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## **Renormalization of Interacting Diffusions**

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# RENORMALIZATION OF INTERACTING DIFFUSIONS

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## **Abstract**

This is a mini-review of some recent results on the large space-time behavior of infinite systems of interacting diffusions. Although the techniques that have been used come from stochastic analysis, the results are driven by renormalization ideas from statistical physics, centered around the notion of universality.

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# 1 Introduction

Statistical physics is the bridge between the microscopic world of interacting particles and the macroscopic world of physical phenomena. Its goal is to describe the behavior on large space-time scales of stochastic systems with a large number of locally interacting components and to explain the different types of phenomena occurring in such systems. A central task of statistical physics is to explain *universality*, i.e., the experimentally observed fact that for many systems the large scale behavior is to a certain extent independent of the precise nature of the local interaction. One way to approach this task is via the method of *renormalization*, which will be demonstrated here on a specific class of models.

The fact that on large space-time scales stochastic systems show universality should come as no surprise. Intuitively one expects ‘averaging principles’ similar in spirit to the law of large numbers to be in force. However, it turns out to be very hard to make this intuition precise in concrete examples, even though it should be quite general.<sup>1</sup>

Renormalization is the idea to look at the system along a *sequence of increasing space-time scales* and to show that *block averages* along this sequence have a dynamics that converges to a simple limiting dynamics as one moves up in the hierarchy of scales. Each step up in this hierarchy is associated with a *renormalization transformation*, acting on appropriate quantities, in which some information on the previous scale is averaged out. In this way the iterates of the renormalization transformation form the link between the microscopic and the macroscopic world. The universal limiting dynamics arises as the attracting fixed point or attracting orbit of the renormalization transformation.<sup>2</sup>

In what follows we will make the above intuition precise for a class of models of mutually attracting diffusions, namely, we will identify explicitly all the objects appearing in italics above.

## 2 The model

We consider a collection of diffusions

$$\{X_i(t): t \geq 0\}_{i \in \mathcal{L}} \tag{1}$$

indexed by a *lattice*  $\mathcal{L}$  and taking values in a *state space*  $\mathcal{S} \subseteq \mathbb{R}$ . A typical choice is

$$\mathcal{L} = \mathbb{Z}^N (N \geq 1), \quad \mathcal{S} = \mathbb{R}. \tag{2}$$

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<sup>1</sup>A well-known example is the behavior of fluids, which is described by the so-called hydrodynamic equations. These equations are the same for all fluids, except for certain transport coefficients, like shear and bulk viscosity, thermal conductivity, etc.

<sup>2</sup>This short introduction focusses on renormalization in *non-equilibrium* statistical physics. Renormalization is also successfully applied in *equilibrium* statistical physics, for instance, to study Gibbs measures, percolation, etc. Moreover, various types of renormalization transformations other than block averages are possible, like majority rules, dilution rules, etc.

The diffusions evolve according to the following system of coupled SDE's:

$$dX_i(t) = \sum_{j \in \mathcal{L}} a(j-i)[X_j(t) - X_i(t)] dt + \sqrt{b(X_i(t))} dW_i(t) \quad (3)$$

$$(i \in \mathcal{L}, t \geq 0).$$

Here,  $a(\cdot)$  is a nonnegative function on  $\mathcal{L}$  controlling the interacting strength between different components,  $b(\cdot)$  is a nonnegative function on  $\mathcal{S}$  controlling the diffusion rate of single components, and  $\{W_i(t): t \geq 0\}_{i \in \mathcal{L}}$  is an i.i.d. collection of standard Brownian motions on  $\mathcal{S}$ . As the initial condition for (3) we choose the constant configuration:

$$X_i(0) = \theta \quad \forall i \in \mathcal{L} \quad \text{for some } \theta \in \mathcal{S}. \quad (4)$$

The right-hand side of (3) consists of two parts:

- an *interactive* part, where different components attract each other with a strength depending on their relative location;
- a *noninteractive* part, where each component follows an autonomous Brownian motion with a local diffusion rate.

What makes the system in (3) interesting is that these parts compete: the first tends to make the configuration flat, the second tends to make it noisy.

