The version of the following full text has not yet been defined or was untraceable and may differ from the publisher's version.

For additional information about this publication click this link. http://hdl.handle.net/2066/18723

Please be advised that this information was generated on 2019-10-22 and may be subject to change.
On the continuity property for an attractor of a semidynamical system with a parameter

Andrei A. Kornev

Report No. 9917 (April 1999)
On the continuity property for an attractor of a semidynamical system with a parameter

Andrei A. Kornev *

Key words: Dynamical system, attractors

AMS (MOS) subject classification: 35B40, 58F12, 58F39, 34D45, 34C35

Abstract

An approach to verify continuity of a global attractor of a semidynamical system with a parameter is presented. This approach makes it possible to establish a connection between upper and lower semicontinuity of a global attractor and boundedness as a function of rate of attraction to an attractor. The obtained results were used for the scalar Lorenz wave equation in 3D space, the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

1 Introduction

In the present paper for a semigroup \( \{S_\lambda(t, \cdot)\} \) corresponding, for example, to an evolution equation, where \( \lambda \) is a problem parameter, and having global attractor the problem of stability of \( \mathcal{M}_\lambda \) as \( \lambda \rightarrow \lambda_0 \) is considered. The most powerful test for the attractor \( \mathcal{M}_\lambda \) being in \( O_\varepsilon(\mathcal{M}_{\lambda_0}) \), with \( O_\varepsilon(\mathcal{M}_{\lambda_0}) \) the \( \varepsilon \)-neighborhood of \( \mathcal{M}_{\lambda_0} \), was proved by Kapitanski and Kostin.

The main purpose of the current work is to consider a criterion when in addition \( \mathcal{M}_{\lambda_0} \subset O_\varepsilon(\mathcal{M}_\lambda) \) and \( \varepsilon \rightarrow 0 \) as \( \lambda \rightarrow \lambda_0 \). The basis of this theorem are the characteristics of the function of rate of attraction \( \Psi(\lambda, t) \) in some small \( \delta \)-neighborhood of \( \lambda_0 \)

\[
dist(S_\lambda(t, B_0), \mathcal{M}_\lambda) \leq \Psi(\lambda, t) \dist(B_0, \mathcal{M}_\lambda), \quad t \geq 0, \quad \lambda \in O_\delta(\lambda_0).
\]

Here \( B_0 \) is a bounded absorbing set and \( S_\lambda(t, \cdot) \) is an approximation of the given nonlinear operator \( S_{\lambda_0}(t, \cdot) \).

These results are applied in Section 4 to the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

* Faculty of Mathematics and Informatics, University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands; Permanent: Dept. Mech. & Math., Moscow State University, Moscow 119899, Russia; E-mail: kornev@big.lkm.msu.ru
2 Asymptotically compact semigroups

Let $X$ be a Banach space with norm $\| \cdot \|$, $\Theta$ be a nontrivial subgroup of real number $R$ and let $\Theta_+ = \Theta \cap [0, +\infty[$ be the intersection of $\Theta$ and $R_+$. We shall deal with the abstract semigroup $\{X, \Theta_+, S(\cdot)\}$ of nonlinear operator $S : X \times \Theta_+ \to X$. The term semigroup or semidynamical system refers to any family of singlevalued continuous operator $S$ depending on a parameter $t \in \Theta_+$ and enjoying the semigroup property:

$$S(t_1, S(t_2, u)) = S(t_1 + t_2, u), \quad \forall t_1, t_2 \in \Theta_+, \forall u \in X$$

A Banach space $X$ is a phase space of a semigroup, $\Theta_+$ is a time space and $S(\cdot)$ is an evolution operator. When $\Theta = R$ a semigroup is a semigroup with continuous time.

Let $B$ and $M$ be bounded subsets of $X$. We say that $B$ is attracted to $M$ by the semigroup $S(\cdot)$ if

$$\text{dist}(S(t, B), M) \to 0 \quad \text{as } t \to \infty$$

Here

$$\text{dist}(A, B) = \sup_{y \in A} \{\text{dist}(y, B)\}, \quad \text{dist}(y, B) = \inf_{x \in B} \| x - y \|$$

A set $M$ is called an attracting set of the semigroup if $M$ attracts each bounded $B \subset X$. The minimal among the closed attracting sets is called the global attractor [9] (minimal global $B$-attractor [3]). The global attractor of a semigroup is defined as the set $M$ which is compact in $X$, invariant for $S(\cdot)$, i.e.

$$S(t, M) = M, \quad t \geq 0$$

and which attracts all the bounded sets of $X$.

