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Foundations of the calculus in the Netherlands, 1840–1870

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Abstract

The foundations of analysis offered by Cauchy and Riemann were not immediately welcomed by the mathematical community. Before 1870 the foundations of mathematics were considered more or less a national affair. In this paper Dutch ideas of rigour in analysis between 1840 and 1870 will be discussed.

When in 1823 Cauchy published his Résumé des leçons données à l’école Royale Polytechnique sur le calcul infinitésimal his ideas on the use of limits to obtain rigour in the calculus were not immediately welcomed by all European mathematicians. In Germany, for example, his work was not appreciated: übermäßig kompliziert was the opinion of Martin Ohm and A.L. Crelle. In England it was only after 1835 that Cauchy’s ideas became accepted by mathematicians, and his Calcul Infinitésimal was never translated. Even in France, at the École Polytechnique where Cauchy worked, his views met with a cold response: when Cauchy left the École, to follow his king in exile, his textbooks were banned from the academy.

The foundations of the calculus by Riemann were not immediately universally adopted either. Riemann’s students and devotees as late as 1863 found reasons to complain about how little of Riemann’s theories was known outside their small circle. In general, there was much controversy as to how the refinement in analysis should be pursued. Counterexamples that were produced were refuted as not being functions. The extra information that was deemed necessary by Cauchy and Riemann in order to formulate a precise theorem was regarded as obscuring the matter. Many alternative views on the foundations of analysis existed before 1880.

In this paper I will discuss the Dutch views on the foundations of the calculus from 1840 until 1870. This will allow us a glimpse at one of the alternative (national) notions on mathematical rigour that were, consciously or not, set by mathematicians.

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5 M. Kline, Mathematical Thought from Ancient to Modern Times, New York (1972), pp. 972–973
in the past. After shortly discussing Cauchy’s and Riemann’s contributions and setting the Dutch scene, I will sketch the Dutch perspective on the foundations of the calculus around the middle of the nineteenth century. Finally, I will give an overview of the cultural settings that may have contributed to this situation.

1 Cauchy and Riemann

Nowadays the *Calcul Infinitésimal* by A.L. Cauchy is considered the first step to the most common rigorous approach to the calculus, as it was finally developed by Riemann and Weierstrass. Historians consider Cauchy as the first mathematician who thoroughly understood the limit concept: he was very cautious in situations where the existence of limits was not guaranteed. His definition of the derivative for example, was by means of the limit of the ordinary difference quotient \( \frac{\Delta y}{\Delta x} \), but he added *lorsqu’elle existe*, which illustrates his understanding of the limit.

But there was more: Cauchy was the first to give a rigorous definition of “continuous function”, and a fine \( \varepsilon \delta \)-proof of the intermediate value theorem. He was very clear in rejecting the ideas of Lagrange about how to obtain rigour. In the preface to his calculus textbook he mentioned what he was going to do: he was trying to provide a rigorous calculus and this calculus could not be founded on algebraic notions.

His interpretation of the limit remained however somewhat ambivalent. Infinitesimals still played a role in his reasoning. About the derivative of a sum of functions, for example, he stated:

\[
\text{Soient toujours } x \text{ la variable indépendante et } \Delta x = \alpha h = adx \text{ un accroissement infiniment petit attribué à cette variable. Si l'on désigne par } s, u, v, w, \ldots \text{ plusieurs fonctions de } x, \text{ et par } \Delta s, \Delta u, \Delta v, \Delta w, \ldots \text{ les accroissements simultanés qu'elles reçoivent, tandis que l'on fait croître } x \text{ de } \Delta x, \text{ les différentielles } ds, du, dv, dw, \ldots \text{ seront, d’après leurs définition mêmes, respectivement égales aux limites des rapports}
\]

\[
\frac{\Delta s}{\alpha}, \frac{\Delta u}{\alpha}, \frac{\Delta v}{\alpha}, \frac{\Delta w}{\alpha}, \ldots
\]

Cela posé, concevons d’abord que la fonction \( s \) soit la somme de toutes les autres, en sorte qu’on ait

\[
s = u + v + w + \ldots
\]

On trouvera successivement

\[
\Delta s = \Delta u + \Delta v + \Delta w + \ldots,
\]

\[
\frac{\Delta s}{\alpha} = \frac{\Delta u}{\alpha} + \frac{\Delta v}{\alpha} + \frac{\Delta w}{\alpha} + \ldots,
\]

puis, en passant aux limites,

\[
ds = du + dv + dw + \ldots
\]

---


Although he sometimes made use of infinitesimals in his reasoning, his work as a whole may be viewed as a thorough and successful moulding of the methods of the calculus into the Euclidean form. There was some unfinished business, but in general existence proofs were linked beautifully to the 18th century achievements, and many things that had been taken for granted were now explored to greater depth. For example, for the first time a reasonably acceptable proof for the fundamental theorem of the calculus was presented, while before it had been hidden in a view of integration as the inverse operation of differentiation. In fact, Cauchy defined the (definite) integral as the limit of the area of the rectangles under a curve, resulting from partitioning the interval. He even showed that this definition was sound, that is: he showed that for a continuous function on a closed interval the (definite) integral existed, and was in fact independent of the sequence of partitions. After this proof, he extended his definition of the integral to improper integrals.9