For the system in (3) to make sense, some restrictions must be placed on  $a(\cdot)$  and  $b(\cdot)$ . For instance, we must require that  $0 < \sum_{i \in \mathcal{L}} a(i) < \infty$ . Furthermore, in order for (3) to have a unique strong solution we must require that  $b(\cdot)$  is sufficiently smooth on the interior of  $\mathcal{S}$ . In addition,  $b(\cdot)$  should behave properly near the boundary of  $\mathcal{S}$ , in order for the components not to leave  $\mathcal{S}$  in a finite time.

### 3 Interpretation of the model

The system in (3) is a model for population dynamics, where we imagine that each site of  $\mathcal{L}$  carries a large number of individuals.

Suppose that  $\mathcal{S} = [0, 1]$ . Then we have a model where each individual can be of two ‘types’:  $X_i(t)$  denotes the fraction of individuals at site  $i$  at time  $t$  of type 1 and  $1 - X_i(t)$  the fraction of individuals of type 2. The states at different sites change due to two random mechanisms:

- *migration*: individuals move randomly between different sites;
- *resampling*: individuals reproduce randomly at single sites.

The system in (3) is the *continuum limit* where the number of individuals per site tends to infinity. The migration is controlled by  $a(\cdot)$ , the resampling by  $b(\cdot)$ . For more background the reader is referred to Sawyer and Felsenstein [9] and to Ethier and Kurtz [7] Chapter 10.

Suppose that  $\mathcal{S} = [0, \infty)$ . Then we have a model where each individual can have a certain ‘mass’:  $X_i(t)$  describes the mass of the population at site  $i$  at time  $t$ . Again the states at different sites change due to migration and resampling.

## 4 Block averages and renormalization

As explained in Section 1, we want to look at the system in (3) along a sequence of increasing space-time scales. To that end we define the following *block averages*. For  $n \in \mathbb{N}_0$ , let

$$X_i^{(n)}(t) = \frac{1}{|B_i^{(n)}|} \sum_{j \in B_i^{(n)}} X_j(s_n t) \quad (i \in \mathcal{L}, t \geq 0) \quad (5)$$

with

$$B_i^{(n)} = \{j \in \mathcal{L}: \|j - i\| \leq n\}. \quad (6)$$

This is the average of the components in a block of radius  $n$  centered at  $i \in \mathcal{L}$ , with time scaled up by a factor  $s_n$ . For the distance  $\|\cdot\|$  we may choose some appropriate norm on  $\mathcal{L}$ , e.g. the Euclidean norm in case  $\mathcal{L} = \mathbb{Z}^N$  ( $N \geq 1$ ). The function  $n \mapsto s_n$  is called the *space-time scaling relation* and needs to be chosen appropriately depending on the model. Typically  $s_n$  increases with  $n$ , e.g.  $s_n = |B_0^{(n)}|$ .

If we fix  $n$  and substitute (5) into (3), then we find that the collection

$$\{X_i^{(n)}(t): t \geq 0\}_{i \in \mathcal{L}} \quad (7)$$

again satisfies a system of coupled SDE’s, but with a more complicate structure. As  $n$  increases this system becomes gradually more involved, because as the blocks get larger more information is summed out. However, the idea is that in the limit as  $n \rightarrow \infty$  the behavior of the system should simplify because of some underlying ‘averaging principle’. This leads us to the following:

- PROGRAM: Show that

$$\{X_0^{(n)}(t): t \geq 0\} \implies \{Y(t): t \geq 0\} \quad (n \rightarrow \infty) \quad (8)$$

and identify the limiting process explicitly.

Here  $\implies$  denotes weak convergence on path space, and 0 is chosen as a typical center of the  $n$ -blocks. The process  $\{Y(t): t \geq 0\}$  is obviously important, because it describes our system in (3) on a large space-time scale: the limit  $n \rightarrow \infty$  effectuates the link between the microscopic and the macroscopic level of description. Typically we may expect this limiting process to display universality properties: it should be to a certain extent independent of the particular choice of the coefficients  $a(\cdot)$  and  $b(\cdot)$  in (3).

In the sequel we will demonstrate that the above program can indeed be carried out successfully in a number of cases, with interesting conclusions.