We need the following lemmas (see [5, 6]):

**Lemma 1** Let $M$ be a compact attractor of the semigroup $\{X, \Theta_+, S(\cdot)\}$, $\tilde{M}$ be a compact attractor of the semigroup $\{X, \tilde{\Theta}_+, \tilde{S}(\cdot)\}$. Let $\tilde{\Theta}_+$ be a subsemigroup of $\Theta_+$ and let for some points $t, \tilde{t}$

$$S(t, u) = S(\tilde{t}, u) \quad \text{for all } u \in X$$

Then $M = \tilde{M}$.

Lemma 1 implies

**Lemma 2** Let $M$ be a compact attractor of the semigroup $\{X, R_+, S(\cdot)\}$ and let $t_0 > 0$. Then $M$ is attractor of the semigroup $\{X, t_0 Z_+, S(\cdot)\}$. Here $t_0 Z_+ \equiv \{kt_0, k \in Z_+\}$.

Later on we need the following definitions, see [3].

A set $B_\varepsilon$ is called absorbing if for each bounded $B \subset X$ and for each $\varepsilon > 0$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset B_\varepsilon, \quad \forall t \geq T$$
If a semigroup possesses a nonempty bounded attractor $\mathcal{M}$ then for arbitrary $\varepsilon > 0$ the set $O_\varepsilon(\mathcal{M})$ is an absorbing set. Here $O_\varepsilon(\mathcal{M})$ is the $\varepsilon$-neighbourhood of $\mathcal{M}$, i.e.

$$O_\varepsilon(\mathcal{M}) = \{u : \exists v \in \mathcal{M}, \|u - v\| < \varepsilon\}$$

A semigroup is called bounded if for each bounded $B$ the set $S(t, B)$ is bounded for any $t > 0$.

A semigroup is called pointwise dissipative if it has a pointwise absorbing set $B_0$

$$\forall x \in X, \exists T(x) : S(t, x) \subset B_0, \text{ for any } t \geq T(x)$$

A semigroup is called asymptotically compact if for each bounded $B$ such that $S(t, B)$ is bounded for any $t > 0$ each sequence of the form

$$\{S(t_k, u_k)\}_{k=1}^\infty, t_k \uparrow \infty, u_k \in B$$

is precompact.

The following theorem holds, see [3].

**Theorem 1** Let the semigroup $\{X, \Theta_, \Theta(\cdot)\}$ be a continuous bounded pointwise dissipative asymptotically compact semigroup. Then there exists a non-empty attractor $\mathcal{M}$

$$\mathcal{M} = \bigcap_{t \geq 0} [S(t, B_0)]_X$$

$\mathcal{M}$ is compact and invariant. If $X$ is connected then $\mathcal{M}$ is also connected.

We now summarize the results:

**Lemma 3** Let $S(t) : X \rightarrow X, t \in \mathbb{R}_+$ be a continuous semigroup possessing a non-empty compact attractor $\mathcal{M}$. Then this semigroup is an asymptotically compact semigroup.

**Proof.** We shall prove that each sequence of the form

$$\{S(t_k, u_k)\}_{k=1}^\infty, t_k \uparrow \infty, u_k \in B$$

can be covered by a finite $\varepsilon$-network where $\varepsilon$ is arbitrary small positive number. The set $O_{\varepsilon/2}(\mathcal{M})$ is an absorbing set and for any bounded $B$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset O_{\varepsilon/2}(\mathcal{M}) \text{ for any } t \geq T.$$ 

Let us choose $N$ so large that the set $\{S(t_k, u_k)\}_{k=N}^\infty, u_k \in B$ belongs to $O_{\varepsilon/2}(\mathcal{M})$.

