POSITIVE THINKING
Lacroix’s theory on negative numbers
in the Netherlands

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Positive Thinking
Lacroix’s theory on negative numbers in the Netherlands

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Abstract
The beginning of the 19th century witnessed the emergence of several new theories regarding negative numbers. New notions of rigour made the 18th century conception of negative quantities unacceptable. This paper discusses Dutch theories on negative numbers in the early 19th century. Dutch mathematicians opted for another approach than their counterparts in Germany and France. Focussing on the Dutch translation of Lacroix’s *Éléments d’algèbre* it will be shown what constituted the Dutch notion of rigour.

Recently attention has grown for the reception of mathematical theories. One of the surprising results so far has been that mathematics in the various European countries faced quite different epistemological approaches. According to Gert Schubring the very different institutional contexts where mathematical knowledge was pursued in France and Germany gave rise to different notions of rigour.

Some of the interesting things to look at in this respect are the textbooks by S.F. Lacroix, since they were very popular in France and kept in line with the ideas on rigour that were prevalent there. Also, they were translated into several European languages and during the process of translation the text was adapted to fit the national taste for rigour. In this paper the Dutch translation of Lacroix’s algebra textbook will be discussed, to see which ideas on negative numbers were prevailing in the Netherlands. The European background and 18th century Dutch work on negative numbers will serve as an introduction.

1 European background
During the 18th century mathematicians on the continent, the French and Germans first, began viewing algebra as something like a universal language. Whereas the British chose a more careful approach to algebra and, for example, adhered strongly

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to the geometrically inspired Newtonian theory of fluxions, Leibnitzian calculus can be viewed as an exponent of continental algebraic belief. The proof of the rule \( \frac{dy}{dx} = \frac{du}{dz} \frac{dz}{dx} \), for example, relied heavily on the notation\(^2\). Subtracting “the larger from the smaller” was not an obvious thing to do in geometry, so to the British negative quantities were suspect. Frend and Maseres downright denounced the use of algebra beyond universal arithmetic\(^3\). Usually, negative quantities were regarded as being “less than nothing”, metaphorically linked to debt, as opposed to possession. By the end of the 18th century this view of negative quantities was no longer satisfying.

The first half of the 19th century witnessed a more abstract attitude towards algebra. In fact, various algebras emerged, with Hamilton, Peacock, Gauß and Galois as the most noteworthy contributors\(^4\). These new theories, although intended to treat objects of some kind or other and not intended to be mere structures exhibited by arbitrary laws\(^5\), many regarded as pointless philosophical play\(^6\). Therefore, symbolical algebra could hardly serve as a foundation for negative quantities, as it does nowadays.

In France, Carnot’s ideas about negative quantities were to become very influential. Bumping into negative quantities while solving equations, he stated that negative solutions could only be the result of an unsolvable problem: they might have some interpretation, but reckoning with them was a dangerous business. He preferred the terms “directes” and “inverses” as it came to interpreting negative solutions. Thus he avoided d’Alembert’s paradox: that the larger could be to the smaller as the smaller to the larger in \( 1 : -1 = -1 : 1 \). Carnot’s ideas found their way, for example, into the textbooks by S.F. Lacroix. In his textbook on elementary algebra, Lacroix avoided the use of negative numbers. If solving a problem (an equation) resulted in a negative solution, Lacroix simply re-formulated the initial problem (this topic will be further elaborated on in section 4). This made solving equations a highly complicated and very annoying business.

In Germany, on the other hand, quantities were conceived as provided not only with a quantitative, but also with a qualitative (positive or negative) attribute. For every quantity \( a \) there was also a quantity \( \bar{a} \) such that \( a + \bar{a} = 0 \). A German translation of the algebra textbook by Lacroix was ruthlessly adapted to the German point of view\(^7\).

\(^5\) Ibidem, p. 434
\(^7\) G. Schubring, ‘Changing cultural and epistemological …’, pp. 368–369
2 Negative numbers in the Netherlands

Simon Stevin (1548-1620) introduced and developed mathematics courses at Leyden university as early as 1600. These were very practical courses, focussing on surveying and fortress building, and they shaped Dutch mathematics until well into the 19th century. Dutch mathematicians during the 18th century were mainly interested in physical and applied research. They were quite capable of reading foreign work, and many foreign books were translated. It must be mentioned that many of these translations were textbooks, put into the vernacular by engineers or schoolteachers. Most of them saw absolutely no foundational problems in algebra: mathematical theories were destined to be applied, and as long as the application worked well, there was no reason to look into the details. For example: A.B. Strabbe (1741-1805) translated the textbooks by A.C. Clairaut and the trigonometry by Th. Simpson.