The theory of real and complex functions was to be extended largely by Riemann in the following decades. He built on Cauchy’s definition, and with help from Cauchy himself, Dirichlet, Seidel and Jacobi, elementary analysis was cast in a form resembling the calculus of today. By the early 1850s, distinction was made between continuity and uniform continuity, integrals were introduced by means of Riemann sums, and the entire theory was solidly based on εδ-proofs.10 Weierstass would in the 1860s, with the help of Kronecker and Casorati, restructure analysis in arithmetical terms, which they thought would be more suitable than the rather “unrigorous path Riemann had followed”. But they extended the definition of function beyond the geometrical curve representation which was what earlier mathematicians had in mind. Moreover, they were making use of real number theory.11 This view was not present at all in the Netherlands, so these contributions will not be discussed here.

2 Dutch contributions

In the same year that Cauchy published his Calcul infinitésimal, the Dutch mathematician Jacob de Gelder (1765–1848) published his book on calculus. De Gelder treated the foundation of calculus in a remotely Lagrangian style. He linked the differential of a function to the series of differences

\[ x_2 - x_1, x_3 - x_2, x_4 - x_3, x_5 - x_4, x_6 - x_5, \ldots. \]

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8 A.-L. Cauchy, Calcul Infinitésimal, Paris (1823), pp. 32–33
9 J. Grabiner, The origin’s of Cauchy’s rigorous calculus, pp. 145–148, 164–166
Derived from a series $x_1, x_2, x_3, x_4, x_5, \ldots$. Applying all sorts of results from the theory of finite differences, De Gelder looked at the series $f(x), f(x + \Delta x), f(x + 2\Delta x), f(x + 3\Delta x) \ldots$ and defined the derivative by taking $\Delta x = 0$ in the difference quotient. This approach was considered absolutely rigorous by Dutch mathematicians and remained the common view on the foundations of calculus, until the early 1840s. The work done by Cauchy was known, but largely neglected\textsuperscript{12}.

In 1850 De Gelder’s publisher decided to publish a sequel to his calculus book. It was largely written by G.J. Verdam (1802–1866), a student of De Gelder and his successor at Leyden university. It received a positive review, but the reviewer made a very critical remark:

There is no need to say anything in favour of a book that already distinguishes itself by its title and by the two people who worked on it. Without doubt, everybody who reads it will find it very useful and thus again have reason to mourn the loss of De Gelder for the mathematical community. Our respect for the author, however, may not refrain us from pointing out an obscurity, which could lead to a false interpretation of the author. We mean his way of using infinity. For quite some time science has acknowledged the fact that the infinite can be no subject of human calculation. The author was of the same opinion, which can clearly be seen from the motto on the title page: *L’infini est le gouffre, où se perdent nos pensées.* [...] Everything is founded on the comparison of the different numbers which occur in a calculation. A magnitude so big, that all other magnitudes in the calculation no longer matter, could be called *infinitely large*, and on the other hand, a magnitude so small that neglecting it in the calculation will not change the result, *infinitely small*\textsuperscript{13}.

As can be read from this review, by the end of the 1840s, views on the foundations of the calculus had somewhat shifted in favour of a new approach. A better understanding of infinitely small and large quantities was favoured. Two years before the sequel to De Gelder’s book, a calculus textbook by J. Badon Ghyben (1798–1870)

\textsuperscript{12}D.J. Beckers, ‘Lagrange in the Netherlands’ accepted for publication in *Historia Mathematica*. There is one reference to Cauchy by Lobatto dating from 1837, which will be discussed in section 3. Lobatto thanked the Utrecht professor of mathematics Van Rees for pointing out the idea of using Cauchy’s solution in this particular case. De Gelder, also, was familiar with Cauchy’s work. For some reason the foundations of calculus as presented by Cauchy were simply not taken seriously.

\textsuperscript{13}Recensent, ook der Recensenten, 1850-1, pp. 431–432. Literally: “Wij behoeven niets ter aanbeveling van een werk te zeggen, dat zich reeds door zijn’ titel, maar vooral door de twee namen der bearbeiders zoo gunstig aanbeveelt. Ongetwijfeld zal ieder, die het ter hand neemt, het nut er van ondervinden, en daarin eene nieuwe reden vinden, om het verlies te betreuren, dat de Wiskunde in den kundigen De Gelder geleden heeft. Onze eerbied voor den hooggeachte Schrijver mag ons intusschen niet wederhouden om eene enkele aanmerking in het midden te brengen, welke somtyds tot eene verkeerde beoordeling van den Schrijver zou kunnen aanleiding geven. Wij bedoelen zijne aanwending in de rekenkunde van het oneindige. Sedert lang heeft de wetenschap erkend, dat het oneindige geen functie der menschelijke berekeningen zijn kan. Dat de Schrijver volstrekt van het zelfde gevoel was, blijkt boven allen twijfel uit het motto op den titel: *L’infini est le gouffre, où se perdent nos pensées.* […] Alles berust alleen op de vergelijking der deelen, die in eene rekening voorkomen, onderling. Eene grootheid zoo groot, dat alle andere, die in de onderwerpelijke rekening kunnen voorkomen, daar bij in geene aanmerking kunnen komen, kan men, zoo men wil, oneindig groot noemen, en omgekeerd die, welke bij geene der overige in aanmerking kunnen komen, oneindig klein.” — smallcaps and italics as in the original.
was published. The notions of mutual comparability mentioned by the anonymous
reviewer above, can easily be recognized here:

