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On the continuity property for an attractor of a semidynamical system with a parameter

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Report No. 9917 (April 1999)
On the continuity property for an attractor of a semidynamical system with a parameter

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Key words: Dynamical system, attractors
AMS (MOS) subject classification: 35B40, 58F12, 58F39, 34D45, 34C35

Abstract

An approach to verify continuity of a global attractor of a semidynamical system with a parameter is presented. This approach makes it possible to establish a connection between upper and lower semicontinuity of a global attractor and boundedness as a function of rate of attraction to an attractor. The obtained results were used for the scalar Lorenz wave equation in 3D space, the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

1 Introduction

In the present paper for a semigroup \( \{S_\lambda(t, \cdot)\} \) corresponding, for example, to an evolution equation, where \( \lambda \) is a problem parameter, and having global attractor the problem of stability of \( M_{\lambda_0} \) as \( \lambda \to \lambda_0 \) is considered. The most powerful test for the attractor \( M_\lambda \) being in \( O_\varepsilon(M_{\lambda_0}) \), with \( O_\varepsilon(M_{\lambda_0}) \) the \( \varepsilon \)-neighborhood of \( M_{\lambda_0} \), was proved by Kapitanskii and Kostin.

The main purpose of the current work is to consider a criterion when in addition \( M_{\lambda_0} \subset O_\varepsilon(M_\lambda) \) and \( \varepsilon \to 0 \) as \( \lambda \to \lambda_0 \). The basis of this theorem are the characteristics of the function of rate of attraction \( \Psi(\lambda, t) \) in some small \( \delta \)-neighborhood of \( \lambda_0 \)

\[
\text{dist}(S_\lambda(t, B_\delta), M_\lambda) \leq \Psi(\lambda, t)\text{dist}(B_\delta, M_\lambda), \quad t \geq 0, \quad \lambda \in O_\delta(\lambda_0).
\]

Here \( B_\delta \) is a bounded absorbing set and \( S_\lambda(t, \cdot) \) is an approximation of the given nonlinear operator \( S_{\lambda_0}(t, \cdot) \).

These results are applied in Section 4 to the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

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2 Asymptotically compact semigroups

Let $X$ be a Banach space with norm $\| \cdot \|$, $\Theta$ be a nontrivial subgroup of real number $R$ and let $\Theta_+ = \Theta \cap [0, +\infty[$ be the intersection of $\Theta$ and $R_+$. We shall deal with the abstract semigroup $\{X, \Theta_+, S(\cdot)\}$ of nonlinear operator $S : X \times \Theta_+ \to X$. The term semigroup or semidynamical system refers to any family of singlevalued continuous operator $S$ depending on a parameter $t \in \Theta_+$ and enjoying the semigroup property:

$$S(t_1, S(t_2, u)) = S(t_1 + t_2, u), \quad \forall t_1, t_2 \in \Theta_+, \forall u \in X$$

A Banach space $X$ is a phase space of a semigroup, $\Theta_+$ is a time space and $S(\cdot)$ is an evolution operator. When $\Theta = R$ a semigroup is a semigroup with continuous time.

Let $B$ and $M$ be bounded subsets of $X$. We say that $B$ is attracted to $M$ by the semigroup $S(\cdot)$ if

$$\text{dist}(S(t, B), M) \to 0 \quad \text{as } t \to \infty$$

Here

$$\text{dist}(A, B) = \sup \{ \text{dist}(y, B) \} , \quad \text{dist}(y, B) = \inf_{x \in B} \| x - y \|$$

A set $M$ is called an attracting set of the semigroup if $M$ attracts each bounded $B \subset X$. The minimal among the closed attracting sets is called the global attractor [9] (minimal global B-attractor [3]). The global attractor of a semigroup is defined as the set $\mathcal{M}$ which is compact in $X$, invariant for $S(\cdot)$, i.e.

$$S(t, \mathcal{M}) = \mathcal{M}, \quad t \geq 0$$

and which attracts all the bounded sets of $X$.