## 5 The hierarchical lattice

It would be nice if we could work with the lattice  $\mathcal{L} = \mathbb{Z}^N$  ( $N \geq 1$ ). Unfortunately, this is at present too difficult. However, a complete story can be told when

$$\mathcal{L} = \Omega_N = \text{the } N\text{-dimensional hierarchical lattice} \quad (9)$$

and  $N$  is very large. The hierarchical lattice is defined as

$$\Omega_N = \left\{ i = (i_1, i_2, \dots) : i_k \in \{0, 1, \dots, N-1\}, i_k \neq 0 \text{ finitely often} \right\}. \quad (10)$$

Think of  $\Omega_N$  as the collection of infinite telephone numbers on  $N$  symbols ending with all 0's. With componentwise addition modulo  $N$ ,  $\Omega_N$  is a countable group.

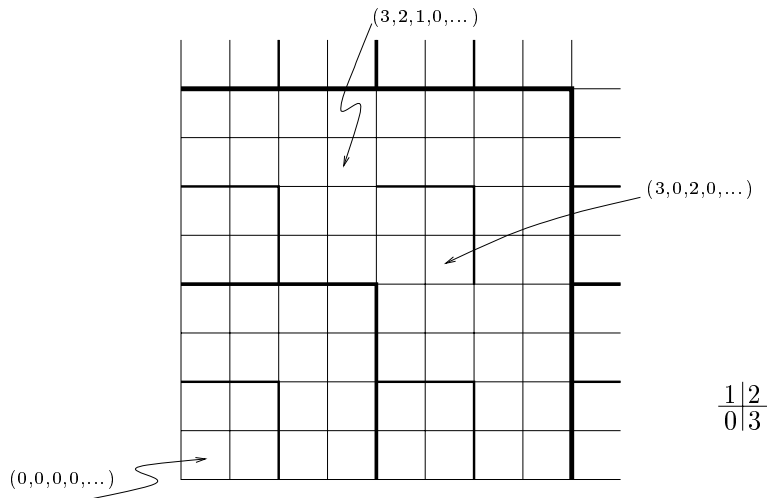


FIG. 1. The hierarchical lattice  $\Omega_N$  for  $N = 4$

On  $\Omega_N$  we have a natural norm:

$$\|i - j\| = \inf \{ l \geq 0 : i_k = j_k \forall k > l \}, \quad (11)$$

called the *hierarchical distance*. Note that for each site in  $\Omega_N$  the number of sites within distance  $l$  is  $N^l$  ( $l \geq 0$ ).

At first sight the choice  $\mathcal{L} = \Omega_N$  may seem a bit artificial. However, this choice arises naturally in an application from genetics. What  $\Omega_N$  does is organize the sites according to their ‘genetic information’. The population is organized into families, clans, villages, etc. Site  $i$  is a member of family  $i_1$ , clan  $i_2$ , village  $i_3$ , etc. See Sawyer and Felsenstein [9].

For the state space  $\mathcal{S}$  we will consider the following three choices:

$$\mathcal{S} = [0, 1], [0, \infty), \mathbb{R}. \quad (12)$$

For the interaction function  $a(\cdot)$  on  $\Omega_N$  we choose:

$$a(0) = 0, \quad a(i) = \sum_{l \geq \|i\|} c_l N^{1-2l} \quad (i \neq 0), \quad (13)$$

where  $c_1, c_2, \dots \in (0, \infty)$  are constants that are assumed to satisfy

$$\sum_{l \geq 1} \frac{1}{c_l} = \infty. \quad (14)$$

The reason for (14) will become clear later on.<sup>3</sup> We also need to require that  $\sum_{l \geq 1} c_l N^{-l} < \infty$ , as can be seen from (16) below. Since  $|\{i \in \Omega_N: \|i\| \leq l\}| = N^l$  ( $l \geq 1$ ), what (13) says is that the total interaction strength between a given component and all the components at a given hierarchical distance decays by a factor  $N^{-1}$  per unit in the hierarchical distance. For the diffusion function  $b(\cdot)$  on  $\mathcal{S}$  we assume that it satisfies the following conditions:

•  $\mathcal{S} = [0, 1]$ :

- (i)  $b(0) = b(1) = 0$
- (ii)  $b(x) > 0 \forall x \in (0, 1)$
- (iii)  $b$  is Lipschitz on  $[0, 1]$ .

