Generalized augmented matrix preconditioning approach and its application to iterative solution of ill-conditioned algebraic systems *

Alexander Padiy† Owe Axelsson‡ Ben Polman‡

Abstract

The present work is devoted to a class of preconditioners based on the augmented matrix approach considered earlier by two of the present authors. It presents some generalizations of the subspace-correction schemes studied earlier and gives a brief comparison of the developed technique with a somewhat similar "deflation" algorithm.

The developed preconditioners are able to improve significantly an eigenvalue distribution of certain severely ill-conditioned algebraic systems by using properly chosen projection matrices, which correct the low-frequency components in the spectrum. One of the main advantages of the proposed approach is the possibility to use inexact solvers within the projectors. Another attractive feature of the developed method is that it can be easily combined with other preconditioners, for instance those which correct the high-frequency eigenmodes.

KEY WORDS iterative solvers, preconditioning, subspace correction

1 Introduction

In many problems the convergence of iterative schemes can be significantly slowed down by a presence of several very small eigenvalues in the spectrum of the algebraic system to be solved. This occurs, for example, when the conjugate gradient (CG) method is applied to algebraic problems arising from discretization of second order elliptic problems, especially in the case of strongly discontinuous and/or anisotropic problem coefficients.

One of the ways to improve the convergence rate of the CG method is to "deflate" certain components of the residual by using the projector

$$B = I - V(V^T A V)^{-1} V^T A$$

as a (right) preconditioner, see e.g. [12]. Here A is the original system matrix and V is a rectangular matrix constructed in such a way that the Rayleigh quotient $(\mathbf{x}^T\mathbf{x})/(\mathbf{x}^T A^{-1}\mathbf{x})$

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†Institute of Numerical Mathematics, 8 Gubkina Str., GSP-1, 117333 Moscow, Russia and Department of Mathematics, University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

‡Department of Mathematics, University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

does not take extremely small/large values on the subspace orthogonal to the image of V. Note, that a projector of a similar structure appears also in the multigrid setting. If V is chosen to be a coarse-to-fine prolongation operator then B is normally referred to as a *coarse-grid correction operator* (an overview of the multigrid framework can be found in e.g. [8], [16], [17]).

A nice feature of the algorithm is that the convergence rate of the "deflated" preconditioned conjugate (PCG) method depends on the ratio $\tilde{\kappa}$,

$$\tilde{\kappa} = \frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}}, \qquad \tilde{\lambda}_{\max} = \sup_{\mathbf{x} \perp \operatorname{Im} V} \frac{\mathbf{x}^T \mathbf{x}}{\mathbf{x}^T A^{-1} \mathbf{x}}, \qquad \tilde{\lambda}_{\min} = \inf_{\mathbf{x} \perp \operatorname{Im} V} \frac{\mathbf{x}^T \mathbf{x}}{\mathbf{x}^T A^{-1} \mathbf{x}}$$

rather than on the condition number κ,

$$\kappa = \frac{\lambda_{\max}}{\lambda_{\min}}, \qquad \lambda_{\max} = \sup \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}, \qquad \lambda_{\min} = \inf \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

Since $\tilde{\lambda}_{\max} \leq \lambda_{\max}$, $\tilde{\lambda}_{\min} \geq \lambda_{\min}$, the convergence rate of the "deflated" PCG method is always better than the convergence rate of the unpreconditioned one. As was shown in [12] for a class of second order elliptic problems with smooth isotropic coefficients, the dimension of the low-frequency eigencluster is normally relatively small and, therefore, the number of columns in V can also be chosen small as compared to the dimension of A. This makes efficient implementations of the "deflation" procedure possible, see [10, 11, 12]. However, the algorithm requires the system with the matrix $A_V = V^T A V$ to be solved exactly on every iteration of the PCG method; if the action of A_V^{-1} is computed inaccurately then the convergence of the iterative scheme can be slow or even divergence can occur. This is a drawback of the "deflation" procedure since the computation of A_V^{-1} can be costly if its dimension is not small; the system with A_V can be efficiently solved only if it has a simple sparsity structure (preferably block-diagonal) and, thus, the choice of V is severely restricted.

Following the idea suggested in [2], but strongly extending and improving that method, in the present paper we consider an alternative approach to tackle the low-frequency eigenmodes. Instead of "deflating" the small eigenvalues we propose to "move" them to the vicinity of the largest eigenvalue by using a preconditioner B in the form

$$B = I + \sigma V B_V^{-1} V^T,$$

where B_V is an easily invertible approximation of A_V; we also refer to [9], where a somewhat similar algorithm was studied.

One of the main advantages of the proposed algorithm is the possibility to avoid exact solving of systems with A_V. This relaxes the restrictions posed on the choice of V and often leads to more efficient implementations of the solver. Moreover, the algorithm involves no extra multiplication with the system matrix A (as required in the "deflation" method) and can be easily combined with another preconditioner M which bounds the largest eigenvalues:

$$B = M^{-1} + \sigma V B_V^{-1} V^T.$$

The developed algorithm belongs to the additive Schwarz framework. When applied recursively with particular choices of M, V and B_V it leads to a number of known methods such as I-AMLI [1, 3], BPX [4] or MDS [20]. This issue is addressed in Sections 3 and 4.

We will first introduce the method as a generalization of the augmented matrix preconditioning approach [2] and then discuss its application to the problems arising from finite-element discretization of second order elliptic equations with highly discontinuous and/or anisotropic coefficients.

2 The augmented matrix preconditioning approach

Let the matrices A and V be of order $n \times n$ and $n \times m$ respectively. Assume that $\operatorname{rank} V = m$. Consider the augmented matrix

$$\widetilde{A} = \begin{bmatrix} A & -AV \\ -V^T A & V^T A V \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -V^T & I_m \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_n & -V \\ 0 & I_m \end{bmatrix} \tag{1}$$

of order $(n+m) \times (n+m)$.

Theorem 2.1 *[2] The following relations between the eigenvalues of $\widetilde{A}$ and A hold.*

a) $\widetilde{A}$ has at least m zero eigenvalues. The rest of the spectrum of $\widetilde{A}$ coincides with the spectrum of $(I + VV^T)A$.

b) If A is symmetric positive definite then for every eigenvalue λ_i of A there exists an eigenvalue $\widetilde{\lambda}_i$ of $\widetilde{A}$ such that $\widetilde{\lambda}_i \geq \lambda_i$.

c) If A is nonsingular and symmetric and V is constructed as $V = [\alpha_1 \mathbf{v}_1, \ldots, \alpha_m \mathbf{v}_m]$, where $\mathbf{v}_i$ are the normalized eigenvectors of A corresponding to λ_i, $i = 1, \ldots, m$, then the nonzero eigenvalues of $\widetilde{A}$ are the following:

$$\widetilde{\lambda}_i = \begin{cases} (1 + \alpha_i^2)\lambda_i, & i = 1, \ldots, m, \\ \lambda_i, & i = m+1, \ldots, n. \end{cases}$$

Proof It follows from (1) that $\widetilde{A}$ is similar to

$$\begin{bmatrix} I_n & -V \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -V^T & I_m \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (I_n + VV^T)A & 0 \\ -V^T A & 0 \end{bmatrix}.$$

This shows part a). The eigenvalues of $(I_n + VV^T)A$ are equal to those of $A + A^{\frac{1}{2}} VV^T A^{\frac{1}{2}}$ so part b) follows from $\mathbf{x}^T A \mathbf{x} + (V^T A^{\frac{1}{2}} \mathbf{x})^T (V^T A^{\frac{1}{2}} \mathbf{x}) \geq \mathbf{x}^T A \mathbf{x}$ for any x. Part c) immediately follows from the the orthonormality of the eigenvectors $\mathbf{v}_i$ of A, $i = 1, \ldots, n$. □

Assume that A is s.p.d. with an ordered set of eigenvalues $\{\lambda_i\}_{i=1}^n$, $\lambda_1 \leq \cdots \leq \lambda_n = \lambda_{\max}$. In this case the above theorem implies that in order to improve the condition number of $\widetilde{A}$ one can define the matrix V by using the eigenvectors $\mathbf{v}_i$, $i = 1, \ldots, m$ of A. If the scaling factors α_i are chosen as $\alpha_i = \sqrt{\lambda_n/\lambda_i - 1}$ or $\alpha_i = \sqrt{\lambda_n/\lambda_i}$, then the smallest eigenvalues λ_i of A are "moved" to $\tilde{\lambda}_i = \lambda_n = \lambda_{\max}$ or to $\tilde{\lambda}_i = \lambda_n + \lambda_i \leq 2\lambda_{\max}$, respectively. As was pointed out in [2], instead of using the matrix $\widetilde{A}$ in the iterative scheme one can alternatively use the matrix $(I + VV^T)A$, i.e. one can use $I + VV^T$ as a preconditioner to A. The preconditioner can also be written in the form $I + VD^{-1}V^T$ where $D = \operatorname{diag}(\alpha_i^2)$ and $V = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$.

There are two problems associated with the practical implementation of this method, namely, in general the eigenvectors $\mathbf{v}_i$ are not known and likewise the scaling factors α_i are not known. To handle this we consider the case when $\mathbf{v}_i$ are only assumed to be linearly independent vectors spanning a proper subspace and introduce a more general scaling matrix. Moreover, as it turns out, it is the subspace spanned by $\{\mathbf{v}_i\}_1^m$ which matters and not the particular basis vectors used. Further we study a preconditioner in a more general form $I + VD^{-1}V^T$, where the matrix D is no longer assumed to be diagonal.

Lemma 2.1 *Let A be s.p.d. Then*

$$P = A^{\frac{1}{2}} V (V^T A V)^{-1} V^T A^{\frac{1}{2}}$$

is an orthogonal projection, i.e. $P^2 = P$. Therefore, 0 and 1 are the only eigenvalues of P.

Proof $P^2 = A^{\frac{1}{2}} V (V^T A V)^{-1} V^T A V (V^T A V)^{-1} V^T A^{\frac{1}{2}} = A^{\frac{1}{2}} V (V^T A V)^{-1} V^T A^{\frac{1}{2}} = P.$ □

Lemma 2.2 *[13] (Monotonicity). Let A and $\hat{A}$ be symmetric positive definite matrices of order $n \times n$ and let V_k be rectangular matrices of order $n \times m_k$, $k = 1, 2$ such that* $\operatorname{rank} V_k = m_k$, *$k = 1, 2$. If* $\operatorname{Im} V_1 \subseteq \operatorname{Im} V_2$ *then for all i, $1 \le i \le n$ the following inequality holds*

$$\lambda_i\big((I + V_2 (V_2^T \hat{A} V_2)^{-1} V_2^T) A\big) \ge \lambda_i\big((I + V_1 (V_1^T \hat{A} V_1)^{-1} V_1^T) A\big).$$

Proof It is readily seen that the proposition holds if $F = V_2 (V_2^T \hat{A} V_2)^{-1} V_2^T - V_1 (V_1^T \hat{A} V_1)^{-1} V_1^T$ is nonnegative definite. But since $\operatorname{Im} V_1 \subseteq \operatorname{Im} V_2$, there exists some matrix Q of order $m_2 \times m_1$ such that $V_1 = V_2 Q$. Then with $D_k = V_k^T \hat{A} V_k$ we have

$$F = V_2 (D_2^{-1} - Q D_1^{-1} Q^T) V_2^T = V_2 D_2^{-\frac{1}{2}} (I - D_2^{\frac{1}{2}} Q D_1^{-1} Q^T D_2^{\frac{1}{2}}) D_2^{-\frac{1}{2}} V_2^T,$$

where

$$D_2^{\frac{1}{2}} Q D_1^{-1} Q^T D_2^{\frac{1}{2}} = D_2^{\frac{1}{2}} Q (Q^T D_2 Q)^{-1} Q^T D_2^{\frac{1}{2}}$$

is an orthogonal projector, whose only eigenvalues are 0 and 1. □

Corollary 2.1 If $\operatorname{Im} V_1 = \operatorname{Im} V_2$ then $I + V_2 D_2^{-1} V_2^T = I + V_1 D_1^{-1} V_1^T$.

