

and it has a normal distribution with mean  $m$  and covariance matrix given by Arató, M. [1]. Complex-valued Ornstein-Uhlenbeck process with zero start can be handled similarly.

The stationary Ornstein-Uhlenbeck sheet  $\{\tilde{X}(s, t) : s, t \in \mathbb{R}\}$  is a zero mean Gaussian process with

$$\mathbb{E}\tilde{X}(s_1, t_1)\tilde{X}(s_2, t_2) = \frac{\sigma^2}{4\alpha\beta} e^{-\alpha|s_2-s_1|-\beta|t_2-t_1|},$$

where  $\alpha > 0$ ,  $\beta > 0$ ,  $\sigma > 0$ . Consider the process  $\tilde{Y}(s, t) := \tilde{X}(s, t) + m$ ,  $s, t \in \mathbb{R}$ . Arató, N.M. [3] proved by the help of partial stochastic differential equations that in case of  $\alpha = \beta = -1$  the MLE of  $m$  based on the observation of  $\{\tilde{Y}(s, t) : s, t \in [0, T]\}$  is given by

$$\tilde{m} = \frac{\tilde{Y}(0, 0) + \tilde{Y}(0, T) + \tilde{Y}(T, 0) + \tilde{Y}(T, T) + \int_{\partial G} \tilde{Y} + \int_G \tilde{Y}}{(2 + T)^2},$$

where  $G := [0, T]^2$  and  $\partial G$  denotes the boundary of  $G$ .

The random field

$$X(s, t) = \sigma \int_0^s \int_0^t e^{\alpha(u-s)+\beta(v-t)} dW(u, v), \quad s, t \geq 0,$$

where  $\alpha \in \mathbb{R}$ ,  $\beta \in \mathbb{R}$ ,  $\sigma > 0$  and  $\{W(s, t) : s, t \geq 0\}$  is a standard Wiener sheet, can be considered as the Ornstein-Uhlenbeck sheet with zero initial condition on the axes. We can consider the shifted random field  $Y(s, t) := X(s, t) + m$ ,  $s, t \geq 0$ .

The purpose of the present paper is to derive the MLE of  $m$  based on the observation of  $\{\tilde{Y}(s, t) : s \in [S_1, S_2], t \in [T_1, T_2]\}$  or  $\{Y(s, t) : s \in [S_1, S_2], t \in [T_1, T_2]\}$ . It turns out that the MLE is in each case a weighted linear combination of the values at the vertices, integrals on the edges and the integral on the whole rectangular of the observed process. We do not use partial stochastic differential equations, we apply direct discrete time approach instead. We give the proofs only in the nonstationary case, because the stationary case can be handled similarly.

## 2 Case of nonstationary Ornstein-Uhlenbeck sheets

Consider the Ornstein-Uhlenbeck sheet

$$X(s, t) = \sigma \int_0^s \int_0^t e^{\alpha(u-s)+\beta(v-t)} dW(u, v), \quad s, t \geq 0, \quad (2)$$

where  $\alpha \in \mathbb{R}$ ,  $\beta \in \mathbb{R}$  and  $\sigma > 0$  are known parameters and  $\{W(u, v) : u, v \geq 0\}$  denotes a standard Wiener sheet. The assumption that the parameters  $\alpha \in \mathbb{R}$ ,  $\beta \in \mathbb{R}$  and  $\sigma > 0$  are known can be explained by the fact that they can be estimated in a strongly consistent way (see Arató, Pap, Zuijlen [2]). Let  $Y(s, t) = X(s, t) + m$ ,  $s, t \geq 0$ , where  $m$  is an unknown parameter. Consider a rectangle  $[S_1, S_2] \times [T_1, T_2] \subset (0, \infty) \times (0, \infty)$ . Take a grid on it induced by the points  $S_1 =: s_1 < s_2 < \dots < s_M := S_2$  and  $T_1 =: t_1 < t_2 < \dots < t_N := T_2$ , and let  $s_0 := 0$ ,  $t_0 := 0$ .

First we will deal with the problem of estimation of the shift parameter  $m$  in the case when the process  $Y$  is observed in the lattice points  $(s_i, t_j)$ ,  $i = 1, \dots, M$ ,  $j = 1, \dots, N$ . Let  $\Delta s_i = s_i - s_{i-1}$ ,  $\Delta t_j = t_j - t_{j-1}$  and denote by  $\tilde{\Delta}_1 X(s_i, t_j)$  and  $\tilde{\Delta}_2 X(s_i, t_j)$  the modified increments of the process  $X$  given by

$$\begin{aligned} \tilde{\Delta}_1 X(s_i, t_j) &:= X(s_i, t_j) - e^{-\alpha \Delta s_i} X(s_{i-1}, t_j), \\ \tilde{\Delta}_2 X(s_i, t_j) &:= X(s_i, t_j) - e^{-\beta \Delta t_j} X(s_i, t_{j-1}). \end{aligned}$$

