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On the continuity property for an attractor of a semidynamical system with a parameter

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On the continuity property for an attractor of a semidynamical system with a parameter

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Abstract

An approach to verify continuity of a global attractor of a semidynamical system with a parameter is presented. This approach makes it possible to establish a connection between upper and lower semicontinuity of a global attractor and boundedness as a function of rate of attraction to an attractor. The obtained results were used for the scalar Lorenz wave equation in 3D space, the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

1 Introduction

In the present paper for a semigroup \( \{ S_\lambda(t, \cdot) \} \) corresponding, for example, to a evolution equation, where \( \lambda \) is a problem parameter, and having global attractor the problem of stability of \( M_{\lambda o} \) as \( \lambda \to \lambda_0 \) is considered. The most powerful test for the attractor \( M_{\lambda} \) being in \( O_\varepsilon(M_{\lambda_0}) \), with \( O_\varepsilon(M_{\lambda_0}) \) the \( \varepsilon \)-neighborhood of \( M_{\lambda_0} \), was proved by Kapitanskii and Kostin.

The main purpose of the current work is to consider a criterion when in addition \( M_{\lambda_0} \subset O_\varepsilon(M_{\lambda}) \) and \( \varepsilon \to 0 \) as \( \lambda \to \lambda_0 \). The basis of this theorem are the characteristics of the function of rate of attraction \( \Psi(\lambda, t) \) in some small \( \delta \)-neighborhood of \( \lambda_0 \)

\[
dist(S_\lambda(t, B_\delta), M_{\lambda}) \leq \Psi(\lambda, t)dist(B_\delta, M_{\lambda}), \quad t \geq 0, \ \lambda \in O_\varepsilon(\lambda_0).
\]

Here \( B_\delta \) is a bounded absorbing set and \( S_\lambda(t, \cdot) \) is an approximation of the given nonlinear operator \( S_{\lambda_0}(t, \cdot) \).

These results are applied in Section 4 to the 1D Chafee-Infante problem and the 2D Navier-Stokes equation.

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2 Asymptotically compact semigroups

Let $X$ be a Banach space with norm $\| \cdot \|$, $\Theta$ be a nontrivial subgroup of real number $R$ and let $\Theta_+ = \Theta \cap [0, +\infty[$ be the intersection of $\Theta$ and $R_+$. We shall deal with the abstract semigroup $\{X, \Theta_+, S(\cdot)\}$ of nonlinear operator $S : X \times \Theta_+ \to X$. The term semigroup or semidynamical system refers to any family of singlevalued continuous operator $S$ depending on a parameter $t \in \Theta_+$ and enjoying the semigroup property:

$$S(t_1, S(t_2, u)) = S(t_1 + t_2, u), \quad \forall t_1, t_2 \in \Theta_+, \forall u \in X$$

A Banach space $X$ is a phase space of a semigroup, $\Theta_+$ is a time space and $S(\cdot)$ is an evolution operator. When $\Theta = R$ a semigroup is a semigroup with continuous time.

Let $B$ and $M$ be bounded subsets of $X$. We say that $B$ is attracted to $M$ by the semigroup $S(\cdot)$ if

$$\lim_{t \to \infty} \text{dist}(S(t, B), M) = 0$$

Here

$$\text{dist}(A, B) = \sup_{y \in A} \{\text{dist}(y, B)\}, \quad \text{dist}(y, B) = \inf_{x \in B} \|x - y\|$$

A set $M$ is called an attracting set of the semigroup if $M$ attracts each bounded $B \subset X$. The minimal among the closed attracting sets is called the global attractor [9] (minimal global $B$-attractor [3]). The global attractor of a semigroup is defined as the set $\mathcal{M}$ which is compact in $X$, invariant for $S(\cdot)$, i.e.

$$S(t, \mathcal{M}) = \mathcal{M}, \quad t \geq 0$$

and which attracts all the bounded sets of $X$.