•  $\mathcal{S} = [0, \infty)$ :

- (i)  $b(0) = 0$
- (ii)  $b(x) > 0 \forall x > 0$
- (iii)  $b$  is locally Lipschitz on  $[0, \infty)$
- (iv)  $b(x) = o(x^2)$  ( $x \rightarrow \infty$ ).

•  $\mathcal{S} = \mathbb{R}$ :

- (i)  $b$  is bounded away from 0
- (ii)  $b$  is locally Lipschitz on  $\mathbb{R}$
- (iii)  $b(x) = o(x^2)$  ( $|x| \rightarrow \infty$ ).

The class of such functions will be denoted by  $\mathcal{B}$  in each of the three cases. For the space-time scaling relation we pick

$$s_n = N^n = \{i \in \Omega_N: \|i\| \leq n\}. \quad (15)$$

---

<sup>3</sup>Equation (14) is precisely the condition under which the random walk on  $\Omega_N$  with jump rate  $a(\cdot)$  is recurrent. This fact plays an important role in the universality properties. See also the last paragraph of Section 9.

To exhibit the  $N$ -dependence, we write the components of (3) as  $X_i^N(t)$ , and the  $n$ -blocks in (5) as  $X_i^{N,(n)}(t)$ . In terms of the block averages the system in (3) with initial condition in (4) can be rewritten as

$$dX_i^N(t) = \sum_{n \geq 1} c_n N^{1-n} [X_i^{N,(n)}(t/N^n) - X_i^N(t)] dt + \sqrt{b(X_i^N(t))} dW_i(t) \quad (16)$$

$$X_i^N(0) = \theta \quad (i \in \Omega_N, t \geq 0).$$

This representation shows that each component is linearly attracted to a weighted average of the block averages. It also explains why we wrote the interaction function in the form given in (13).

Apart from  $\mathcal{S}$  and  $N$ , there are three parameters appearing in (16):  $\theta$ ,  $b$  and  $(c_l)_{l \in \mathbb{N}}$ . These parameters will play a key role in the renormalization approach that is described next.

## 6 The renormalization transformation

Let us have a look at equation (16) for  $i = 0$  and see what happens when we pass to the limit  $N \rightarrow \infty$ , at least heuristically. Clearly, only the term with  $n = 1$  survives. Moreover, the term  $X_0^{N,(1)}(t/N)$  converges to  $\theta$  for all  $t \geq 0$ . We therefore find that

$$\{X_0^N(t): t \geq 0\} \Longrightarrow \{Y(t): t \geq 0\} \quad (17)$$

with the right-hand side given by the SDE

$$dY(t) = c_1[\theta - Y(t)] dt + \sqrt{b(Y(t))} dW(t), \quad Y(t) = \theta. \quad (18)$$

In other words, in the limit as  $N \rightarrow \infty$  the single components decouple and follow a simple diffusion equation parametrized by  $\theta$ ,  $b$  and  $c_1$ .

It turns out that something similar happens with the  $n$ -blocks for  $n \geq 1$ , as is seen through the following result due to Dawson and Greven [3], [4], [5], [6].

**Theorem 1** Fix  $\theta \in \mathcal{S}$ ,  $b \in \mathcal{B}$ ,  $(c_l)_{l \in \mathbb{N}} \subset (0, \infty)$  and  $n \in \mathbb{N}_0$ . Then as  $N \rightarrow \infty$

$$\{X_0^{N,(n)}(t): t \geq 0\} \Longrightarrow \{Y^{\theta, F^{(n)}b, c_{n+1}}(t): t \geq 0\}, \quad (19)$$

where  $\{Y^{y,b,c}(t): t \geq 0\}$  is the unique strong solution of the SDE

$$dY(t) = c[y - Y(t)] dt + \sqrt{b(Y(t))} dW(t), \quad Y(0) = y. \quad (20)$$

In (19),  $F^{(n)} = F_{c_n} \circ \dots \circ F_{c_1}$  is the  $n$ -th iterate of a renormalization transformation  $F_c$ , acting on  $b$ , which is identified in Theorem 2.