It is easy to see, that

$$\tilde{\Delta}_1 \tilde{\Delta}_2 X(s_i, t_j) = \sigma \int_{s_{i-1}}^{s_i} \int_{t_{j-1}}^{t_j} e^{\alpha(u-s_i)+\beta(v-t_j)} dW(u, v). \quad (3)$$

Since the Wiener sheet has independent plane increments, it follows that  $\{\tilde{\Delta}_1 \tilde{\Delta}_2 X(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  are independent random variables having a normal distribution with zero mean and covariance

$$\sigma_{i,j}^2 := \begin{cases} \sigma^2 \frac{(1-e^{-2\alpha \Delta s_i})(1-e^{-2\beta \Delta t_j})}{4\alpha\beta}, & \text{if } \alpha \neq 0, \beta \neq 0, \\ \sigma^2 \frac{1-e^{-2\beta \Delta t_j}}{2\beta} \Delta s_i & \text{if } \alpha = 0, \beta \neq 0, \\ \sigma^2 \frac{1-e^{-2\alpha \Delta s_i}}{2\alpha} \Delta t_j & \text{if } \alpha \neq 0, \beta = 0, \\ \sigma^2 \Delta s_i \Delta t_j & \text{if } \alpha = 0, \beta = 0. \end{cases} \quad (4)$$

We will give the calculations only in case of  $\alpha \neq 0$ ,  $\beta \neq 0$ , since the other cases can be handled similarly. The random variables  $\{X(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  can be represented as a linear transform of the variables  $\{\tilde{\Delta}_1 \tilde{\Delta}_2 X(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  with Jacoby determinant 1. Hence, the joint density of  $\{X(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  has the form

$$f(x_{1,1}, x_{1,2}, \dots, x_{M,N}) = c \exp \left\{ -\frac{1}{2} \left( \frac{x_{1,1}^2}{\sigma_{1,1}^2} + \sum_{i=2}^M \frac{(\tilde{\Delta}_1 x_{i,1})^2}{\sigma_{i,1}^2} + \sum_{j=2}^N \frac{(\tilde{\Delta}_2 x_{1,j})^2}{\sigma_{1,j}^2} + \sum_{i=2}^M \sum_{j=2}^N \frac{(\tilde{\Delta}_1 \tilde{\Delta}_2 x_{i,j})^2}{\sigma_{i,j}^2} \right) \right\}, \quad (5)$$

where  $c$  is a norming constant.

Substituting  $y_{i,j} - m$  for  $x_{i,j}$ ,  $i = 1, \dots, M$ ,  $j = 1, \dots, N$ , we get the joint density of  $\{Y(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$ :

$$g(y_{1,1}, y_{1,2}, \dots, y_{M,N}) = c \exp \left\{ -\frac{1}{2} \left( \frac{(y_{1,1} - m)^2}{\sigma_{1,1}^2} + \sum_{i=2}^M \frac{(\tilde{\Delta}_1 y_{i,1} - (1 - e^{-\alpha \Delta s_i})m)^2}{\sigma_{i,1}^2} + \sum_{j=2}^N \frac{(\tilde{\Delta}_2 y_{1,j} - (1 - e^{-\beta \Delta t_j})m)^2}{\sigma_{1,j}^2} + \sum_{i=2}^M \sum_{j=2}^N \frac{(\tilde{\Delta}_1 \tilde{\Delta}_2 y_{i,j} - (1 - e^{-\alpha \Delta s_i})(1 - e^{-\beta \Delta t_j})m)^2}{\sigma_{i,j}^2} \right) \right\}. \quad (6)$$

Hence, the MLE  $\hat{m}_{M,N}$  of  $m$  based on the observations  $\{Y(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  has the form  $\hat{m}_{M,N} = \zeta_{M,N}/A_{M,N}$ , where

$$\zeta_{M,N} = \frac{Y(s_1, t_1)}{\sigma_{1,1}^2} + \sum_{i=2}^M \frac{(1 - e^{-\alpha \Delta s_i}) \tilde{\Delta}_1 Y(s_i, t_1)}{\sigma_{i,1}^2} + \sum_{j=2}^N \frac{(1 - e^{-\beta \Delta t_j}) \tilde{\Delta}_2 Y(s_1, t_j)}{\sigma_{1,j}^2} + \sum_{i=2}^M \sum_{j=2}^N \frac{(1 - e^{-\alpha \Delta s_i})(1 - e^{-\beta \Delta t_j}) \tilde{\Delta}_1 \tilde{\Delta}_2 Y(s_i, t_j)}{\sigma_{i,j}^2},$$

and

$$A_{M,N} = \frac{1}{\sigma_{1,1}^2} + \sum_{i=2}^M \frac{(1 - e^{-\alpha \Delta s_i})^2}{\sigma_{i,1}^2} + \sum_{j=2}^N \frac{(1 - e^{-\beta \Delta t_j})^2}{\sigma_{1,j}^2} + \sum_{i=2}^M \sum_{j=2}^N \frac{(1 - e^{-\alpha \Delta s_i})^2 (1 - e^{-\beta \Delta t_j})^2}{\sigma_{i,j}^2}.$$