We need the following lemmas (see [5, 6]):

**Lemma 1** Let $\mathcal{M}$ be a compact attractor of the semigroup $\{X, \Theta_+, S(\cdot)\}$, $\mathcal{N}$ be a compact attractor of the semigroup $\{X, \check{\Theta}_+, \check{S}(\cdot)\}$. Let $\check{\Theta}_+$ be a subsemigroup of $\Theta_+$ and let for some points $t, \check{t}$

$$S(t, u) = \check{S}(\check{t}, u) \quad \text{for all } u \in X$$

Then $\mathcal{M} = \mathcal{N}$.

Lemma 1 implies

**Lemma 2** Let $\mathcal{M}$ be a compact attractor of the semigroup $\{X, R_+, S(\cdot)\}$ and let $t_0 > 0$. Then $\mathcal{M}$ is attractor of the semigroup $\{X, t_0Z_+, S(\cdot)\}$. Here $t_0Z_+ \equiv \{kt_0, k \in Z_+\}$.

Later on we need the following definitions, see [3].

A set $B_\varepsilon$ is called absorbing if for each bounded $B \subset X$ and for each $\varepsilon > 0$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset B_\varepsilon, \quad \forall t \geq T$$

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If a semigroup possesses a nonempty bounded attractor $\mathcal{M}$ then for arbitrary $\varepsilon > 0$ the set $O_{\varepsilon}(\mathcal{M})$ is an absorbing set. Here $O_{\varepsilon}(\mathcal{M})$ is the $\varepsilon$-neighbourhood of $\mathcal{M}$, i.e.

$$O_{\varepsilon}(\mathcal{M}) = \{ u : \exists v \in \mathcal{M}, \| u - v \| < \varepsilon \}$$

A semigroup is called bounded if for each bounded $B$ the set $S(t, B)$ is bounded for any $t > 0$.

A semigroup is called pointwise dissipative if it has a pointwise absorbing set $B_0$

$$\forall x \in X, \exists T(x) : S(t, x) \subset B_0, \text{ for any } t \geq T(x)$$

A semigroup is called asymptotically compact if for each bounded $B$ such that $S(t, B)$ is bounded for any $t > 0$ each sequence of the form

$$\{ S(t_k, u_k) \}_{k=1}^{\infty}, t_k \uparrow \infty, u_k \in B$$

is precompact.

The following theorem holds, see [3].

**Theorem 1** Let the semigroup $\{ X, \Theta, S(\cdot) \}$ be a continuous bounded pointwise dissipative asymptotically compact semigroup. Then there exists a non-empty attractor $\mathcal{M}$

$$\mathcal{M} = \bigcap_{t \geq 0} [S(t, B_0)]_X$$

$\mathcal{M}$ is compact and invariant. If $X$ is connected then $\mathcal{M}$ is also connected.

We now summarize the results:

**Lemma 3** Let $S(t) : X \to X, t \in \mathbb{R}_+$ be a continuous semigroup possessing a non-empty compact attractor $\mathcal{M}$. Then this semigroup is an asymptotically compact semigroup.

**Proof.** We shall prove that each sequence of the form

$$\{ S(t_k, u_k) \}_{k=1}^{\infty}, t_k \uparrow \infty, u_k \in B$$

can be covered by a finite $\varepsilon$-network where $\varepsilon$ is arbitrary small positive number. The set $O_{\varepsilon/2}(\mathcal{M})$ is an absorbing set and for any bounded $B$ there exists $T = T(\varepsilon, B)$ such that

$$S(t, B) \subset O_{\varepsilon/2}(\mathcal{M}) \text{ for any } t \geq T.$$ 

Let us choose $N$ so large that the set $\{ S(t_k, u_k) \}_{k=N}^{\infty}, u_k \in B$ belongs to $O_{\varepsilon/2}(\mathcal{M})$. The set

$$V^\mathcal{M} = \{ v_k^\mathcal{M} \in \mathcal{M} : \| v_k^\mathcal{M} - u_k \| \leq \varepsilon/2, k \geq N \} \subset \mathcal{M}$$