If no number suffices to express the ratio $\frac{A}{a}$ of two numbers or comparable
magnitudes; in other words: if the largest number that someone could mentions
would still be too small to express the number of times $a$ is contained in $A,$
then $a$ is called infinitely small compared to $A,$ and $A$ is called infinitely large
compared to $a.$ In this situation, if $A$ is finite, than $a$ is called infinitely small
in the strict sense. However, if $a$ is finite, then $A$ is called infinitely large in the
strict sense. If two numbers are both infinitely small or large in the strict sense,
but are not infinitely small or large compared to one another, then one is called
finite compared to the other.14

When it was absolutely necessary to express these infinitely large or small quantities
in a symbol, then 0 and $\infty$ were used. Infinitely small and large quantities could
not be measured with ordinary quantities, but could very well be measured among
each other. Badon Ghyben’s view on these huge and small quantities was probably
inspired by popular theories in geometry: here, for example, a definition of the angle
was quite common which linked the angle to the infinite surface between its legs15.
He explicitly put forward this connection16, which also had been made before him by
Jacob de Gelder, to explain why a division like $\frac{\infty}{\infty}$ could sometimes be represented by
a finite expression17.

To be able to reason properly with infinitesimal quantities, Badon Ghyben intro­
duced the volstrekte gelijkheid (unconditional equality) next to the ordinary equality,
the first being stronger than the latter. When, for example, he spoke about an in­
finitely small quantity being equal to zero, he didn’t mean the unconditional equality,
but the ordinary equality. He illustrated this with the following example:

$$\frac{\sin \varphi}{\tan \varphi} = \cos \varphi \quad \Rightarrow \quad \frac{\sin 0}{\tan 0} = 1.$$ 

The last expression being correct, nothing seemed to go wrong. But analogously:

$$\frac{\sin 2\varphi}{\tan \varphi} = 2 \cos^2 \varphi \quad \Rightarrow \quad \frac{\sin(2 \cdot 0)}{\tan 0} = 2.$$ 

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14 J. Badon Ghyben, Beginselen der differentiaal- en integraal-rekening, Breda (1848), p. 2. Literally: “Wanneer geen eindig getal groot genoeg is, om de verhouding $\frac{A}{a}$ van twee andere getallen of gelijksoortige grootheden $A$ en $a$ te kunnen uitdrukken; dat is met ander woorden: wanneer het grootste getal, dat iemand in staat zou zijn op te geven, altijd nog te klein is, om aan te duiden hoe dikwijls $a$ in $A$ begrepens is, wordt $a$ oneindig klein met betrekking tot $A,$ en $A$ oneindig groot met
betrekking tot $a$ genoemd. Is hierbij $A$ eindig, dan noemt men $a$ oneindig klein in de volstreken zin; is echter $a$ eindig, dan noemt men $A$ oneindig groot in de volstreken zin. Wanneer twee getallen of gelijksoortige grootheden beide in de volstreken zijn, worden beide oneindig groot, maar niet oneindig klein of groot met betrekking tot elkander zijn, wordt de eene
oneindig met betrekking tot de andere genoemd.” —italics as in the original


16 J. Badon Ghyben, Beginselen der differentiaal- en integraal-rekening, p. 2

17 J. de Gelder, Beginselen der differentiaal-, integraal- en variatierekening part II (1850)
Something peculiar happened here. One would be inclined to say that \( 2 \cdot 0 = 0 \), so in the special case \( \varphi = 0 \) reckoning seemed to go wrong. Badon Ghyben solved this paradox by looking more closely at the infinitely small quantities involved. They were measurable among each other, so the equality signs did not represent unconditional equalities (as De Gelder had suggested). For some infinitesimal quantity \( \delta \), which could be neglected in comparison to 1, \( \cos \varphi \) in the first equation could be linked to \( 1 - \delta \), which indeed would be equal to 1 in the limit, whereas \( 2 \cos^2 \varphi \) would have to be read as \( 2 \cdot (1 - \delta)^2 \). The equality signs represented ordinary equalities, and certainly not unconditional ones. This saved the entire theory: 0 was not really zero in these equations, but could not be distinguished from it\(^{18}\).

In the 1840s the infinitesimals had thus returned in Dutch analysis. The idea behind the infinitesimals was that they could somehow be explained by a peculiar mixture of geometrical insight and algebraic knowledge. From geometry the notion of infinite magnitudes arose, and the possibility of very small increments on curves representing a function was somehow perceivable. On the other hand, algebra taught that \( \frac{0}{0} \) could have some meaning. The expressions \( \frac{x^2 - 1}{x - 1} \) and \( x + 1 \) clearly indicated the same thing, and thus would also represent the same number if \( x \) was equal to one.