We need the following lemmas (see [5, 6]):

**Lemma 1** Let $\mathcal{M}$ be a compact attractor of the semigroup $\{X, \Theta_+, S(\cdot)\}$, $\mathcal{M}$ be a compact attractor of the semigroup $\{X, \hat{\Theta}_+, \hat{S}(\cdot)\}$. Let $\Theta_+$ be a subsemigroup of $\hat{\Theta}_+$ and let for some points $t, \hat{t}$

$$S(t, u) = \hat{S}(\hat{t}, u) \quad \text{for all } u \in X$$

Then $\mathcal{M} = \mathcal{M}$.

Lemma 1 implies

**Lemma 2** Let $\mathcal{M}$ be a compact attractor of the semigroup $\{X, R_+, S(\cdot)\}$ and let $t_0 > 0$. Then $\mathcal{M}$ is attractor of the semigroup $\{X, t_0Z_+, S(\cdot)\}$. Here $t_0Z_+ \equiv \{kt_0, k \in Z_+ \}$.

Later on we need the following definitions, see [3].

A set $B_\varepsilon$ is called absorbing if for each bounded $B \subset X$ and for each $\varepsilon > 0$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset B_\varepsilon, \quad \forall t \geq T$$
If a semigroup possesses a nonempty bounded attractor $\mathcal{M}$ then for arbitrary $\varepsilon > 0$ the set $O_\varepsilon(\mathcal{M})$ is an absorbing set. Here $O_\varepsilon(\mathcal{M})$ is the $\varepsilon$-neighbourhood of $\mathcal{M}$, i.e.,

$$O_\varepsilon(\mathcal{M}) = \{ u : \exists v \in \mathcal{M}, \| u - v \| < \varepsilon \}$$

A semigroup is called bounded if for each bounded $B$ the set $S(t, B)$ is bounded for any $t > 0$.

A semigroup is called pointwise dissipative if it has a pointwise absorbing set $B_0$

$$\forall x \in X, \exists T(x) : S(t, x) \subset B_0, \text{ for any } t \geq T(x)$$

A semigroup is called asymptotically compact if for each bounded $B$ such that $S(t, B)$ is bounded for any $t > 0$ each sequence of the form

$$\{ S(t_k, u_k) \}_{k=1}^\infty, t_k \uparrow \infty, u_k \in B$$

is precompact.

The following theorem holds, see [3].

**Theorem 1** Let the semigroup $\{ X, \Theta_t, S(\cdot) \}$ be a continuous bounded pointwise dissipative asymptotically compact semigroup. Then there exists a non-empty attractor $\mathcal{M}$

$$\mathcal{M} = \bigcap_{t \geq 0} [S(t, B_0)]_X$$

$\mathcal{M}$ is compact and invariant. If $X$ is connected then $\mathcal{M}$ is also connected.

We now summarize the results:

**Lemma 3** Let $S(t) : X \to X, t \in \mathbb{R}_+$ be a continuous semigroup possessing a nonempty compact attractor $\mathcal{M}$. Then this semigroup is an asymptotically compact semigroup.

**Proof.** We shall prove that each sequence of the form

$$\{ S(t_k, u_k) \}_{k=1}^\infty, t_k \uparrow \infty, u_k \in B$$

can be covered by a finite $\varepsilon$-network where $\varepsilon$ is arbitrary small positive number. The set $O_{\varepsilon/2}(\mathcal{M})$ is an absorbing set and for any bounded $B$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset O_{\varepsilon/2}(\mathcal{M}) \text{ for any } t \geq T.$$ 

Let us choose $N$ so large that the set $\{ S(t_k, u_k) \}_{k=1}^N, u_k \in B$ belongs to $O_{\varepsilon/2}(\mathcal{M})$.