**Proposition 1** *If*

$$\max_{2 \leq i \leq M} |\Delta s_i| \rightarrow 0, \quad \max_{2 \leq j \leq N} |\Delta t_j| \rightarrow 0$$

as  $M, N \rightarrow \infty$ , then

$$\zeta_{M,N} \rightarrow \alpha\beta\zeta/\sigma^2 \quad (7)$$

in  $L^2$ -sense and

$$A_{M,N} \rightarrow \alpha\beta A/\sigma^2, \quad (8)$$

where

$$\begin{aligned} \zeta &= \coth(\alpha S_1) \coth(\beta T_1) Y(S_1, T_1) + \coth(\alpha S_1) Y(S_1, T_2) + \coth(\beta T_1) Y(S_2, T_1) \\ &+ Y(S_2, T_2) + \alpha \coth(\beta T_1) \int_{S_1}^{S_2} Y(u, T_1) du + \alpha \int_{S_1}^{S_2} Y(u, T_2) du \\ &+ \beta \coth(\alpha S_1) \int_{T_1}^{T_2} Y(S_1, v) dv + \beta \int_{T_1}^{T_2} Y(S_2, v) dv + \alpha\beta \int_{S_1}^{S_2} \int_{T_1}^{T_2} Y(u, v) dudv, \end{aligned}$$

and

$$A = \left( \frac{2}{1 - e^{-2\alpha S_1}} + \alpha(S_2 - S_1) \right) \left( \frac{2}{1 - e^{-2\beta T_1}} + \beta(T_2 - T_1) \right).$$

**Proof.** As  $\mathbb{E}\zeta_{M,N} = mA_{M,N}$  and  $\mathbb{E}\zeta = mA$ , it is sufficient to prove (8) and that

$$\begin{aligned} &\mathbb{E}(\zeta_{M,N} - mA_{M,N} - \alpha\beta(\zeta - mA)/\sigma^2)^2 \\ &= \mathbb{E}(\zeta_{M,N} - mA_{M,N})^2 - 2\alpha\beta \text{cov}(\zeta_{M,N}, \zeta)/\sigma^2 + \alpha^2\beta^2\mathbb{E}(\zeta - mA)^2/\sigma^4 \rightarrow 0 \quad (9) \end{aligned}$$

as  $M, N \rightarrow \infty$ . By the independence of the random variables  $Y(s_1, t_1)$ ,  $\tilde{\Delta}_1 Y(s_i, t_1)$ ,  $\tilde{\Delta}_2 Y(s_1, t_j)$  and  $\tilde{\Delta}_1 \tilde{\Delta}_2 Y(s_i, t_j)$ ,  $i = 2, \dots, M$ ,  $j = 2, \dots, N$ , it is easy to see, that  $\mathbb{E}(\zeta_{M,N} - mA_{M,N})^2 = A_{M,N}$ . Using (4), after a short calculation one gets the following form of  $A_{M,N}$ :

$$A_{M,N} = \frac{\alpha\beta}{\sigma^2} \left( \frac{2}{1 - e^{-2\alpha s_1}} + 2 \sum_{i=2}^M \frac{1 - e^{-\alpha \Delta s_i}}{1 + e^{-\alpha \Delta s_i}} \right) \left( \frac{2}{1 - e^{-2\beta t_1}} + 2 \sum_{j=2}^N \frac{1 - e^{-\beta \Delta t_j}}{1 + e^{-\beta \Delta t_j}} \right).$$

To find the limit of  $\sum_{i=2}^M \frac{1 - e^{-\alpha \Delta s_i}}{1 + e^{-\alpha \Delta s_i}}$  we will use the Taylor expansion of  $e^{-\alpha \Delta s_i}$  in  $\Delta s_i$ :

$$e^{-\alpha \Delta s_i} = 1 - \alpha \Delta s_i + (\alpha^2/2)(\Delta s_i)^2 e^{-\alpha \vartheta_i},$$

where  $0 < \vartheta_i < \Delta s_i$ ,  $i = 2, \dots, M$ . Hence

$$\begin{aligned} \sum_{i=2}^M \frac{1 - e^{-\alpha \Delta s_i}}{1 + e^{-\alpha \Delta s_i}} &= \sum_{i=2}^M \frac{\alpha \Delta s_i - (\alpha^2/2)(\Delta s_i)^2 e^{-\alpha \vartheta_i}}{1 + e^{-\alpha \Delta s_i}} \\ &= \frac{\alpha}{2}(S_2 - S_1) + \frac{\alpha}{2} \sum_{i=2}^M \frac{(1 - e^{-\alpha \Delta s_i}) \Delta s_i - \alpha e^{-\alpha \vartheta_i} (\Delta s_i)^2}{1 + e^{-\alpha \Delta s_i}}. \quad (10) \end{aligned}$$