is a precompact set and may be covered by a finite $\varepsilon/2$-network $\tilde{V}^\mathcal{M} = \{ \tilde{v}_i \in V^\mathcal{M} \}_{i=1}^{n}$ and for any $k$ there exists $i$ such that:

$$\| v_k^\mathcal{M} - \tilde{v}_i \| \leq \varepsilon/2, \quad k \geq N$$
Thus the set $\tilde{V}^M$ is a finite $\varepsilon$-network of the set $\{S(t_k, u_k)\}_{k=1}^\infty$ and then the set $\{S(t_k, u_k)\}_{k=1}^\infty$ is a compact set.

Note, if a semigroup possesses a nonempty attractor $\mathcal{M}$ then the semigroup is pointwise dissipative.

### 3 Semigroups with a parameter

The current paragraph deals with the problem of stability of $\mathcal{M}$ with respect to perturbations of the original operator $S(\cdot)$.

Let us consider a semigroup $S_\lambda(\cdot) : X \to X$ which depends on a parameter $\lambda \in \Lambda$. The main purpose is to consider a criterion when the set $\mathcal{M}\lambda_0$ and $\mathcal{M}\lambda$ are close to each other in the Hausdorff metric, that is

$$\lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}\lambda, \mathcal{M}\lambda_0) = 0 \quad (1)$$

$$\lim_{\lambda \to \lambda_0} \text{dist}(\mathcal{M}\lambda_0, \mathcal{M}\lambda) = 0 \quad (2)$$

The relations (1) and (2) are usually referred to as the upper and lower semicontinuity of the attractor $\mathcal{M}\lambda$ in $\lambda_0$.

The most powerful test for the attractor $\mathcal{M}\lambda$ being in $O_\varepsilon(\mathcal{M}\lambda_0)$ with $O_\varepsilon(\mathcal{M}\lambda_0)$ the $\varepsilon$-neighborhood of $\mathcal{M}\lambda_0$, was proved in [5].

We consider when in addition $\mathcal{M}\lambda_0 \subset O_\varepsilon(\mathcal{M}\lambda)$ and $\varepsilon \to 0$ as $\lambda \to \lambda_0$. The basis of this theorem [2] are the characteristics of the function of rate of attraction in a some small $\delta$-neighborhood of $\lambda_0$.

We assume that the following conditions (a) hold:

1. $\Lambda$ is compact with metric $\|\cdot\|_\Lambda$ and $\lambda_0$ is a nonisolated point of $\Lambda$.
2. For each $\lambda \in \Lambda$ the semigroup $\{X, \Theta_+, S_\lambda(\cdot)\}$ possesses a pointwise absorbing set $B_\lambda$ and non-empty attractor $\mathcal{M}\lambda$.
3. There exists a bounded absorbing set $B_\sigma$ and for each $\lambda \in \Lambda$ a set $B_\lambda$ belongs to the set $B_\sigma$.

By definition, each $\varepsilon$-neighbourhood $O_\varepsilon(\mathcal{M}\lambda)$ is an absorbing set. Assume that we know a function $\Theta(\lambda, \varepsilon) = \Theta(\lambda, \varepsilon, B_\sigma)$ such that

$$\text{dist}(S_\lambda(t, B_\sigma), M) \leq \varepsilon, \quad \text{as } t \geq \Theta(\varepsilon, \lambda)$$

**Lemma 4** Under the assumptions (a) let there exist $\varepsilon > 0$, $\lambda_1, \lambda_2 \in \Lambda$ and a point $T \geq \Theta(\lambda_1, \varepsilon)$ such that for the operators $S_{\lambda_1}(t)$ and $S_{\lambda_2}(t)$ the following estimate is valid

$$\|S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)\| < \varepsilon \quad \text{for any } u \in B_\sigma$$

Then

$$\mathcal{M}_{\lambda_2} \subset O_{2\varepsilon}(\mathcal{M}_{\lambda_1})$$
Remark. Without loss of generality, assume that $O_{2\varepsilon}(M_{\lambda_1}) \subset B_\delta$.