### 3 Lobatto’s textbook on the calculus

In 1851–1852 the Jewish professor of mathematics at the Dutch polytechnical institute, Rehuel Lobatto (1797–1866), published a textbook on calculus. His book was intended as an introduction for Dutch students. To allow the students to understand foreign literature, Lobatto, between parentheses, mentioned terms current in French. The French translation was really necessary since Lobatto, for example, used the Dutch “onafgebroken functie” for the French “fonction continue” —while the Dutch equivalent “continue functie” had been used for several decades. Since his definitions were not very formal, for a student it would not have been easy to see what the French or German equivalent would be.

In the preface Lobatto explicitly mentioned that he was following the way “Cauchy and other contemporary authors” had indicated\(^{19}\); a comparison of the chapter titles reveal quite some resemblance to Cauchy’s *Calcul infinitésimal*. The content, on a number of occasions, bears remarkable resemblance to Cauchy’s work, but it had been altered on essential points, and the status of the remarks had changed in numerous places. Lobatto did not copy Cauchy’s \( \varepsilon \delta \)-proofs, for a start. The intermediate value theorem was proved with intuitive reasoning.

The careful phrase by Cauchy “if this limit exists” while defining the derivative was omitted by Lobatto. He defined the derivative about the same way as Cauchy, but than suddenly changed his language:

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f_1(x). \quad \text{The derivative } f_1(x) \text{ thus represents the value of the limit of the quotient of the infinitely small changes } dy, \ dx, \text{ which any function}
\]

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\(^{18}\) J. Badon Ghyben, *Beginselen der differentiaal- en integraalrekening*, pp. 15–16

and its variable element $x$ undergo simultaneously. This quotient \( \frac{dy}{dx} \) is called the differential quotient or differential coefficient of the function $y$. For some particular values of $x$ this limit may be 0 or $\infty$, which obviously depends on the changing direction of the tangent in the various points of the curve. In the first case the tangent is parallel to, in the second it is perpendicular to the $x$-axis.\(^{20}\)

The tangents were also mentioned by Cauchy, but not in his definition of the derivative. In Cauchy’s textbook a passage similar to the quote above may be found in an example of the calculation of a tangent to the curve of a function\(^{21}\). So for Cauchy this certainly did not come close to a defining property. Lobatto’s link to the geometric curve representation of a(ny) function not only included the horizontal and vertical tangents, but immediately ruled out non-existing limits: these monstrosities he did not consider to be functions.

In fact, Lobatto was far more concerned with infinitesimals: he spent a lot of pages on the problem of the several orders of the infinitely small, thus linking his work to that of Badon Glyben. At the point where Cauchy was treating the limits of functions that appeared under the form $\frac{\partial}{\partial}, \frac{\infty}{\infty}$, etc., Lobatto constructed an entire theory of the infinitely small. The several orders of the infinitely small were linked to limits of series he knew: an infinitesimal of order $i$ was identified with $\frac{\partial}{\partial^i}$ for $\alpha$ tending to infinity and $a$ some finite (real) number. From this definition he derived that these infinitesimals obeyed several rules. For example: if $a$ was an infinitesimal of order $n$ and $\beta$ was an infinitesimal of order $n + m$ then $\frac{\beta}{\alpha}$ was an infinitesimal of order $m$ and was still infinitely small compared to 1. If, however, $\beta$ was of order $n$ and $\alpha$ of order 1, then $\frac{\beta}{\alpha}$ was a finite number.

If $A$ and $B$ are two infinitely small quantities of the same order, of which the difference $\delta$ is infinitely small of a higher order, then their quotient will have limit 1; because one has $\frac{\delta}{\beta} = 1 + \delta$.\(^{22}\)

This allowed Lobatto to change some of Cauchy’s proofs. Infinitesimal reasoning for example provided the proof for the product rule in a manner reminiscent of the Leibnizian proof: neglecting the term $dXdX_i$ as being infinitely small compared to the other terms in $(X + dX)(X_i + dX_i)$\(^{23}\). Cauchy had proved this theorem by applying the theorem on the derivative of the sum of two functions to the derivative of the logarithm function. The latter he had calculated directly from the definition.

\(^{20}\)R. Lobatto, Differentialrekening dl. 1 (1851), p. 6. Literally: “lim $\frac{dy}{dx} = \frac{d^i}{dx^i} = f(x)$. De afgeleide functie $f(x)$ stelt alzoo de waarde voor van de limiet der verhouding tusschen de oneindig kleine veranderingen, $dy, dx$, welke eenige functie $y$ en haar veranderlijk element $x$ gelijktijdig ondergaan. Men noemt deze verhouding $\frac{dy}{dx}$ veerdel het differentiaal quotient of de differentiaal coefficient der functie $y$. Voor sommige byzondere waarden van $x$, kan deze limiet $0$ of $\infty$ worden; hetgeen bijzonders van de veranderlijke rigting der raaklijn in de onderscheidene punten der kromme afhankelijk is. In het eerste geval loopt die lijn evenwijdig aan, en in het tweede geval staat zij loodreg tegen de as der abcissen.”