Translating these books which were stylistically so distinct, indicates that Strabbe did not really care about what foundations mathematics was built upon.

Some surveyors' textbooks treated algebra, but only by means of recipes. For example, Pibo Steenstra (ca. 1730-1780) treated some algebra in his geometry textbook of 1763: first he taught the pupil to distinguish the algebraic magnitude from the coefficient: in $3ab$, for example, 3 was the coefficient and $ab$ was the algebraic magnitude. Then the pupil was taught how to add a number of comparable algebraic expressions of the same sign, by putting the sum of the coefficients in front of the common magnitude, and putting the sign (plus or minus) in front of it. Then he taught the pupil what to do if the signs in front of the coefficients were not all the same:

First find the sums of the positive and negative expressions, the way it was done before, and subtract the smallest coefficient from the largest. Put the sign of the largest of the two coefficients in front of the result, and the entire result in front of the magnitude.

Some examples, such as $3x - 4x - 2x + 5x = 2x$, helped to clarify the procedure. Of

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11A.B. Strabbe (transl.), *Gronden der Algebra*, Amsterdam (1760); A.B. Strabbe (transl.), *Beginzelen der Geometrie ... en met eene korte Trigonometrie, meerdereeds gevolgt na het werkje van den Heer Simpson, vermeerderd*, Amsterdam (1760)
course, there were ideas on education involved that made Steenstra write his text like this. Since algebra rarely was anything more than elementary reckoning — solving equations, mostly — ideas behind the computational rules for negative numbers remained hidden in the recipes and a few words on the interpretation of a negative solution. Higher level textbooks by E. de Joncourt (ca. 1675-ca. 1770), J.J. Blassière (1736-1795) and W.J. 's Gravesande (1688-1742) did not go beyond solving equations either, and more or less accepted the negative numbers as self-evident. For the Dutch, negative numbers were quite acceptable. Negative quantities were used without taking notice of British rejections.\footnote{Imaginary numbers were quite another story. No imaginary numbers whatsoever can be found in Dutch mathematical texts during the 18th century, except for one case. In his algebra textbook J.J. Blassière devoted a few pages to imaginary numbers. He noted that sometimes the solution of equations of degree two gave impossible solutions. These solutions, impossible as they might have seemed, could be used in reckoning. Blassière's problem was that they didn't seem to obey the law $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, since $-1 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1$. He solved this by stating that the imaginaries seemed to have another way of reckoning, and gave some examples. After these examples he quickly closed the section, because: "le nombre d'examples que nous venons de donner, étant plus que suffisant dans une matière qui est plutôt curieuse qu'utile, nous renvoyons ceux, qui ont envie de le voir traiter plus amplement, au Traité d'Algèbre de Maclaurin, Part II. dans les §. 73. &c.". See: J.J. Blassière, Institution du Calcul Numérique et littéral, Leiden (1770), p. 145}

By the end of the 18th century algebra was viewed as a very promising theory to provide a rigorous foundation of calculus, and as a highly interesting part of recreational mathematics. Apart from that, it was considered to be a very valuable piece of reckoning equipment, and the trust in its results was considered well-founded. It was known to a relatively small group of people only. Slowly, however, mathematics became an important subject as well at the Military Academy\footnote{J.A.M.M. Janssen, Op weg naar Breda, Den Haag (1989)} as in engineering studies\footnote{N.B. Goudswaard, Vijf-en-zestig jaren nijverheidsonderwijs, Assen (1981)}. Since the 1780s, institutions were founded where one could learn "serious mathematics" — the middle class parents who sent their children to such schools, no longer considered the old set of rules appropriate in math instruction\footnote{D.J. Beckers, 'Het is al Mathesis dat de klok slaat' in: De Negentiende Eeuw 22 nr. 4 (december 1998), pp. 220–234}. In 1815, elementary algebra became an obligatory subject at secondary school\footnote{H.J. Smid, Een onbekookte Nieuwigheid, Delft (1997)}. Thus, it is not that strange that interest in the foundations of algebra grew. In the next section, the major contributions will be discussed. Special attention will be devoted to the proof of the rule $(-a) \times (-b) = ab$, since this is the most obviously problematic theorem.