The set

$$V^\mathcal{M} = \{v^\mathcal{M} \in \mathcal{M} : \|v^\mathcal{M}_k - u_k\| \leq \varepsilon/2, k \geq N\} \subset \mathcal{M}$$

is a precompact set and may be covered by a finite $\varepsilon/2$-network $\bar{V}^\mathcal{M} = \{\bar{v}_i \in V^\mathcal{M}\}_{i=1}^n$ and for any $k$ there exists $i$ such that:

$$\|v^\mathcal{M}_k - \bar{v}_i\| \leq \varepsilon/2, \quad k \geq N$$
This implies
\[ \|u_k - \bar{v}_i\| \leq \|u_k - v^M_k\| + \|v^M_k - \bar{v}_i\| \leq \varepsilon. \]
Thus the set \( \bar{V}^M \) is a finite \( \varepsilon \)-network of the set \( \{S(t_{k},u_{k})\}_{k=1}^{\infty} \) and then the set \( \{S(t_{k},u_{k})\}_{k=1}^{\infty} \) is a compact set.

Note, if a semigroup possesses a nonempty attractor \( \mathcal{M} \) then the semigroup is pointwise dissipative.

## 3 Semigroups with a parameter

The current paragraph deals with the problem of stability of \( \mathcal{M} \) with respect to perturbations of the original operator \( S(\cdot) \).

Let us consider a semigroup \( S_{\lambda}(\cdot) : X \to X \) which depends on a parameter \( \lambda \in \Lambda \). The main purpose is to consider a criterion when the set \( \mathcal{M}_{\lambda_0} \) and \( \mathcal{M}_\lambda \) are close to each other in the Hausdorff metric, that is

\[ \lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}_{\lambda}, \mathcal{M}_{\lambda_0}) = 0 \quad (1) \]
\[ \lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = 0 \quad (2) \]

The relations (1) and (2) are usually referred to as the upper and lower semicontinuity of the attractor \( \mathcal{M}_\lambda \) in \( \lambda_0 \).

The most powerful test for the attractor \( \mathcal{M}_\lambda \) being in \( O_{\varepsilon}(\mathcal{M}_{\lambda_0}) \) with \( O_{\varepsilon}(\mathcal{M}_{\lambda_0}) \) the \( \varepsilon \)-neighborhood of \( \mathcal{M}_{\lambda_0} \), was proved in [5].

We consider when in addition \( \mathcal{M}_{\lambda_0} \subset O_{\varepsilon}(\mathcal{M}_\lambda) \) and \( \varepsilon \to 0 \) as \( \lambda \to \lambda_0 \). The basis of this theorem [2] are the characteristics of the function of rate of attraction in a some small \( \delta \)-neighborhood of \( \lambda_0 \).

We assume that the following conditions (a) hold:
1. \( \Lambda \) is compact with metric \( \| \cdot \|_\Lambda \) and \( \lambda_0 \) is a nonisolated point of \( \Lambda \).
2. For each \( \lambda \in \Lambda \) the semigroup \( \{X, \Theta_+ , S(\cdot)\} \) possesses a pointwise absorbing set \( B_\lambda \) and non-empty attractor \( \mathcal{M}_\lambda \).
3. There exists a bounded absorbing set \( B_\alpha \) and for each \( \lambda \in \Lambda \) a set \( B_\lambda \) belongs to the set \( B_\alpha \).

By definition, each \( \varepsilon \)-neighbourhood \( O_{\varepsilon}(\mathcal{M}_\lambda) \) is an absorbing set. Assume that we know a function \( \Theta(\lambda, \varepsilon) = \Theta(\lambda, \varepsilon, B_\alpha) \) such that

\[ \text{dist}(S(\lambda, B_\alpha), \mathcal{M})) \leq \varepsilon, \quad \text{as } t \geq \Theta(\varepsilon, \lambda) \]

**Lemma 4** Under the assumptions (a) let there exist \( \varepsilon > 0 \), \( \lambda_1, \lambda_2 \in \Lambda \) and a point \( T \geq \Theta(\lambda_1, \varepsilon) \) such that for the operators \( S(\lambda_1, t) \) and \( S(\lambda_2, t) \) the following estimate is valid

\[ ||S(\lambda_1, T, u) - S(\lambda_2, T, u)|| < \varepsilon \quad \text{for any } u \in B_\alpha \quad (3) \]

Then

\[ \mathcal{M}_{\lambda_2} \subset O_{2\varepsilon}(\mathcal{M}_{\lambda_1}) \]
Remark. Without loss of generality, assume that $O_{2\varepsilon}(M_{\lambda_1}) \subset B_\alpha$.

