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Report No. 9917 (April 1999)

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Key words: Dynamical system, attractors

AMS (MOS) subject classification: 35B40, 58F12, 58F39, 34D45, 34C35

Abstract

An approach to verify continuity of a global attractor of a semidynamical system with a parameter is presented. This approach makes it possible to establish a connection between upper and lower semicontinuity of a global attractor and boundedness as a function of rate of attraction to a attractor. The obtained results were used for the scalar Lorenz wave equation in 3D space, the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

1 Introduction

In the present paper for a semigroup $\{S_\lambda(t,\cdot)\}$ corresponding, for example, to a evolution equation, where λ is a problem parameter, and having global attractor the problem of stability of $\mathcal{M}_\lambda$ as $\lambda \to \lambda_0$ is considered. The most powerful test for the attractor $\mathcal{M}_\lambda$ being in $O_\varepsilon(\mathcal{M}_{\lambda_0})$, with $O_\varepsilon(\mathcal{M}_{\lambda_0})$ the ε-neighborhood of $\mathcal{M}_{\lambda_0}$, was proved by Kapitanskii and Kostin.

The main purpose of the current work is to consider a criterion when in addition $\mathcal{M}_{\lambda_0} \subset O_\varepsilon(\mathcal{M}_\lambda)$ and $\varepsilon \to 0$ as $\lambda \to \lambda_0$. The basis of this theorem are the characteristics of the function of rate of attraction $\Psi(\lambda,t)$ in some small δ-neighborhood of λ_0

$$dist(S_\lambda(t,B_a),\mathcal{M}_\lambda) \leq \Psi(\lambda,t) dist(B_a,\mathcal{M}_\lambda), \quad t \geq 0,\ \lambda \in O_\delta(\lambda_0).$$

Here B_a is a bounded absorbing set and $S_\lambda(t,\cdot)$ is an approximation of the given nonlinear operator $S_{\lambda_0}(t,\cdot)$.

These results are applied in Section 4 to the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

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2 Asymptotically compact semigroups

Let X be a Banach space with norm $\|\cdot\|$, Θ be a nontrivial subgroup of real number R and let $\Theta_+ = \Theta \cap [0, +\infty[$ be the intersection of Θ and R_+. We shall deal with the abstract semigroup $\{X, \Theta_+, S(\cdot)\}$ of nonlinear operator $S : X \times \Theta_+ \to X$. The term semigroup or semidynamical system refers to any family of singlevalued continuous operator S depending on a parameter $t \in \Theta_+$ and enjoying the semigroup property:

$$S(t_1, S(t_2, u)) = S(t_1 + t_2, u), \quad \forall t_1, t_2 \in \Theta_+, \forall u \in X$$

A Banach space X is a phase space of a semigroup, Θ_+ is a time space and $S(\cdot)$ is an evolution operator. When $\Theta = R$ a semigroup is a semigroup with continuous time.

Let B and M be bounded subsets of X. We say that B is attracted to M by the semigroup $S(\cdot)$ if

$$dist(S(t, B), M) \to 0 \quad as\ t \to \infty$$

Here

$$dist(A, B) = \sup_{y \in A} \{dist(y, B)\}, \quad dist(y, B) = \inf_{x \in B} \|x - y\|$$

A set M is called an attracting set of the semigroup if M attracts each bounded $B \subset X$. The minimal among the closed attracting sets is called the global attractor [9] (minimal global B-attractor [3]). The global attractor of a semigroup is defined as the set $\mathcal{M}$ which is compact in X, invariant for $S(\cdot)$, i.e.

$$S(t, \mathcal{M}) = \mathcal{M}, \quad t \geq 0$$

and which attracts all the bounded sets of X.