\(^{22}\)R. Lobatto, Differential-Rekening, p. 10; literally: “Indien $A$ en $B$ twee oneindig kleinen van dezelfde orde voorstellen, waarvan het verschil $\delta$ een oneindig kleine van eene hogere orde is, zal hare verhouding ongelyks de eenheid tot limiet hebben; want men heeft wederom $\frac{\delta}{\beta} = 1 + \delta$.”

\(^{23}\)Ibidem, p. 18
The theory of finite differences, which De Gelder had used as a foundation of calculus, was now degraded to a piece of algebra which was of considerable use in calculus, but could no longer serve as a foundation. The Taylor series, however, were introduced by substituting \( \infty \) for \( n \) in the formula for the Taylor polynomials. With help of the Taylor series it was then deduced that \( F(a + h) - F(a) = h \cdot F'(a + ih) \) for some \( i \) in the interval \( (0,1) \). In a footnote Lobatto made the remark that this formula could also be “proved geometrically”: his proof consisted of looking at two pictures. These pictures represented an ascending and a descending function respectively; the chords, indicated by the formula, were drawn. By comparing these chords to the part of the function between the endpoints of the chords, it could easily be seen, according to Lobatto, that the tangent to the curve would in some point be parallel to the chord.

In his definition of the integral Lobatto leaned on infinitesimals too. He used the (Leibnizian / Newtonian) definition of integrating as the inverse operation of differentiation. He mentioned that the integral sign was derived from the “S” of “Summa”, and that he would later explain why this was in fact a correct name for the procedure. Lobatto reminded his readers that \( dF(x) = d(F(x) + C) \). In practice, this meant that the integral was only determined up to a constant. Then he defined the definite integral \( \int_{a'}^{a} f(x)dx \) as “the integral” of \( f \) for the specific value \( x = a \) which was equal to zero for \( a = a' \) — a very sensible definition since he wanted it to represent the area underneath the curve. Of course this integral would be equal to \( F(a) + C \), where \( F'(x) = f(x) \). From this observation he derived:

\[
\int_{a'}^{a} f(x)dx = F(a) - F(a')
\]

Then Lobatto showed that the integral indeed represented the area underneath the curve of \( f \) (and could thus be linked to a sum), with help of a proof clearly based on Cauchy’s proof of the uniqueness of his definite integral: he partitioned the interval \((a', a)\) by means of \( x_0, x_1, \ldots, x_n \) where, by definition, \( x_0 = a' \) and \( x_n = a \). This yielded

\[
\int_{a'}^{a} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx
\]

By definition all these terms could be written as \( F(x_m) - F(x_{m-1}) \), with \( f \) the derivative of \( F \). Since \( f \) was supposed to be continuous, for some some \( i \) between 0 and 1:

\[
F(x_m) - F(x_{m-1}) = (x_m - x_{m-1}) \cdot (f(x_{m-1} + i \cdot (x_m - x_{m-1})) = (x_m - x_{m-1}) \cdot (f(x_{m-1}) + \varepsilon)
\]

\footnote{In his early years Lobatto had used De Gelder’s theories, as can be seen, for example, in his “Sur les développement des coefficients différentiels d’une fonction au moyen de ses différences finies, et réciproquement” in: Journal für die Reine und Angewandte Mathematik \textbf{XVI} (1837), pp. 11–20.}

\footnote{Lobatto in fact also treated the theory of finite differences in his \textit{Lessen over de hoogere algebra} (1845), pp. 187–198, meant for the students of the polytechnic institute at Delft.}

\footnote{Lobatto, \textit{Differentiaalrekening}, pp. 70–73}
were $\varepsilon$ represented a number that was supposed to “disappear together with the difference” of $x_m$ and $x_{m-1}$.

Hence the definite integral could be written as:

$$\int_{x_0}^{x_n} f(x)dx = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \ldots + (x_n - x_{n-1})f(x_{n-1}) +$$

$$+ (x_1 - x_0)\varepsilon_0 + (x_2 - x_1)\varepsilon_1 + \ldots + (x_n - x_{n-1})\varepsilon_n$$

and by choosing for $\varepsilon$ some intermediate value of all the $\varepsilon_i$'s:

$$\int_{x_0}^{x_n} f(x)dx = (x_1 - x_0)f(x_0) + \ldots (x_n - x_{n-1})f(x_{n-1}) + \varepsilon \cdot (x_n - x_0)$$

Lobatto’s concluding remarks, except for noting —not proving!— that the procedure was independent of the chosen (series of) partitions, were as follows:

The larger we choose the number of parts $n$, the smaller each of the values $x_1 - x_0, x_2 - x_1, \ldots$ becomes, and so also the values $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ and thereby also their mean value $\varepsilon$. If we choose $n = \infty$, that is, if we make all the succeeding values of $x$ between $x_0$ and $x_n$, increase with infinitely small differences, then the first part of the above formula becomes equal to an infinite sum, whose individual terms tend to zero, and whose sum tends to the value of the definite integral.

In this way he completely turned the work of Cauchy upside down. Cauchy had introduced the integral the other way around: first the definite integral, and then the integral as a function. Cauchy had linked the integral and derivative in proof, not in definition. Where Cauchy was a bit ambivalent in his attitude towards infinitesimals, Lobatto clearly chose for reasoning with infinitesimals. His infinitesimals were introduced in a new way, by means of limits of clearly defined series, and from the quotation above it is clear that he wanted his readers to apply his theorems on these infinitesimals to the formulæ he had derived.