Let us deal with the sum on the right hand side of (10).

$$\begin{aligned} \left| \sum_{i=2}^M \frac{(1-e^{-\alpha\Delta s_i})\Delta s_i - \alpha e^{-\alpha\tilde{\vartheta}_i}(\Delta s_i)^2}{1+e^{-\alpha\Delta s_i}} \right| &\leq \max_{2 \leq i \leq M} |\Delta s_i| \sum_{i=2}^M \left| 1 - e^{-\alpha\Delta s_i} - \alpha e^{-\alpha\tilde{\vartheta}_i} \Delta s_i \right| \\ &= |\alpha| \max_{2 \leq i \leq M} |\Delta s_i| \sum_{i=2}^M \left| (e^{-\alpha\tilde{\vartheta}_i} - e^{-\alpha\vartheta_i}) \Delta s_i \right|, \end{aligned} \quad (11)$$

where again,  $0 < \tilde{\vartheta}_i < \Delta s_i$ ,  $i = 2, \dots, M$ . Since  $\max_{2 \leq i \leq M} |\Delta s_i| \rightarrow 0$  as  $M \rightarrow \infty$ , both  $e^{\alpha\vartheta_i}$  and  $e^{\alpha\tilde{\vartheta}_i}$  are bounded. Thus the right hand side of (11) tends to zero as  $M \rightarrow \infty$ . Hence

$$\lim_{M \rightarrow \infty} \sum_{i=2}^M \frac{1 - e^{-\alpha\Delta s_i}}{1 + e^{-\alpha\Delta s_i}} = \frac{\alpha}{2}(S_2 - S_1), \quad (12)$$

and similarly,

$$\lim_{N \rightarrow \infty} \sum_{j=2}^N \frac{1 - e^{-\beta\Delta t_j}}{1 + e^{-\beta\Delta t_j}} = \frac{\beta}{2}(T_2 - T_1), \quad (13)$$

which proves (8).

To calculate  $\text{cov}(\zeta_{M,N}, \zeta)$  and  $\mathbb{E}(\zeta - mA)^2$  we need the following formula giving the covariance structure of  $Y$ :

$$\text{cov}(Y(u, v), Y(w, z)) = \sigma^2 \frac{(e^{-\alpha(u+w)} - e^{-\alpha|u-w|})(e^{-\beta(v+z)} - e^{-\beta|v-z|})}{4\alpha\beta}. \quad (14)$$

To make the calculations more transparent we introduce a notation for the components of  $\zeta$ , namely  $\zeta = \zeta^{(1)} + \zeta^{(2)} + \zeta^{(3)} + \zeta^{(4)}$ , where

$$\begin{aligned} \zeta^{(1)} &= \coth(\alpha S_1) \coth(\beta T_1) Y(S_1, T_1) + \coth(\alpha S_1) Y(S_1, T_2) \\ &\quad + \coth(\beta T_1) Y(S_2, T_1) + Y(S_2, T_2), \\ \zeta^{(2)} &= \alpha\beta \int_{S_1}^{S_2} \int_{T_1}^{T_2} Y(u, v) \, dudv, \\ \zeta^{(3)} &= \alpha \coth(\beta T_1) \int_{S_1}^{S_2} Y(u, T_1) \, du + \alpha \int_{S_1}^{S_2} Y(u, T_2) \, du, \\ \zeta^{(4)} &= \beta \coth(\alpha S_1) \int_{T_1}^{T_2} Y(S_1, v) \, dv + \beta \int_{T_1}^{T_2} Y(S_2, v) \, dv. \end{aligned}$$

Using (14) and the Fubini theorem we get:

$$\begin{aligned}\text{cov}(\zeta^{(1)}, \zeta_{M,N}) &= \frac{4}{(1 - e^{-2\alpha S_1})(1 - e^{-2\beta T_1})}, \\ \text{cov}(\zeta^{(2)}, \zeta_{M,N}) &= 4 \sum_{i=2}^M \sum_{j=2}^N \frac{1 - e^{-\alpha \Delta s_i}}{1 + e^{-\alpha \Delta s_i}} \frac{1 - e^{-\beta \Delta t_j}}{1 + e^{-\beta \Delta t_j}}, \\ \text{cov}(\zeta^{(3)}, \zeta_{M,N}) &= \frac{4}{1 - e^{-2\beta T_1}} \sum_{i=2}^M \frac{1 - e^{-\alpha \Delta s_i}}{1 + e^{-\alpha \Delta s_i}}, \\ \text{cov}(\zeta^{(4)}, \zeta_{M,N}) &= \frac{4}{1 - e^{-2\alpha S_1}} \sum_{j=2}^N \frac{1 - e^{-\beta \Delta t_j}}{1 + e^{-\beta \Delta t_j}}.\end{aligned}$$