Proof. Suppose that estimate (3) holds for a $T \geq \Theta(\lambda_1, \varepsilon)$ with some $\varepsilon > 0$. The set $M_{\lambda_1}$ is the attractor for $\{X, \Theta_+, S_{\lambda_1}()\}$ and for the $B_\delta$ we can find an attraction time $T = \Theta(\lambda_1, \varepsilon)$ to the $\varepsilon$-neighbourhood of $M_{\lambda_1}$

$$S_{\lambda_1}(T, u) \subset O_\varepsilon(M_{\lambda_1}), \forall u \in B_\delta$$

Furthermore, we obtain $S_{\lambda_1}(T + t, u) \subset O_\varepsilon(M_{\lambda_1})$ for any $t \geq 0$. Due to (3) we have $||S_{\lambda_1}(T, u) - S_{\lambda_2}(T, u)|| < \varepsilon$. Hence

$$S_{\lambda_2}(T, u) \in O_{2\varepsilon}(M_{\lambda_1})$$

The above injection and $S_{\lambda_2}(T, u) \in B_\delta$ gives us

$$||S_{\lambda_1}(T, S_{\lambda_2}(T, u)) - S_{\lambda_2}(T, S_{\lambda_2}(T, u))|| < \varepsilon$$

for any $u \in B_\delta$

Since $S_{\lambda_1}(T, v) \in O_\varepsilon(M_{\lambda_1})$ for each $v \in B_\delta$ we have $S_{\lambda_2}(T, S_{\lambda_2}(T, u)) = S_{\lambda_2}(2T, u) \in O_{2\varepsilon}(M_{\lambda_1})$. After a finite number of steps we obtain $S_{\lambda_2}(nT, u) \in O_{2\varepsilon}(M_{\lambda_1})$ as $n = 1, 2, ...$ for any $u \in B_\delta$. This, together with Lemma 1, implies

$$M_{\lambda_2} \subset O_{2\varepsilon}(M_{\lambda_1})$$

This completes the proof.

The next theorem provides an estimate for the distance between two attractors.

Theorem 2 Under the assumptions (a)

(i) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ and a point $T_{\lambda_0} \geq \Theta(\lambda_0, \varepsilon)$ such that

$$||S_{\lambda}(T_{\lambda_0}, u) - S_{\lambda_0}(T_{\lambda_0}, u)|| < \varepsilon \forall u \in B_\delta, \forall \lambda \in O_\delta(\lambda_0)$$

(4)

Then the attractor $M_{\lambda}$ is upper semicontinuous in the point $\lambda_0$ and the following estimate holds

$$\text{dist}(M_{\lambda}, M_{\lambda_0}) \leq 2\varepsilon.$$  (5)

(ii) Assume, that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

for arbitrary $\lambda \in O_\delta(\lambda_0)$ there exists a point $T_{\lambda} = T(\lambda) \geq \Theta(\lambda, \varepsilon)$ satisfies the following estimate

$$||S_{\lambda}(T_{\lambda}, u) - S_{\lambda_0}(T_{\lambda}, u)|| < \varepsilon \forall u \in B_\delta$$

(6)

Then the attractor $M_{\lambda}$ is lower and upper semicontinuous in the point $\lambda_0$ and the following estimate holds

$$\max \{\text{dist}(M_{\lambda_0}, M_{\lambda}), \text{dist}(M_{\lambda}, M_{\lambda_0})\} \leq 2\varepsilon.$$  (7)

Proof. According to (4) and assumption (i) we have that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for a $\lambda \in O_\delta(\lambda_0)$ we have $M_{\lambda} \subset O_{2\varepsilon}(M_{\lambda_0})$. Thus,

$$\text{dist}(M_{\lambda}, M_{\lambda_0}) = \sup_{u \in M_{\lambda}} (u, M_{\lambda_0}) \leq 2\varepsilon.$$
As $\varepsilon \to 0$ we obtain $\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \to 0$. Thus, inequality (5) is proved.