One begins to wonder why Lobatto had linked his work to that of Cauchy (almost 30 years after its publication!), while he used so little of the theories that Cauchy, Riemann and their followers had based calculus on. Lobatto already knew Cauchy’s work in 1837. At that time he published a paper in which he noted that if for some

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27 Lobatto, *Differentiaalrekening* II, p. 7; literally: “zijnde $\varepsilon_0$ eene grootheid, welke tegelijk met het verschil $x_1 - x_0$ verdwijnt”. Of course Lobatto wrote down this formula for $m = 0, 1, 2$ and then put in some dots to indicate that it would hold for all $m$.

28 Ibidem, p. 8; literally: “Hoe groter het aantal deelen $n$ genomen wordt, hoe kleiner de waarde van elk der verschillen $x_1 - x_0, x_2 - x_1$ enz. en dus ook die van elk der grootheiden $\varepsilon_0, \varepsilon_1, \varepsilon_2$ wordt, hetgeen met de middelwaarde of tusschenwaarde $\varepsilon$ evenzeer het geval moet zijn. Stelt men dan $n = \infty$, dat is, laat men al de opvolgende waarden van $x$ tusschen $x_0$ en $x_n$, met oneindig kleine verschillen onaangelicht toenemen, zoo gaat het voorste lid van verg. (5) in eene oneindige reeks over, zamensteld uit termen, die elk in het byzonder tot nul naderen, en waarvan de som tot limiet heeft de waarde der bepaalde integraal $f^{(2)}_x f(x)dx.$"
continuous function $\phi(x)$ and for some $c$ with $a < c < b$ you were interested in the integral of $\frac{\phi(x)}{x-c}$ over the interval $[a, b]$ you would have to split the interval “as suggested by Cauchy”:

$$\int_a^b \frac{\phi(x)}{x-c} \, dx = \int_a^{c-\varepsilon} \frac{\phi(x)}{x-c} \, dx + \int_{c+\varepsilon}^b \frac{\phi(x)}{x-c} \, dx$$

The quantity $\varepsilon$ was not seen as a way to introduce a limit, but was simply called a “quantité évanescente”\textsuperscript{29}. Where Cauchy had tried to get a firmer grip on the concept of “limit”, Lobatto had interpreted “limit” in a more 18th century sense of the word. We find this expressed in his calculus textbook, too.

## 4 D. Bierens de Haan and his calculus

In 1865 the Leyden professor of mathematics D. Bierens de Haan (1822-1895) published a calculus textbook. This book is somewhere between Lobatto’s foundations of the calculus and the ideas of Riemann. Bierens de Haan was aware of the fact that he offered a totally different introduction to calculus than his colleagues abroad. In the introduction to his book he mentioned that French and German authors used other words, and even slightly different ideas. However, he assumed that most of the students would not read foreign literature, and therefore he only once mentioned that the Dutch word “grens”, for example, in foreign books was called “limit”, and the Dutch “doorlopend” usually was called “continuous”. Both notions were defined intuitively, as in Lobatto’s book\textsuperscript{30}. Like Lobatto, Bierens de Haan—in the preface—explicitly remarked that he had given the French and German terms only to allow students to understand foreign literature on more advanced subjects\textsuperscript{31}.

Although strikingly different from the ideas offered by Cauchy or Riemann, the textbook had been brought up to date in a number of ways. Bierens de Haan corrected some of the mistakes that had been made by Cauchy—as he undoubtedly had picked up from foreign literature. The mistake Cauchy had made in allowing an infinite number of functions in the law

$$(f + g + h + \ldots)' = f' + g' + h' + \ldots$$

was corrected by Bierens de Haan, since “an infinte number of epsilons could not be guaranteed to equal zero”\textsuperscript{32}. The epsilonics was however less strict; it was more intuitively applied, as may be read from the above example. In fact, Bierens de Haan’s $\varepsilon$ was more like an infinitesimal quantity, as the one of Cauchy in his most old-fashioned moments.

\textsuperscript{29}R. Lobatto, ‘Mémoire sur la théorie des caractéristiques’ in: Nieuwe Verhandelingen der Eerste Klasse van het Koninklijk Nederlandsche Instituut van Wetenschappen VI (1837), p. 49

\textsuperscript{30}D. Bierens de Haan, Overzigt van de Differentiaalrekening, Leiden (1865), pp. 3–5

\textsuperscript{31}D. Bierens de Haan, Overzigt van de Differentiaalrekening, Leiden (1865), p. V

\textsuperscript{32}D. Bierens de Haan, Overzigt van de Differentiaalrekening, p. 12; literally: “onder de voorwaarde, dat het aantal termen niet oneindig groot worde, omdat alsdan [ \ldots ] de som der $\varepsilon$ niet meer noodzakelijk aan nul gelijk behoefte te zijn”. With Lobatto (part I, p. 16) it is still the loose: as many as one pleases: “een willekeurig aantal”.
Bierens de Haan warned against diverging series\textsuperscript{33} and was more careful proving the existence of the Taylor series expansion of a function. He said: if \( f(x + h) = F_0(x) + F_1(x)h + F_2(x)h^2 + \ldots + R_n h^n \) with \( F_p(x) \) certain functions in \( x \), then \( F_0(x) = f(x) \), which he “proved” by choosing \( h = 0 \). Now he knew \( \frac{f(x+h) - f(x)}{h} = F_1(x) + F_2(x)h + F_3(x)h^2 + \ldots \) and choosing \( h = 0 \) again, \( F_1(x) \) was obtained. The other functions were determined accordingly. He concluded that his proof immediately illustrated why you could leave out all the terms of order larger than \( n \) in a differential equation if you were only interested in a \( n \)-th order approximation. All the higher order terms contained a factor \( h \) or \( \Delta x \), which were “infinitesimals of a higher order”\textsuperscript{34}.