We need the following lemmas (see[5, 6]):

Lemma 1 *Let $\mathcal{M}$ be a compact attractor of the semigroup $\{X, \Theta_+, S(\cdot)\}$, $\tilde{\mathcal{M}}$ be a compact attractor of the semigroup $\left\{X, \tilde{\Theta}_+, \tilde{S}(\cdot)\right\}$. Let $\tilde{\Theta}_+$ be a subsemigroup of Θ_+ and let for some points t, $\tilde{t}$*

$$S(t, u) = \tilde{S}(\tilde{t}, u) \quad \text{for all } u \in X$$

Then $\mathcal{M} = \tilde{\mathcal{M}}$.

Lemma 1 implies

Lemma 2 *Let $\mathcal{M}$ be a compact attractor of the semigroup $\{X, R_+, S(\cdot)\}$ and let $t_0 > 0$. Then $\mathcal{M}$ is attractor of the semigroup $\{X, t_0 Z_+, S(\cdot)\}$. Here $t_0 Z_+ \equiv \{kt_0, k \in Z_+\}$.*

Later on we need the following definitions, see [3].

A set B_a is called absorbing if for each bounded $B \subset X$ and for each $\varepsilon > 0$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset B_a, \quad \forall t \geq T$$

If a semigroup possesses a nonempty bounded attractor $\mathcal{M}$ then for arbitrary $\varepsilon > 0$ the set $O_\varepsilon(\mathcal{M})$ is an absorbing set. Here $O_\varepsilon(\mathcal{M})$ is the ε-neighbourhood of $\mathcal{M}$, i.e.

$$O_\varepsilon(\mathcal{M}) = \{u : \exists v \in M, \|u - v\| < \varepsilon\}$$

A semigroup is called bounded if for each bounded B the set $S(t, B)$ is bounded for any $t > 0$.

A semigroup is called pointwise dissipative if it has a pointwise absorbing set B_0

$$\forall x \in X,\ \exists T(x): \quad S(t, x) \subset B_a, \ \text{ for any } t \geq T(x)$$

A semigroup is called asymptotically compact if for each bounded B such that $S(t, B)$ is bounded for any $t > 0$ each sequence of the form

$$\{S(t_k, u_k)\}_{k=1}^{\infty}, t_k \uparrow \infty, u_k \in B$$

is precompact.

The following theorem holds, see [3].

Theorem 1 *Let the semigroup $\{X, \Theta_+, S(\cdot)\}$ be a continuous bounded pointwise dissipative asymptotically compact semigroup. Then there exists a non-empty attractor $\mathcal{M}$*

$$\mathcal{M} = \bigcap_{t \geq 0} [S(t, B_a)]_X$$

$\mathcal{M}$ is compact and invariant. If X is connected then $\mathcal{M}$ is also connected.

We now summarize the results:

Lemma 3 *Let $S(t): X \to X, t \in R_+$ be a continuous semigroup possessing a non-empty compact attractor $\mathcal{M}$. Then this semigroup is an asymptotically compact semigroup.*

Proof. We shall prove that each sequence of the form

$$\{S(t_k, u_k)\}_{k=1}^{\infty}, t_k \uparrow \infty, u_k \in B$$

can be covered by a finite ε-network where ε is arbitrary small positive number. The set $O_{\varepsilon/2}(\mathcal{M})$ is an absorbing set and for any bounded B there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset O_{\varepsilon/2}(\mathcal{M}) \quad \text{for any } t \geq T.$$

Let us choose N so large that the set $\{S(t_k, u_k)\}_{k=N}^{\infty}$, $u_k \in B$ belongs to $O_{\varepsilon/2}(\mathcal{M})$. The set

$$V^{\mathcal{M}} = \left\{v_k^{\mathcal{M}} \in \mathcal{M} : \|v_k^{\mathcal{M}} - u_k\| \leq \varepsilon/2, k \geq N\right\} \subset \mathcal{M}$$

is a precompact set and may be covered by a finite $\varepsilon/2$-network $\bar{V}^{\mathcal{M}} = \left\{\bar{v}_i \in V^{\mathcal{M}}\right\}_{i=1}^{n}$ and for any k there exists i such that:

$$\|v_k^{\mathcal{M}} - \bar{v}_i\| \leq \varepsilon/2, \quad k \geq N$$

This implies

$$\|u_k - \bar{v}_i\| \le \|u_k - v_k^{\mathcal{M}}\| + \|v_k^{\mathcal{M}} - \bar{v}_i\| \le \varepsilon.$$