Summing up the above terms we can see that  $\alpha\beta \text{cov}(\zeta_{M,N}, \zeta)/\sigma^2 = A_{M,N}$ , so its limit as  $M, N \rightarrow \infty$  equals  $\alpha\beta A/\sigma^2$ . Similarly,

$$\begin{aligned}\text{cov}(\zeta^{(1)}, \zeta) &= \frac{4\sigma^2}{\alpha\beta(1 - e^{-2\alpha S_1})(1 - e^{-2\beta T_1})}, \\ \text{cov}(\zeta^{(2)}, \zeta) &= (S_2 - S_1)(T_2 - T_1)\sigma^2, \\ \text{cov}(\zeta^{(3)}, \zeta) &= \frac{2\sigma^2}{\beta(1 - e^{-2\beta T_1})}(S_2 - S_1), \\ \text{cov}(\zeta^{(4)}, \zeta) &= \frac{2\sigma^2}{\alpha(1 - e^{-2\alpha S_1})}(T_2 - T_1).\end{aligned}$$

Hence,  $\alpha^2\beta^2\mathbb{E}(\zeta - mA)^2/\sigma^4 = \alpha\beta A/\sigma^2$ , which means that (9) is proved.  $\square$

Denote by  $\mathbb{P}_Y$  and  $\mathbb{P}_X$  the measures generated on  $C([S_1, S_2] \times [T_1, T_2])$  by the processes  $Y$  and  $X$ , respectively.

**Theorem 1** *Suppose that  $\alpha \neq 0$ ,  $\beta \neq 0$ . The measures  $\mathbb{P}_Y$  and  $\mathbb{P}_X$  are equivalent and the Radon-Nikodym derivative of  $\mathbb{P}_Y$  with respect to  $\mathbb{P}_X$  equals*

$$\frac{d\mathbb{P}_Y}{d\mathbb{P}_X}(Y) = \exp \left\{ -\frac{\alpha\beta}{2\sigma^2}(m^2A - 2m\zeta) \right\}. \quad (15)$$

*The maximum likelihood estimator of the shift parameter  $m$  based on the observations  $\{Y(s, t) : s \in [S_1, S_2], t_2 \in [T_1, T_2]\}$  has the form  $\hat{m} = \zeta/A$  and it has a normal distribution with mean  $m$  and variance  $\sigma^2/(\alpha\beta A)$ .*

**Proof.** Let  $\mathbb{P}_Y^{M,N}$  and  $\mathbb{P}_X^{M,N}$  denote the measures generated by  $\{Y(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  and  $\{X(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  on  $\mathbb{R}^{MN}$ , respectively. Using (5) and (6) we can give the Radon-Nikodym derivative of  $\mathbb{P}_Y^{M,N}$  with respect to  $\mathbb{P}_X^{M,N}$ :

$$\frac{d\mathbb{P}_Y^{M,N}}{d\mathbb{P}_X^{M,N}}(Y(s_1, t_1), Y(s_1, t_2), \dots, Y(s_M, t_N)) = \exp \left\{ -\frac{1}{2}(m^2 A_{M,N} - 2m\zeta_{M,N}) \right\}.$$

By Proposition 1

$$\frac{d\mathbb{P}_Y^{M,N}}{d\mathbb{P}_X^{M,N}}(Y(s_1, t_1), Y(s_1, t_2), \dots, Y(s_M, t_N)) \rightarrow \exp\left\{-\frac{\alpha\beta}{2\sigma^2}(m^2 A - 2m\zeta)\right\}$$

in probability. The expectation of the above limit is 1, since  $\zeta$  has a normal distribution with mean  $Am$  and variance  $A\sigma^2/(\alpha\beta)$ . Hence we obtain (15) (see, e.g., Arató [1]).  $\square$

**Remark 1** The maximum likelihood estimator  $\hat{m}_{M,N} = \zeta_{N,M}/A_{N,M}$  of  $m$  based on the discrete sample  $\{Y(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  converges to  $\hat{m}$  in quadratic mean as  $M, N \rightarrow \infty$ .

**Remark 2** In case of  $\alpha = \beta = 0$  we have  $Y(s, t) = W(s, t) + m$ ,  $s, t \geq 0$ , and the joint density of  $\{Y(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  has the following form.

$$f(y_{1,1}, y_{1,2}, \dots, y_{M,N}) = c \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{(y_{1,1}-m)^2}{s_1 t_1} + \sum_{i=2}^M \frac{(\Delta_1 y_{i,1})^2}{t_1 \Delta s_i} + \sum_{j=2}^N \frac{(\Delta_2 y_{1,j})^2}{s_1 \Delta t_j} + \sum_{i=2}^M \sum_{j=2}^N \frac{(\Delta_1 \Delta_2 y_{i,j})^2}{\Delta s_i \Delta t_j}\right)\right\}.$$

Consequently, the MLE of  $m$  based on the observations  $\{Y(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  is simply  $\hat{m}_{MN} = Y(s_1, t_1)$ . Hence, the MLE of  $m$  based on the sample  $\{Y(s, t) : s \in [S_1, S_2], t \in [T_1, T_2]\}$  is also  $\hat{m} = Y(S_1, T_1)$ . We remark that it is in fact the limit of  $\zeta/A$  as  $\alpha, \beta \rightarrow 0$ . The cases  $\alpha = 0, \beta \neq 0$  and  $\alpha \neq 0, \beta = 0$  can be handled similarly.

