A NOTE ON POSSIBLE COUNTEREXAMPLES TO THE ABHYANKAR-SATHAYE CONJECTURE CONSTRUCTED BY SHPILRAIN AND YU

Arno van den Essen, Peter van Rossum

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Arno van den Essen     Peter van Rossum

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In [SY99], Shpilrain and Yu construct a class of candidate counterexamples to the Embedding Conjecture of Abhyankar and Sathaye (see [Sat76] and [Abh78]).

Proposition 1 ([SY99], Proposition 1.5). Let \( q \) be a polynomial over \( \mathbb{C} \) in four variables and let \( k \) be a positive integer. In \( \mathbb{C}[x,y,z,t,u] \), define \( \omega := x^k - tz^2 - uz \), \( \alpha := \omega - (w^2 + yz)z \), and \( \beta := y - (\omega^2 + yz)^2z + 2\omega(\omega^2 + yz) \) and consider the following polynomial \( p \):

\[
p := x - q(\beta + \alpha^2, z, t + \beta, u - \beta).
\]

Then the zero fiber of this polynomial is isomorphic to a coordinate hyperplane, i.e., \( \mathbb{C}[x,y,z,t,u]/(p) \cong \mathbb{C}[x,y,z,t] \).

If the Embedding Conjecture were true, then these polynomials \( p \) are all coordinates, i.e., component of a polynomial automorphism \( \mathbb{C}[x,y,z,t,u] \to \mathbb{C}[x,y,z,t,u] \). Shpilrain and Yu explain why this class of polynomials \( p \) (for \( k \geq 2 \)) could contain counterexamples to this conjecture.

However, this note shows that these polynomials \( p \) are all coordinates. They even turn out to be tame.

Proposition 2. In the situation of the previous proposition, \( p \) is a tame coordinate.

Proof. Write \( \nu := \omega^2 + yz \), \( A := \beta + \alpha^2 \), \( B := z \), \( C := t + \beta \), and \( D := u - \beta \). Note that \( \alpha = \omega - \nu z \), \( \beta = y - \nu^2 z + 2\omega \nu \), and \( \nu = x^k - z(u + tz) \). Applying the (elementary) polynomial automorphism

\[
\begin{align*}
    u &\mapsto u - tz \quad (\text{and} \, x \mapsto x, \, y \mapsto y, \, z \mapsto z, \, t \mapsto t)
\end{align*}
\] (1)
to the polynomials $\omega, \nu, \alpha, \beta, A, B, C,$ and $D$ transforms them into

\[
\begin{align*}
\omega_1 & := x^k - zu, \\
\nu_1 & := \omega_1^2 + yz = x^{2k} + z(y - 2x^k u + zu^2), \\
\alpha_1 & := \omega_1 - \nu_1 z, \\
\beta_1 & := y - \nu_1^2 z + 2\omega_1 \nu_1, \\
A_1 & := \beta_1 + \alpha_1^2, \\
B_1 & := z, \\
C_1 & := t + \beta_1, \quad \text{and} \\
D_1 & := u - tz - \beta_1
\end{align*}
\]

respectively. Now applying the polynomial automorphism

\[ y \mapsto y + 2x^k u - zu^2, \quad (2) \]

transforms these polynomials into

\[
\begin{align*}
\omega_2 & := x^k - zu, \\
\nu_2 & := x^{2k} + zy, \\
\alpha_2 & := \omega_2 - \nu_2 z \\
& = x^k - z(u + x^{2k} + zy), \\
\beta_2 & := y - \nu_2^2 z + 2\omega_2 \nu_2, \\
A_2 & := \beta_2 + \alpha_2^2, \\
B_2 & := z, \\
C_2 & := t + \beta_2, \quad \text{and} \\
D_2 & := u - tz - \beta_2,
\end{align*}
\]

respectively. Now applying the polynomial automorphism

\[ u \mapsto u - x^{2k} - zy \quad (3) \]
transforms these polynomials into
\[\omega_3 := x^k - z(u - x^2 - zy),\]
\[\nu_3 := x^k + zy,\]
\[\alpha_3 := x^k - zu,\]
\[\beta_3 := y + 2x^k(u - x^2 - zy) - z(u - x^2 - zy)^2 - \nu_3^2 z + 2\omega_3 \nu_3,\]
\[A_3 := \beta_3 + \alpha_3^2,\]
\[B_3 := z,\]
\[C_3 := t + \beta_3, \quad \text{and}\]
\[D_3 := (u - x^2 - zy) - tz - \beta_3\]
\[= u - y - x^2 - 2x^k u - yz - tz + u^2 z\]
\[= u - y - x^2 - 2x^k u - z(t + y - u^2),\]
respectively. Now applying the polynomial automorphism
\[t \mapsto t - y + u^2\] transforms these polynomials into
\[\omega_4 := x^k - z(u - x^2 - zy),\]
\[\nu_4 := x^k + zy,\]
\[\alpha_4 := x^k - zu,\]
\[\beta_4 := y + 2x^k(u - x^2 - zy) - z(u - x^2 - zy)^2 - \nu_4^2 z + 2\omega_4 \nu_4,\]
\[A_4 := \beta_4 + \alpha_4^2,\]
\[B_4 := z,\]
\[C_4 := t - y + u^2 + \beta_4, \quad \text{and}\]
\[D_4 := u - y - x^2 - 2x^k u - zt,\]
respectively. Now applying the polynomial automorphism
\[y \mapsto -y + u - x^2 - tz - 2x^k u\] transforms the polynomials \(A_4, B_4, C_4,\) and \(D_4\) into
\[A_5 := -y + u - 2x^k uz - tz + u^2 z^2 - u^2 z,\]
\[B_5 := z,\]
\[C_5 := t + 2x^k u + u^2 - u^2 z, \quad \text{and}\]
\[D_5 := y,\]
respectively. Now applying the polynomial automorphism
\[t \mapsto t - 2x^k u - u^2 + u^2 z\]
transforms these polynomials into
\[ A_6 := -y + u - 2x^k uz - tz + 2x^k uz + u^2 z - u^2 z^2 + u^2 z^2 - u^2 z \]
\[ = u - y - tz, \]
\[ B_6 := z, \]
\[ C_6 := t, \]
\[ D_6 := y, \]
respectively. Now applying the polynomial automorphism
\[ u \mapsto u + y + tz \] (7)
transforms these polynomials into \( u, z, t, \) and \( y \) respectively. Hence the polynomial \( p = x - q(A,B,C,D) \) is transformed into \( x - q(u,z,t,y) \) by successively applying these automorphisms. Finally applying the polynomial automorphism
\[ x \mapsto x + q(u,z,t,y) \] (8)
then transforms it into \( x \). Since the polynomial automorphisms (1)-(8) are all elementary, \( p \) is a tame coordinate. \( \square \)

References


Arno van den Essen <essen@sci.kun.nl> Department of Mathematics
Peter van Rossum <petervr@sci.kun.nl> University of Nijmegen
Toernooiveld 1
6525 ED Nijmegen
The Netherlands