Thus the set $\bar{V}^{\mathcal{M}}$ is a finite ε-network of the set $\{S(t_k, u_k)\}_{k=N}^{\infty}$ and then the set $\{S(t_k, u_k)\}_{k=1}^{\infty}$ is a compact set.

Note, if a semigroup possesses a nonempty attractor $\mathcal{M}$ then the semigroup is pointwise dissipative.

3 Semigroups with a parameter

The current paragraph deals with the problem of stability of $\mathcal{M}$ with respect to perturbations of the original operator $S(\cdot)$.

Let us consider a semigroup $S_\lambda(\cdot) : X \to X$ which depends on a parameter $\lambda \in \Lambda$. The main purpose is to consider a criterion when the set $\mathcal{M}_{\lambda_0}$ and $\mathcal{M}_\lambda$ are close to each other in the Hausdorff metric, that is

$$\lim_{\lambda \to \lambda_0} dist(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) = 0 \tag{1}$$

$$\lim_{\lambda \to \lambda_0} dist(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = 0 \tag{2}$$

The relations (1) and (2) are usually referred to as the *upper* and *lower* semicontinuity of the attractor $\mathcal{M}_\lambda$ in λ_0.

The most powerful test for the attractor $\mathcal{M}_\lambda$ being in $O_\varepsilon(\mathcal{M}_{\lambda_0})$ with $O_\varepsilon(\mathcal{M}_{\lambda_0})$ the ε-neighborhood of $\mathcal{M}_{\lambda_0}$, was proved in [5].

We consider when in addition $\mathcal{M}_{\lambda_0} \subset O_\varepsilon(\mathcal{M}_\lambda)$ and $\varepsilon \to 0$ as $\lambda \to \lambda_0$. The basis of this theorem [2] are the characteristics of the function of rate of attraction in a some small δ-neighborhood of λ_0.

We assume that the following conditions (α) hold:

1. Λ is compact with metric $\|\cdot\|_\Lambda$ and λ_0 is a nonisolated point of Λ.

2. For each $\lambda \in \Lambda$ the semigroup $\{X, \Theta_+, S_\lambda(\cdot)\}$ possesses a pointwise absorbing set B_λ and non-empty attractor $\mathcal{M}_\lambda$.

3. There exists a bounded absorbing set B_a and for each $\lambda \in \Lambda$ a set B_λ belongs to the set B_a.

By definition, each ε-neighbourhood $O_\varepsilon(\mathcal{M}_\lambda)$ is an absorbing set. Assume that we know a function $\Theta(\lambda, \varepsilon) = \Theta(\lambda, \varepsilon, B_a)$ such that

$$dist(S_\lambda(t, B_a), \mathrm{M})) \le \varepsilon, \qquad \text{as } t \ge \Theta(\varepsilon, \lambda)$$

Lemma 4 *Under the assumptions (α) let there exist $\varepsilon > 0$, $\lambda_1, \lambda_2 \in \Lambda$ and a point $T \ge \Theta(\lambda_1, \varepsilon)$ such that for the operators $S_{\lambda_1}(t)$ and $S_{\lambda_2}(t)$ the following estimate is valid*

$$\|S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)\| < \varepsilon \quad for\ any\ u \in B_a \tag{3}$$

Then

$$\mathcal{M}_{\lambda_2} \subset O_{2\varepsilon}(\mathcal{M}_{\lambda_1})$$

Remark. Without loss of generality, assume that $O_{2\varepsilon}(\mathcal{M}_{\lambda_1}) \subset B_a$.