**Theorem 2** Fix  $c \in (0, \infty)$ . Then  $F_c$  acts on  $b \in \mathcal{B}$  as follows:

$$(Fb)(y) = \int_{\mathcal{S}} b(x) \nu^{y,b,c}(dx) \quad (y \in \mathcal{S}). \quad (21)$$

Here  $\nu^{y,b,c}$  is the unique equilibrium associated with (20), which is given by the formula

$$\nu^{y,b,c}(dx) = \frac{\mu^{y,b,c}(x) dx}{\int_{\mathcal{S}} \mu^{y,b,c}(x) dx}, \quad (22)$$

where

$$\mu^{y,b,c}(x) = \frac{1}{b(x)} \exp \left[ - \int_y^x \frac{c(z-y)}{b(z)} dz \right]. \quad (23)$$

The formula in (23) is the solution (modulo normalization) of the equation

$$c(y-x)\mu(x) = \frac{\partial}{\partial x} [b(x)\mu(x)], \quad (24)$$

which expresses the equilibrium property.

Here is a heuristic explanation of what is behind Theorems 1 and 2. By summing (16) over the components in a 1-block and using the special properties of the hierarchical distance, we get the system of SDE's:

$$\begin{aligned} dX_i^{N,(1)}(t) &= \sum_{n \geq 2} c_n N^{2-n} [X_i^{N,(n)}(t/N^{n-1}) - X_i^{N,(1)}(t)] dt \\ &\quad + \frac{1}{\sqrt{N}} \sum_{j: \|j-i\| \leq 1} \sqrt{b(X_j^N(Nt))} dW_j(t) \end{aligned} \quad (25)$$

$$X_i^N(0) = \theta \quad (i \in \Omega_N, t \geq 0).$$

Here we have scaled up time by a factor  $N$ , both in the 1-block and in the Brownian motions. Three observations are now crucial:

- If a 1-block has value  $y$ , then each of the  $N$  components in this block is approximately attracted towards  $y$ , since the attraction towards the values of the  $n$ -blocks with  $n \geq 2$  is weak when  $N$  is large (recall (16)).
- On a time scale smaller than  $N$  each of the components in the 1-block approximately reaches the equilibrium  $\nu^{y,b,c_1}$ .
- On the time scale  $N$  the value of the  $n$ -blocks with  $n \geq 2$  is still very close to the initial value  $\theta$ .

By combining these three observations we see that the diffusion rate of a 1-block at value  $y$  is given by  $(F_{c_1}b)(y)$ , the *average diffusion rate of a component* in the equilibrium  $\nu^{y,b,c_1}$ , and that the drift is towards the initial value  $\theta$  with drift strength  $c_2$ . This is precisely what (19-20) says for  $n = 1$ .

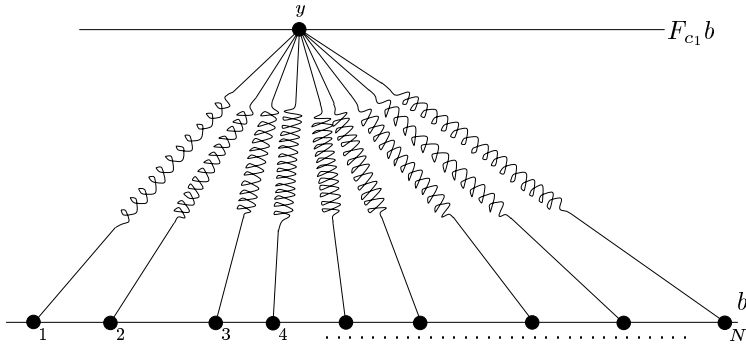


FIG. 2. The renormalization transformation  $F_{c_1}$

The same type of argument applies when we go from  $n$ -blocks to  $(n + 1)$ -blocks, which is why the iterates  $F^{(n)}$  appear.