Proof In this case Q in $V_1 = V_2 Q$ is invertible. Thus, $D_2^{\frac{1}{2}} Q (Q^T D_2 Q)^{-1} Q^T D_2^{\frac{1}{2}} = I.$ □

Remark 2.1 The above corollary shows that the individual eigenvectors of A are not needed when constructing the matrix V; we are rather interested in the subspace spanned by them.

Next we consider a specific version of the preconditioner $B = I + \sigma V A_V^{-1} V^T$ with the scaling matrix $A_V = V^T A V$. The following theorem is similar to a theorem from [13].

Theorem 2.2 *Let A be an $n \times n$ symmetric positive semidefinite matrix and let a rectangular matrix V of order $n \times m$ be defined as $V = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$. Assume that* $\operatorname{rank} V = m$. *Further, define $\widetilde{A}$ as $\widetilde{A} = (I + \sigma V A_V^{-1} V^T) A$, where $A_V = V^T A V$. Then the following statements hold:*

a) $\lambda_{\max}(\tilde{A}) \le \sigma + \lambda_{\max}(A)$;

b) if for some i, $1 \le i \le m$, $\mathbf{v}_i$ is an eigenvector of A with eigenvalue λ_i, then it is also an eigenvector of $\widetilde{A}$ with eigenvalue $\lambda_i + \sigma$;

c) let $(\lambda_i, \mathbf{v}_i)$ be the eigenpairs of A and assume that $V = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$ contains m eigenvectors. Then

$$\widetilde{A}\mathbf{v}_i = \begin{cases} (\lambda_i + \sigma)\mathbf{v}_i, & i = 1, \ldots, m \\ \lambda_i \mathbf{v}_i, & i = m+1, \ldots, n \end{cases}$$

Proof Clearly,

$$\begin{aligned}\lambda_{\max}(\tilde{A}) &\leq \lambda_{\max}(A) + \sigma \sup \frac{\mathbf{x}^T V (V^T A V)^{-1} V^T \mathbf{x}}{\mathbf{x}^T A^{-1} \mathbf{x}} \\ &= \lambda_{\max}(A) + \sigma \sup \frac{\mathbf{x}^T A^{\frac{1}{2}} V (V^T A V)^{-1} V^T A^{\frac{1}{2}} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \\ &= \lambda_{\max}(A) + \sigma,\end{aligned}$$

where the last equality follows from Lemma 2.1. This proves part a).

Let $\mathbf{w}_i = (V^T V)^{-1} V^T \mathbf{v}_i$. Then for $i = 1, \ldots, m$

$$\begin{aligned}\widetilde{A}\mathbf{v}_i &= \lambda_i \mathbf{v}_i + \sigma V (V^T A V)^{-1} V^T A \mathbf{v}_i \\ &= \lambda_i \mathbf{v}_i + \sigma V (V^T A V)^{-1} V^T A V \mathbf{w}_i \\ &= \lambda_i \mathbf{v}_i + \sigma V \mathbf{w}_i = (\lambda_i + \sigma)\mathbf{v}_i\end{aligned}$$

which shows part b). To prove part c) note that the eigenvectors are orthogonal so $V^T A \mathbf{v}_i = 0$, $i = m+1, \ldots, n$. □

Corollary 2.2 Let V be such that $\operatorname{Im} V$ is spanned by the m eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of A corresponding to the cluster of m smallest eigenvalues $\lambda_1, \ldots, \lambda_m$. Then the eigenvalues of $\widetilde{A} = (I + \sigma V (V^T A V)^{-1} V^T) A$ are $\tilde{\lambda}_i = \sigma + \lambda_i$ for $i = 1, \ldots, m$ and $\tilde{\lambda}_i = \lambda_i$ for $i = m+1, \ldots, n$, which implies that

$$\min\{\sigma + \lambda_1, \lambda_{m+1}\} \leq \lambda_i(\widetilde{A}) \leq \max\{\sigma + \lambda_m, \lambda_{\max}(A)\}. \tag{2}$$

Proof Use Corollary 2.1 and Theorem 2.2. □

Corollary 2.3 Assume that

$$\operatorname{Im} V \supseteq \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}. \tag{3}$$

Then the following estimate holds:

$$\min\{\sigma + \lambda_1, \lambda_{m+1}\} \leq \lambda_i(\widetilde{A}) \leq \sigma + \lambda_{\max}(A).$$

Proof Use Corollary 2.2 and Lemma 2.1. □

As follows from the above corollaries, the preconditioner

$$B = I + \sigma V (V^T A V)^{-1} V^T, \quad \sigma = \lambda_{\max}(A), \quad \operatorname{Im} V \supseteq \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \tag{4}$$

efficiently scales the smallest eigenvalues λ_i of A as they are "moved" to $\tilde{\lambda}_i = \lambda_{\max}(A) + \lambda_i$. Since $\tilde{\lambda}_i \leq 2\lambda_{\max}(A)$, this leads to the following condition number estimate

$$\kappa(BA) \leq \frac{2\lambda_{\max}(A)}{\lambda_{m+1}}. \tag{5}$$

However, the preconditioner (4) is normally expensive to apply because of the need to invert the matrix A_V. In the following section we show that the action of A_V^{-1} can be replaced by the action of a preconditioner B_V^{-1} to A_V^{-1}. We also discuss there the possibility to relax the condition (3).

3 Generalized version with inexact projectors

Theorem 3.1 *Define the preconditioner* $\hat{B}$ *as*

$$\hat{B} = I + \hat{\sigma} V B_V^{-1} V^T, \quad \hat{\sigma} = \lambda_{\max}(A)/\lambda_{\max}(B_V^{-1} A_V), \quad \operatorname{Im} V \supseteq \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}, \tag{6}$$

where B_V *is an* $m \times m$ *symmetric positive definite approximation of* A_V*. The eigenvalues* $\lambda(\hat{B}A)$ *of* $\hat{B}A$ *are bounded as follows:*

$$\min\left\{\frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)} + \lambda_1, \lambda_{m+1}\right\} \le \lambda(\hat{B}A) \le 2\lambda_{\max}(A). \tag{7}$$

Proof The minimal eigenvalue of $\hat{B}A$ can be estimated as

$$\begin{aligned}
\lambda_{\min}(\hat{B}A) &= \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T(I + \hat{\sigma} V B_V^{-1} V^T)\,\mathbf{x}}{\mathbf{x}^T A^{-1}\mathbf{x}}\right\} \\
&= \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^T A^{-1}\mathbf{x}} + \hat{\sigma}\cdot\frac{\mathbf{x}^T V B_V^{-1} V^T\mathbf{x}}{\mathbf{x}^T A^{-1}\mathbf{x}}\right\} \\
&= \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^T A^{-1}\mathbf{x}} + \hat{\sigma}\cdot\frac{\mathbf{x}^T V B_V^{-1} V^T\mathbf{x}}{\mathbf{x}^T V A_V^{-1} V^T\mathbf{x}}\cdot\frac{\mathbf{x}^T V A_V^{-1} V^T\mathbf{x}}{\mathbf{x}^T A^{-1}\mathbf{x}}\right\} \\
&\ge \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^T A^{-1}\mathbf{x}} + \hat{\sigma}\lambda_{\min}(B_V^{-1}A_V)\cdot\frac{\mathbf{x}^T V A_V^{-1} V^T\mathbf{x}}{\mathbf{x}^T A^{-1}\mathbf{x}}\right\} \\
&\ge \min\left\{\frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)} + \lambda_1, \lambda_{m+1}\right\},
\end{aligned}$$

where the last inequality follows from Corollary 2.2 with $\sigma = \hat{\sigma}\lambda_{\min}(B_V^{-1}A_V) = \dfrac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}$.

Analogously, $\lambda_{\max}(\hat{B}A) \le 2\lambda_{\max}(A)$. □

Remark 3.1 The value of $\hat{\sigma}$ in (6) was chosen as $\hat{\sigma} = \lambda_{\max}(A)/\lambda_{\max}(B_V^{-1}A_V)$ for the ease of presentation. The optimal value of $\hat{\sigma}$ (the value of $\hat{\sigma}$ which minimizes the condition number of $\hat{B}A$) corresponds to the case when $\hat{\sigma}\lambda_{\min}(B_V^{-1}A_V) + \lambda_1 = \lambda_{m+1}$. Note, that λ_{m+1} is not known in general.

Remark 3.2 As follows from (7), if $\kappa(B_V^{-1}A_V) \le \lambda_{\max}(A)/\lambda_{m+1}$ then the bounds for $\kappa(\hat{B}A)$ and $\kappa(BA)$ coincide.

Remark 3.3 As follows from the above remark, if $\kappa(A_V) \le \lambda_{\max}(A)/\lambda_{m+1}$ then one can define B_V simply as $B_V = I$ or $B_V = \operatorname{diag} A_V$.

As follows from Theorem 3.1, the preconditioner (6) is able to improve the spectrum of A even in the case when the action of A_V^{-1} is replaced by the action of a preconditioner B_V^{-1}. It should be noted, however, that the preconditioner (6) is still difficult to implement in practice since the condition $\operatorname{Im} V \supseteq \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is not easy to satisfy. Later, in Theorem 3.2, we show that this condition can be significantly relaxed.

In the following we use the notation $\cos(W_1, W_2)$ for $\cos(\varphi(W_1, W_2))$, where φ denotes the angle between the vector subspaces W_1 and W_2, namely,

$$\cos(W_1, W_2) = \cos(\varphi(W_1, W_2)) = \sup_{\substack{\mathbf{x} \in W_1 \\ \mathbf{y} \in W_2}} \frac{\mathbf{x}^T \mathbf{y}}{(\mathbf{x}^T \mathbf{x})^{\frac{1}{2}} (\mathbf{y}^T \mathbf{y})^{\frac{1}{2}}}.$$

Lemma 3.1 *Consider two arbitrary matrices $V_1 \in \mathbb{R}_{n \times m_1}$, $\operatorname{rank} V_1 = m_1$ and $V_2 \in \mathbb{R}_{n \times m_2}$, $\operatorname{rank} V_2 = m_2$. If there exists $\gamma < 1$ such that*

$$\cos(\operatorname{Im} V_1, \operatorname{Im} V_2) = \sup_{\substack{\mathbf{x} \in \operatorname{Im} V_1 \\ \mathbf{y} \in \operatorname{Im} V_2}} \frac{\mathbf{x}^T \mathbf{y}}{(\mathbf{x}^T \mathbf{x})^{\frac{1}{2}} (\mathbf{y}^T \mathbf{y})^{\frac{1}{2}}} \leq \gamma$$

then

$$\lambda_{\max} \left(\sum_{i=1}^{2} V_i (V_i^T V_i)^{-1} V_i^T \right) \leq 1 + \gamma.$$

Proof Define an auxiliary $(m_1 + m_2) \times n$ matrix R as

$$R = \begin{bmatrix} (V_1^T V_1)^{-\frac{1}{2}} V_1^T \\ (V_2^T V_2)^{-\frac{1}{2}} V_2^T \end{bmatrix}.$$

The matrix R exists since V_i are full rank matrices and, thus, the matrices $V_i^T V_i$ are s.p.d. Since $\lambda_{\max}(Q^T Q) = \lambda_{\max}(Q Q^T)$ for all Q,

$$\lambda_{\max} \left(\sum_{i=1}^{2} V_i (V_i^T V_i)^{-1} V_i^T \right) = \sup_{\mathbf{x}} \frac{\mathbf{x}^T R^T R \, \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \lambda_{\max}(R^T R) = \lambda_{\max}(R R^T).$$

Taking into account the explicit form of $R R^T$ we have

$$\lambda_{\max}(R R^T) = \lambda_{\max} \left(\begin{bmatrix} I & W_1^T W_2 \\ W_2^T W_1 & I \end{bmatrix} \right) = 1 + \cos(\operatorname{Im} W_1, \operatorname{Im} W_2),$$

where $W_i = V_i (V_i^T V_i)^{-\frac{1}{2}}$, $i = 1, 2$. Clearly, $\operatorname{Im} V_i = \operatorname{Im} W_i$.