In order to prove (7), from (5) and (ii) we have

$$\text{dist}(\mathcal{M}_\lambda, \mathcal{M}_{\lambda_0}) \leq 2\varepsilon$$

To obtain the reverse inequality

$$\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) \leq 2\varepsilon \to 0 \text{ as } \lambda \to \lambda_0$$

use the assumption (ii) for arbitrary $\varepsilon > 0$ and give $O_\delta(\lambda_0)$ such that for any $\lambda \in O_\delta(\lambda_0)$ inequality (6) holds in a point $T_\lambda \geq \Theta(\lambda, \varepsilon)$. This, together with Lemma 4 implies

$$\mathcal{M}_{\lambda_0} \subset O_{2\varepsilon}(\mathcal{M}_\lambda), \forall \lambda \in O_\delta(\lambda_0).$$

As $\varepsilon \to 0$ we have

$$\text{dist}(\mathcal{M}_{\lambda_0}, \mathcal{M}_\lambda) = \sup_{u \in \mathcal{M}_{\lambda_0}} \langle u, \mathcal{M}_\lambda \rangle \to 0.$$

This completes the proof.

**Remark 1.** For a semigroup with compact attractor *upper semicontinuity* was proved in [5] when (5) is valid in an *some* point $T_{\lambda_0} > 0$. But, we cannot find the rate of convergence for $\mathcal{M}_\lambda \to \mathcal{M}_{\lambda_0}$. When inequality (4) holds for $t \in [\tau, \tau + T_{\lambda_0}]$ for any $\tau > 0$, uniformly in $\tau$, then (5) was proved in [10].

**Remark 2.** The assumption (ii) allows $T_\lambda \to \infty$ as $\lambda \to \lambda_0$.

**Remark 3.** A function rate of attraction $\Theta(\cdot)$ can be approximate, for example see [11], for some nontrivial PDE system with nontrivial attractor.

A sequence of operators $S_\lambda(\cdot)$ is called *locally converging* in a point $\lambda_0$ on a bounded $B$ if for any $\varepsilon > 0$ and for each $T > 0$ there exists $\delta$ such that

$$||S_\lambda(T, u) - S_{\lambda_0}(T, u)|| \leq \varepsilon \quad \forall \lambda \in O_\delta(\lambda_0), \forall u \in B.$$

Theorem 2 and the above definition imply

**Theorem 3** Under the assumptions (a) let the sequence of operators $S_\lambda(\cdot)$ be locally converging in a point $\lambda_0$ on a set $B_{\delta}$. Let the function $\Theta(\lambda, \varepsilon)$ be uniformly bounded on $\lambda$ for each fixed $\varepsilon > 0$, i.e.

$$\forall \varepsilon > 0 \exists \delta > 0 : \sup_{\lambda \in O_\delta(\lambda_0)} \Theta(\lambda, \varepsilon, B_{\delta}) \leq T_\varepsilon < \infty.$$

Then, the attractor $\mathcal{M}_\lambda$ depends on $\lambda$ in the point $\lambda_0$ continuously.

**Remark.** In this way, if we cannot calculate a function $\Theta(\lambda, \varepsilon)$, but in a some small neighbourhood of the $\lambda_0$ we may find above estimate for $\Theta(\cdot)$ and sequence of operators $S_\lambda$ is locally converging to the $S_{\lambda_0}$ as $\lambda \to \lambda_0$ then the attractor $\mathcal{M}_{\lambda_0}$ is continuous in $\lambda_0$.