Proof. Suppose that estimate (3) holds for a $T \geq \Theta(\lambda_1, \varepsilon)$ with some $\varepsilon > 0$. The set $M_{\lambda_1}$ is the attractor for $\{X_\lambda, \Theta_\lambda, S_{\lambda_1}(\cdot)\}$ and for the $B_\alpha$ we can find an attraction time $T = \Theta(\lambda_1, \varepsilon)$ to the $\varepsilon$-neighbourhood of $M_{\lambda_1}$.

$$S_{\lambda_1}(T, u) \subset O_\varepsilon(M_{\lambda_1}), \forall u \in B_\alpha$$

Furthermore, we obtain $S_{\lambda_1}(T + t, u) \subset O_\varepsilon(M_{\lambda_1})$ for any $t \geq 0$. Due to (3) we have $\|S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)\| < \varepsilon$. Hence

$$S_{\lambda_2}(T, u) \in O_{2\varepsilon}(M_{\lambda_1})$$

The above injection and $S_{\lambda_2}(T, u) \in B_\alpha$ gives us

$$\|S_{\lambda_1}(T, S_{\lambda_2}(T, u)) - S_{\lambda_2}(T, S_{\lambda_2}(T, u))\| < \varepsilon \quad \text{for any } u \in B_\alpha$$

Since $S_{\lambda_1}(T, v) \in O_{\varepsilon}(M_{\lambda_1})$ for each $v \in B_\alpha$ we have $S_{\lambda_2}(T, S_{\lambda_2}(T, u)) = S_{\lambda_2}(2T, u) \in O_{2\varepsilon}(M_{\lambda_1})$. After a finite number of steps we obtain $S_{\lambda_2}(nT, u) \in O_{2\varepsilon}(M_{\lambda_1})$ as $n = 1, 2, \ldots$ for any $u \in B_\alpha$. This, together with Lemma 1, implies

$$M_{\lambda_2} \subset O_{2\varepsilon}(M_{\lambda_1})$$

This completes the proof.

The next theorem provides an estimate for the distance between two attractors.

Theorem 2 Under the assumptions (a)

(i) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ and a point $T_{\lambda_0} \geq \Theta(\lambda_0, \varepsilon)$ such that

$$\|S_\lambda(T_{\lambda_0}, u) - S_{\lambda_0}(T_{\lambda_0}, u)\| < \varepsilon \quad \forall u \in B_\alpha, \forall \lambda \in O_{\delta}(\lambda_0) \quad (4)$$

Then the attractor $M_\lambda$ is upper semicontinuous in the point $\lambda_0$ and the following estimate holds

$$\text{dist}(M_\lambda, M_{\lambda_0}) \leq 2\varepsilon. \quad (5)$$

(ii) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

for arbitrary $\lambda \in O_{\delta}(\lambda_0)$ there exists a point $T_\lambda = T(\lambda) \geq \Theta(\lambda, \varepsilon)$ satisfies the following estimate

$$\|S_\lambda(T_\lambda, u) - S_{\lambda_0}(T_\lambda, u)\| < \varepsilon \quad \forall u \in B_\alpha \quad (6)$$

Then the attractor $M_\lambda$ is lower and upper semicontinuous in the point $\lambda_0$ and the following estimate holds

$$\text{max}\{\text{dist}(M_{\lambda_0}, M_\lambda), \text{dist}(M_\lambda, M_{\lambda_0})\} \leq 2\varepsilon. \quad (7)$$

Proof. According to (4) and assumption (i) we have that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for a $\lambda \in O_{\delta}(\lambda_0)$ we have $M_\lambda \subset O_{2\varepsilon}(M_{\lambda_0})$. Thus,

$$\text{dist}(M_\lambda, M_{\lambda_0}) = \sup_{u \in M_\lambda} (u, M_{\lambda_0}) \leq 2\varepsilon.$$
As \( \varepsilon \to 0 \) we obtain \( \text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \to 0 \). Thus, inequality (5) is proved.