Thus, $\lambda_{\max} \left(\sum_{i=1}^{2} V_i (V_i^T V_i)^{-1} V_i^T \right) = \lambda_{\max}(R R^T) = 1 + \cos(\operatorname{Im} V_1, \operatorname{Im} V_2) \leq 1 + \gamma.$ □

Theorem 3.2 *Consider the preconditioner $\hat{B}$*

$$\hat{B} = I + \hat{\sigma} V B_V^{-1} V^T, \qquad \hat{\sigma} = \lambda_{\max}(A) / \lambda_{\max}(B_V^{-1} A_V). \tag{8}$$

Assume that $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^n$ is an ordered set of eigenpairs of A such that $\lambda_1 \leq \cdots \leq \lambda_n$. If V is such that the subspaces $\mathcal{W} = (\operatorname{Im} A^{\frac{1}{2}} V)^{\perp}$ and $\mathcal{V}_e = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ satisfy the condition

$$\cos(\mathcal{W}, \mathcal{V}_e) = \sup_{\substack{\mathbf{x} \in \mathcal{W} \\ \mathbf{y} \in \mathcal{V}_e}} \frac{\mathbf{x}^T \mathbf{y}}{(\mathbf{x}^T \mathbf{x})^{\frac{1}{2}} (\mathbf{y}^T \mathbf{y})^{\frac{1}{2}}} \leq \gamma \tag{9}$$

for some $\gamma < 1$, then the minimal eigenvalue of $\hat{B}A$ is bounded as

$$\lambda_{\min}(\hat{B}A) \geq \max\left\{\lambda_1, (1-\gamma)\cdot\min\left\{\frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}, \lambda_{m+1}\right\}\right\} \tag{10}$$

while $\lambda_{\max}(\hat{B}A)$ is bounded as $\lambda_{\max}(\hat{B}A) \leq 2\lambda_{\max}(A)$ for any choice of V and B_V.

Proof As follows from Theorem 2.2, the maximal eigenvalue $\lambda_{\max}(\hat{B}A)$ can be estimated as

$$\begin{aligned}
\lambda_{\max}(\hat{B}A) &= \sup_{\mathbf{x}}\left\{\frac{\mathbf{x}^T(I+\hat{\sigma}VB_V^{-1}V^T)\,\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}}\right\} \\
&\leq \sup_{\mathbf{x}}\left\{\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}} + \hat{\sigma}\lambda_{\max}(B_V^{-1}A_V)\cdot\frac{\mathbf{x}^TVA_V^{-1}V^T\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}}\right\} \\
&= \sup_{\mathbf{x}}\left\{\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}} + \lambda_{\max}(A)\cdot\frac{\mathbf{x}^TVA_V^{-1}V^T\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}}\right\} \;\leq\; 2\lambda_{\max}(A).
\end{aligned}$$

If we take into account that the eigensubspaces $\mathcal{V}_e$ and $(\mathcal{V}_e)^\perp$ of A are A-orthogonal, then the minimal eigenvalue of $\hat{B}A$ can be estimated as follows

$$\begin{aligned}
\lambda_{\min}(\hat{B}A) &= \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T(I+\hat{\sigma}VB_V^{-1}V^T)\,\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}}\right\} \\
&\geq \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}} + \frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}\cdot\frac{\mathbf{x}^TVA_V^{-1}V^T\mathbf{x}}{\mathbf{x}^TA^{-1}\mathbf{x}}\right\} \\
&= \inf_{\mathbf{y}}\left\{\frac{\mathbf{y}^TA\mathbf{y}}{\mathbf{y}^T\mathbf{y}} + \frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}\cdot\frac{\mathbf{y}^TA^{\frac{1}{2}}VA_V^{-1}V^TA^{\frac{1}{2}}\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right\} \\
&= \inf_{\mathbf{y}}\left\{\frac{\mathbf{y}^TP_e^\perp AP_e^\perp\mathbf{y} + \mathbf{y}^TP_eAP_e\mathbf{y}}{\mathbf{y}^T\mathbf{y}} + \frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}\cdot\frac{\mathbf{y}^TP_w\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right\} \\
&\geq \inf_{\mathbf{y}}\left\{\frac{\mathbf{y}^TP_e^\perp AP_e^\perp\mathbf{y}}{\mathbf{y}^T\mathbf{y}} + \frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}\cdot\frac{\mathbf{y}^TP_w\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right\} \\
&\geq \inf_{\mathbf{y}}\left\{\lambda_{m+1}\cdot\frac{\mathbf{y}^TP_e^\perp\mathbf{y}}{\mathbf{y}^T\mathbf{y}} + \frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}\cdot\frac{\mathbf{y}^TP_w\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right\} \\
&\geq \min\left\{\lambda_{m+1}, \frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}\right\}\cdot\inf_{\mathbf{y}}\left\{\frac{\mathbf{y}^TP_e^\perp\mathbf{y} + \mathbf{y}^TP_w\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right\} \\
&= \min\left\{\lambda_{m+1}, \frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}\right\}\cdot\left(2 - \sup_{\mathbf{y}}\left\{\frac{\mathbf{y}^TP_e\mathbf{y} + \mathbf{y}^TP_w^\perp\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\right\}\right),
\end{aligned}$$

where P_e, $P_e^\perp$, P_w and $P_w^\perp$ are the orthogonal projectors onto $\mathcal{V}_e$, $\mathcal{V}_e^\perp$, $\operatorname{Im}A^{\frac{1}{2}}V$ and $(\operatorname{Im}A^{\frac{1}{2}}V)^\perp$ respectively. As follows from Lemma 3.1 with the matrices V_1 and V_2 chosen such that $\operatorname{Im}V_1 = \mathcal{V}_e$ and $\operatorname{Im}V_2 = (\operatorname{Im}A^{\frac{1}{2}}V)^\perp$,

$$\lambda_{\min}(\hat{B}A) \;\geq\; (1-\gamma)\cdot\min\left\{\frac{\lambda_{\max}(A)}{\kappa(B_V^{-1}A_V)}, \lambda_{m+1}\right\}.$$

Finally noting that $VB_V^{-1}V^T$ is positive semidefinite we conclude the proof of (10). $\square$

Remark 3.4 As follows from (7), $\kappa(\hat{B}A) \leq 2\kappa(A)$ for all choices of V and B_V. Thus, for all V and B_V the convergence rate of the $\hat{B}$–preconditioned iterative scheme is not worse as of the same order as of the unpreconditioned one. In particular, no divergence of the iterative scheme can occur (if we assume that the round-off effects are neglected). This is a nice feature of the developed algorithm as compared to the "deflation" procedure [12], since the latter can be divergent if the matrix B_V is chosen inappropriately.

Lemma 3.2 *If the eigensubspace* $\mathcal{V}_e = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ *of* A *is known then the value of* $\cos(\mathcal{W}, \mathcal{V}_e)$ *in (9) is readily computable. It can be evaluated as*

$$\cos(\mathcal{W}, \mathcal{V}_e) = \lambda_{\max}(ZA), \qquad Z = V_e(V_e^T A V_e)^{-1} V_e^T - V(V^T A V)^{-1} V^T,$$

where the matrix V_e *is chosen such that* $\operatorname{rank} V_e = m$, $\operatorname{Im} V_e = \mathcal{V}_e$.

Proof Introduce two auxiliary matrices V_e and W such that $\operatorname{Im} V_e = \mathcal{V}_e$, $\operatorname{rank} V_e = \dim \mathcal{V}_e$, $V_e^T V_e = I$, $\operatorname{Im} W = \mathcal{W}$, $\operatorname{rank} W = \dim \mathcal{W}$, $\mathcal{W} = (\operatorname{Im} A^{\frac{1}{2}} V)^{\perp}$, $W^T W = I$. Similarly to the proof of Lemma 3.1, we have

$$\begin{aligned}
\cos(\mathcal{W}, \mathcal{V}_e) &= \lambda_{\max}\left(\begin{bmatrix} I & W^T V_e \\ V_e^T W & I \end{bmatrix}\right) - 1 \\
&= \lambda_{\max}\left(V_e V_e^T + W W^T\right) - 1 \\
&= \lambda_{\max}\left(V_e V_e^T + I - A^{\frac{1}{2}} V (V^T A V)^{-1} V^T A^{\frac{1}{2}}\right) - 1 \\
&= \sup_{\mathbf{x}} \frac{\mathbf{x}^T V_e (V_e^T V_e)^{-1} V_e^T \mathbf{x} - \mathbf{x}^T A^{\frac{1}{2}} V (V^T A V)^{-1} V^T A^{\frac{1}{2}} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \\
&= \sup_{\mathbf{y}} \frac{\mathbf{y}^T V_e (V_e^T A V_e)^{-1} V_e^T \mathbf{y} - \mathbf{y}^T V (V^T A V)^{-1} V^T \mathbf{y}}{\mathbf{y}^T A^{-1} \mathbf{y}},
\end{aligned}$$

where the last equality follows from Corollary 2.1 by taking into account that the subspace $\mathcal{V}_e$ is $A^{\frac{1}{2}}$-invariant. Since

$$\lambda_{\max}(QA) = \sup_{\mathbf{x}} \frac{\mathbf{x}^T Q \mathbf{x}}{\mathbf{x}^T A^{-1} \mathbf{x}}, \qquad Q = Q^T \geq 0, \quad A = A^T > 0,$$

we conclude that $\cos(\mathcal{W}, \mathcal{V}_e) = \lambda_{\max}(ZA)$, $Z = V_e(V_e^T A V_e)^{-1} V_e^T - V(V^T A V)^{-1} V^T$. $\square$

Remark 3.5 Since the eigensubspace $\mathcal{V}_e = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is $A^{\frac{1}{2}}$-invariant, it follows from $\operatorname{Im} V \supseteq \mathcal{V}_e$ that $\operatorname{Im} A^{\frac{1}{2}} V \supseteq \mathcal{V}_e$. This implies that $\gamma = 0$ if $\operatorname{Im} V \supseteq \mathcal{V}_e$.

As follows from Theorem 3.2, there is no need to approximate the subspace spanned by $\{\mathbf{v}_i\}_{i=1}^m$ with a very high accuracy. For the isotropic second order elliptic problem with a smooth coefficient function we can let, for instance, $V = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$, where $\mathbf{w}_i$ are the pointwise nodal values of the coarse mesh finite element basis functions φ_i^H. An iterative scheme based on such choice of V was constructed in [12]. Another choice of $\mathbf{w}_1, \ldots, \mathbf{w}_m$ could be the basis

vectors of a known eigensubspace of a similar problem, such as of a problem with a different coefficient function. This approach was taken in [3] and [14] where the low-frequency subspace of a strongly anisotropic diffusion operator was approximated using the eigenvectors of the limit problem with a degenerate diffusion tensor.

It should also be noted that the algorithm (8) can be applied recursively, i.e. we can consider a nested sequence of preconditioners $\{\hat{B}_k\}_{k=0}^{J}$ defined as

$$\hat{B}_k = I + \hat{\sigma}_k V_{k,k-1} \hat{B}_{k-1} V_{k,k-1}^T. \tag{11}$$

According to the classification introduced in [17] the above recursive algorithm belongs to the class of *parallel subspace correction* methods.

Remark 3.6 In the case when $B_0 = I$ and $V_{k,k-1}$ are prolongation operators in the standard multigrid setting the developed algorithm corresponds to the BPX method [4].