## 7 Analysis of the orbit

Theorems 1 and 2 show that on hierarchical space-time scale  $n$  the block averages behave like a diffusion with drift towards  $\theta$  (the initial value in (4)), with drift strength  $c_n$ , and with diffusion function  $F^{(n)}b$  (the  $n$ -th renormalized diffusion function), provided  $N$  is large enough. Thus, to study the large space-time behavior of our system it remains to study the orbit

$$(F^{(n)}b)_{n \in \mathbb{N}_0} \quad (26)$$

and to see whether this orbit converges to an attracting fixed point or an attracting orbit. This is not an easy task, because  $F_c$  is a *nonlinear integral transform*, as can be seen from (21–23). Fortunately,  $F_c$  turns out to have certain nice order-preserving properties, which can be exploited to analyze the asymptotics of the orbit. The following results are due to Baillon *et al.* [1], [2] and show that for each of the three choices of  $\mathcal{S}$  there is indeed convergence of the orbit after appropriate scaling.

**Theorem 3**  $F_c \mathcal{B} \subseteq \mathcal{B}$  for all  $c \in (0, \infty)$  and each of the three classes  $\mathcal{B}$  for  $\mathcal{S} = [0, 1], [0, \infty), \mathbb{R}$ .

This result says that the iterates  $F^{(n)}$  are well-defined on  $\mathcal{B}$ , so that indeed we have the right to speak about an orbit. In the following we abbreviate

$$\sigma_n = \sum_{l=1}^n \frac{1}{c_l} \quad (n \geq 1). \quad (27)$$

- $\mathcal{S} = [0, 1]$ :

**Theorem 4** Suppose that  $b \in \mathcal{B}$ . Then  $\lim_{n \rightarrow \infty} \sigma_n F^{(n)} b = b^*$  pointwise and uniformly, with  $b^*(x) = x(1-x)$ .

This result says that the  $n$ -blocks have a diffusion function that is close to  $\frac{1}{\sigma_n} b^*$  for large  $n$ , irrespective of the diffusion function  $b$  for single components. In other words, there is *complete universality* within the class  $\mathcal{B}$ , with the parabola  $b^*$  acting as the global attractor. The diffusion in (20) with  $b = b^*$  is called the *Fisher-Wright diffusion*.

•  $\mathcal{S} = [0, \infty)$ :

**Theorem 5** (a) Suppose that  $b \in \mathcal{B}$  satisfies  $b(x) \sim dx$  ( $x \rightarrow \infty$ ) for some  $d > 0$ . Then

$$\lim_{n \rightarrow \infty} F^{(n)} b = db^* \quad \text{pointwise and locally uniformly,} \quad (28)$$

with  $b^*(x) = x$ .

(b) Suppose that  $b \in \mathcal{B}$  satisfies  $b(x) \sim x^\alpha L(x)$  ( $x \rightarrow \infty$ ) with  $\alpha \in (0, 2) \setminus \{1\}$  and with  $L$  slowly varying at infinity. Then there exist constants  $0 < K_1 \leq K_2 < \infty$ , depending only on  $\alpha$ , such that

$$K_1 b^* \leq \liminf_{\sigma_n \rightarrow \infty} \frac{\sigma_n}{e_n} F^{(n)} b \leq \limsup_{\sigma_n \rightarrow \infty} \frac{\sigma_n}{e_n} F^{(n)} b \leq K_2 b^* \quad (29)$$

pointwise and locally uniformly,

with  $b^*(x) = x$ . Here  $(e_n)_{n \in \mathbb{N}}$  is defined by

$$\frac{1}{\sigma_n} = \frac{g(e_n)}{e_n^2}. \quad (30)$$

This result says that the  $n$ -blocks have a diffusion function that, modulo a constant, is close to  $\frac{e_n}{\sigma_n} b^*$  for large  $n$ , with a linear  $b^*$ . So again  $b^*$  is the global attractor within the class  $\mathcal{B}$ , except that this time the scaling with  $n$  depends on the growth rate of  $b$  at infinity through the solution  $(e_n)_{n \in \mathbb{N}}$  of (30). Although this solution is in general not unique, all solutions are asymptotically equivalent. Thus there is universality within the subclass of  $\mathcal{B}$  consisting of all  $b$  with the same  $\alpha$  and  $L$ . The diffusion in (20) with  $b = b^*$  is called *Feller's branching diffusion*.