3 Dutch concern with the foundation of negative quantities

Jacobus P. Tholen (1764-1824) — in 1797 to become the successor of Nicolaas Ypey (1714-1785), professor of mathematics at Franeker university — was in 1784 awarded a doctorate. The first and best presented subject of his thesis was the question of Dutch concern with the foundation of negative quantities.
why multiplication of two negative numbers would have a positive result. Tholen started his dissertation by stating that a multiplication \( a \times b \) could best be viewed as finding the number \( x \) that satisfied \( x : a = b : 1 \). Interpreting the construction of \( x \) geometrically, and Euclidean geometry being considered the high light of rigour, this definition was as sound as one could wish.

Now Tholen began looking into the nature of negative numbers. In geometry the interpretation of a negative quantity was obvious: one simply imagined a line to have a point “zero” somewhere, and points at one side of this line were at the positive, the others at the negative side. In algebra, Tholen used an arithmetical series to define negative quantities. Looking at

\[
\ldots, a - 3b, a - 2b, a - b, a + b, a + 2b, a + 3b, \ldots
\]

he took \( a = b \) and noted that now \( a - b \) was equal to zero. Also, he stated, \( a - 2b \) was negative, namely \(-b\), and this indicated that the next term in the series would be zero. In the same way \(-nb\) indicated the zero term could be reached in \( n \) steps.

Having shown these examples, Tholen concluded that negative numbers were thus real objects that could be algebraically manipulated, and therefore he could apply his definition of multiplication. The geometrical interpretation of multiplication, probably perceived in the way of Descartes’ *Géométrie*, now indeed left no doubt about the product of two negative numbers being positive (see figure below)\(^{18}\).

\[+\]

\[+ab\]

\[-\]

\[-b\]  \[1\]  \[+\]

\[-a\]

\[-\]

In 1798 an anonymous paper was published in the middle class magazine *Nieuwe Vaderlandsche Bibliotheek* (New Dutch Library) in which the author also dealt with the problem of negative numbers. He choose to tackle the problem by defining “subtraction” in a new way:

\(^{18}\)J.P. Tholen, *Theses Philosophicae*, Franeker (1784), pp. 3–7
Subtracting we call investigating which number we should add to one of the numbers —what we should add to the quantity we must subtract, to make it equal to the other.\textsuperscript{19}

The number was conceived as a line segment in a certain direction: one side being positive, the other being negative. Now the author looked at the expression $m - s$ in three cases: $m$ and $s$ both positive, both negative, or one positive and one negative. In the first case he again distinguished three cases: $m > s, m = s$ or $m < s$. The addition being conceived as a sort of vector-addition (without formal definition, of course), these three subcases yielded a positive, zero or negative result respectively. In the second case there was not really a problem either: the definition of subtraction made clear what had to happen. The author felt he had to explain the last case, though. The last case was subdivided in two subcases: if $m$ was negative and $s$ positive that would result in an expression that clearly resembled the first case and could be treated likewise. If, however, $m$ was positive and $s$ was negative, $m - s$ represented a number, which, if added to the negative number $s$, would result in the positive number $m$: that, of course, was (in modern terms) the positive number $m + |s|\textsuperscript{20}$.

Although it is a strange approach, one thing is clear: from the end of the 18th century Dutch mathematicians began to feel uncomfortable with the notion of negative quantity, and they looked for a reliable way of treating these obscure numbers. In one of his courses, the Amsterdam professor of mathematics J.H. van Swinden (1746-1823) said:

> It is very inaccurate to say that a negative number is less than nothing, which is what many authors do. A negative number is a positive number, but in another sense, and therefore relative.\textsuperscript{21}

It seems that Van Swinden opted for the German notion of negative numbers, in which quantities had both a quantitative and a qualitative attribute.

In 1815, Jacob de Gelder (1765-1848) published a treatise on negative quantities. De Gelder was an engineer, who had made his career during the French occupation, and in 1815 taught mathematics at the Military Academy\textsuperscript{22}. His paper was published, accompanied by a letter of approval from the Dutch Royal Institute of Arts and Sciences, the Dutch equivalent of the Académie des Sciences. This paper was particularly appreciated by several Dutch mathematicians\textsuperscript{23}. Since it received so much praise and attention, it will be discussed in some more detail.