Remark 3.7 The matrices $V_{k,k-1}$ in the multilevel preconditioner (11) can be constructed using the matrix-dependent prolongation operators developed in [6], [19] or [5, 15]. We also refer to [7], where a number of preconditioners of a similar structure was studied.

If we additionally introduce a polynomial stabilization procedure to bound the condition number of $\kappa(B_k A_k)$ (see [1], for instance), then we arrive at the BPX-like preconditioner of the W-cycle type:

$$\bar{B}_k = \left(I - P_{\nu_k}(\hat{B}_k A_k)\right) A_k^{-1}, \qquad \hat{B}_k = I + \bar{\sigma}_k V_{k,k-1} \bar{B}_{k-1} V_{k,k-1}^T,$$

where

$$A_l = A, \qquad A_{k-1} = V_{k,k-1}^T A_k V_{k,k-1}$$

and P_{ν_k} denotes a Chebyshev polynomial of degree ν_k normalized at the origin.

4 Incorporation of an "external" smoother

The method presented above improves the condition number of the preconditioned system by "moving" the smallest eigenvalues to the upper part of the spectrum. Next we show how it can be combined with a smoother, which essentially improves the conditioning by making the largest eigenvalues smaller.

The preconditioner is constructed as

$$\tilde{B} = M^{-1} + \tilde{\sigma} V B_V^{-1} V^T, \qquad \tilde{\sigma} = \lambda_{\max}(M^{-1}A)/\lambda_{\max}(B_V^{-1} A_V), \tag{12}$$

where M and B_V are symmetric positive definite preconditioners for A and A_V respectively.

Remark 4.1 Preconditioners in this form appear within the additive Schwarz framework (with application to domain decomposition methods). The term $\tilde{\sigma} V B_V^{-1} V^T$ then normally corresponds to the coarse mesh correction operator while the smoother M^{-1} corresponds to a series of subdomain solves.

Remark 4.2 When applied recursively in the standard multigrid setting with $M_k = \mathrm{diag}\,(A_k)$, the algorithm (12) corresponds to the MDS method [20].

Remark 4.3 Consider the case when the matrices A_k are generated using the hierarchical basis of finite elements. If the smoother M_k is defined as $M_k^{-1} = I - V_{k,k-1}(V_{k,k-1}^T V_{k,k-1})^{-1} V_{k,k-1}^T$ then the algorithm corresponds to the multilevel method developed in [18]. If the smoother M_k is extended to the form

$$M_k^{-1} = (I - V_{k,k-1}(V_{k,k-1}^T V_{k,k-1})^{-1} V_{k,k-1}^T)\, M_{11}^{(k)}\, (I - V_{k,k-1}(V_{k,k-1}^T V_{k,k-1})^{-1} V_{k,k-1}^T)$$

and, additionally, the polynomial stabilization procedure is used to bound the condition number of $\kappa(\hat{B}_k A_k)$ then the method reduces to the additive version of the algebraic multilevel iterations (AMLI) method [1, 3].

Theorem 4.1 *Consider the preconditioner (12). Assume that $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^n$ is an ordered set of eigenpairs of $M^{-1}A$ such that $\lambda_1 \le \cdots \le \lambda_n$. Define the matrix V_e as $V_e = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$. If V is such that the subspaces $\mathcal{W} = (\operatorname{Im} A^{\frac{1}{2}} V)^\perp$ and $\mathcal{V}_e = \operatorname{Im} A^{\frac{1}{2}} V_e$ satisfy the condition (9) for some $\gamma < 1$ then the minimal eigenvalue of $\tilde{B}A$ is bounded as*

$$\lambda_{\min}(\tilde{B}A) \ge \max\left\{\lambda_1, (1-\gamma)\cdot \min\left\{\frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)}, \lambda_{m+1}\right\}\right\}. \tag{13}$$

The maximal eigenvalue of $\tilde{B}A$ is bounded as

$$\lambda_{\max}(\tilde{B}A) \le 2\lambda_{\max}(M^{-1}A) \tag{14}$$

for any choice of V and B_V.

Proof The maximal eigenvalue $\lambda_{\max}(\tilde{B}A)$ can be estimated as in the proof of Theorem 3.2. Taking into account that $\mathcal{V}_e$ is the eigensubspace of $A^{\frac{1}{2}} M^{-1} A^{\frac{1}{2}}$ the minimal eigenvalue can be estimated as

$$\begin{aligned}
\lambda_{\min}(\tilde{B}A) &= \inf_{\mathbf{x}} \left\{ \frac{\mathbf{x}^T A^{\frac{1}{2}}\, M^{-1} A^{\frac{1}{2}} \mathbf{x} + \tilde{\sigma} \mathbf{x}^T A^{\frac{1}{2}} V B_V^{-1} V^T A^{\frac{1}{2}} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \right\} \\
&\ge \inf_{\mathbf{x}} \left\{ \frac{\mathbf{x}^T A^{\frac{1}{2}}\, M^{-1} A^{\frac{1}{2}} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} + \frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)} \cdot \frac{\mathbf{x}^T A^{\frac{1}{2}} V A_V^{-1} V^T A^{\frac{1}{2}} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \right\} \\
&\ge \min\left\{ \lambda_{m+1}, \frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)} \right\} \cdot \inf_{\mathbf{x}} \left\{ \frac{\mathbf{x}^T P_e^\perp \mathbf{x} + \mathbf{x}^T P_* \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \right\} \\
&= \min\left\{ \lambda_{m+1}, \frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)} \right\} \cdot \left(2 - \sup_{\mathbf{x}} \left\{ \frac{\mathbf{x}^T P_e \mathbf{x} + \mathbf{x}^T P_*^\perp \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \right\} \right),
\end{aligned}$$

where P_e, $P_e^\perp$, P_* and $P_*^\perp$ are the orthogonal projectors onto $\mathcal{V}_e = \operatorname{Im} A^{\frac{1}{2}} V_e$, $\mathcal{V}_e^\perp = (\operatorname{Im} A^{\frac{1}{2}} V_e)^\perp$, $\operatorname{Im} A^{\frac{1}{2}} V$ and $(\operatorname{Im} A^{\frac{1}{2}} V)^\perp$ respectively. As follows from Lemma 3.1 with the matrices V_1 and V_2 chosen such that $\operatorname{Im} V_1 = \mathcal{V}_e$ and $\operatorname{Im} V_2 = (\operatorname{Im} A^{\frac{1}{2}} V)^\perp$,

$$\lambda_{\min}(\tilde{B}A) \ge (1-\gamma)\cdot \min\left\{ \frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)}, \lambda_{m+1} \right\}.$$

Now (13) follows by taking into account that $V B_V^{-1} V^T$ is positive semidefinite. □

Remark 4.4 The value of $\tilde{\sigma}$ in (12) was chosen as $\tilde{\sigma} = \lambda_{\max}(M^{-1}A)/\lambda_{\max}(B_V^{-1}A_V)$ for the ease of presentation. The optimal value of $\tilde{\sigma}$ (the value of $\tilde{\sigma}$ which minimizes the condition number of $\tilde{B}A$) corresponds to the case when $\tilde{\sigma}\lambda_{\min}(B_V^{-1}A_V) + \lambda_1 = \lambda_{m+1}$.

Lemma 4.1 *If the eigensubspace* $\mathcal{V}_e = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ *of* $M^{-1}A$ *is known then the value of* $\cos(\mathcal{W}, \mathcal{V}_e) = \cos((\operatorname{Im} A^{\frac{1}{2}}V)^{\perp}, \operatorname{Im} A^{\frac{1}{2}}V_e)$ *is readily computable. It can be evaluated as*

$$\cos(\mathcal{W}, \mathcal{V}_e) = \lambda_{\max}(ZA), \qquad Z = V_e(V_e^T A\, V_e)^{-1}V_e^T - V(V^T A\, V)^{-1}V^T,$$

where the matrix V_e *is chosen such that* $\operatorname{rank} V_e = m$, $\operatorname{Im} V_e = \mathcal{V}_e$.

Proof Similar to the proof of Lemma 3.2. □

In the following Theorem 4.2 the assumptions of Theorem 4.1 are slightly relaxed.

Theorem 4.2 *Consider the preconditioner (12). Assume that there exist two matrices* $\hat{M}$ *and* $\hat{A}$ *such that* $\hat{M} = \hat{M}^T > 0$, $\hat{A} = \hat{A}^T \geq 0$, $\hat{M} \geq M$ *and* $\hat{A} \leq A$ *(all inequalities here are meant in positive definite sense). Assume also that* $\{(\hat{\lambda}_i, \hat{\mathbf{v}}_i)\}_{i=1}^n$ *is an ordered set of eigenpairs of* $\hat{M}^{-1}\hat{A}$ *such that* $\hat{\lambda}_1 \leq \cdots \leq \hat{\lambda}_n$. *Define the matrix* $\hat{V}_e$ *as* $\hat{V}_e = [\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_m]$. *If the subspaces* $\mathcal{W} = (\operatorname{Im} A^{\frac{1}{2}}V)^{\perp}$ *and* $\hat{\mathcal{V}}_e = \operatorname{Im} A^{\frac{1}{2}}\hat{V}_e$ *satisfy the condition* $\cos(\mathcal{W}, \hat{\mathcal{V}}_e) \leq \hat{\gamma}$ *for some* $\hat{\gamma} < 1$ *then the minimal eigenvalue of* $\tilde{B}A$ *is bounded as follows:*

$$\lambda(\tilde{B}A) \geq \max\left\{\lambda_{\min}(M^{-1}A), (1-\hat{\gamma})\cdot\min\left\{\frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)}, \hat{\lambda}_{m+1}\right\}\right\}. \tag{15}$$

Proof Similarly to the proof of Theorem 4.1,

$$\begin{aligned}
\lambda_{\min}(\tilde{B}A) &= \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T A^{\frac{1}{2}}\, M^{-1}A^{\frac{1}{2}}\mathbf{x} + \tilde{\sigma}\mathbf{x}^T A^{\frac{1}{2}}VB_V^{-1}V^T A^{\frac{1}{2}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\right\} \\
&\geq \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T A^{\frac{1}{2}}\, M^{-1}A^{\frac{1}{2}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} + \frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)}\cdot\frac{\mathbf{x}^T A^{\frac{1}{2}}VA_V^{-1}V^T A^{\frac{1}{2}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\right\} \\
&\geq \inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T \hat{A}^{\frac{1}{2}}\, \hat{M}^{-1}\hat{A}^{\frac{1}{2}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} + \frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)}\cdot\frac{\mathbf{x}^T A^{\frac{1}{2}}VA_V^{-1}V^T A^{\frac{1}{2}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\right\} \\
&\geq \min\left\{\hat{\lambda}_{m+1}, \frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)}\right\}\cdot\inf_{\mathbf{x}}\left\{\frac{\mathbf{x}^T\hat{P}_e^{\perp}\mathbf{x} + \mathbf{x}^T P_*\mathbf{x}}{\mathbf{x}^T\mathbf{x}}\right\} \\
&\geq (1-\hat{\gamma})\cdot\min\left\{\frac{\lambda_{\max}(M^{-1}A)}{\kappa(B_V^{-1}A_V)}, \hat{\lambda}_{m+1}\right\},
\end{aligned}$$

where $\hat{P}_e$, $\hat{P}_e^{\perp}$, P_* and $P_*^{\perp}$ are the orthogonal projectors onto $\hat{\mathcal{V}}_e = \operatorname{Im} A^{\frac{1}{2}}\hat{V}_e$, $\hat{\mathcal{V}}_e^{\perp} = (\operatorname{Im} A^{\frac{1}{2}}\hat{V}_e)^{\perp}$, $\operatorname{Im} A^{\frac{1}{2}}V$ and $(\operatorname{Im} A^{\frac{1}{2}}V)^{\perp}$ respectively. Combining the above estimate with the result of Theorem 4.1 we obtain (15). □

The above theorem shows that there is no need to know the eigenvectors of the matrix $M^{-1}A$ to construct the matrix V in the preconditioner (12). It suffices to find the matrices $\hat{M}$ and $\hat{A}$

such that the low-frequency eigensubspace $\hat{\mathcal{V}}_e = \text{span}\{\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_m\}$ of $\hat{M}^{-1}\hat{A}$ is known. Then the matrix V can be defined simply as $V = [\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_m]$. This approach is taken in the next section where we construct a number of preconditioners for a class of singularly perturbed elliptic problems by using the known eigensubspace of a degenerate limit problem.