In order to prove (7), from (5) and (ii) we have

\[
\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon
\]

To obtain the reverse inequality

\[
\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) \leq 2\varepsilon \to 0 \quad \text{as } \lambda \to \lambda_0
\]

use the assumption (ii) for arbitrary \( \varepsilon > 0 \) and give \( O_\varepsilon(\lambda_0) \) such that for any \( \lambda \in O_\varepsilon(\lambda_0) \) inequality (6) holds in a point \( T_\lambda \geq \Theta(\lambda, \varepsilon) \). This, together with Lemma 4 implies

\[
\mathcal{M}_{\lambda_0} \subset O_{2\varepsilon}(\mathcal{M}_\lambda) \quad \forall \lambda \in O_{\varepsilon}(\lambda_0).
\]

As \( \varepsilon \to 0 \) we have

\[
\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = \sup_{u \in \mathcal{M}_{\lambda_0}} \langle u, \mathcal{M}_\lambda \rangle \to 0.
\]

This completes the proof.

**Remark 1.** For a semigroup with compact attractor upper semicontinuity was proved in [5] when (5) is valid in an some point \( T_{\lambda_0} > 0 \). But, we can not find the rate of convergence for \( \mathcal{M}_\lambda \to \mathcal{M}_{\lambda_0} \). When inequality (4) holds for \( t \in [\tau, \tau + T_{\lambda_0}] \) for any \( \tau > 0 \), uniformly in \( \tau \), then (5) was proved in [10].

**Remark 2.** The assumption (ii) allows \( T_\lambda \to \infty \) as \( \lambda \to \lambda_0 \).

**Remark 3.** A function rate of attraction \( \Theta(\cdot) \) can be approximate, for example see [11], for some nontrivial PDE system with nontrivial attractor.

A sequence of operators \( S_\lambda(\cdot) \) is called locally converging in a point \( \lambda_0 \) on a bounded \( B \) if for any \( \varepsilon > 0 \) and for each \( T > 0 \) there exists \( \delta \) such that

\[
\|S_\lambda(T, u) - S_{\lambda_0}(T, u)\| \leq \varepsilon \quad \forall \lambda \in O_\varepsilon(\lambda_0), \forall u \in B.
\]

Theorem 2 and the above definition imply

**Theorem 3** Under the assumptions (a) let the sequence of operators \( S_\lambda(\cdot) \) be locally converging in a point \( \lambda_0 \) on a set \( B_\alpha \). Let the function \( \Theta(\lambda, \varepsilon) \) be uniformly bounded on \( \lambda \) for each fixed \( \varepsilon > 0 \), i.e.

\[
\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \sup_{\lambda \in O_\varepsilon(\lambda_0)} \Theta(\lambda, \varepsilon, B_\alpha) \leq T_\varepsilon < \infty.
\]

Then, the attractor \( \mathcal{M}_\lambda \) depends on \( \lambda \) in the point \( \lambda_0 \) continuously.

**Remark.** In this way, if we cannot calculate a function \( \Theta(\lambda, \varepsilon) \), but in a some small neighbourhood of the \( \lambda_0 \) we may find above estimate for \( \Theta(\cdot) \) and sequence of operators \( S_\lambda \) is locally converging to the \( S_{\lambda_0} \) as \( \lambda \to \lambda_0 \) then the attractor \( \mathcal{M}_{\lambda_0} \) is continuous in \( \lambda_0 \).

A sequence of operators \( S_\lambda(\cdot) \) is called globally converging in a point \( \lambda_0 \) on a bounded set \( B \) if for any \( \varepsilon > 0 \) and for any \( T > 0 \) there exists \( \delta \) such that the following estimate holds

\[
\|S_\lambda(t, u) - S_{\lambda_0}(t, u)\| \leq \varepsilon \quad \forall \lambda \in O_\varepsilon(\lambda_0), \forall u \in B, \ 0 \leq t \leq T.
\]
Remark. For the ODE \( y'(t) = -\alpha y(t) \), as \( \alpha \to 0^+ \), we have globally converging operators. It is easy to verify that the attractor \( \mathcal{M}_0 \) in \( \lambda_0 = 0 \) is \textit{upper semicontinuous} but not \textit{lower semicontinuous}.

