A NOTE ON POSSIBLE COUNTEREXAMPLES TO THE ABHYANKAR-SATHAYE CONJECTURE CONSTRUCTED BY SHPILRAIN AND YU

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In [SY99], Shpilrain and Yu construct a class of candidate counterexamples to the
Embedding Conjecture of Abhyankar and Sathaye (see [Sat76] and [Abh78]).

**Proposition 1 ([SY99], Proposition 1.5).** Let q be a polynomial over \( \mathbb{C} \) in four
variables and let k be a positive integer. In \( \mathbb{C}[x,y,z,t,u] \), define \( \omega := x^k - tz^2 - uz \),
\( \alpha := \omega - (w^2 + yz)z \), and \( \beta := y - (\omega^2 + yz)^2z + 2\omega(\omega^2 + yz) \) and consider the following
polynomial \( p \):

\[
p := x - q(\beta + \alpha^2, z, t + \beta, u - \beta).
\]

Then the zero fiber of this polynomial is isomorphic to a coordinate hyperplane, i.e.,
\( \mathbb{C}[x,y,z,t,u]/(p) \cong \mathbb{C}[x,y,z,t] \).

If the Embedding Conjecture were true, then these polynomials \( p \) are all coordinates,
i.e., component of a polynomial automorphism \( \mathbb{C}[x,y,z,t,u] \rightarrow \mathbb{C}[x,y,z,t,u] \).
Shpilrain and Yu explain why this class of polynomials \( p \) (for \( k \geq 2 \)) could contain
counterexamples to this conjecture.

However, this note shows that these polynomials \( p \) are all coordinates. They even
turn out to be tame.

**Proposition 2.** In the situation of the previous proposition, \( p \) is a tame coordinate.

**Proof.** Write \( \nu := \omega^2 + yz \), \( A := \beta + \alpha^2 \), \( B := z \), \( C := t + \beta \), and \( D := u - \beta \). Note that
\( \alpha = \omega - \nu z \), \( \beta = y - \nu^2 z + 2\omega \nu \), and \( \omega = x^k - z(u + tz) \). Applying the (elementary)
polynomial automorphism

\[
u \mapsto u - tz \quad \text{and} \quad x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t
\]
to the polynomials $\omega, \nu, \alpha, \beta, A, B, C,$ and $D$ transforms them into

$$
\omega_1 := x^k - zu,
\nu_1 := \omega_1^2 + yz = x^{2k} + z(y - 2x^k u + zu^2),
\alpha_1 := \omega_1 - \nu_1 z,
\beta_1 := y - \nu_1^2 z + 2\omega_1 \nu_1,
A_1 := \beta_1 + \alpha_1^2,
B_1 := z,
C_1 := t + \beta_1, \quad \text{and}
D_1 := u - tz - \beta_1
$$

respectively. Now applying the polynomial automorphism

$$
y \mapsto y + 2x^k u - zu^2, \quad (2)
$$

transforms these polynomials into

$$
\omega_2 := x^k - zu,
\nu_2 := x^{2k} + zy,
\alpha_2 := \omega_2 - \nu_2 z
\quad = x^k - z(u + x^{2k} + zy),
\beta_2 := y - \nu_2^2 z + 2\omega_2 \nu_2,
A_2 := \beta_2 + \alpha_2^2,
B_2 := z,
C_2 := t + \beta_2, \quad \text{and}
D_2 := u - tz - \beta_2,
$$

respectively. Now applying the polynomial automorphism

$$
u \mapsto u - x^{2k} - zy \quad (3)
$$
transforms these polynomials into

\[ \omega_3 := x^k - z(u - x^{2k} - zy), \]
\[ \nu_3 := x^{2k} + zy, \]
\[ \alpha_3 := x^k - zu, \]
\[ \beta_3 := y + 2x^k(u - x^{2k} - zy) - z(u - x^{2k} - zy)^2 - \nu_3^2 z + 2\omega_3 \nu_3, \]
\[ A_3 := \beta_3 + \alpha_3^2, \]
\[ B_3 := z, \]
\[ C_3 := t + \beta_3, \quad \text{and} \]
\[ D_3 := (u - x^{2k} - zy) - tz - \beta_3 \]
\[ = u - y - x^{2k} - 2x^k u - yz - tz + u^2 z \]
\[ = u - y - x^{2k} - 2x^k u - z(t + y - u^2), \]
respectively. Now applying the polynomial automorphism
\[ t \mapsto t - y + u^2 \]
transforms these polynomials into

\[ \omega_4 := x^k - z(u - x^{2k} - zy), \]
\[ \nu_4 := x^{2k} + zy, \]
\[ \alpha_4 := x^k - zu, \]
\[ \beta_4 := y + 2x^k(u - x^{2k} - zy) - z(u - x^{2k} - zy)^2 - \nu_4^2 z + 2\omega_4 \nu_4, \]
\[ A_4 := \beta_4 + \alpha_4^2, \]
\[ B_4 := z, \]
\[ C_4 := t - y + u^2 + \beta_4, \quad \text{and} \]
\[ D_4 := u - y - x^{2k} - 2x^k u - zt, \]
respectively. Now applying the polynomial automorphism
\[ y \mapsto -y + u - x^{2k} - tz - 2x^k u \]
transforms the polynomials \( A_4, B_4, C_4, \) and \( D_4 \) into

\[ A_5 := -y + u - 2x^k u z - tz + u^2 z^2 - u^2 z, \]
\[ B_5 := z, \]
\[ C_5 := t + 2x^k u + u^2 - u^2 z, \quad \text{and} \]
\[ D_5 := y, \]
respectively. Now applying the polynomial automorphism
\[ t \mapsto t - 2x^k u - u^2 + u^2 z \]
transforms these polynomials into

\[ A_0 := -y + u - 2x^kuz - tz + 2x^ku z + u^2 z - u^2 z^2 + u^2 z^2 - u^2 z = u - y - tz, \]

\[ B_0 := z, \]

\[ C_0 := t, \] and

\[ D_0 := y, \]

respectively. Now applying the polynomial automorphism

\[ u \mapsto u + y + tz \]

transforms these polynomials into \( u, z, t, \) and \( y \) respectively. Hence the polynomial

\[ p = x - q(A, B, C, D) \]

is transformed into \( x - q(u, z, t, y) \) by successively applying these automorphisms. Finally applying the polynomial automorphism

\[ x \mapsto x + q(u, z, t, y) \]

then transforms it into \( x \). Since the polynomial automorphisms (1)-(8) are all elementary, \( p \) is a tame coordinate. □

References


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