The derivative was defined in a way which reminds of Cauchy:

\[
f'(x) = \text{Grens} \frac{f(x + \delta) - f(x)}{\delta} \quad \text{meaning:} \quad \frac{f(x + \delta) - f(x)}{\delta} = f'(x) + \varepsilon
\]

But here the similarity ends. Grens \( \delta = 0 \) implied, according to Bierens de Haan, that Grens \( \varepsilon \) had to be zero as well. His notion of the derivative bore much resemblance to Lobatto’s — be it written down more formally.

In defining the integral Bierens de Haan started with the definite integral, which was indeed defined as Riemann had done: by making over and under estimates. But Bierens de Haan’s method was, to our taste, less precise and immediately invoked the derivative. The interval \([a, b]\) was partitioned into \( n \) parts of length \( \delta_1, \delta_2, \delta_3, \ldots, \delta_n \). Then Bierens de Haan stated:

\[
f(a + b) - f(a) = \delta_1 f'(a) + \delta_2 f'(a + \delta_1) + \ldots + \delta_n f'(a + b) + \frac{\delta_1 \varepsilon_1 + \delta_2 \varepsilon_2 + \ldots + \delta_n \varepsilon_n}{A}
\]

where all the \( \varepsilon \)'s tended to zero if the \( \delta \)'s became smaller (then also \( n \) became larger, but Bierens de Haan saw no problem there). This could be concluded from the definition of the derivative. Among all the \( \varepsilon \)'s there would be a largest \( \varepsilon_g \) and a smallest \( \varepsilon_k \textsuperscript{35} \), so that the term indicated by \( A \) satisfied:

\[
\varepsilon_g (b - a) > A > \varepsilon_k (b - a)
\]

Since both \( \varepsilon_g \) and \( \varepsilon_k \) tended to zero it was obvious that for any function \( f'(x) \), continuous on the interval \([a, b]\), the definite integral defined as

\[
\int_a^b f'(x)dx = \text{Grens} \sum_{a}^{b} f'(x)\Delta x \quad (\text{Grens} \Delta x = 0)
\]

existed and was equal to \( f(b) - f(a) \). It is clear that Bierens de Haan knew of the developments abroad, but he was very particular when it came to using these new theories in his textbook.

\textsuperscript{33}ibidem, p. 87
\textsuperscript{34}ibidem, pp. 88-90
\textsuperscript{35}The \( g \) and \( k \) are derived from the Dutch words for “largest” and “smallest”. 

11
5 Mathematical background and reception

What did the Dutch mathematical community think of these textbooks? There were no serious alternatives in the vernacular, apart from the much older book by De Gelder, which probably still was in use at some places. The textbook by J. Badon Ghyben was written for the students of the Military Academy in Breda where he was teaching. There, his book was used at least until 1889, when a new calculus book by N.C. Grootendorst was published\textsuperscript{36}. This textbook also based the entire theory on a rather loose concept of infinitesimals, but the Military Academy published its own textbooks, which since the 1850s clearly deviated from what was offered outside the academy. Lobatto’s textbook was originally written for his students at the Polytechnic school in Delft, but it was also welcomed by at least some of the professors of mathematics at the Dutch universities.

The famous physicist C.H.D. Buys Ballot (1817–1890), at the time professor of mathematics at Utrecht, wrote a review on Lobatto’s textbook. He stated that in general there were two ways of writing a treatise on a mathematical subject. The first was the scientific way: in such books, the theory should be offered compendium-wise. All relations should be mentioned and made clear to the reader, who was supposed to be familiar with the matter. In this way the expert was offered an overview over the existing body of knowledge. The second way had the same goal but another public:

The other way is for him who learns while reading. He who seeks knowledge and wants to bring scientific order into it, should be led by the most even road; one should have him look around and back time and time again, from each level he has reached, so that everything falls into place. Professor Lobatto has found this way and has, to our opinion, succeeded much better than Schlömilch, whose calculus textbook in many respects remains far behind his other (very popular) treatises.\textsuperscript{37}

The mr. Schlömilch mentioned was Oskar Schlömilch, professor of mathematics at the Dresden Polytechnische Schule. In a way he might be called a colleague of Lobatto. Schlömilch’s work on calculus somewhat resembled Lobatto’s: the reasoning with infinitesimals, and the “proof” of the fundamental theorem of integration, bear remarkable resemblance to Lobatto’s proofs\textsuperscript{38}. This could lead us to think that in the opinion of the mathematicians, Lobatto’s foundations of the calculus must be