**Proposition 2** *The random variable  $\zeta$  can also be expressed by the help of integrals with respect to the Wiener sheet, namely*

$$\begin{aligned} \zeta &= \frac{4\sigma}{(1 - e^{-2\alpha S_1})(1 - e^{-2\beta T_1})} \int_0^{S_1} \int_0^{T_1} e^{\alpha(u-S_1)+\beta(v-T_1)} dW(u, v) \\ &+ \frac{2\sigma}{(1 - e^{-2\alpha S_1})} \int_0^{S_1} \int_{T_1}^{T_2} e^{\alpha(u-S_1)} dW(u, v) \\ &+ \frac{2\sigma}{(1 - e^{-2\beta T_1})} \int_{S_1}^{S_2} \int_0^{T_1} e^{\beta(v-T_1)} dW(u, v) + \sigma \int_{S_1}^{S_2} \int_{T_1}^{T_2} dW(u, v) + mA \end{aligned} \quad (16)$$

with probability one.

**Proof.** Denote by  $\tilde{\zeta}$  the expression on the right hand side of (16). Obviously,  $\mathbb{E}\tilde{\zeta} = mA$ . To prove the proposition it suffices to show that  $\zeta_{M,N} \rightarrow \alpha\beta\tilde{\zeta}/\sigma^2$  in  $L^2$ -sense as  $M, N \rightarrow \infty$ , that is

$$\begin{aligned} &\mathbb{E}(\zeta_{M,N} - mA_{M,N} - \alpha\beta(\tilde{\zeta} - mA)/\sigma^2)^2 \\ &= \mathbb{E}(\zeta_{M,N} - mA_{M,N})^2 - 2\alpha\beta \text{cov}(\zeta_{M,N}, \tilde{\zeta})/\sigma^2 + \alpha^2\beta^2\mathbb{E}(\tilde{\zeta} - mA)^2/\sigma^4 \rightarrow 0 \end{aligned} \quad (17)$$

as  $M, N \rightarrow \infty$ .

In the proof of Proposition 1 we have shown that  $\mathbb{E}(\zeta_{M,N} - mA_{M,N})^2 = A_{M,N}$ . As the terms of  $\tilde{\zeta} - mA$  are independent we get

$$\begin{aligned} \mathbb{E}(\tilde{\zeta} - mA)^2 &= \frac{16\sigma^2}{(1 - e^{-2\alpha S_1})^2(1 - e^{-2\beta T_1})^2} \int_0^{S_1} \int_0^{T_1} e^{2\alpha(u-S_1)+2\beta(v-T_1)} dudv \quad (18) \\ &+ \frac{4\sigma^2}{(1 - e^{-2\alpha S_1})^2} \int_0^{S_1} \int_{T_1}^{T_2} e^{2\alpha(u-S_1)} dudv \\ &+ \frac{4\sigma^2}{(1 - e^{-2\beta T_1})^2} \int_{S_1}^{S_2} \int_0^{T_1} e^{2\beta(v-T_1)} dudv + \sigma^2 \int_{S_1}^{S_2} \int_{T_1}^{T_2} dudv \\ &= \frac{\sigma^2}{\alpha\beta} \left( \frac{2}{1 - e^{-2\alpha S_1}} + \alpha(S_2 - S_1) \right) \left( \frac{2}{1 - e^{-2\beta T_1}} + \beta(T_2 - T_1) \right) = \frac{\sigma^2}{\alpha\beta} A. \end{aligned}$$

By the help of (3)  $\zeta_{M,N} - mA_{M,N}$  can be expressed in the following form:

$$\begin{aligned} \zeta_{M,N} - mA_{M,N} &= \frac{4\alpha\beta}{\sigma(1 - e^{-2\alpha s_1})(1 - e^{-2\beta t_1})} \int_0^{s_1} \int_0^{t_1} e^{\alpha(u-s_1)+\beta(v-t_1)} dW(u, v) \\ &+ \frac{4\alpha\beta}{\sigma} \sum_{i=2}^M \sum_{j=2}^N \frac{1}{1 + e^{-\alpha\Delta s_i}} \frac{1}{1 + e^{-\beta\Delta t_j}} \int_{s_{i-1}}^{s_i} \int_{t_{j-1}}^{t_j} e^{\alpha(u-s_i)+\beta(v-t_j)} dW(u, v) \\ &+ \frac{4\alpha\beta}{\sigma(1 - e^{-2\beta t_1})} \sum_{i=2}^M \frac{1}{1 + e^{-\alpha\Delta s_i}} \int_{s_{i-1}}^{s_i} \int_0^{t_1} e^{\alpha(u-s_i)+\beta(v-t_1)} dW(u, v) \\ &+ \frac{4\alpha\beta}{\sigma(1 - e^{-2\alpha s_1})} \sum_{j=2}^N \frac{1}{1 + e^{-\beta\Delta t_j}} \int_0^{s_1} \int_{t_{j-1}}^{t_j} e^{\alpha(u-s_1)+\beta(v-t_j)} dW(u, v). \end{aligned}$$