Bounds for  $K_1$  and  $K_2$  are known, but they are not sharp. It is conjectured that

$$K_1 = K_2 = (\alpha!)^{\frac{1}{2-\alpha}} 2^{\frac{1-\alpha}{2-\alpha}}. \quad (31)$$

For the special case where  $c_l \equiv 1$  and  $L \equiv 1$ , we find that  $\sigma_n = n$  and  $e_n \sim n^{1/(2-\alpha)}$  ( $n \rightarrow \infty$ ) and so

$$F^{(n)} b \asymp n^{-\frac{1-\alpha}{2-\alpha}} b^* \quad (n \rightarrow \infty). \quad (32)$$

Thus, for  $\alpha \in (1, 2)$  the iterates  $F^{(n)} b$  grow with  $n$ , while for  $\alpha \in (0, 1)$  they shrink with  $n$ .

•  $\mathcal{S} = \mathbb{R}$ :

**Theorem 6** Suppose that  $b \in \mathcal{B}$  satisfies  $b(x) \rightarrow d$  ( $|x| \rightarrow \infty$ ) for some  $d > 0$ . Then

$$\lim_{n \rightarrow \infty} F^{(n)}b = db^* \quad \text{pointwise and locally uniformly,} \quad (33)$$

with  $b^*(x) = 1$ .

Now the diffusion function of the  $n$ -blocks is close to  $db^*$  for large  $n$ , with a constant  $b^*$ . So this time  $b^*$  is the attractor for the subclass of  $\mathcal{B}$  consisting of all  $b$  that are asymptotically constant. This is the analogue of Theorem 5(a). There is an analogue of Theorem 5(b) as well, but we will not discuss this here.<sup>4</sup> The diffusion in (20) with  $b = b^*$  is called the *Ornstein-Uhlenbeck diffusion*.

The above three examples show that the program outlined in Section 4 can be made explicit for one-dimensional diffusions indexed by the hierarchical lattice in the limit of large dimension. The outcome shows that indeed there is universality, and that the limiting processes correspond to certain *special* types of diffusions.

## 8 Higher-dimensional state spaces

What happens if  $\mathcal{S}$  is not one-dimensional but, say, is chosen to be

$$\mathcal{S} \subset \mathbb{R}^d \quad (d \geq 2) \text{ convex and compact?} \quad (34)$$

This is an important generalization, because now the state  $X_i^N(t)$  at site  $i \in \Omega_N$  at time  $t$  can be used to describe  $d + 1$  different ‘types’ of individuals (compare with the interpretation in Section 3). The typical example is when  $\mathcal{S}$  is the  $d$ -dimensional simplex:

$$\mathcal{S} = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_m \geq 0, \sum_{m=1}^d x_m \leq 1 \right\}. \quad (35)$$

If  $X_i(t) = x$ , then  $x_m$  is the fraction of individuals at site  $i$  at time  $t$  of type  $m = 1, \dots, d$ , while  $1 - \sum_{m=1}^d x_m$  is the fraction of individuals of type  $d + 1$ . The class  $\mathcal{B}$  is now specified by:

- (i)  $b(x) = 0 \quad \forall x \in \partial\mathcal{S}$
- (ii)  $b(x) > 0 \quad \forall x \in \text{int}(\mathcal{S})$
- (iii)  $b$  is Lipschitz on  $\mathcal{S}$ .

The rest of the model is unchanged.<sup>5</sup> The question we address is: Do Theorems 1, 2, 3 and 4 carry over?

It turns out that the generalization to (34) causes major problems. The reason is two-fold:

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<sup>4</sup>The result in Theorem 6 is not actually stated in Baillon *et al.* [2] but can be easily deduced from the formalism developed there.

<sup>5</sup>In particular, the diffusion term in (3) remains a scalar and is not replaced by a  $d \times d$  diffusion matrix. This choice corresponds to an isotropic model. Generalization to anisotropic models is possible. See Swart [10].