Proof. Suppose that estimate (3) holds for a $T \geq \Theta(\lambda_1, \varepsilon)$ with some $\varepsilon > 0$. The set $\mathcal{M}_{\lambda_1}$ is the attractor for $\{X, \Theta_+, S_{\lambda_1}(\cdot)\}$ and for the B_a we can find an attraction time $T = \Theta(\lambda_1, \varepsilon)$ to the ε-neighbourhood of $\mathcal{M}_{\lambda_1}$

$$S_{\lambda_1}(T, u) \subset O_\varepsilon(\mathcal{M}_{\lambda_1}), \forall u \in B_a$$

Furthermore, we obtain $S_{\lambda_1}(T + t, u) \subset O_\varepsilon(\mathcal{M}_{\lambda_1})$ for any $t \geq 0$. Due to (3) we have $\|S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)\| < \varepsilon$. Hence

$$S_{\lambda_2}(T, u) \in O_{2\varepsilon}(\mathcal{M}_{\lambda_1})$$

The above injection and $S_{\lambda_2}(T, u) \in B_a$ gives us

$$\|S_{\lambda_1}(T, S_{\lambda_2}(T, u)) - S_{\lambda_2}(T, S_{\lambda_2}(T, u))\| < \varepsilon \quad \text{for any } u \in B_a$$

Since $S_{\lambda_1}(T, v) \in O_\varepsilon(\mathcal{M}_{\lambda_1})$ for each $v \in B_a$ we have $S_{\lambda_2}(T, S_{\lambda_2}(T, u)) = S_{\lambda_2}(2T, u) \in O_{2\varepsilon}(\mathcal{M}_{\lambda_1})$. After a finite number of steps we obtain $S_{\lambda_2}(nT, u) \in O_{2\varepsilon}(\mathcal{M}_{\lambda_1})$ as $n = 1, 2, ...$ for any $u \in B_a$. This, together with Lemma 1, implies

$$\mathcal{M}_{\lambda_2} \subset O_{2\varepsilon}(\mathcal{M}_{\lambda_1}).$$

This completes the proof.

The next theorem provides an estimate for the distance between two attractors.

Theorem 2 *Under the assumptions* (α)

(i) Assume, that for any $\varepsilon > 0$ *there exists* $\delta > 0$ *and a point* $T_{\lambda_0} \geq \Theta(\lambda_0, \varepsilon)$ *such that*

$$\|S_\lambda(T_{\lambda_0}, u) - S_{\lambda_0}(T_{\lambda_0}, u)\| < \varepsilon \quad \forall u \in B_a, \ \forall \lambda \in O_\delta(\lambda_0) \tag{4}$$

Then the attractor $\mathcal{M}_\lambda$ *is upper semicontinuous in the point* λ_0 *and the following estimate holds*

$$dist(M_\lambda, M_{\lambda_0}) \leq 2\varepsilon. \tag{5}$$

(ii) Assume, that for any $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

for arbitrary $\lambda \in O_\delta(\lambda_0)$ *there exists a point* $T_\lambda = T(\lambda) \geq \Theta(\lambda, \varepsilon)$ *satisfies the following estimate*

$$\|S_\lambda(T_\lambda, u) - S_{\lambda_0}(T_\lambda, u)\| < \varepsilon \quad \forall u \in B_a \tag{6}$$

Then the attractor $\mathcal{M}_\lambda$ *is lower and upper semicontinuous in the point* λ_0 *and the following estimate holds*

$$\max\{dist(M_{\lambda_0}, M_\lambda), dist(M_\lambda, M_{\lambda_0})\} \leq 2\varepsilon. \tag{7}$$

Proof. According to (4) and assumption (i) we have that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for a $\lambda \in O_\delta(\lambda_0)$ we have $\mathcal{M}_\lambda \subset O_{2\varepsilon}(\mathcal{M}_{\lambda_0})$. Thus,

$$dist(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) = \sup_{u \in \mathcal{M}_\lambda} (u, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon.$$

As $\varepsilon \to 0$ we obtain $dist(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \to 0$. Thus, inequality (5) is proved.

