A NOTE ON POSSIBLE COUNTEREXAMPLES TO THE ABHYANKAR-SATHAYE CONJECTURE CONSTRUCTED BY SHPILRAIN AND YU

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In [SY99], Shpilrain and Yu construct a class of candidate counterexamples to the Embedding Conjecture of Abhyankar and Sathaye (see [Sat76] and [Abh78]).

Proposition 1 ([SY99], Proposition 1.5). Let $q$ be a polynomial over $\mathbb{C}$ in four variables and let $k$ be a positive integer. In $\mathbb{C}[x,y,z,t,u]$, define $\omega := x^k - t z^2 - u z$, $\alpha := \omega - (w^2 + y z) z$, and $\beta := y - (\omega^2 + y z)^2 z + 2 \omega (\omega^2 + y z)$ and consider the following polynomial $p$:

$$p := x - q(\beta + \alpha^2, z, t + \beta, u - \beta).$$

Then the zero fiber of this polynomial is isomorphic to a coordinate hyperplane, i.e., $\mathbb{C}[x,y,z,t,u]/(p) \cong \mathbb{C}[x,y,z,t]$. If the Embedding Conjecture were true, then these polynomials $p$ are all coordinates, i.e., component of a polynomial automorphism $\mathbb{C}[x,y,z,t,u] \rightarrow \mathbb{C}[x,y,z,t,u]$. Shpilrain and Yu explain why this class of polynomials $p$ (for $k \geq 2$) could contain counterexamples to this conjecture.

However, this note shows that these polynomials $p$ are all coordinates. They even turn out to be tame.

Proposition 2. In the situation of the previous proposition, $p$ is a tame coordinate.

Proof. Write $\nu := \omega^2 + y z$, $A := \beta + \alpha^2$, $B := z$, $C := t + \beta$, and $D := u - \beta$. Note that $\alpha = \omega - \nu z$, $\beta = y - \nu^2 z + 2 \omega \nu$, and $\omega = x^k - z(u + tz)$. Applying the (elementary) polynomial automorphism

$$u \mapsto u - tz \quad (\text{and} \quad x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t)$$

(1)
to the polynomials $\omega, \nu, \alpha, \beta, A, B, C,$ and $D$ transforms them into

$$
\omega_1 := x^k - zu,
$$
$$
\nu_1 := \omega_1^2 + yz = x^{2k} + z(y - 2x^k u + zu^2),
$$
$$
\alpha_1 := \omega_1 - \nu_1 z,
$$
$$
\beta_1 := y - \nu_1^2 z + 2\omega_1 \nu_1,
$$
$$
A_1 := \beta_1 + \alpha_1^2,
$$
$$
B_1 := z,
$$
$$
C_1 := t + \beta_1, \quad \text{and}
$$
$$
D_1 := u - tz - \beta_1
$$

respectively. Now applying the polynomial automorphism

$$
y \mapsto y + 2x^k u - zu^2, \quad (2)
$$

transforms these polynomials into

$$
\omega_2 := x^k - zu,
$$
$$
\nu_2 := x^{2k} + zy,
$$
$$
\alpha_2 := \omega_2 - \nu_2 z
$$
$$
= x^k - z(u + x^{2k} + zy),
$$
$$
\beta_2 := y - \nu_2^2 z + 2\omega_2 \nu_2,
$$
$$
A_2 := \beta_2 + \alpha_2^2,
$$
$$
B_2 := z,
$$
$$
C_2 := t + \beta_2, \quad \text{and}
$$
$$
D_2 := u - tz - \beta_2
$$

respectively. Now applying the polynomial automorphism

$$
u \mapsto u - x^{2k} - zy \quad (3)$$
transforms these polynomials into

\[
\begin{align*}
\omega_3 &:= x^k - z(u - x^2 - zy), \\
\nu_3 &:= x^{2k} + zy, \\
\alpha_3 &:= x^k - zu, \\
\beta_3 &:= y + 2x^k(u - x^2 - zy) - z(u - x^2 - zy)^2 - \nu_3^2 z + 2\omega_3 \nu_3, \\
A_3 &:= \beta_3 + \alpha_3^2, \\
B_3 &:= z, \\
C_3 &:= t + \beta_3, \quad \text{and} \\
D_3 &:= (u - x^2 - zy) - tz - \beta_3 \\
&= u - y - x^2 - 2x^k u - yz - tz + u^2 z \\
&= u - y - x^2 - 2x^k u - z(t + y - u^2),
\end{align*}
\]
respectively. Now applying the polynomial automorphism

\[
t \mapsto t - y + u^2
\]
transforms these polynomials into

\[
\begin{align*}
\omega_4 &:= x^k - z(u - x^2 - zy), \\
\nu_4 &:= x^{2k} + zy, \\
\alpha_4 &:= x^k - zu, \\
\beta_4 &:= y + 2x^k(u - x^2 - zy) - z(u - x^2 - zy)^2 - \nu_4^2 z + 2\omega_4 \nu_4, \\
A_4 &:= \beta_4 + \alpha_4^2, \\
B_4 &:= z, \\
C_4 &:= t - y + u^2 + \beta_4, \quad \text{and} \\
D_4 &:= u - y - x^2 - 2x^k u - zt,
\end{align*}
\]
respectively. Now applying the polynomial automorphism

\[
y \mapsto -y + u - x^2 - tz - 2x^k u
\]
transforms the polynomials \(A_4, B_4, C_4, \) and \(D_4\) into

\[
\begin{align*}
A_5 &:= -y + u - 2x^k uz - tz + u^2 z^2 - u^2 z, \\
B_5 &:= z, \\
C_5 &:= t + 2x^k u + u^2 - u^2 z, \quad \text{and} \\
D_5 &:= y,
\end{align*}
\]
respectively. Now applying the polynomial automorphism

\[
t \mapsto t - 2x^k u - u^2 + u^2 z
\]
transforms these polynomials into

\[ A_6 := -y + u - 2x^k uz - tz + 2x^k uz + u^2 z - u^2 z^2 + u^2 z^2 - u^2 z = u - y - tz, \]

\[ B_6 := z, \]

\[ C_6 := t, \quad \text{and} \]

\[ D_6 := y, \]

respectively. Now applying the polynomial automorphism

\[ u \mapsto u + y + tz \quad \text{(7)} \]

transforms these polynomials into \( u, z, t, \) and \( y \) respectively. Hence the polynomial \( p = x - q(A, B, C, D) \) is transformed into \( x - q(u, z, t, y) \) by successively applying these automorphisms. Finally applying the polynomial automorphism

\[ x \mapsto x + q(u, z, t, y) \quad \text{(8)} \]

then transforms it into \( x \). Since the polynomial automorphisms (1)-(8) are all elementary, \( p \) is a tame coordinate. \( \square \)

References


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