- Diffusions in higher-dimensional domains with boundaries are presently rather ill understood. For instance, it is not known whether (3) has a unique strong solution for all  $b \in \mathcal{B}$ , nor is it known whether (20) has a unique and ergodic equilibrium<sup>6</sup>. These questions are settled only for a subclass of  $\mathcal{B}$ , but this subclass is too small for our purposes (e.g. it does not contain the  $b^*$  in (36) below).
- Even in the case of a unique and ergodic equilibrium there is no explicit formula for the equilibrium measure  $\nu^{y,b}$ , like in (22–23) for the one-dimensional case. This means that there is no explicit formula for the renormalization transformation  $F_c$  either, like in (21) for the one-dimensional case. Consequently, one has to use somewhat indirect techniques to get a handle on the renormalization analysis.

The generalization to (34) has been analyzed by den Hollander and Swart [8]. There Theorems 1, 2 and 3 are essentially *assumed* to be true, and the work focusses on the analogue of Theorem 4. The following is the main result.

**Theorem 7**  $\lim_{n \rightarrow \infty} \sigma_n F^{(n)} b = b^*$  *pointwise for all  $b \in \mathcal{B}$ , with  $b^*$  the unique solution of the equation*

$$\begin{cases} \frac{1}{2} \Delta b &= -1 & \text{on } \text{int}(\mathcal{S}) \\ b &= 0 & \text{on } \partial \mathcal{S}, \end{cases} \quad (36)$$

where  $\Delta$  is the Laplacian on  $\mathbb{R}^d$ .

Thus we again get complete universality, this time with  $b^*$  as the global attractor. The diffusion in (20) with  $b = b^*$  is the analogue of the Fisher-Wright diffusion on  $[0, 1]$  and depends on the shape of  $\mathcal{S}$ .

There are as yet no analogues of Theorem 7 for noncompact domains, like  $\mathcal{S} = [0, \infty)^d$ . The problems alluded to above are even worse for this case, and the scaling with  $n$  is not known either. Hopefully some progress will come in the near future.

## 9 Open problems

The above results lead us to the following open problems:

1. Try to prove Theorems 1 and 2 for the choice in (34). This task has been taken up with limited success in den Hollander and Swart [8], namely, only for a certain subclass of  $\mathcal{B}$ .
2. Try to extend Theorem 7 to  $\mathcal{S} = [0, \infty)^d$ , i.e, try to prove the analogue of Theorem 5 for higher dimensions.

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<sup>6</sup>In den Hollander and Swart [8] it is shown though that if (3) has a unique weak solution for a given  $b \in \mathcal{B}$ , then (20) has a unique and ergodic equilibrium for this  $b$ .

3. Try to replace  $\Omega_N$  by a different lattice, e.g.  $\mathbb{Z}^N$ . Do similar results hold in the limit as  $N \rightarrow \infty$ ? The answer will certainly depend on the choice of the function  $a(\cdot)$  controlling the interaction strength between different components. A different space-time scaling relation  $n \mapsto s_n$  will be needed too.

Swart [10] has proved universality in the *clustering* properties of the system in (3) under very general conditions. For any lattice  $\mathcal{L}$  that is a finite or countable Abelian group, for any state space  $\mathcal{S}$  that is a convex and compact subset of  $\mathbb{R}^d$  ( $d \geq 1$ ), for any diffusion function  $b(\cdot)$  that is continuous on  $\mathcal{S}$  with an appropriate behavior near  $\partial\mathcal{S}$ , and for any interaction function  $a(\cdot)$  such that the symmetrized function  $a(\cdot) + a(-\cdot)$  is the jump rate of a *recurrent* random walk on  $\mathcal{L}$ , the result is that the system clusters: it develops large blocks in which the components are almost identical. The limiting distribution of the single components can be identified explicitly and turns out to be independent of  $a(\cdot)$  and  $b(\cdot)$ . In the proof the function  $b^*(\cdot)$  solving (36) again plays a major role.

For certain special choices of the state space  $\mathcal{S}$  and the diffusion function  $b(\cdot)$  the limiting distribution of the single components can actually be identified explicitly. This analysis considerably enhances our insight into what drives the universality.

## 10 Conclusion

What is described above is the implementation of renormalization ideas from statistical physics for a particular class of interacting diffusions. We have managed to actually renormalize the whole dynamics explicitly. There are very few examples of interacting particle systems for which a picture like here has been corroborated in full detail, although physicists believe it should be quite general.

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