The set

$$V^{\mathcal{M}} = \{ v_k^{\mathcal{M}} \in \mathcal{M} : \| v_k^{\mathcal{M}} - u_k \| \leq \varepsilon/2, k \geq N \} \subset \mathcal{M}$$

is a precompact set and may be covered by a finite $\varepsilon/2$-network $\tilde{V}^{\mathcal{M}} = \{ \tilde{v}_i \in V^{\mathcal{M}} \}_{i=1}^n$ and for any $k$ there exists $i$ such that:

$$\| v_k^{\mathcal{M}} - \tilde{v}_i \| \leq \varepsilon/2, \quad k \geq N$$

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Thus the set \(\mathcal{V}^M\) is a finite \(\epsilon\)-network of the set \(\{S(t_k, u_k)\}_{k=1}^{\infty}\) and then the set \(\{S(t_k, u_k)\}_{k=1}^{\infty}\) is a compact set.

Note, if a semigroup possesses a nonempty attractor \(\mathcal{M}\) then the semigroup is pointwise dissipative.

## 3 Semigroups with a parameter

The current paragraph deals with the problem of stability of \(\mathcal{M}\) with respect to perturbations of the original operator \(S(\cdot)\).

Let us consider a semigroup \(S_\lambda(\cdot): X \to X\) which depends on a parameter \(\lambda \in \Lambda\). The main purpose is to consider a criterion when the set \(\mathcal{M}_{\lambda_0}\) and \(\mathcal{M}_\lambda\) are close to each other in the Hausdorff metric, that is

\[
\lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) = 0 \quad (1)
\]

\[
\lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = 0 \quad (2)
\]

The relations (1) and (2) are usually referred to as the upper and lower semicontinuity of the attractor \(\mathcal{M}_\lambda\) in \(\lambda_0\).

The most powerful test for the attractor \(\mathcal{M}_\lambda\) being in \(O_\epsilon(\mathcal{M}_{\lambda_0})\) with \(O_\epsilon(\mathcal{M}_{\lambda_0})\) the \(\epsilon\)-neighborhood of \(\mathcal{M}_{\lambda_0}\), was proved in [5].

We consider when in addition \(\mathcal{M}_{\lambda_0} \subset O_\epsilon(\mathcal{M}_\lambda)\) and \(\epsilon \to 0\) as \(\lambda \to \lambda_0\). The basis of this theorem [2] are the characteristics of the function of rate of attraction in a some small \(\delta\)-neighborhood of \(\lambda_0\).

We assume that the following conditions (a) hold:

1. \(\Lambda\) is compact with metric \(\|\cdot\|_\Lambda\) and \(\lambda_0\) is a nonisolated point of \(\Lambda\).

2. For each \(\lambda \in \Lambda\) the semigroup \(\{X, \Theta_{\cdot}, S_\lambda(\cdot)\}\) possesses a pointwise absorbing set \(B_\lambda\) and non-empty attractor \(\mathcal{M}_\lambda\).

3. There exists a bounded absorbing set \(B_\lambda\) and for each \(\lambda \in \Lambda\) a set \(B_\lambda\) belongs to the set \(B_\lambda\).

By definition, each \(\epsilon\)-neighbourhood \(O_\epsilon(\mathcal{M}_\lambda)\) is an absorbing set. Assume that we know a function \(\Theta(\lambda, \epsilon) = \Theta(\lambda, \epsilon, B_\lambda)\) such that

\[
\text{dist}(S_\lambda(t, B_\lambda), \mathcal{M}) \leq \epsilon, \quad \text{as } t \geq \Theta(\epsilon, \lambda)
\]

**Lemma 4** Under the assumptions (a) let there exist \(\epsilon > 0, \lambda_1, \lambda_2 \in \Lambda\) and a point \(T \geq \Theta(\lambda_1, \epsilon)\) such that for the operators \(S_{\lambda_1}(t)\) and \(S_{\lambda_2}(t)\) the following estimate is valid

\[
\|S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)\| < \epsilon \quad \text{for any } u \in B_\lambda
\]