In order to prove (7), from (5) and (ii) we have

$$dist(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon$$

To obtain the reverse inequality

$$dist(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) \leq 2\varepsilon \to 0 \quad as\ \lambda \to \lambda_0$$

use the assumption (ii) for arbitrary $\varepsilon > 0$ and give $O_\delta(\lambda_0)$ such that for any $\lambda \in O_\delta(\lambda_0)$ inequality (6) holds in a point $T_\lambda \geq \Theta(\lambda, \varepsilon)$. This, together with Lemma 4 implies

$$\mathcal{M}_{\lambda_0} \subset O_{2\varepsilon}(\mathcal{M}_\lambda) \quad \forall \lambda \in O_\delta(\lambda_0).$$

As $\varepsilon \to 0$ we have

$$dist(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = \sup_{u \in \mathcal{M}_{\lambda_0}} (u, \mathcal{M}_\lambda) \to 0.$$

This completes the proof.

Remark 1. For a semigroup with compact attractor *upper semicontinuity* was proved in [5] when (5) is valid in an *some* point $T_{\lambda_0} > 0$. But, we can not find the rate of convergence for $\mathcal{M}_\lambda \to \mathcal{M}_{\lambda_0}$. When inequality (4) holds for $t \in [\tau, \tau + T_{\lambda_0}]$ for any $\tau > 0$, uniformly in τ, then (5) was proved in [10].

Remark 2. The assumption (ii) allows $T_\lambda \to \infty$ as $\lambda \to \lambda_0$.

Remark 3. A function rate of attraction $\Theta(\cdot)$ can be approximate, for example see [11], for some nontrivial PDE system with nontrivial attractor.

A sequence of operators $S_\lambda(\cdot)$ is called *locally converging* in a point λ_0 on a bounded B if for any $\varepsilon > 0$ and for each $T > 0$ there exists δ such that

$$||S_\lambda(T, u) - S_{\lambda_0}(T, u)|| \leq \varepsilon \quad \forall \lambda \in O_\delta\left(\lambda_0\right), \, \forall u \in B.$$

Theorem 2 and the above definition imply

Theorem 3 *Under the assumptions* (α) *let the sequence of operators* $S_\lambda(\cdot)$ *be locally converging in a point* λ_0 *on a set* B_a. *Let the function* $\Theta(\lambda, \varepsilon)$ *be uniformly bounded on* λ *for each fixed* $\varepsilon > 0$, *i.e.*

$$\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0 : \sup_{\lambda \in O_\delta(\lambda_0)} \Theta(\lambda, \varepsilon, B_a) \leq T_\varepsilon < \infty.$$

Then, the attractor $\mathcal{M}_\lambda$ *depends on* λ *in the point* λ_0 *continuously.*

Remark. In this way, if we cannot calculate a function $\Theta(\lambda, \varepsilon)$, but in a some small neighbourhood of the λ_0 we may find above estimate for $\Theta(\cdot)$ and sequence of operators S_λ is locally converging to the S_{λ_0} as $\lambda \to \lambda_0$ then the attractor $\mathcal{M}_{\lambda_0}$ is continuous in λ_0.

A sequence of operators $S_\lambda(\cdot)$ is called *globally converging* in a point λ_0 on a bounded set B if for any $\varepsilon > 0$ and for any $T > 0$ there exists δ such that the following estimate holds

$$||S_\lambda(t, u) - S_{\lambda_0}(t, u)|| \leq \varepsilon \quad \forall \lambda \in O_{\delta_\varepsilon}\left(\lambda_0\right), \, \forall u \in B, \; 0 \leq t \leq T. \tag{8}$$

Remark. For the ODE $y'(t) = -\alpha_n y(t)$, as $\alpha_n \to 0^+$, we have globally converging operators. It is easy to verify that the attractor $\mathcal{M}_0$ in $\lambda_0 = 0$ is *upper semicontinuous* but not *lower semicontinuous.*

Let us prove a criterion (see [8]) when an attractor for sequence of globally converging operators is continuous (lower and upper semicontinuous) in a point λ_0 .