Let us prove a criterion (see [8]) when an attractor for sequence of globally converging operators is continuous (lower and upper semicontinuous) in a point \( \lambda_0 \).

**Theorem 4** Under the assumptions (a) assume, that a sequence of operators \( S_\lambda(\cdot) \) is globally converging in \( \lambda_0 \) on the set \( B_\varepsilon \). Then the attractor \( \mathcal{M}_\lambda \) depends continuously on \( \lambda \) in the point \( \lambda_0 \) if and only if a function \( \Theta(\lambda, \varepsilon) \) is uniformly bounded on \( \lambda \) for each fixed \( \varepsilon > 0 \), i.e.

\[
\forall \varepsilon > 0 \, \exists \delta_\varepsilon > 0 : \sup_{\lambda \in O_{\delta_\varepsilon}(\lambda_0)} \Theta(\lambda, \varepsilon, B_\varepsilon) \leq T_\varepsilon < \infty. 
\]

(9)

**Proof.** Theorem 2 together with (8), (9) implies the continuity of the attractor. Suppose, that attractor depends continuously on \( \lambda \) in \( \lambda_0 \). This implies

\[
\forall \varepsilon > 0 \, \exists \delta_1(\varepsilon) > 0 : \mathcal{M}_\lambda \subset O_\varepsilon(M_\lambda), \, M_\lambda \subset O_\varepsilon(M_{\lambda_0}), \, \forall \lambda \in O_{\delta_1}(\lambda_0). 
\]

(10)

Under the definition of a function \( \Theta(\cdot) \) for \( T = \Theta(\lambda_0, \varepsilon) \) we have

\[
S_{\lambda_0}(T + \tau, u) \subset O_\varepsilon(M_{\lambda_0}), \, \forall \tau > 0.
\]

This, together with (8), implies that there exists \( \delta_2(\varepsilon, T) \) such that

\[
\| S_\lambda(t, u) - S_{\lambda_0}(t, u) \| \leq \varepsilon \, \forall \lambda \in O_{\delta_2}(\lambda_0), \, \forall u \in B, t \leq T.
\]

Choose \( \delta \) in the following way \( \delta = \min\{\delta_1, \delta_2\} \). Then

\[
S_{\lambda_0}(T, B_\delta) \subset O_\varepsilon(M_{\lambda_0}) \subset O_{\delta_2}(M_\lambda) \, \forall \lambda \in O_\delta(\lambda_0).
\]

This, together with (10), gives us

\[
S_\lambda(T, B_\delta) \subset O_{\delta_2}(M_\lambda) \, \forall \lambda \in O_\delta(\lambda_0). 
\]

(11)

Combine (11) with (10) for any \( \tau > 0 \) and use the same arguments as in the proof of Theorem 2 to obtain

\[
S_\lambda(T + \tau, B_\delta) \subset O_{\delta_2}(M_\lambda) \, \forall \lambda \in O_\delta(\lambda_0), \, \tau > 0.
\]

The last estimate means that function \( \Theta(\lambda_0, \varepsilon) \) is a function of attraction to the \( 3\varepsilon \)-neighbourhood of the attractor \( M_\lambda \) for any \( \lambda \in O_\delta(\lambda_0) \). Thus, the function \( \Theta(\lambda, \varepsilon) \leq \Theta(\lambda_0, \varepsilon/3) < \infty \) for \( \forall \lambda \in O_\delta(\lambda_0) \), which proved the Theorem.

### 4 On a function of rate of attraction to a attractor

Here we consider the example of a semigroup having global attractor and known function of rate of attraction to attractor, namely, 1D Chafee-Infante problem

\[
u_t = u_{xx} + bu - f(u), \quad b > 0, \quad f(\cdot) \in C^1(\cdot)
\]
\[ u(0,t) = u(\tau,t) = 0; \quad u(x,0) = u_0(x) \in H^1_0(0,\pi) \]
\[ f(0) = f'(0) = 0, \quad s^{-1}f(s) < f'(s), s \neq 0; \quad S^{-1}f(s) \to +\infty, \text{ as } |s| \to \infty. \]

The Chafee-Infante problem generated semigroup in the phase space \( H^1_0(0,\pi) \). The following theorem was given in [6].