A sequence of operators $S_\lambda(\cdot)$ is called *globally converging* in a point $\lambda_0$ on a bounded set $B$ if for any $\varepsilon > 0$ and for any $T > 0$ there exists $\delta$ such that the following estimate holds

$$||S_\lambda(t, u) - S_{\lambda_0}(t, u)|| \leq \varepsilon \quad \forall \lambda \in O_{\delta_\varepsilon}(\lambda_0), \forall u \in B, 0 \leq t \leq T.$$  (8)
Remark. For the ODE $y'(t) = -\alpha_n y(t)$, as $\alpha_n \to 0^+$, we have globally converging operators. It is easy to verify that the attractor $M_0$ in $\lambda_0 = 0$ is upper semicontinuous but not lower semicontinuous.

Let us prove a criterion (see [8]) when an attractor for sequence of globally converging operators is continuous (lower and upper semicontinuous) in a point $\lambda_0$.

**Theorem 4** Under the assumptions (a) assume, that a sequence of operators $S_\lambda(\cdot)$ is globally converging in $\lambda_0$ on the set $B_a$. Then the attractor $M_\lambda$ depends continuously on $\lambda$ in the point $\lambda_0$ if and only if a function $\Theta(\lambda, \varepsilon)$ is uniformly bounded on $\lambda$ for each fixed $\varepsilon > 0$, i.e.

$$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \sup_{\lambda \in \Omega(\lambda_0)} \Theta(\lambda, \varepsilon, B_a) \leq T_\varepsilon < \infty.$$  \hfill (9)

**Proof.** Theorem 2 together with (8), (9) implies the continuity of the attractor. Suppose, that attractor depends continuously on $\lambda$ in $\lambda_0$. This implies

$$\forall \varepsilon > 0 \exists \delta_1(\varepsilon) > 0 : M_{\lambda_0} \subset O_\varepsilon(M_{\lambda_0}), \ M_\lambda \subset O_\varepsilon(M_{\lambda_0}), \ \forall \lambda \in \Omega(\lambda_0).$$  \hfill (10)

Under the definition of a function $\Theta(\cdot)$ for $T = \Theta(\lambda_0, \varepsilon)$ we have

$$S_{\lambda_0}(T + \tau, u) \subset O_\varepsilon(M_{\lambda_0}), \ \forall \tau > 0.$$  

This, together with (8), implies that there exists $\delta_2(\varepsilon, T)$ such that

$$\|S_\lambda(t, u) - S_{\lambda_0}(t, u)\| \leq \varepsilon \ \forall \lambda \in \Omega(\lambda_0), \ \forall \lambda \in B, t \leq T.$$  

Choose $\delta$ in the following way $\delta = \min\{\delta_1, \delta_2\}$. Then

$$S_{\lambda_0}(T, B_a) \subset O_\varepsilon(M_{\lambda_0}) \subset O_{2\varepsilon}(M_{\lambda}) \forall \lambda \in \Omega(\lambda_0).$$  

This, together with (10), gives us

$$S_{\lambda_0}(T, B_a) \subset O_{3\varepsilon}(M_{\lambda}) \forall \lambda \in \Omega(\lambda_0).$$  \hfill (11)

Combine (11) with (10) for any $\tau > 0$ and use the same arguments as in the proof of Theorem 2 to obtain

$$S_\lambda(T + \tau, B_a) \subset O_{3\varepsilon}(M_{\lambda}) \forall \lambda \in \Omega(\lambda_0), \tau > 0.$$  

The last estimate means that function $\Theta(\lambda_0, \varepsilon)$ is a function of attraction to the $3\varepsilon$-neighbourhood of the attractor $M_\lambda$ for any $\lambda \in \Omega(\lambda_0)$. Thus, the function $\Theta(\lambda, \varepsilon) \leq \Theta(\lambda_0, \varepsilon/3) < \infty$ for $\forall \lambda \in \Omega(\lambda_0)$, which proved the Theorem.