Then

\[
\mathcal{M}_{\lambda_2} \subset O_{2\epsilon}(\mathcal{M}_{\lambda_1})
\]
Remark. Without loss of generality, assume that \( O_{2\varepsilon}(M_{\lambda_1}) \subset B_\alpha. \)

Proof. Suppose that estimate (3) holds for a \( T \geq \Theta(\lambda_1, \varepsilon) \) with some \( \varepsilon > 0 \). The set \( M_{\lambda_1} \) is the attractor for \( \{ X, \Theta_+, S_{\lambda_1}(\cdot) \} \) and for the \( B_\alpha \) we can find an attraction time \( T = \Theta(\lambda_1, \varepsilon) \) to the \( \varepsilon \)-neighbourhood of \( M_{\lambda_1} \):

\[
S_{\lambda_1}(T, u) \subset O_\varepsilon(M_{\lambda_1}), \forall u \in B_\alpha
\]

Furthermore, we obtain \( S_{\lambda_1}(T + t, u) \subset O_\varepsilon(M_{\lambda_1}) \) for any \( t \geq 0 \). Due to (3) we have \( \|S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)\| < \varepsilon \). Hence

\[
S_{\lambda_2}(T, u) \in O_{2\varepsilon}(M_{\lambda_1})
\]

The above injection and \( S_{\lambda_2}(T, u) \in B_\alpha \) gives us

\[
\|S_{\lambda_1}(T, S_{\lambda_2}(T, u)) - S_{\lambda_2}(T, S_{\lambda_2}(T, u))\| < \varepsilon \quad \text{for any} \quad u \in B_\alpha
\]

Since \( S_{\lambda_1}(T, v) \in O_\varepsilon(M_{\lambda_1}) \) for each \( v \in B_\alpha \) we have \( S_{\lambda_2}(T, S_{\lambda_2}(T, u)) = S_{\lambda_2}(2T, u) \in O_{2\varepsilon}(M_{\lambda_1}) \). After a finite number of steps we obtain \( S_{\lambda_2}(nT, u) \in O_{2\varepsilon}(M_{\lambda_1}) \) as \( n = 1, 2, ... \) for any \( u \in B_\alpha \). This, together with Lemma 1, implies

\[
M_{\lambda_2} \subset O_{2\varepsilon}(M_{\lambda_1}).
\]

This completes the proof.

The next theorem provides an estimate for the distance between two attractors.

**Theorem 2** Under the assumptions \( (\alpha) \)

(i) Assume, that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) and a point \( T_{\lambda_0} \geq \Theta(\lambda_0, \varepsilon) \) such that

\[
\|S_{\lambda}(T_{\lambda_0}, u) - S_{\lambda_0}(T_{\lambda_0}, u)\| < \varepsilon \quad \forall u \in B_\alpha, \forall \lambda \in O_\delta(\lambda_0)
\]

Then the attractor \( M_\lambda \) is upper semicontinuous in the point \( \lambda_0 \) and the following estimate holds

\[
\text{dist}(M_\lambda, M_{\lambda_0}) \leq 2\varepsilon.
\]

(ii) Assume, that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

for arbitrary \( \lambda \in O_\delta(\lambda_0) \) there exists a point \( T_\lambda = T(\lambda) \geq \Theta(\lambda, \varepsilon) \) satisfies the following estimate

\[
\|S_{\lambda}(T_\lambda, u) - S_{\lambda_0}(T_\lambda, u)\| < \varepsilon \quad \forall u \in B_\alpha
\]

Then the attractor \( M_\lambda \) is lower and upper semicontinuous in the point \( \lambda_0 \) and the following estimate holds

\[
\max \{\text{dist}(M_{\lambda_0}, M_\lambda), \text{dist}(M_\lambda, M_{\lambda_0})\} \leq 2\varepsilon.
\]

Proof. According to (4) and assumption (i) we have that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for a \( \lambda \in O_\delta(\lambda_0) \) we have \( M_\lambda \subset O_{2\varepsilon}(M_{\lambda_0}) \). Thus,

\[
\text{dist}(M_\lambda, M_{\lambda_0}) = \sup_{u \in M_\lambda} (u, M_{\lambda_0}) \leq 2\varepsilon.
\]
As $\varepsilon \to 0$ we obtain $\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \to 0$. Thus, inequality (5) is proved.