Hence,

$$\begin{aligned} \text{cov}(\zeta_{M,N}, \tilde{\zeta}) &= \frac{16\alpha\beta}{(1 - e^{-2\alpha s_1})^2(1 - e^{-2\beta t_1})^2} \int_0^{s_1} \int_0^{t_1} e^{2\alpha(u-s_1)+2\beta(v-t_1)} dudv \\ &+ 4\alpha\beta \sum_{i=2}^M \sum_{j=2}^N \frac{1}{1 + e^{-\alpha\Delta s_i}} \frac{1}{1 + e^{-\beta\Delta t_j}} \int_{s_{i-1}}^{s_i} \int_{t_{j-1}}^{t_j} e^{\alpha(u-s_i)+\beta(v-t_j)} dudv \\ &+ \frac{8\alpha\beta}{(1 - e^{-2\beta t_1})^2} \sum_{i=2}^M \frac{1}{1 + e^{-\alpha\Delta s_i}} \int_{s_{i-1}}^{s_i} \int_0^{t_1} e^{\alpha(u-s_i)+2\beta(v-t_1)} dudv \quad (19) \\ &+ \frac{8\alpha\beta}{(1 - e^{-2\alpha s_1})^2} \sum_{j=2}^N \frac{1}{1 + e^{-\beta\Delta t_j}} \int_0^{s_1} \int_{t_{j-1}}^{t_j} e^{2\alpha(u-s_1)+\beta(v-t_j)} dudv \\ &= \left( \frac{2}{1 - e^{-2\alpha s_1}} + 2 \sum_{i=2}^M \frac{1 - e^{-\alpha\Delta s_i}}{1 + e^{-\alpha\Delta s_i}} \right) \left( \frac{2}{1 - e^{-2\beta t_1}} + 2 \sum_{j=2}^N \frac{1 - e^{-\beta\Delta t_j}}{1 + e^{-\beta\Delta t_j}} \right) \\ &= \frac{\sigma^2}{\alpha\beta} A_{M,N}. \end{aligned}$$

Now, (8), (12), (13), (18) and (19) imply (17) which completes the proof.  $\square$

### 3 Case of stationary Ornstein-Uhlenbeck sheets

Consider the stationary Ornstein-Uhlenbeck sheet  $\{\tilde{X}(s, t) : s, t \in \mathbb{R}\}$  which is a zero mean Gaussian process with

$$\mathbb{E}\tilde{X}(s_1, t_1)\tilde{X}(s_2, t_2) = \frac{\sigma^2}{4\alpha\beta}e^{-\alpha|s_2-s_1|-\beta|t_2-t_1|},$$

where  $\alpha > 0$ ,  $\beta > 0$ ,  $\sigma > 0$ . Let  $\tilde{Y}(s, t) := \tilde{X}(s, t) + m$ ,  $s, t \in \mathbb{R}$ . Consider a grid on the rectangle  $[S_1, S_2] \times [T_1, T_2] \subset \mathbb{R}^2$  induced by the points  $S_1 =: s_1 < s_2 < \dots < s_M := S_2$  and  $T_1 =: t_1 < t_2 < \dots < t_N := T_2$ . It is easy to check that the random variables  $\tilde{Y}(s_1, t_1)$ ,  $\tilde{\Delta}_1\tilde{Y}(s_i, t_1)$ ,  $\tilde{\Delta}_2\tilde{Y}(s_1, t_j)$  and  $\tilde{\Delta}_1\tilde{\Delta}_2\tilde{Y}(s_i, t_j)$ ,  $i = 2, \dots, M$ ,  $j = 2, \dots, N$ , are independent and have a normal distribution with zero mean and variances

$$\frac{\sigma^2}{4\alpha\beta}, \quad \frac{\sigma^2}{4\alpha\beta}(1-e^{-2\alpha\Delta s_i}), \quad \frac{\sigma^2}{4\alpha\beta}(1-e^{-2\beta\Delta t_j}), \quad \frac{\sigma^2}{4\alpha\beta}(1-e^{-2\alpha\Delta s_i})(1-e^{-2\beta\Delta t_j}),$$

respectively. Consequently, the joint density of  $\{\tilde{Y}(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  has the form