### 4 On a function of rate of attraction to a attractor

Here we consider the example of a semigroup having global attractor and known function of rate of attraction to attractor, namely, 1D Chafee-Infante problem

$$u_t = u_{xx} + bu - f(u), \quad b > 0, \ f(\cdot) \in C^1(\cdot)$$
\[u(0,t) = u(\pi, t) = 0; \quad u(x,0) = u_0(x) \in H_0^1(0, \pi)\]

\[f(0) = f'(0) = 0, \quad s^{-1}f(s) < f'(s), s \neq 0; \quad S^{-1}f(s) \to +\infty, \text{ as } |s| \to \infty.\]

The Chafee-Infante problem generated semigroup in the phase space \(H_0^1(0, \pi)\). The following theorem was given in [6].

**Theorem 5** Under the above assumptions, if \(b \neq m^2, m = 1, 2, \ldots\), then the attractor \(\mathcal{M}\) attracts its neighbourhood exponentially

\[\text{dist}(S(t, B), \mathcal{M}) \leq Ce^{-at}, \quad C, a = \text{const} > 0\]

If \(b = m^2\) and in addition

\[K^{-1}(|s| + |t|)^q(s-t)^2 \leq (f(s) - f(t))(s-t) \leq K(|s| + |t|)^q(s-t)^2\]

for some positive \(K, q, s_0\) and all \(s, t \in [-s_0, s_0]\), then the attractor \(\mathcal{M}\) attracts its neighbourhood polynomially

\[\text{dist}(S(t, B), \mathcal{M}) \leq D(1 + t)^{-\alpha}, \quad D, \alpha = \text{const} > 0.\]

This, together with Theorem 4, implies continuity of the attractor \(\mathcal{M}\) under different types of perturbations of the original operator \(S(\cdot)\). For example

\[u_t = u_{xx} + (b + \delta)u - f(u) + \delta_1(u)\]

or difference approximations with steps \(\tau, h\), when the constant \(a, \alpha, C, D\) are bounded as \(\delta, \delta_1(\cdot), \tau, h \to 0\) (see [12]).

Let us consider the Navier-Stokes equations

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u &= -\nabla p + f, \quad \nu \neq 0 \\
\text{div} u &= 0 \\
u |_{\partial \Omega} &= 0; \quad u(t = 0, x) = u_0(x)
\end{align*}
\]

in an arbitrary bounded \(\Omega \subset \mathbb{R}^2\). Here \(x = (x_1, x_2)\), \(u = (u_1, u_2)\) is the velocity vector, \(p\) is the pressure, \(\nu = \text{const} > 0\). Existence of the global attractor \(\mathcal{M}\) for (12),(13) was first studied by Ladyzhenskaya in [2]. In [1] it was proved, that some approximations of (12),(13) have global attractors \(\mathcal{M}_{\tau, h}\) in \(\varepsilon\)-neighborhoods of \(\mathcal{M}\). Babin an Vishik in [4] established a theorem on upper semicontinuity of the attractor \(\mathcal{M}_\lambda\) for \(\lambda = (\nu, f)\). Following [2, 1] it is easy to verify the assumptions (\(\alpha\)) when \(\lambda = (\nu, f)(\omega \lambda = (\tau, h))\) and globally converging \(S_{\lambda}(\cdot)\) to \(S_{\lambda_0}\) when \(\lambda \to \lambda_0\). The properties of the function \(\Theta(\cdot)\) for the problem (12),(13) is unknown. However, when the operator \(S_{\lambda}(\cdot)\) is upper semicontinuous on a compact set \(\Lambda\) then it is continuous on a some everywhere dense subset \(\Lambda \subset \Lambda\). Thus, for any \(\lambda\) and each \(\varepsilon > 0\) there exists \(\lambda_0\) and \(\delta = \delta(\varepsilon)\) such that

\[\|\lambda - \lambda_0\| < \varepsilon \quad \text{and} \quad \sup_{\lambda_n \in O_{\delta}(\lambda_0)} \Theta(\lambda_n, \varepsilon) \leq T_{\varepsilon} < \infty.\]
The above results are valid (see [7]) for some modifications of the systems (12), (13) in 3D case.

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