**Theorem 5** Under the above assumptions, if \( b \neq m^2, m = 1,2,\ldots \), then the attractor \( \mathcal{M} \) attracts its neighbourhood exponentially
\[
dist(S(t,B),\mathcal{M}) \leq Ce^{-\alpha t}, \quad C, \alpha = \text{const} > 0
\]

If \( b = m^2 \) and in addition
\[
K^{-1}(|s| + |t|)^q(s - t)^2 \leq (f(s) - f(t))(s - t) \leq K(|s| + |t|)^q(s - t)^2
\]
for some positive \( K,q,s_0 \) and all \( s,t \in [-s_0,s_0] \), then the attractor \( \mathcal{M} \) attracts its neighbourhood polynomially
\[
dist(S(t,B),\mathcal{M}) \leq D(1 + t)^{-\alpha}, \quad D, \alpha = \text{const} > 0.
\]

This, together with Theorem 4, implies continuity of the attractor \( \mathcal{M} \) under different types of perturbations of the original operator \( S(\cdot) \). For example
\[
u_t = u_{xx} + (b + \delta)u - f(u) + \delta_1(u)
\]
or difference approximations with steps \( \tau,h \), when the constant \( a,\alpha,C,D \) are bounded as \( \delta,\delta_1(\cdot),\tau,h \to 0 \) (see [12]).

Let us consider the Navier-Stokes equations
\[
\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u = -\nabla p + f, \quad \nu \neq 0
\]
\[ \text{div } u = 0 \]
\[ u|_{\partial\Omega} = 0; \quad u(t = 0,x) = u_0(x) \quad (13)
\]
in an arbitrary bounded \( \Omega \subset \mathbb{R}^2 \). Here \( x = (x_1,x_2) \), \( u = (u_1,u_2) \) is the velocity vector, \( p \) is the pressure, \( \nu = \text{const} > 0 \). Existence of the global attractor \( \mathcal{M} \) for (12),(13) was first studied by Ladyzhenskaya in [2]. In [1] it was proved, that some approximations of (12),(13) have global attractors \( \mathcal{M}_{\tau,h} \) in \( \varepsilon \)-neighborhoods of \( \mathcal{M} \). Babin an Vishik in [4] established a theorem on upper semicontinuity of the attractor \( \mathcal{M}_\lambda \) for \( \lambda = (\nu,f) \). Following [2,1] it is easy to verify the assumptions (a) when \( \lambda = (\nu,f)(or\lambda = (\tau,h)) \) and globally converging \( S_\lambda(\cdot) \) to \( S_{\lambda_0} \) when \( \lambda \to \lambda_0 \). The properties of the function \( \Theta(\cdot) \) for the problem (12),(13) is unknown. However, when the operator \( S_\lambda(\cdot) \) is upper semicontinuous on a compact set \( \Lambda \) then it is continuous on a some everywhere dense subset \( \Lambda \subset \Lambda \). Thus, for any \( \lambda \) and each \( \varepsilon > 0 \) there exists \( \lambda_0 \) and \( \delta = \delta(\varepsilon) \) such that
\[
||\lambda - \lambda_0|| < \varepsilon \quad \text{and} \quad \sup_{\lambda_0 \in O_\delta(\lambda_0)} \Theta(\lambda_0,\varepsilon) \leq T_\varepsilon < \infty.
\]
The above results are valid (see [7]) for some modifications of the systems (12), (13) in 3D case.

Acknowledgments. This research was supported in part by the NWO-RFBR programme doss.nr. 047.003.017 and RFBR 99-01-00263.

I would like to express my gratitude to Professor A.O.H. Axelsson for his support.

References


[7] A.A. Kornev , On new a priori estimates for some modification of the Navier-Stokes equations in domains with non smooth boundaries in three dimensional case. Submitted to "Fundamental and Applied Mathematics".