$$\begin{aligned} & f(y_{1,1}, y_{1,2}, \dots, y_{M,N}) \\ &= c \exp \left\{ -\frac{2\alpha\beta}{\sigma^2} \left( (y_{1,1} - m)^2 + \sum_{i=2}^M \frac{(\tilde{\Delta}_1 y_{i,1} - (1 - e^{-\alpha\Delta s_i})m)^2}{1 - e^{-2\alpha\Delta s_i}} \right. \right. \\ & \quad + \sum_{j=2}^N \frac{(\tilde{\Delta}_2 y_{1,j} - (1 - e^{-\beta\Delta t_j})m)^2}{1 - e^{-2\beta\Delta t_j}} \\ & \quad \left. \left. + \sum_{i=2}^M \sum_{j=2}^N \frac{(\tilde{\Delta}_1 \tilde{\Delta}_2 y_{i,j} - (1 - e^{-\alpha\Delta s_i})(1 - e^{-\beta\Delta t_j})m)^2}{(1 - e^{-2\alpha\Delta s_i})(1 - e^{-2\beta\Delta t_j})} \right) \right\}. \end{aligned}$$

Hence the MLE  $\hat{m}_{M,N}$  of  $m$  based on the observations  $\{\tilde{Y}(s_i, t_j) : i = 1, \dots, M; j = 1, \dots, N\}$  has the form  $\hat{m}_{M,N} = \tilde{\zeta}_{M,N}/\tilde{A}_{M,N}$ , where

$$\tilde{\zeta}_{M,N} = \tilde{Y}(s_1, t_1) + \sum_{i=2}^M \frac{\tilde{\Delta}_1 \tilde{Y}(s_i, t_1)}{1 + e^{-\alpha\Delta s_i}} + \sum_{j=2}^N \frac{\tilde{\Delta}_2 \tilde{Y}(s_1, t_j)}{1 + e^{-\beta\Delta t_j}} + \sum_{i=2}^M \sum_{j=2}^N \frac{\tilde{\Delta}_1 \tilde{\Delta}_2 \tilde{Y}(s_i, t_j)}{(1 + e^{-\alpha\Delta s_i})(1 + e^{-\beta\Delta t_j})}$$

and

$$\tilde{A}_{M,N} = 1 + \sum_{i=2}^M \frac{1 - e^{-\alpha\Delta s_i}}{1 + e^{-\alpha\Delta s_i}} + \sum_{j=2}^N \frac{1 - e^{-\beta\Delta t_j}}{1 + e^{-\beta\Delta t_j}} + \sum_{i=2}^M \sum_{j=2}^N \frac{(1 - e^{-\alpha\Delta s_i})(1 - e^{-\beta\Delta t_j})}{(1 + e^{-\alpha\Delta s_i})(1 + e^{-\beta\Delta t_j})}.$$

The following statements can be proved as Proposition 1 and Theorem 1, respectively.

**Proposition 3** *If  $\max_{2 \leq i \leq M} |\Delta s_i| \rightarrow 0$ ,  $\max_{2 \leq j \leq N} |\Delta t_j| \rightarrow 0$  as  $M, N \rightarrow \infty$  then  $\tilde{\zeta}_{M,N} \rightarrow \tilde{\zeta}/4$  in  $L^2$ -sense and  $\tilde{A}_{M,N} \rightarrow \tilde{A}/4$ , where*

$$\begin{aligned} \tilde{\zeta} = & \tilde{Y}(S_1, T_1) + \tilde{Y}(S_1, T_2) + \tilde{Y}(S_2, T_1) + \tilde{Y}(S_2, T_2) + \alpha \int_{S_1}^{S_2} (\tilde{Y}(u, T_1) + \tilde{Y}(u, T_2)) du \\ & + \beta \int_{T_1}^{T_2} (\tilde{Y}(S_1, v) + \tilde{Y}(S_2, v)) dv + \alpha\beta \int_{S_1}^{S_2} \int_{T_1}^{T_2} \tilde{Y}(u, v) dudv, \end{aligned}$$

and

$$\tilde{A} = (2 + \alpha(S_2 - S_1))(2 + \beta(T_2 - T_1)).$$

**Theorem 2** *The measures  $\mathbb{P}_{\tilde{Y}}$  and  $\mathbb{P}_{\tilde{X}}$  are equivalent and the Radon-Nikodym derivative of  $\mathbb{P}_{\tilde{Y}}$  with respect to  $\mathbb{P}_{\tilde{X}}$  equals*

$$\frac{d\mathbb{P}_{\tilde{Y}}}{d\mathbb{P}_{\tilde{X}}}(\tilde{Y}) = \exp \left\{ -\frac{\alpha\beta}{2\sigma^2} (m^2 \tilde{A} - 2m\tilde{\zeta}) \right\}.$$

*The maximum likelihood estimator of the shift parameter  $m$  based on the observations  $\{\tilde{Y}(s, t) : s \in [S_1, S_2], t \in [T_1, T_2]\}$  has the form  $\tilde{m} = \tilde{\zeta}/\tilde{A}$  and it has normal distribution with mean  $m$  and variance  $\sigma^2/(\alpha\beta\tilde{A})$ .*

We remark that Theorem 2 corresponds to the result obtained by N.M. Arató [3].

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