In order to prove (7), from (5) and (ii) we have

$$\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon$$

To obtain the reverse inequality

$$\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) \leq 2\varepsilon \to 0 \quad \text{as } \lambda \to \lambda_0$$

use the assumption (ii) for arbitrary $\varepsilon > 0$ and give $O_\delta(\lambda_0)$ such that for any $\lambda \in O_\delta(\lambda_0)$ inequality (6) holds in a point $T_\lambda \geq \Theta(\lambda, \varepsilon)$. This, together with Lemma 4 implies

$$\mathcal{M}_{\lambda_0} \subset O_{2\varepsilon}(\mathcal{M}_\lambda) \quad \forall \lambda \in O_\delta(\lambda_0).$$

As $\varepsilon \to 0$ we have

$$\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = \sup_{u \in \mathcal{M}_{\lambda_0}} \langle u, \mathcal{M}_\lambda \rangle \to 0.$$

This completes the proof.

**Remark 1.** For a semigroup with compact attractor upper semicontinuity was proved in [5] when (5) is valid in an some point $T_{\lambda_0} > 0$. But, we can not find the rate of convergence for $\mathcal{M}_\lambda \to \mathcal{M}_{\lambda_0}$. When inequality (4) holds for $t \in [\tau, \tau + T_{\lambda_0}]$ for any $\tau > 0$, uniformly in $\tau$, then (5) was proved in [10].

**Remark 2.** The assumption (ii) allows $T_\lambda \to \infty$ as $\lambda \to \lambda_0$.

**Remark 3.** A function rate of attraction $\Theta(\cdot)$ can be approximate, for example see [11], for some nontrivial PDE system with nontrivial attractor.

A sequence of operators $S_{\lambda}(\cdot)$ is called locally converging in a point $\lambda_0$ on a bounded $B$ if for any $\varepsilon > 0$ and for each $T > 0$ there exists $\delta$ such that

$$\|S_\lambda(T, u) - S_{\lambda_0}(T, u)\| \leq \varepsilon \quad \forall \lambda \in O_\delta(\lambda_0), \forall u \in B.$$

Theorem 2 and the above definition imply

**Theorem 3** Under the assumptions (a) let the sequence of operators $S_{\lambda}(\cdot)$ be locally converging in a point $\lambda_0$ on a set $B_\alpha$. Let the function $\Theta(\lambda, \varepsilon)$ be uniformly bounded on $\lambda$ for each fixed $\varepsilon > 0$, i.e.

$$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \sup_{\lambda \in O_\delta(\lambda_0)} \Theta(\lambda, \varepsilon, B_\alpha) \leq T_\varepsilon < \infty.$$

Then, the attractor $\mathcal{M}_\lambda$ depends on $\lambda$ in the point $\lambda_0$ continuously.

**Remark.** In this way, if we cannot calculate a function $\Theta(\lambda, \varepsilon)$, but in a some small neighbourhood of the $\lambda_0$ we may find above estimate for $\Theta(\cdot)$ and sequence of operators $S_{\lambda}$ is locally converging to the $S_{\lambda_0}$ as $\lambda \to \lambda_0$ then the attractor $\mathcal{M}_{\lambda_0}$ is continuous in $\lambda_0$.

A sequence of operators $S_{\lambda}(\cdot)$ is called globally converging in a point $\lambda_0$ on a bounded set $B$ if for any $\varepsilon > 0$ and for any $T > 0$ there exists $\delta$ such that the following estimate holds

$$\|S_{\lambda}(t, u) - S_{\lambda_0}(t, u)\| \leq \varepsilon \quad \forall \lambda \in O_{\delta_\varepsilon}(\lambda_0), \forall u \in B, \ 0 \leq t \leq T.$$

(8)
Remark. For the ODE $y'(t) = -\alpha_n y(t)$, as $\alpha_n \to 0^+$, we have globally converging operators. It is easy to verify that the attractor $M_0$ in $\lambda_0 = 0$ is \textit{upper semicontinuous} but not \textit{lower semicontinuous}.