\textsuperscript{19}Literally: “Subtraheren heet onderzoeken, wat men bij ééne van twee grootheden —wat men bij de grootheid die men aftrekken moet, behoort te voegen, (addeeren) om ze aan de andere gelijk te maken.”

\textsuperscript{20}Iets over het denkbeeld van subtractie of aftrekking, met opzicht tot de algebra’ in: Nieuwe Vaderlandsche Bibliotheek II.2 (1798), pp. 466–470

\textsuperscript{21}lecture notes by A.J. Deiman made during a course by J.H. van Swinden, ca. 1800. University of Utrecht, archival collection, inv.nr. VIII G 12, p. 270; literally: “Zeer onnaauwkeurig is de uitdrukking van veelle schrijvers dat een negatief Getal minder is dan 0. Een negatief getal is een Positiev in een andere zin & dus relatief.”

\textsuperscript{22}D.J. Beckers, ‘Mathematics as a way of life —a Biography of the mathematician Jacob de Gelder’ in: Nieuw Archief voor Wiskunde (IV) 14 nr. 2 (juli 1996), pp. 275–297

\textsuperscript{23}A praising review, in which he expressed the hope that a French translation would appear soon,
De Gelder explicitly announced that he wanted to solve d’Alembert’s paradox. He stated that most authors had only complicated matters by giving terrible philosophical explanations for negative quantities. He wanted to introduce them clearly and simply, for he was of the opinion that they constituted an important part of algebra\textsuperscript{24}. He made the paradox by d’Alembert a bit more absurd—he thought—by rewriting it as $8 : -6 = -4 : 3$.

De Gelder started with simple counting. Once one could count, numbers could easily be represented on a line. On the line there was a relation $<$, defined as: $a < b \iff a$ is to the left of $b$. Adding an item to a stack was linked to going one step to the right on a line. The addition operation was defined in terms of “going to the right”. Analogously, going one step to the left was identified with taking one item from the stack, and subtraction was defined in terms of “going to the left”. Thus $a - b$ had been defined for all $b < a$. Multiplication was perceived as a short operation for adding the same number repeatedly. Division was perceived analogously. The geometrical representation of a number on a line was practical here, since now both multiplication and division could be identified with their respective geometrical constructions. Defining $a^b$, De Gelder remarked that the meaning of all these operations could easily be understood for all rational numbers, and for $a$ or $b$ equal to zero as well:

No philosopher we know of, ever opposed to the meaning of the words and symbols discussed, nor to the very peculiar expression $a^0 = 1$. This is very natural, since all these expressions, no matter how peculiar, are captured within the general expressions $a \times b$ and $a^b$, and they follow from them as naturally as do the notions of positive and negative from the expression $a - b$\textsuperscript{25}.

This is an important remark: although De Gelder is about to extend the notion of quantity with negative quantities, he says that he is not really making an extension of the number system. All these numbers, like the rational numbers, were already present for him. They simply had to be understood in the correct sense.

Now the negative numbers were introduced, for according to De Gelder there was no reason to end the line at 0: one could easily extend (by the Euclidean postulate) and go on counting $-1, -2, -3, \text{etc}$. These numbers were called “negative” to indicate that they were to the left of zero. In fact, the terms positive and negative were more

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\textsuperscript{24}J. de Gelder, \textit{Proeve over den waren aard van den positieven en negatieven toestand der grootheden etc.} Amsterdam (1815), p. V.

\textsuperscript{25}ibidem, pp. 42-43; literally: “Geen wijsgeer, zoo veel ons bekend is, heeft zich tegen deze beteekenis van woorden en teekens, bij onderlinge overeenkomst, vastgesteld, immer verzet, noch zich aan de zeer oneigenlijke uitdrukking $a^0 = 1$ gestoet; en dit is ook zeer natuurlijk; omdat alle deze eigenlijke en oneigenlijke wijze van zeggen, in de algemene formule $a \times b$ en $a^b$, in dezelfder algemeenste uitgestrektheid genomen, liggen opgesloten en uit dezelve even zoo natuurlijk voortvloeijen, als, uit de algemeenheid der formule $a - b$, het begrip van positief en negatief”. 