Theorem 4 *Under the assumptions* (α) *assume, that a sequence of operators* $S_\lambda(\cdot)$ *is globally converging in* λ_0 *on the set* B_a. *Then the attractor* $\mathcal{M}_\lambda$ *depends continuously on* λ *in the point* λ_0 *if and only if a function* $\Theta(\lambda, \varepsilon)$ *is uniformly bounded on* λ *for each fixed* $\varepsilon > 0$, *i.e.*

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \sup_{\lambda \in O_{\delta_\varepsilon}(\lambda_0)} \Theta(\lambda, \varepsilon, B_a) \leq T_\varepsilon < \infty. \tag{9}$$

Proof. Theorem 2 together with (8), (9) implies the continuity of the attractor. Suppose, that attractor depends continuously on λ in λ_0. This implies

$$\forall \varepsilon > 0 \ \exists \delta_1(\varepsilon) > 0 : \mathcal{M}_{\lambda_0} \subset O_\varepsilon(\mathcal{M}_\lambda), \ \ \mathcal{M}_\lambda \subset O_\varepsilon(\mathcal{M}_{\lambda_0}), \ \forall \lambda \in O_{\delta_1}(\lambda_0). \tag{10}$$

Under the definition of a function $\Theta(\cdot)$ for $T = \Theta(\lambda_0, \varepsilon)$ we have

$$S_{\lambda_0}(T + \tau, u) \subset O_\varepsilon(\mathcal{M}_{\lambda_0}), \quad \forall \tau > 0.$$

This, together with (8), implies that there exists $\delta_2(\varepsilon, T)$ such that

$$\|S_\lambda(t, u) - S_{\lambda_0}(t, u)\| \leq \varepsilon \quad \forall \lambda \in O_{\delta_2}(\lambda_0), \ \forall u \in B, t \leq T.$$

Choose δ in the following way $\delta = \min\{\delta_1, \delta_2\}$. Then

$$S_{\lambda_0}(T, B_a) \subset O_\varepsilon(\mathcal{M}_{\lambda_0}) \subset O_{2\varepsilon}(\mathcal{M}_\lambda) \ \forall \lambda \in O_\delta(\lambda_0).$$

This, together with (10), gives us

$$S_\lambda(T, B_a) \subset O_{3\varepsilon}(\mathcal{M}_\lambda) \ \ \forall \lambda \in O_\delta(\lambda_0). \tag{11}$$

Combine (11) with (10) for any $\tau > 0$ and use the same arguments as in the proof of Theorem 2 to obtain

$$S_\lambda(T + \tau, B_a) \subset O_{3\varepsilon}(\mathcal{M}_\lambda) \ \ \forall \lambda \in O_\delta(\lambda_0), \tau > 0.$$

The last estimate means that function $\Theta(\lambda_0, \varepsilon)$ is a function of attraction to the 3ε-neighbourhood of the attractor $\mathcal{M}_\lambda$ for any $\lambda \in O_\delta(\lambda_0)$. Thus, the function $\Theta(\lambda, \varepsilon) \leq \Theta(\lambda_0, \varepsilon/3) < \infty$ for $\forall \lambda \in O_\delta(\lambda_0)$, which proved the Theorem.