5 Application to second order elliptic problems with strongly discontinuous and/or anisotropic coefficients

In this section we illustrate the application of the algorithm (12) on a number of severely ill-conditioned model problems involving a parameter.

Example 5.1 Consider the following one-dimensional diffusion problem:

$$\begin{aligned} (a(x)u')' &= f, \quad x \in [0,1], \\ u'(0) &= u'_0, \\ u(1) &= u_1 \end{aligned} \tag{16}$$

discretized by means of conforming piecewise-linear finite-elements on a uniform Cartesian mesh with stepsize h. Assume that $a(x) = 1$ for $x < 1/2$ and $a(x) = \varepsilon$ for $x \geq 1/2$. The stiffness matrix of the discrete problem has the following structure:

$$A = h^2 \begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & \cdots & \cdots & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 1+\varepsilon & -\varepsilon & & & \\ & & & -\varepsilon & 2\varepsilon & -\varepsilon & & \\ & & & & \cdots & \cdots & & \\ & & & & -\varepsilon & 2\varepsilon & -\varepsilon & \\ & & & & & -\varepsilon & 2\varepsilon & \end{bmatrix}.$$

For this problem the matrices $\hat{A}$ and $\hat{M}$ in Theorem 4.2 can be defined as follows:

$$\hat{A} = \hat{A}_1 = h^2 \begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & \cdots & \cdots & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 1 & & & & \\ & & & & \varepsilon & -\varepsilon & & \\ & & & & \cdots & \cdots & & \\ & & & & -\varepsilon & 2\varepsilon & -\varepsilon & \\ & & & & & -\varepsilon & 2\varepsilon & \end{bmatrix}, \quad \hat{M} = \hat{M}_1 = 2\,\text{diag}\,\hat{A}_1. \tag{17}$$

As can be easily verified, the following statements hold:

a) $\hat{A}_1 \leq A$, $\hat{M}_1 \geq \text{diag}\,A$,

b) the null-space of $\hat{M}_1^{-1}\hat{A}_1$ consists of the single vector $\mathbf{v}_0 = (1, \ldots, 1, 0, \ldots, 0)$,

c) on the subspace orthogonal to the null-space of $\hat{M}_1^{-1}\hat{A}_1$ the spectrum of $\hat{M}_1^{-1}\hat{A}_1$ is contained in an interval $[O(h^2), O(1)]$.

Thus, the preconditioner (12) for A can be constructed as

$$\tilde{B} = M^{-1} + \tilde{\sigma} V B_V^{-1} V^T, \qquad M = \operatorname{diag} A, \qquad V = [\mathbf{v}_0].$$

As follows from Theorem 4.2, the spectrum of $\tilde{B}A$ is contained in the interval $[O(h^2), O(1)]$, and the bounds for $\lambda(\tilde{B}A)$ are independent on ε.

Remark 5.1 The above algorithm for constructing the preconditioner (12) can be straightforwardly extended to the case when the dimension of the space is greater than 1.

Remark 5.2 The problems with multiple jumps in the coefficient function and with other types of boundary conditions can be treated analogously. In this case the null-space of $\hat{M}_1^{-1}\hat{A}_1$ contains the vectors $\mathbf{v}_i$ which are constant along the "subdomains" Ω_i with constant value of $a(x)$, which are not adjacent to the Dirichlet part of the boundary. The matrix V in (12) is then constructed as

$$V = [\mathbf{v}_0, \ldots, \mathbf{v}_m]. \tag{18}$$

Remark 5.3 Alternatively to (17), we can define the matrices $\hat{A}$ and $\hat{M}$ as

$$\hat{A} = \hat{A}_2 = h^2 \begin{bmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & \cdots & \cdots & \cdots & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 1 & & & & \\ & & & & \varepsilon & -\varepsilon & & \\ & & & & \cdots & \cdots & \cdots & \\ & & & & -\varepsilon & 2\varepsilon & -\varepsilon & \\ & & & & & -\varepsilon & \varepsilon \end{bmatrix}, \quad \hat{M} = \hat{M}_2 = 2 \operatorname{diag} \hat{A}_2,$$

i.e. we can replace all the boundary conditions of Dirichlet type by boundary conditions of Neumann type. In this case the null-space of $\hat{M}_2^{-1}\hat{A}_2$ contains the vectors which are constant along all the "subdomains" Ω_i (even along the "subdomains" which are adjacent to the Dirichlet part of the boundary). If the matrix V in (12) is chosen such that $\operatorname{Im} V = \ker \hat{M}_2^{-1}\hat{A}_2$ then the preconditioner "shifts" the low-frequency eigencluster at least as well (or even better) as the preconditioner with the matrix V defined by (18). The improvement is possible since the smallest nonzero eigenvalue $\lambda_{\min}^{+}(\hat{M}_2^{-1}\hat{A}_2)$ of $\hat{M}_2^{-1}\hat{A}_2$ is at least as large as the smallest nonzero eigenvalue $\lambda_{\min}^{+}(\hat{M}_1^{-1}\hat{A}_1)$ of $\hat{M}_1^{-1}\hat{A}_1$ (and in some cases it can be even significantly larger).

Example 5.2 Consider the following two-dimensional diffusion problem

$$\begin{aligned} \nabla K(\mathbf{x}) \nabla u(\mathbf{x}) &= f(\mathbf{x}) && \text{in} \quad \mathbf{x} \in \Omega \\ u(\mathbf{x}) &= 0 && \text{on} \quad \mathbf{x} \in \Gamma_D \subseteq \partial\Omega, \\ \partial u(\mathbf{x})/\partial \mathbf{n} &= 0 && \text{on} \quad \mathbf{x} \in \Gamma_N = \partial\Gamma / \Gamma_D \end{aligned}$$

with highly anisotropic diffusion tensor $K(\mathbf{x})$

$$K(\mathbf{x}) = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \qquad 0 < \varepsilon \ll 1$$

discretized by means of conforming piecewise-linear finite elements on a uniform Cartesian grid. For this problem the auxiliary matrices $\hat{A}$ and $\hat{M}$ in Theorem 4.2 can be chosen as follows: the matrix $\hat{A}$ can be defined to be the stiffness matrix of the degenerate limit problem with $\varepsilon = 0$, the matrix $\hat{M}$ can be defined as $\hat{M} = 2 \operatorname{diag} \hat{A}$. Clearly, $\hat{A} \leq A$, $\hat{M} \geq M = \operatorname{diag} A$. With this choice of $\hat{A}$ and $\hat{M}$ the null-space of $\hat{M}^{-1}\hat{A}$ is known, it consists of constant vectors aligned with the y-axis (except the constant vectors which are adjacent with the Dirichlet part of the boundary). The spectrum of $\hat{M}^{-1}\hat{A}$ on the subspace orthogonal to the kernel is contained in the interval $[O(h^2), O(1)]$. Thus, if we choose the matrix V such that $\operatorname{Im} V = \ker \hat{M}^{-1}\hat{A}$, then the spectrum of $\tilde{B}A$ is contained in the interval $[O(h^2), O(1)]$ and is bounded independently on ε.

Remark 5.4 As can be easily verified, with the above choice of V ($\operatorname{Im} V = \ker \hat{M}^{-1}\hat{A}$) the condition number $\kappa(A_V)$ of $A_V = V^T A V$ is of order $O(h^{-2})$, i.e. it is of the same order as the effective condition number $\kappa^+(\hat{M}^{-1}\hat{A})$ of $\hat{M}^{-1}\hat{A}$, $\kappa^+(\hat{M}^{-1}\hat{A}) = \lambda_{\max}(\hat{M}^{-1}\hat{A})/\lambda^+_{\min}(\hat{M}^{-1}\hat{A})$. Thus, the preconditioner B_V for A_V can be constructed as $B_V = I$ or $B_V = \operatorname{diag} A_V$, see Remark 3.3. Such a choice of B_V allows an efficient parallel implementation of the preconditioner (12).

Remark 5.5 A similar approach for constructing the matrices V and B_V can also be applied in the three-dimensional case. If the diffusion tensor $K(\mathbf{x})$ has the form

$$K(\mathbf{x}) = \begin{bmatrix} \varepsilon^\beta & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad 0 < \varepsilon < 1, \qquad \beta \geq 1,$$

then the preconditioner can be constructed by applying the above described algorithm recursively: first to the matrix A and then to the matrix A_V. As in the two-dimensional case, the resulting preconditioner can be efficiently parallelized.

Remark 5.6 We can define the matrix $\hat{A}$ in Theorem 4.2 to be the stiffness matrix which corresponds to the problem not only with $\varepsilon = 0$, but also with $\Gamma_N = \partial\Gamma$. In this case the subspace $\operatorname{Im} V$ contains all the constant vectors aligned with the y-axis.

Remark 5.7 The above algorithm for the anisotropic problems can be easily combined with the algorithm for the discontinuous problems. This allows us to treat the problems where the diffusion tensor is not only anisotropic, but also discontinuous.

6 A purely algebraic algorithm for constructing the matrix V

For the class of diffusion-type problems considered in the previous section the matrix V can be constructed automatically using a heuristic technique developed in [3, 14] (see also [15] for a similar approach). For the sake of completeness a brief description of the algorithm follows.

Consider the diffusion problem as in Example 5.2

$$\begin{array}{rcll} \nabla K(\mathbf{x}) \nabla u(\mathbf{x}) & = & f(\mathbf{x}) & \text{in} \quad \mathbf{x} \in \Omega \\ u(\mathbf{x}) & = & 0 & \text{on} \quad \mathbf{x} \in \Gamma_D \subseteq \partial\Omega, \\ \partial u(\mathbf{x})/\partial \mathbf{n} & = & 0 & \text{on} \quad \mathbf{x} \in \Gamma_N = \partial\Gamma / \Gamma_D \end{array} \tag{19}$$

discretized on a regular finite element mesh. Assume that the diffusion tensor $K(\mathbf{x})$ is piecewise constant and uniformly s.p.d.

Let $A = [a_{i,j}]_{i,j=1}^n$ be the stiffness matrix resulting from the discretization of (19). Define the matrix $Q = [q_{i,j}]_{i,j=1}^n$ which contains a pattern of "strong couplings" within A:

$$q_{i,j} = \begin{cases} 0, & \text{if } |a_{i,j}| < \omega \cdot \min\{ \max\limits_{\substack{k=1,n \\ k \neq i}} |a_{i,k}|, \max\limits_{\substack{k=1,n \\ k \neq j}} |a_{k,j}| \}, \qquad \omega \in (0,1), \\ 1, & \text{otherwise.} \end{cases} \tag{20}$$

Define a symmetric function $\chi(i,j)$ of two integer variables i and j: let the function $\chi(i,j)$ be equal to unity either if $q_{i,j} = 1$ or if there exists a k such that $\chi(i,k) \cdot \chi(k,j) = 1$; otherwise define the function $\chi(i,j)$ to be equal to zero. As can be readily seen, the definition of $\chi(i,j)$ implies that $\chi(i,j) = 1$ if and only if there is a "strong connectivity path" between the unknowns i and j; otherwise $\chi(i,j) = 0$.

Define also a number of sets $G^{(p)}$ of size n_p:

$$G^{(p)} = \{i_1^{(p)}, \ldots, i_{n_p}^{(p)}\}, \quad i_s^{(p)} \in \{1, 2, \ldots, n\}, \quad p = 1, \ldots, \tilde{m},$$

such that they satisfy the following conditions:

- $G^{(p_1)} \bigcap G^{(p_2)} = \{\emptyset\}$ for all $p_1 \neq p_2$,
- for any i and j there exists p such that $i \in G^{(p)}$ and $j \in G^{(p)}$ if and only if $\chi(i,j) = 1$.