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general: if the line was vertical instead of horizontal one could indicate a point 0 and a positive side too. Negative numbers were simply “on the other side” of some point zero. The relation < was still meaningful, addition and subtraction remained possible, and a subtraction like \( a - b \) was no longer restricted to the cases where \( b < a \). Above all, there were numerous practical cases for which these negative numbers could be valuable: debt as negative possession, geometrical position on a line with a point defined as 0. Now De Gelder formulated six axioms regarding the relation < on the positive numbers:

- **axiom 1**: \( a > b \Rightarrow a + c > b + c \)
- **axiom 2**: \( b > c \Rightarrow a - b < a - c \)
- **axiom 3**: \( a > b \) and \( c > d \) \( \Rightarrow \) \( a + c > b + d \)
- **axiom 4**: \( a > b \Rightarrow ac > bc \)
- **axiom 5**: \( a > b \) and \( c > d \) \( \Rightarrow \) \( ac > bd \)
- **axiom 6**: \( a > b \Rightarrow a/c > b/c \)

These axioms were regarded as generally accepted truths: they were illustrated with examples, and De Gelder referred to the definition of > in order to make these axioms plausible\(^{26}\). The first three, De Gelder stated, were obviously also true for the negative numbers, which was illustrated with a few examples. The other three changed if negative numbers were allowed: the inequality might turn around. This, he argued, was logical too, since multiplication with a negative number, for example, would in the sense of his notion of multiplication, turn the whole line over 180 degrees. In fact, the multiplication with a negative number could be viewed as multiplication with the absolute value and changing the sign of the multiplicand.

De Gelder’s proof of the rule that the product of two negative numbers would yield a positive result, came in the form \((-a) \times (-a) = a^2\). To prove this, he studied a square \( ABCD \). The point \( A \) he regarded as 0 on the lines \( AD \) and \( AB \). Now he looked at the square generated by the line segment \( AE \), were \( E \) was between \( A \) and \( B \), moving towards \( A \). This was the square \( AEFG \). With respect to \( ABCD \), it holds that:

\[
AEFG = ABCD - CNFM - DNFG - BMFE
\]

Once \( E \) had actually reached \( A \), the square had degenerated to a point, but \( E \) could keep moving towards \( E' \). The respective terms in the above equation were “directed surfaces”, but in order to determine the sign of the left side of the equation, De Gelder now determined the signs of the terms on the right, and started rewriting the right side of the equation in terms of absolute values. The square \( CNFM \) was positive to begin with, and would during its transformation to \( CN'FM' \) only become larger, so it would certainly remain positive. Both \( DNFG \) and \( BMFE \), however, were zero if \( E \) was in \( A \). After that, one of two generating lines of these squares was negative, and so (by axiom 4 for the negative numbers) the signs of \( DNFG \) and \( BMFE \) would

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\(^{26}\)ibidem, pp. 53–89; Note, however, that they are a very good set of axioms in the modern sense too: axioms 1 and 3 imply the transitivity of the relation >. So do the axioms 4 and 5. De Gelder could have “proved” all these axioms as a result from his definition. By choosing to state them as axioms he makes it particular clear that he thinks foundations of mathematics ought to be explicit—even if they are “generally accepted truths”.  

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change if $G$ or $E$ passed the “zero point” $A$. This meant:

$$AE'F'G' = ABCD - CN'F'M' + DN'F'G' + BM'F'E'$$

The last two terms in this equation were larger than $CN'F'M' - ABCD$, which guaranteed that $AE'F'G'$ would be positive\(^{27}\).

De Gelder solved d’Alembert’s paradox by stating that proportions — being linked to geometrical objects — were only relevant in the sense of absolute values (hoegrotheid). This reduced the meaning of $8 : -6 = -4 : 3$ to $8 \times 3 = 6 \times 4$ (of course the signs had to match somehow). Having dealt with all paradoxes, and claiming to have a firm grasp on negative quantities, De Gelder closed his paper by rejecting Carnot’s quantités directes and quantités inverses. These terms, in his view, simply were equivalent to the ones they were supposed to replace\(^{28}\).

4 Lacroix’s textbook

Lacroix’s textbook was translated by I.R. Schmidt (1782-1826). It was used at the Military Academy. Printed for the first time in 1821, it was used at the Academy until it was replaced in 1838 by the textbook by J. Badon Ghyben (1798-1870) and H. Strootman (1799-1851)\(^{29}\). The subjects and the exercises were all translated rather

\(^{27}\)ibidem, pp. 117-120

\(^{28}\)ibidem, p. 242

\(^{29}\)J. Badon Ghyben & H. Strootman, *Beginselen der Stelkunst*, Breda (1838)
accurately. The theory of negative quantities as it was introduced and applied by Lacroix, however, had been largely replaced by Schmidt’s own words.