4 On a function of rate of attraction to a attractor

Here we consider the example of a semigroup having global attractor and known function of rate of attraction to attractor, namely, 1D Chafee-Infante problem

$$u_t = u_{xx} + bu - f(u), \quad b > 0, \ f(\cdot) \in C^1(\cdot)$$

$$u(0,t) = u(\pi,t) = 0; \quad u(x,0) = u_0(x) \in H_0^1(0,\pi)$$

$$f(0) = f'(0) = 0, \; s^{-1}f(s) < f'(s), s \neq 0; \; S^{-1}f(s) \to +\infty, \text{ as } |s| \to \infty.$$

The Chafee-Infante problem generated semigroup in the phase space $H_0^1(0,\pi)$. The following theorem was given in [6].

Theorem 5 *Under the above assumptions, if $b \neq m^2, m = 1,2,...,$ then the attractor $\mathcal{M}$ attracts its neighbourhood exponentially*

$$dist(S(t,B),\mathcal{M}) \leq Ce^{-at}, \quad C,a = const > 0$$

If $b = m^2$ and in addition

$$K^{-1}(|s|+|t|)^q(s-t)^2 \leq (f(s)-f(t))(s-t) \leq K(|s|+|t|)^q(s-t)^2$$

for some positive K,q,s_0 and all $s,t \in [-s_0,s_0]$, then the attractor $\mathcal{M}$ attracts its neighbourhood polynomially

$$dist(S(t,B),\mathcal{M}) \leq D(1+t)^{-\alpha}, \quad D,\alpha = const > 0.$$

This, together with Theorem 4, implies continuity of the attractor $\mathcal{M}$ under different types of perturbations of the original operator $S(\cdot)$. For example

$$u_t = u_{xx} + (b+\delta)u - f(u) + \delta_1(u)$$

or difference approximations with steps τ, h, when the constant a, α, C, D are bounded as $\delta, \delta_1(\cdot), \tau, h \to 0$ (see [12]).

Let us consider the Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \nu\Delta u + u \cdot \nabla u = -\nabla p + f, \quad \nu \neq 0 \tag{12}$$

$$\text{div u} = 0$$

$$u|_{\partial\Omega} = 0; \quad u(t=0,x) = u_0(x) \tag{13}$$

in an *arbitrary* bounded $\Omega \subset R^2$. Here $x = (x_1,x_2)$, $u = (u_1,u_2)$ is the velocity vector, p is the pressure, $\nu = const > 0$. Existence of the global attractor $\mathcal{M}$ for (12),(13) was first studied by Ladyzhenskaya in [2]. In [1] it was proved, that some approximations of (12),(13) have global attractors $\mathcal{M}_{\tau,h}$ in ε-neighborhoods of $\mathcal{M}$. Babin an Vishik in [4] established a theorem on upper semicontinuity of the attractor $\mathcal{M}_\lambda$ for $\lambda = (\nu, f)$. Following [2, 1] it is easy to verify the assumptions (α) when $\lambda = (\nu,f)(or \lambda = (\tau,h))$ and globally converging $S_\lambda(\cdot)$ to S_{λ_0} when $\lambda \to \lambda_0$. The properties of the function $\Theta(\cdot)$ for the problem (12),(13) is unknown. However, when the operator $S_\lambda(\cdot)$ is upper semicontinuous on a compact set Λ then it is continuous on a some everywhere dense subset $\bar{\Lambda} \subset \Lambda$. Thus, for any λ and each $\varepsilon > 0$ there exists λ_0 and $\delta = \delta(\varepsilon)$ such that

$$||\lambda - \lambda_0|| < \varepsilon \quad \text{and} \quad \sup_{\lambda_n \in O_\delta(\lambda_0)} \Theta(\lambda_n, \varepsilon) \leq T_\varepsilon < \infty.$$

The above results are valid (see [7]) for some modifications of the systems (12), (13) in 3D case.

Acknowledgments. This research was supported in part by the NWO-RFBR programme doss.nr. 047.003.017 and RFBR 99-01-00263.

I would like to express my gratitude to Professor A.O.H. Axelsson for his support.

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