As follows from the above definition, each set $G^{(p)}$ contains a list of "strongly connected unknowns" (with respect to A). The definition of $G^{(p)}$ also implies that if there is no "strong connectivity path" from i to j, then the unknowns i and j belong to different sets. If there is a "strong connectivity path" between the unknowns i and j, then they belong to the same set $G^{(p)}$. As can be readily shown, the sets $G^{(p)}$ can be computed with an arithmetic cost $O(n)$.

Define a set of $\tilde{m}$ sparse vectors $\hat{\mathbf{w}}^{(p)}$ of size n:

$$\hat{\mathbf{w}}_i^{(p)} = 1 \quad \text{if} \quad i \in G^{(p)}; \quad \text{otherwise } \hat{\mathbf{w}}_i^{(p)} = 0. \tag{21}$$

Clearly, the vectors $\hat{\mathbf{w}}^{(p)}$ are L_2-orthogonal to each other. Next, define a vector $\mathbf{h}$ such that

$$\mathbf{h}_i = 1 \quad \text{if} \quad \left| \frac{\sum a_{i,j}}{a_{i,i}} \right| > \omega; \qquad \mathbf{h}_i = 0 \quad \text{otherwise.} \tag{22}$$

Define a set $\mathcal{X}$ of indices p_i, $i = 1, \ldots, m$, $m \leq \tilde{m}$ such that

$$p_i \in \mathcal{X} \quad \text{if and only if} \quad \mathbf{h}^T \hat{\mathbf{w}}^{(p_i)} = 0. \tag{23}$$

Remark 6.1 The algorithm (22)–(23) selects only those sets $G^{(p_i)}$, which are "weakly connected" with the Dirichlet part of the boundary (see the previous section for the motivation).

Define the matrices V_1 and V_2 as follows:

$$\begin{aligned} V_1 &= \left[\tilde{\mathbf{w}}^{(p_1)}, \ldots, \tilde{\mathbf{w}}^{(p_m)}\right], \qquad p_i \in \mathcal{X}, \ i = 1, \ldots, m, \\ V_2 &= \left[\tilde{\mathbf{w}}^{(1)}, \ldots, \tilde{\mathbf{w}}^{(\tilde{m})}\right], \qquad i = 1, \ldots, \tilde{m}. \end{aligned} \tag{24}$$

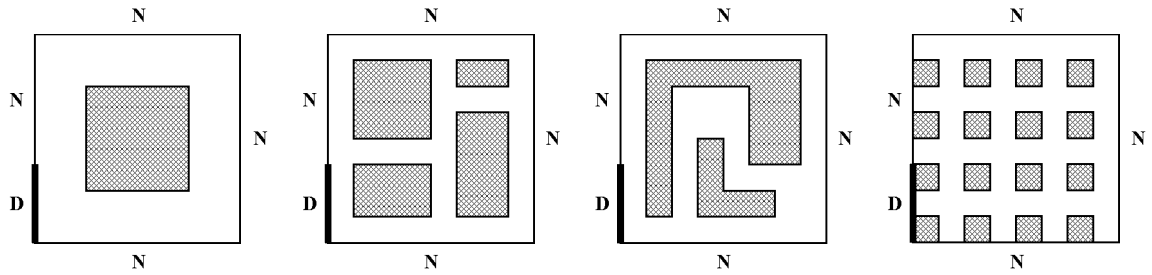


Figure 1: Test problems used in our numerical experiments

Finally, define the matrix V in the preconditioner (12) as either $V = V_1$ or $V = V_2$.

As was demonstrated in the previous section, both $\operatorname{Im} V_1$ and $\operatorname{Im} V_2$ approximate well the low-frequency eigensubspace of nearly degenerate diffusion-type operators. A nice feature of the choice $V = V_1$ is that it leads to a smaller condition number of $A_V = V^T A V$. In many practical cases this allows an easier construction of B_V. It should be noted, however, that in the case $V = V_2$ the smallest eigenvalues of A could be captured more efficiently than in the case $V = V_1$ since $\operatorname{Im} V_1 \subseteq \operatorname{Im} V_2$ (see Lemma 2.2).

As follows from the definition of $\{\tilde{\mathbf{w}}^{(i)}\}$, the matrix V is sparse, it contains at most n nonzero entries. This means that if the action of B_V^{-1} requires $O(n)$ arithmetic operations, then the action of the whole preconditioner (12) also requires only $O(n)$ operations.

Another important feature of the developed preconditioner is that it can be efficiently parallelized since in many practical applications it suffices to use a (block) diagonal preconditioner B_V^{-1} for A_V^{-1}. As can be easily verified, if we distribute the algebraic system between the processors in the multiprocessor system such that the unknowns from the same group $G^{(p)}$ belong to the same processor and distribute the blocks of B_V accordingly, then no interprocessor communications are needed to perform the multiplication with $V B_V^{-1} V^T$ (if the matrix B_V is block-diagonal and the blocks are properly distributed).

7 Numerical experiments

In this section we illustrate the numerical performance of the developed technique on a number of singularly perturbed elliptic problems of the form given in Example 5.2. Namely, we consider piecewise linear conforming finite element discretization of the diffusion equation (19) with $\Omega = [0,1]^2$, $\Gamma_D = \{\mathbf{x} = (x,y) : x = 0,\ 0 \le y < \bar{y} \le 1\}$ and $\Gamma_N = \partial\Omega/\Gamma_D$ on a uniform Cartesian grid. The diffusion tensor $K(\mathbf{x})$ is considered to be of the form

$$K(\mathbf{x}) = a(\mathbf{x}) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \quad a(\mathbf{x}) > 0, \quad \varepsilon > 0.$$

The value of ε is chosen to be equal to 1, 10^3 or 10^6. The coefficient function $a(\mathbf{x})$ is assumed to be the following:

$$a(\mathbf{x}) = \begin{cases} a, & \text{if } \mathbf{x} \text{ belongs to the shaded area (see Figure 1),} \\ 1, & \text{otherwise,} \end{cases}$$

where the value of a is chosen to be either 10^{-6}, 10^{-3}, 1, 10^3 or 10^6.

The main concern is to demonstrate insensitivity of the developed algorithm with respect to the problem parameters. The results of our numerical experiments are presented in Tables 1–8, where we study the performance of the method with respect to variations of a, ε, $\bar{y}$ and different modifications of the preconditioner (12). The performance of the diagonal (pointwise Jacobi) preconditioner is also presented for comparison. The stopping criterion within the PCG method is chosen to be $\|\mathbf{r}^{(k)}\|/\|\mathbf{r}^{(0)}\| < 10^{-6}$, where $\mathbf{r}^{(0)}$ is the initial residual and $\mathbf{r}^{(k)}$ is the residual after the k-th iteration. The right-hand side in the algebraic system is chosen to be random. The matrix V in (12) is constructed automatically using the heuristic algorithm (20)–(24) with $\omega = 0.1$.

Table 1 shows that for the problem with smooth isotropic coefficient function the convergence rate of the diagonally preconditioned PCG method depends mildly on the choice of the boundary conditions, whereas the situation is opposite in the case when the coefficient function is highly anisotropic. Tables 2–8 show that the jumps in the coefficient function have the effect of adding extra (internal) Neumann-type boundary conditions which again leads to a slower convergence of the diagonally preconditioned iterative scheme. To the contrary, the PCG method preconditioned by means of (12) exhibits robust performance in a wide range of a and ε and is insensitive to the choice of the boundary conditions, see Tables 1–8.

Numerical experiments show that the developed subspace-correction technique performs well even if the matrix A_V is replaced by a simple diagonal preconditioner B_V; in many practical cases it suffices to take $B_V = \operatorname{diag} A_V$. However, if the matrix A_V is severely ill-conditioned, special care has to be taken when constructing the preconditioner B_V; one of the possible approaches was mentioned in Remark 5.5, alternatively one can use an incomplete factorization procedure to construct an approximation to A_V^{-1}. In a multilevel setting the matrix B_V can be constructed by using the algorithm (12) recursively; in this case we obtain a preconditioner of the form $B_k = M_k^{-1} + \sigma_k V_{k,k-1} B_{k-1} V_{k,k-1}^T$, see Sections 3 and 4.

In Figures 2 and 3 we also illustrate the eigenvalue distribution of the preconditioned matrix $\hat{B}A$ for different a, ε and $\hat{B}$. As one can see from the above figures, the spectrum of the system preconditioned by (12) is contained in the interval $[O(h^2), O(1)]$, and the bounds are independent of ε and a, whereas in the case of Jacobi preconditioning the spectrum normally contains a number of extremely small eigenvalues, sometimes well separated from the remainder of the spectrum, which may cause slow convergence of the PCG algorithm.

The results of our numerical experiments are in good agreement with the developed theory. Taking into account that the computational overhead associated with the preconditioner is very low (especially in the case when the matrix B_V is chosen to be diagonal) we conclude that the developed algorithm could be viewed as a viable option when constructing efficient solvers for the considered class of ill-conditioned elliptic problems. Note also that the method is even more attractive in a parallel environment, where it can be a serious competitor to more advanced methods (of multigrid/multilevel type, for instance) as it requires only a small amount of interprocessor communications.

8 Acknowledgments

The time invested by Igor Kaporin (Computing Center of Russian Academy of Sciences) during a number of discussions about the developed algorithms is very much appreciated.

	Preconditioner				
		$V = V_2$	$V = V_1$		
	$\hat{B} = (\operatorname{diag} A)^{-1}$	$B_V = A_V$	$B_V = A_V$	$B_V = I$	$B_V = \operatorname{diag} A_V$
$\bar{y}$	Anisotropy: $\varepsilon = 1$ (isotropic case)				
0.125	241 (-)	203 (1)	241 (0)	241 (0)	241 (0)
0.250	239 (-)	238 (1)	239 (0)	239 (0)	239 (0)
0.375	239 (-)	239 (1)	239 (0)	239 (0)	239 (0)
0.500	234 (-)	232 (1)	234 (0)	234 (0)	234 (0)
0.625	225 (-)	223 (1)	225 (0)	225 (0)	225 (0)
0.750	222 (-)	220 (1)	222 (0)	222 (0)	222 (0)
0.875	212 (-)	209 (1)	212 (0)	212 (0)	212 (0)
1.000	179 (-)	117 (1)	179 (0)	179 (0)	179 (0)
$\bar{y}$	Anisotropy: $\varepsilon = 10^3$ (anisotropic case)				
0.125	1910 (-)	237 (65)	331 (57)	346 (57)	347 (57)
0.250	1781 (-)	243 (65)	423 (49)	417 (49)	421 (49)
0.375	1548 (-)	241 (65)	420 (41)	414 (41)	414 (41)
0.500	1290 (-)	235 (65)	405 (33)	407 (33)	410 (33)
0.625	1032 (-)	226 (65)	398 (25)	396 (25)	387 (25)
0.750	805 (-)	221 (65)	389 (17)	385 (17)	384 (17)
0.875	564 (-)	219 (65)	362 (9)	360 (9)	359 (9)
1.000	284 (-)	162 (65)	262 (1)	262 (1)	262 (1)

Table 1: Problems **A**, **B**, **C** and **D**, $h = \dfrac{1}{64}$, $a = 1$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) for different values of ε, $\bar{y}$ and different choices of the preconditioner.