Let us prove a criterion (see [8]) when an attractor for sequence of globally converging operators is continuous (lower and upper semicontinuous) in a point $\lambda_0$.

**Theorem 4** Under the assumptions (a) assume, that a sequence of operators $S_\lambda(\cdot)$ is globally converging in $\lambda_0$ on the set $B_a$. Then the attractor $M_\lambda$ depends continuously on $\lambda$ in the point $\lambda_0$ if and only if a function $\Theta(\lambda, \varepsilon)$ is uniformly bounded on $\lambda$ for each fixed $\varepsilon > 0$, i.e.

$$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \sup_{\lambda \in O_{\delta_\varepsilon}(\lambda_0)} \Theta(\lambda, \varepsilon, B_a) \leq T_\varepsilon < \infty.$$  \hfill (9)

**Proof.** Theorem 2 together with (8), (9) implies the continuity of the attractor. Suppose, that attractor depends continuously on $\lambda$ in $\lambda_0$. This implies

$$\forall \varepsilon > 0 \exists \delta_1(\varepsilon) > 0 : M_{\lambda_0} \subset O_\varepsilon(M_\lambda), \ M_\lambda \subset O_\varepsilon(M_{\lambda_0}), \ \forall \lambda \in O_{\delta_1}(\lambda_0).$$  \hfill (10)

Under the definition of a function $\Theta(\cdot)$ for $T = \Theta(\lambda_0, \varepsilon)$ we have

$$S_{\lambda_0}(T + \tau, u) \subset O_\varepsilon(M_{\lambda_0}), \ \forall \tau > 0.$$  

This, together with (8), implies that there exists $\delta_2(\varepsilon, T)$ such that

$$\|S_\lambda(t, u) - S_{\lambda_0}(t, u)\| \leq \varepsilon \ \forall \lambda \in O_{\delta_2}(\lambda_0), \forall u \in B, t \leq T.$$  

Choose $\delta$ in the following way $\delta = \min\{\delta_1, \delta_2\}$. Then

$$S_{\lambda_0}(T, B_a) \subset O_\varepsilon(M_{\lambda_0}) \subset O_{2\varepsilon}(M_\lambda) \ \forall \lambda \in O_\delta(\lambda_0).$$

This, together with (10), gives us

$$S_{\lambda}(T, B_a) \subset O_{3\varepsilon}(M_\lambda) \ \forall \lambda \in O_\delta(\lambda_0).$$  \hfill (11)

Combine (11) with (10) for any $\tau > 0$ and use the same arguments as in the proof of Theorem 2 to obtain

$$S_{\lambda}(T + \tau, B_a) \subset O_{3\varepsilon}(M_\lambda) \ \forall \lambda \in O_\delta(\lambda_0), \tau > 0.$$  

The last estimate means that function $\Theta(\lambda_0, \varepsilon)$ is a function of attraction to the $3\varepsilon$-neighbourhood of the attractor $M_\lambda$ for any $\lambda \in O_\delta(\lambda_0)$. Thus, the function $\Theta(\lambda, \varepsilon) \leq \Theta(\lambda_0, \varepsilon/3) < \infty$ for $\forall \lambda \in O_\delta(\lambda_0)$, which proved the Theorem.

**4 On a function of rate of attraction to a attractor**

Here we consider the example of a semigroup having global attractor and known function of rate of attraction to attractor, namely, 1D Chafee-Infante problem

$$u_t = u_{xx} + bu - f(u), \ b > 0, \ f(\cdot) \in C^1(\cdot)$$
\[ u(0,t) = u(\pi,t) = 0; \quad u(x,0) = u_0(x) \in H^1_0(0,\pi) \]
\[ f(0) = f'(0) = 0, \quad s^{-1} f(s) < f'(s), s \neq 0; \quad S^{-1} f(s) \to +\infty, \text{ as } |s| \to \infty. \]

The Chafee-Infante problem generated semigroup in the phase space \( H^1_0(0,\pi) \). The following theorem was given in [6].