Lacroix30, “proved” \((-a) \times (-b) = ab\) in a way resembling a much older proof by Laplace31, namely: by looking at the formula \(-a \times (b - b)\). According to the distributive law, this should be equal to \((-a) \times b + (-a) \times (-b)\). The first term was \(-ab\) by the definition of the multiplication. Since the sum had to be equal to zero, the second term could only be equal to \(+ab\), to “compensate”, so to say, for the first term32. For Lacroix, however, this was not a way of reckoning with quantities but, what he called, a “changement de forme”: changing the form of an equation. Like Carnot, Lacroix bumped into negative numbers while solving equations. It was obvious that multiplying with a negative number could be useful if one wanted to solve an equation. But that was really all that Lacroix wanted it to be: he simply wanted a “rule” for transforming an equation into an equivalent one. Multiplying with a negative quantity was not an operation that could actually be performed. A negative solution meant: go back to the original problem, and see if it is possible to restate it, in order to prevent the appearance of the negative number. If this was not possible, the negative solution simply indicated that there were no solutions satisfying the equation.

In contrast Schmidt’s proofs were about negative quantities. He inserted a whole page to explain the nature of these negative quantities. Disregarding Lacroix entirely, he saw a negative quantity —like De Gelder— as a natural thing. Linking to Lacroix’s own proof he called \(-a\) the quantity one had to add to \(a\) to get zero as a result: possession and debt, east and west, left and right, were physical representatives of negative quantities. When an equation turned out to have a negative solution it was quite obvious what to do:

If one looks closely at the definition of negative quantity as it has been given in the last section, it is not necessary to go back to the original equation to look what changes would have to be made, to make these negative solutions disappear.33

Then Schmidt returned to the original text. Lacroix posed a problem: two people start running. One starts from \(A\) and runs \(b\) kilometres per hour. The other starts from \(B\) and runs at a speed of \(c\) km/h. The distance between \(A\) and \(B\) is \(a\), and both runners start running towards each other. The question is at which point they

30Page reference to Lacroix is always from the 11th edition: S.F. Lacroix, Éléments d’algèbre, Paris: Courcier (1815). For the Dutch translation I used the second edition: I.R. Schmidt, Beginselen der Stelkunst, ’s Gravenhage & Amsterdam (1825). The textbook by Lacroix has been revised several times, but on the points that are about to be discussed, later editions resembled the 11th edition. The textbook by Schmidt was never revised. Schmidt died within a year after publication of this edition.
31P.G.J. Vredenduin, De geschiedenis van positief en negatief, p. 102
32S.F. Lacroix, Algèbre p. 93
33I.R. Schmidt, Algebra, p. 94; literally: “Wanneer men de bepaling der negatieve grootheden, in de voorgaande § opgegeven, met aandacht nagaat, is het niet noodzakelijk, bij het verkrijgen van negatieve uitkomsten, tot de oorspronkelijke vergelijking terug te keeren, ten einde de wijzigingen op te speuren, welke men de opgave zou moeten doen ondergaan, om er deze uitkomsten op toepasselijk te maken.” — my emphasis [DB].
will meet.34 The solutions are strikingly different: where Lacroix gives a general solution, avoiding negative quantities and looking in which cases there is no (a negative) solution, Schmidt immediately starts indicating a positive and negative direction.

Lacroix reformulates the problem in the case that both runners are running in the same direction, and in the case that they are running away from each other. In both cases he interprets negative solutions as indicators for the absurdity of the problem posed. Schmidt, on the other hand, uses the solution to the first case, and simply changes the values of \(b\) and \(c\). He interprets the resulting solutions in his framework of negative numbers. In the case that the runners start off in the same direction, and the runner starting “in front” is running faster, Schmidt interprets the resulting negative solution as a virtual common starting point.