	Preconditioner				
		$V = V_2$	$V = V_1$		
	$\hat{B} = (\operatorname{diag} A)^{-1}$	$B_V = A_V$	$B_V = A_V$	$B_V = I$	$B_V = \operatorname{diag} A_V$
$\bar{y}$	Anisotropy: $\varepsilon = 1$ (isotropic case)				
0.125	310 (-)	195 (2)	216 (1)	216 (1)	216 (1)
0.250	307 (-)	194 (2)	214 (1)	214 (1)	214 (1)
0.375	308 (-)	193 (2)	213 (1)	213 (1)	213 (1)
0.500	303 (-)	205 (2)	219 (1)	219 (1)	219 (1)
0.625	298 (-)	205 (2)	217 (1)	217 (1)	217 (1)
0.750	290 (-)	194 (2)	207 (1)	207 (1)	207 (1)
0.875	280 (-)	182 (2)	197 (1)	197 (1)	197 (1)
1.000	260 (-)	160 (2)	171 (1)	171 (1)	171 (1)
$\bar{y}$	Anisotropy: $\varepsilon = 10^3$ (anisotropic case)				
0.125	2114 (-)	275 (131)	415 (123)	905 (123)	477 (123)
0.250	2007 (-)	294 (131)	559 (115)	808 (115)	520 (115)
0.375	1917 (-)	252 (131)	480 (107)	755 (107)	422 (107)
0.500	1884 (-)	246 (131)	480 (99)	700 (99)	421 (99)
0.625	1877 (-)	247 (131)	474 (91)	723 (91)	422 (91)
0.750	1851 (-)	233 (131)	474 (83)	728 (83)	416 (83)
0.875	1648 (-)	251 (131)	471 (75)	595 (75)	429 (75)
1.000	1382 (-)	245 (131)	471 (67)	420 (67)	406 (67)

Table 2: Problem **A**, $h = \dfrac{1}{64}$, $a = 10^3$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) for different values of ε, $\bar{y}$ and different choices of the preconditioner.

	Preconditioner				
		$V = V_2$	$V = V_1$		
	$\hat{B} = (\operatorname{diag} A)^{-1}$	$B_V = A_V$	$B_V = A_V$	$B_V = I$	$B_V = \operatorname{diag} A_V$
$\bar{y}$	Anisotropy: $\varepsilon = 1$ (isotropic case)				
0.125	400 (-)	267 (3)	288 (2)	289 (2)	292 (2)
0.250	397 (-)	263 (3)	284 (2)	281 (2)	283 (2)
0.375	394 (-)	262 (3)	279 (2)	276 (2)	279 (2)
0.500	393 (-)	262 (3)	275 (2)	272 (2)	273 (2)
0.625	389 (-)	260 (3)	271 (2)	268 (2)	272 (2)
0.750	388 (-)	262 (3)	273 (2)	266 (2)	271 (2)
0.875	382 (-)	262 (3)	271 (2)	268 (2)	272 (2)
1.000	380 (-)	260 (3)	268 (2)	265 (2)	268 (2)
$\bar{y}$	Anisotropy: $\varepsilon = 10^3$ (anisotropic case)				
0.125	3410 (-)	270 (261)	423 (253)	611 (253)	445 (253)
0.250	3275 (-)	240 (261)	394 (245)	562 (245)	395 (245)
0.375	3111 (-)	233 (261)	388 (237)	542 (237)	386 (237)
0.500	3087 (-)	222 (261)	380 (229)	544 (229)	377 (229)
0.625	2870 (-)	220 (261)	379 (221)	535 (221)	369 (221)
0.750	2918 (-)	222 (261)	379 (213)	506 (213)	341 (213)
0.875	2766 (-)	225 (261)	380 (205)	506 (205)	341 (205)
1.000	2620 (-)	232 (261)	463 (197)	392 (197)	408 (197)

Table 3: Problem **C**, $h = \dfrac{1}{64}$, $a = 10^3$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) for different values of ε, $\bar{y}$ and different choices of the preconditioner.

	Preconditioner				
		$V = V_2$	$V = V_1$		
	$\hat{B} = (\operatorname{diag} A)^{-1}$	$B_V = A_V$	$B_V = A_V$	$B_V = I$	$B_V = \operatorname{diag} A_V$
$\bar{y}$	Anisotropy: $\varepsilon = 1$ (isotropic case)				
0.125	606 (-)	148 (17)	153 (16)	172 (16)	174 (16)
0.250	606 (-)	179 (17)	191 (15)	186 (15)	165 (15)
0.375	603 (-)	178 (17)	183 (14)	181 (14)	183 (14)
0.500	603 (-)	178 (17)	183 (14)	177 (14)	184 (14)
0.625	563 (-)	173 (17)	180 (13)	176 (13)	179 (13)
0.750	560 (-)	170 (17)	172 (13)	168 (13)	173 (13)
0.875	515 (-)	159 (17)	158 (12)	158 (12)	168 (12)
1.000	512 (-)	154 (17)	156 (12)	156 (12)	156 (12)
$\bar{y}$	Anisotropy: $\varepsilon = 10^3$ (anisotropic case)				
0.125	3709 (-)	199 (317)	207 (309)	965 (309)	508 (309)
0.250	4128 (-)	239 (317)	377 (301)	988 (301)	610 (301)
0.375	3964 (-)	229 (317)	335 (293)	913 (293)	546 (293)
0.500	3760 (-)	247 (317)	464 (285)	831 (285)	612 (285)
0.625	3461 (-)	211 (317)	339 (277)	691 (277)	470 (277)
0.750	2812 (-)	247 (317)	433 (269)	586 (269)	458 (269)
0.875	2111 (-)	205 (317)	334 (261)	455 (261)	326 (261)
1.000	1553 (-)	229 (317)	398 (253)	356 (253)	358 (253)

Table 4: Problem **D**, $h = \dfrac{1}{64}$, $a = 10^3$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) for different values of ε, $\bar{y}$ and different choices of the preconditioner.

	Coefficient jump a				
	10^{-6}	10^{-3}	1	10^{3}	10^{6}
	Anisotropy: $\varepsilon = 1$ (isotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	282 (-)	281 (-)	239 (-)	304 (-)	382 (-)
$B_V = A_V$, V_2	275 (2)	276 (2)	239 (1)	193 (2)	228 (2)
$B_V = A_V$, V_1	285 (1)	282 (1)	239 (0)	224 (1)	232 (1)
$B_V = I$, V_1	271 (1)	288 (1)	239 (0)	214 (1)	240 (1)
$B_V = \operatorname{diag} A_V$, V_1	285 (1)	286 (1)	239 (0)	210 (1)	240 (1)
	Anisotropy: $\varepsilon = 10^{3}$ (anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	2351 (-)	2335 (-)	1548 (-)	1911 (-)	2146 (-)
$B_V = A_V$, V_2	257 (127)	265 (127)	241 (65)	252 (131)	300 (131)
$B_V = A_V$, V_1	455 (103)	467 (103)	420 (41)	480 (107)	582 (107)
$B_V = I$, V_1	425 (103)	428 (103)	414 (41)	758 (107)	1171 (107)
$B_V = \operatorname{diag} A_V$, V_1	429 (103)	428 (103)	414 (41)	422 (107)	658 (107)
	Anisotropy: $\varepsilon = 10^{6}$ (strongly anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	9266 (-)	7916 (-)	4882 (-)	6173 (-)	5923 (-)
$B_V = A_V$, V_2	195 (127)	195 (127)	115 (65)	150 (131)	150 (131)
$B_V = A_V$, V_1	269 (103)	257 (103)	161 (41)	243 (107)	215 (107)
$B_V = I$, V_1	312 (103)	562 (103)	170 (41)	2086 (107)	2501 (107)
$B_V = \operatorname{diag} A_V$, V_1	288 (103)	436 (103)	170 (41)	451 (107)	254 (107)

Table 5: Problem **A**, $h = \dfrac{1}{64}$, $\bar{y} = \dfrac{3}{8}$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) as a function of the coefficient jump a and the anisotropy ratio ε

	Coefficient jump a				
	10^{-6}	10^{-3}	1	10^{3}	10^{6}
	Anisotropy: $\varepsilon = 1$ (isotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	335 (-)	333 (-)	239 (-)	431 (-)	619 (-)
$B_V = A_V$, V_2	317 (5)	320 (5)	239 (1)	177 (5)	197 (5)
$B_V = A_V$, V_1	338 (4)	338 (4)	239 (0)	192 (4)	208 (4)
$B_V = I$, V_1	332 (4)	332 (4)	239 (0)	189 (4)	201 (4)
$B_V = \operatorname{diag} A_V$, V_1	337 (4)	347 (4)	239 (0)	192 (4)	208 (4)
	Anisotropy: $\varepsilon = 10^{3}$ (anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	3064 (-)	2779 (-)	1548 (-)	3015 (-)	4637 (-)
$B_V = A_V$, V_2	239 (217)	242 (217)	241 (65)	232 (233)	278 (233)
$B_V = A_V$, V_1	400 (193)	414 (193)	420 (41)	384 (209)	422 (209)
$B_V = I$, V_1	380 (193)	392 (193)	414 (41)	571 (209)	1260 (209)
$B_V = \operatorname{diag} A_V$, V_1	387 (193)	384 (193)	414 (41)	395 (209)	902 (209)
	Anisotropy: $\varepsilon = 10^{6}$ (strongly anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	13314 (-)	12345 (-)	4882 (-)	10526 (-)	12225 (-)
$B_V = A_V$, V_2	219 (217)	234 (217)	115 (65)	164 (233)	197 (233)
$B_V = A_V$, V_1	307 (193)	303 (193)	161 (41)	249 (209)	279 (209)
$B_V = I$, V_1	318 (193)	801 (193)	170 (41)	2000 (209)	1689 (209)
$B_V = \operatorname{diag} A_V$, V_1	321 (193)	659 (193)	170 (41)	599 (209)	323 (209)

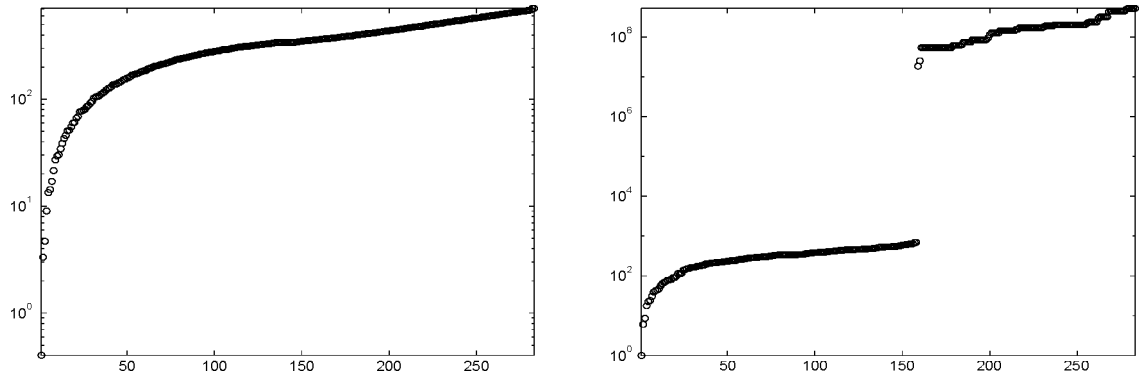
Table 6: Problem **B**, $h = \dfrac{1}{64}$, $\bar{y} = \dfrac{3}{8}$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) as a function of the coefficient jump a and the anisotropy ratio ε