**Theorem 5** Under the above assumptions, if \( b \neq m^2, m = 1, 2, \ldots \), then the attractor \( \mathcal{M} \) attracts its neighbourhood exponentially

\[ \text{dist}(S(t,B),\mathcal{M}) \leq C e^{-\alpha t}, \quad C, \alpha = \text{const} > 0 \]

If \( b = m^2 \) and in addition

\[ K^{-1}(|s| + |t|)^q(s - t)^2 \leq (f(s) - f(t))(s - t) \leq K(|s| + |t|)^q(s - t)^2 \]

for some positive \( K, q, s_0 \) and all \( s, t \in [-s_0, s_0] \), then the attractor \( \mathcal{M} \) attracts its neighbourhood polynomially

\[ \text{dist}(S(t,B),\mathcal{M}) \leq D(1 + t)^{-\alpha}, \quad D, \alpha = \text{const} > 0. \]

This, together with Theorem 4, implies continuity of the attractor \( \mathcal{M} \) under different types of perturbations of the original operator \( S(\cdot) \). For example

\[ u_t = u_{xx} + (b + \delta)u - f(u) + \delta_1(u) \]

or difference approximations with steps \( \tau, h \), when the constant \( a, \alpha, C, D \) are bounded as \( \delta, \delta_1(\cdot), \tau, h \to 0 \) (see [12]).

Let us consider the Navier-Stokes equations

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u &= -\nabla p + f, \quad \nu \neq 0 \\
\text{div } u &= 0 \\
u |_{\partial \Omega} &= 0; \quad u(t = 0, x) = u_0(x)
\end{align*}
\]

(12)

in an arbitrary bounded \( \Omega \subset \mathbb{R}^2 \). Here \( x = (x_1, x_2) \), \( u = (u_1, u_2) \) is the velocity vector, \( p \) is the pressure, \( \nu = \text{const} > 0 \). Existence of the global attractor \( \mathcal{M} \) for (12),(13) was first studied by Ladyzhenskaya in [2]. In [1] it was proved, that some approximations of (12),(13) have global attractors \( \mathcal{M}_\varepsilon \) in \( \varepsilon \)-neighborhoods of \( \mathcal{M} \).

Babin an Vishik in [4] established a theorem on upper semicontinuity of the attractor \( \mathcal{M}_\lambda \) for \( \lambda = (\nu, f) \). Following [2, 1] it is easy to verify the assumptions (\( \alpha \)) when \( \lambda = (\nu, f)(\text{or } \lambda = (\tau, h)) \) and globally converging \( S_\lambda(\cdot) \) to \( S_{\lambda_0} \) when \( \lambda \to \lambda_0 \). The properties of the function \( \Theta(\cdot) \) for the problem (12),(13) is unknown. However, when the operator \( S_\lambda(\cdot) \) is upper semicontinuous on a compact set \( \Lambda \) then it is continuous on a some everywhere dense subset \( \Lambda \subset \Lambda \). Thus, for any \( \lambda \) and each \( \varepsilon > 0 \) there exists \( \lambda_0 \) and \( \delta = \delta(\varepsilon) \) such that

\[ ||\lambda - \lambda_0|| < \varepsilon \quad \text{and} \quad \sup_{\Lambda_\varepsilon(\lambda_0)} \Theta(\lambda_0, \varepsilon) \leq T_\varepsilon < \infty. \]
The above results are valid (see [7]) for some modifications of the systems (12), (13) in 3D case.

Acknowledgments. This research was supported in part by the NWO-RFBR programme doss.nr. 047.003.017 and RFBR 99-01-00263.
I would like to express my gratitude to Professor A.O.H. Axelsson for his support.

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