After discussing the several cases Lacroix concluded:

> Ce qui précède fait voir bien clairement que les solutions algébriques, ou satisfont complètement à l’énoncé du problème, quand il est possible, ou indiquent une modification à faire dans l’énoncé, lorsque les données présentent des contradictions qui peuvent être levées, ou enfin font connaître une impossibilité absolue, lorsque’il n’y a aucun moyen de répondre avec les mêmes données, une question analogue dans un certain sens à la proposée.35

The corresponding sentences in Schmidt’s “translation” are:

> Prom all this we may conclude that the solutions we obtain always are exactly what was asked for, if we only observe closely, what was chosen to be positive when we solved the problem. For if we know this, then the sign [+ or −] of the solution will let us know all the circumstances that may take place.36

Lacroix now turned back to the problem of the two runners, to illustrate what his remarks would mean in practice. Schmidt skipped these pages, and picked up the text where Lacroix ended his exposé.

Schmidt’s “translation” leaves no doubt about his complete rejection of Lacroix’s and Carnot’s views on negative quantities. Staying quite close to the original text, Schmidt changed crucial remarks. He opted for an approach that stood entirely in the tradition of Dutch work on negative quantities as discussed in the preceding section.

5 Concluding remarks

Nowadays a mathematician is trained to build on his definitions and axioms to prove elementary theorems. For the early 19th century Dutch mathematician, this was not

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34S.F. Lacroix, *Algèbre*, p. 94; I.R. Schmidt, *Stelkunst*, pp. 94–95; the Dutch translation mentions two “physical objects” (ligchamen). These objects are running (lopen), however!
35S.F. Lacroix, *Algèbre*, p. 105
36I.R. Schmidt, *Stelkunst*, p. 107; literally: “Uit dit alles blijkt dan, dat de uitkomsten der stelkunstige oplossingen altijd volmaakt aan de opgaaf beantwoorden, indien men slechts met opletendheid nagaat, wat men in de oorspronkelijke oplossing als positief heeft aangemerkt, daar men alsdan, door de teekens, welke de uitkomsten verkrijgen, in alle gevallen, op eene ongedwongene wijze, al de onmogelijkheden zal kunnen verklaren, die er bij de bijzondere gevallen, waarin het vraagstuk verkeeren kan, plaats kunnen grjipen.”
obvious. The definition of a concept, in this case the negative number, did not have
to return in the proof once it had been properly dealt with. The negative numbers
were more or less considered as real objects, and did deserve attention, but only
regarding their meaning: the definition was not used in the proofs. Once a quality (a
mathematical “law”) had been proved, it was generally considered to be true, even
if the set of objects that the law had been devised for was enlarged. If it could be
shown that these objects “naturally” belonged to the same kind of objects as those
for which the law had already been checked, the law would automatically also hold
for them. This type of reasoning —attributing extraordinary power to (or perhaps
faith in) mathematical proof— was noticed in all the papers discussed above, and was
very common in all early nineteenth century Dutch algebra texts.

In fact, “enlarging the class” of objects, as we would call it nowadays, was not
something our early nineteenth century Dutch colleagues did. Once
\[ a + b = b + a \]
had been proved for the natural numbers, it was also true for fractions. The same type
of reasoning may be observed at points where the meaning of \( a^b \) was extended from
integer \( b \) to rational \( b \). The law \( (a^b)^c = a^{(b \cdot c)} \) was beyond doubt: it had been proved
for integers only, but to these mathematicians it was evident that the expression was
also meaningful for rational numbers. The “meaning” of \( a^b \) for rational \( b \) was simply
deduced from this equation. Fractions, after all, were present in the number system
from the instant it was created. Fractions more or less belonged to the same class
of objects as integers —mathematicians probably had nature in mind. Introducing
fractions or negative numbers did not mean that the class of objects was extended:
mathematicians simply got a better grip on the wide variety of objects in the number
system. The “new numbers” had to be there, and necessarily obeyed the laws that
had been proved for the integers.

These notions of proof and definition were shared by a majority of Dutch mathe­
maticians and by several mathematicians abroad. It is tempting to attribute this to
the working climate of these mathematicians. In the Netherlands (as in most other
countries) no mathematical research was carried out. Mathematical texts were largely
produced for educational purposes. Many of the educators saw mathematics merely
as a tool (although an indispensible one) in physical or technical research. However,
there existed in the Netherlands an influential group of people who believed mathe­
matics was valuable on its own accord. They did think foundations of mathematics
were important, and wrote (or translated) textbooks that paid attention to these
foundations. The textbooks by Lacroix are an interesting example. Nowadays these
books may shed light upon the notions of rigour that existed in the early 19th century.