	Coefficient jump a				
	10^{-6}	10^{-3}	1	10^{3}	10^{6}
	Anisotropy: $\varepsilon = 1$ (isotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	384 (-)	381 (-)	239 (-)	394 (-)	527 (-)
$B_V = A_V$, V_2	384 (3)	384 (3)	239 (1)	263 (3)	281 (3)
$B_V = A_V$, V_1	391 (2)	388 (2)	239 (0)	279 (2)	289 (2)
$B_V = I$, V_1	382 (2)	383 (2)	239 (0)	277 (2)	297 (2)
$B_V = \operatorname{diag} A_V$, V_1	390 (2)	385 (2)	239 (0)	279 (2)	294 (2)
	Anisotropy: $\varepsilon = 10^{3}$ (anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	3777 (-)	3168 (-)	1548 (-)	3113 (-)	3771 (-)
$B_V = A_V$, V_2	247 (253)	251 (253)	241 (65)	233 (261)	257 (261)
$B_V = A_V$, V_1	405 (229)	413 (229)	420 (41)	387 (237)	409 (237)
$B_V = I$, V_1	444 (229)	402 (229)	414 (41)	553 (237)	934 (237)
$B_V = \operatorname{diag} A_V$, V_1	434 (229)	394 (229)	414 (41)	387 (237)	663 (237)
	Anisotropy: $\varepsilon = 10^{6}$ (strongly anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	15007 (-)	14853 (-)	4882 (-)	13727 (-)	13274 (-)
$B_V = A_V$, V_2	213 (253)	227 (253)	115 (65)	161 (261)	184 (261)
$B_V = A_V$, V_1	298 (229)	301 (229)	161 (41)	273 (237)	287 (237)
$B_V = I$, V_1	334 (229)	947 (229)	170 (41)	2218 (237)	1653 (237)
$B_V = \operatorname{diag} A_V$, V_1	322 (229)	693 (229)	170 (41)	726 (237)	324 (237)

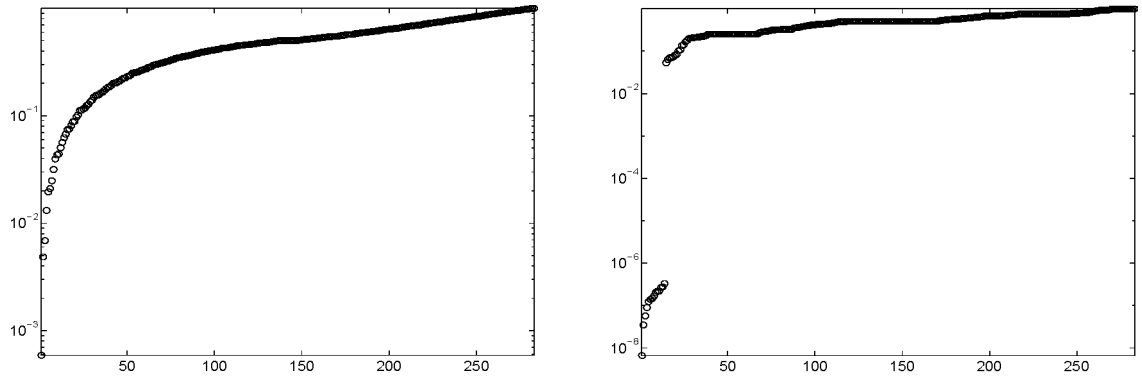
Table 7: Problem **C**, $h = \dfrac{1}{64}$, $\bar{y} = \dfrac{3}{8}$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) as a function of the coefficient jump a and the anisotropy ratio ε

	Coefficient jump a				
	10^{-6}	10^{-3}	1	10^{3}	10^{6}
	Anisotropy: $\varepsilon = 1$ (isotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	292 (-)	259 (-)	239 (-)	602 (-)	984 (-)
$B_V = A_V$, V_2	240 (17)	244 (17)	239 (1)	178 (17)	205 (17)
$B_V = A_V$, V_1	299 (14)	299 (14)	239 (0)	183 (14)	203 (14)
$B_V = I$, V_1	304 (14)	303 (14)	239 (0)	179 (14)	183 (14)
$B_V = \operatorname{diag} A_V$, V_1	311 (14)	312 (14)	239 (0)	183 (14)	181 (14)
	Anisotropy: $\varepsilon = 10^{3}$ (anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	2495 (-)	2432 (-)	1548 (-)	3852 (-)	7422 (-)
$B_V = A_V$, V_2	238 (268)	246 (268)	241 (65)	229 (317)	273 (317)
$B_V = A_V$, V_1	394 (244)	418 (244)	420 (41)	334 (293)	381 (293)
$B_V = I$, V_1	391 (244)	383 (244)	414 (41)	914 (293)	3099 (293)
$B_V = \operatorname{diag} A_V$, V_1	386 (244)	390 (244)	414 (41)	540 (293)	2099 (293)
	Anisotropy: $\varepsilon = 10^{6}$ (strongly anisotropic case)				
$\hat{B} = (\operatorname{diag} A)^{-1}$	10032 (-)	13016 (-)	4882 (-)	18742 (-)	n/a
$B_V = A_V$, V_2	152 (268)	149 (268)	115 (65)	116 (317)	n/a
$B_V = A_V$, V_1	278 (244)	276 (244)	161 (41)	187 (293)	n/a
$B_V = I$, V_1	283 (244)	832 (244)	170 (41)	3559 (293)	n/a
$B_V = \operatorname{diag} A_V$, V_1	302 (244)	588 (244)	170 (41)	785 (293)	n/a

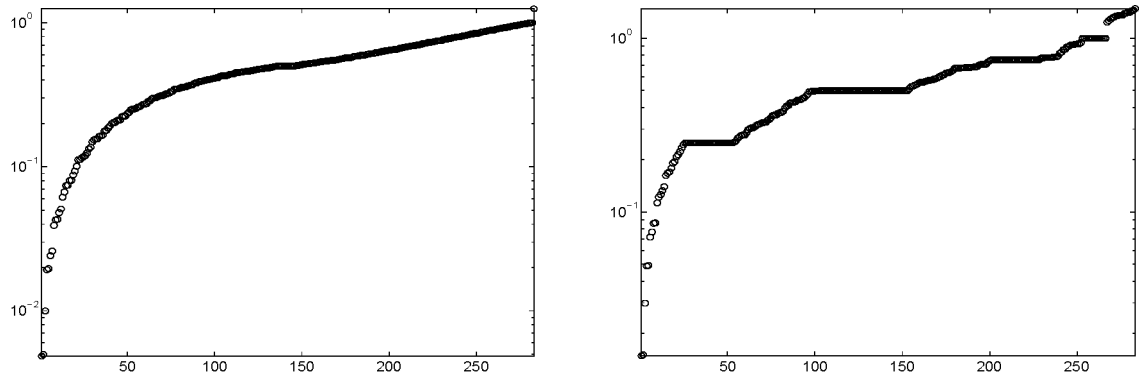
Table 8: Problem **D**, $h = \dfrac{1}{64}$, $\bar{y} = \dfrac{3}{8}$, PCG iteration count and the dimension of $\operatorname{Im} V$ (in brackets) as a function of the coefficient jump a and the anisotropy ratio ε



No preconditioning: $\hat{B} = I$

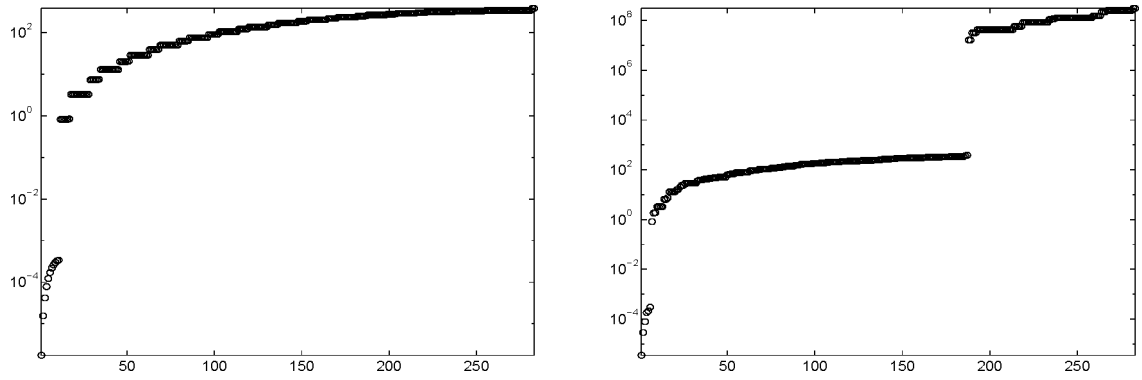


Diagonal scaling: $\hat{B} = (\operatorname{diag} A)^{-1}$

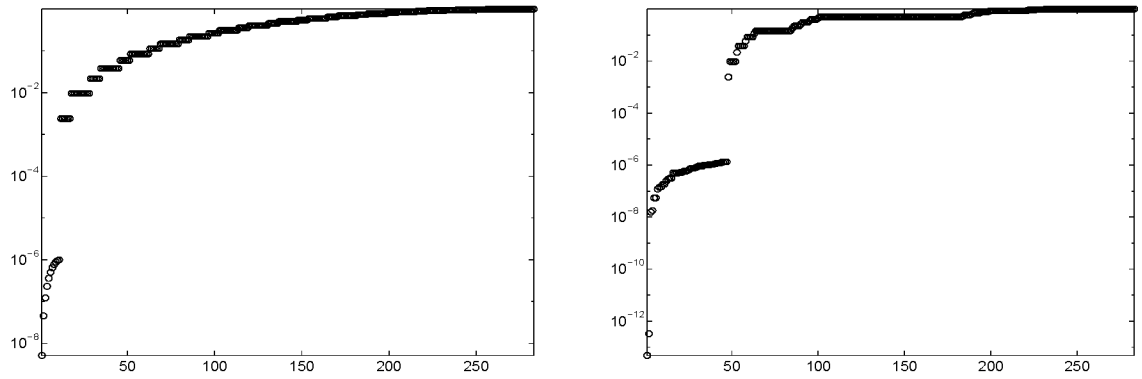


Algorithm (12): $\hat{B} = (\operatorname{diag} A)^{-1} + \sigma V B_V^{-1} V^T,\ B_V = A_V,\ V = V_2$

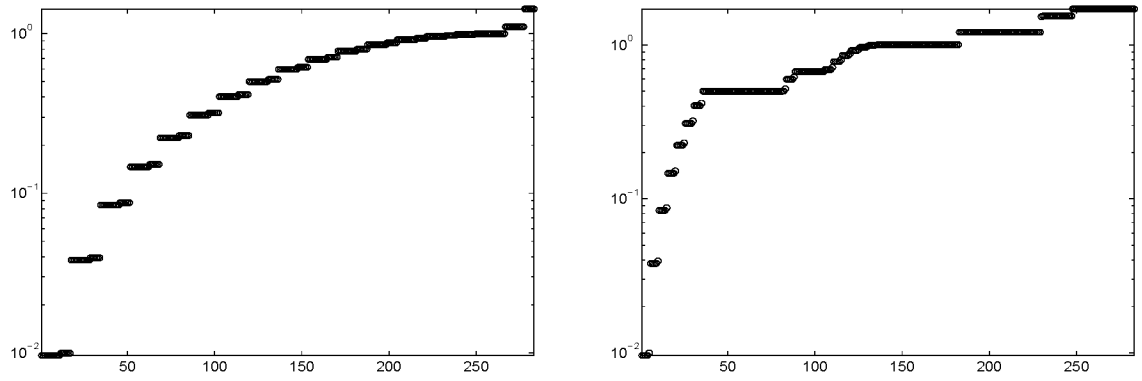
Figure 2: Problem **D**, $h = 1/16$, $\varepsilon = 1$, $\bar{y} = \dfrac{3}{8}$, eigenvalue distribution of $\hat{B}A$ for different preconditioners $\hat{B}$ and different values of the coefficient jump a: $a = 1$ (left) and $a = 10^6$ (right)



No preconditioning: $\hat{B} = I$



Diagonal scaling: $\hat{B} = (\operatorname{diag} A)^{-1}$



Algorithm (12): $\hat{B} = (\operatorname{diag} A)^{-1} + \sigma V B_V^{-1} V^T,\ B_V = A_V,\ V = V_2$

Figure 3: Problem **D**, $h = \dfrac{1}{16}$, $\varepsilon = 10^6$, $\bar{y} = \dfrac{3}{8}$, eigenvalue distribution of $\hat{B}A$ for different preconditioners $\hat{B}$ and different values of the coefficient jump a: $a = 1$ (left) and $a = 10^6$ (right)

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