\textsuperscript{37}\textit{De Recensent, Algemeen Letterliedend Maandschrift} 1852-1, pp. 516–520; quote from p. 517. Literally: “De andere wijze is voor hem, die ook nog de kennis moet verzamelen. Dan geleidt men hem, die de kennis zoekt en tot wetenschap wil ordenen, als aan den hand langs den meest effenen weg; men doet hem telkens rondom zien en terugzien, hem van elk hoogter standpunt uit het geheel in zijne orde vertoonende. De Hoogleeraar Lobatto heeft gelukkig meer dezen weg opgespoord en is, naar onze mening, beter geslaagd dan Schlömilch, wiens differentiaal-rekening in velerlei opzichten als leerboek bij zijne andere zoo populaire verhandelingen achterstaat.”

\textsuperscript{38}O. Schlömilch, \textit{Compendium der Höheren Analysis}, Braunschweig (1853). The calculus text that Buys Ballot referred to in the quotation above was probably the popular \textit{Handbuch der Algebraischen Analysis}, Jena (1845). Two of Schlömilch’s books were translated in Dutch: \textit{Beginselen der Meetkunde}, Sneek (ca. 1870) by H. Onnen; \textit{Leerboek der analytische meetkunde}, Leiden (1872)\textsuperscript{2} by P. van Geer. The latter was a textbook by Schlömilch and O. Fort.
regarded as a version for engineers of the “true mathematical foundations”. A more suitable version of analysis for teaching to engineers, so to say.

The idea that academic mathematics could (or even should) differ from the mathematics taught to engineers might have existed in France and Germany. This, however, was completely incompatible with the ideas of Dutch educators. The review by Buys Ballot, quoted above, and Bierens de Haan’s calculus, written especially for his university students, illustrate this. University students of mathematics were indeed taught analysis in the way it was presented by Lobatto or Bierens de Haan. This was considered a rigorous approach. The university mathematicians and the teachers of the polytechnic institute were united in the Dutch Mathematical Society. Communication between these groups therefore existed naturally, and foundational matters were a topic of interest. In 1859 Bierens de Haan published a lengthy paper in the journal of the Dutch Mathematical Society. In this paper he already uttered many ideas that can be found in his 1865 calculus textbook. For example, his definition of the derivative and his proof with the largest and smallest $\varepsilon$ were presented. His references are all to Cauchy and contemporary Dutch authors\textsuperscript{39}. All this contributes to the image of a more general approval of what the foundations of the calculus should be.

That according to the Dutch mathematical élite, mathematicians and engineering students should be taught alike can also be illustrated by the contempt that mathematicians from both camps showed for the textbooks for engineers in which these requirements were not met. The editorial board of the magazine published by the Mathematical Society in 1859 made it absolutely clear that a certain textbook on perspective for architects was not what they had expected, since its theoretical part was not appropriate\textsuperscript{40}.

Lobatto was highly valued as a mathematician. Mathematics being considered a crucial part in the training of engineers, the courses at the Dutch polytechnic institute were very mathematically oriented. In a review of one of his textbooks, an anonymous reviewer welcomed the appointment of Lobatto to the chair of mathematics at Delft:

> We end this announcement with the remark, that we are very pleased, that the author [Lobatto] is supervising the mathematical part of the education at the Royal Academy for civil engineers, since this guarantees us that our native country may expect excellent mathematicians from this institution.\textsuperscript{41}

During the decades under consideration there are remarkably few Dutch mathematicians among the international celebrities. In fact, there were two: Lobatto and Bierens de Haan. The latter’s activities in the bibliographical field guaranteed many

\textsuperscript{39}D. Bierens de Haan, ‘Gronden van de Theorie der Bepaalde Integralen’ in: \textit{Archief, uitgegeven door het Wiskundig Genootschap I} (1856–1859), pp. 343–422

\textsuperscript{40}Archief, uitgegeven door het Wiskundig Genootschap I (1856–1859), p. 87; the book reviewed was: J.W. Schaap, \textit{Handleiding tot de kennis der perspectief}, Leiden (1856)

\textsuperscript{41}Vaderlandsche Letteroefeningen 1843-I, p. 722. The review was of Lobatto’s \textit{Leerboek der Regt-lijnige en Sphersische Driehoeksmeting} (1843). Literally: “Wij eindigen deze aankondiging met de betuiging, dat het ons verheugt, dat de Schrijver aan het hoofd staat van het wiskundig onderwijs aan de Koninklijke Akademie ter opleiding van burgerlijke Ingenieurs, daar dit ons ten waarborg strekt, dat het vaderland van deze inrigting goede wiskundigen verwachten kan.” — typeface as in the original.
publications in the *Bulletino di bibliografia e di storia delle scienze matematiche e fisiche*, culminating in his *Bibliographie Néerlandaise*42. Although mathematically not very interesting, his work brought Bierens de Haan in contact with a lot of important and productive mathematicians abroad.

Lobatto’s work was on an international level43: he published in both Crelle’s *Journal für die reine und angewandte Mathematik* and in the *Journal des mathématiques pures et appliquées* by Liouville. These papers were mostly contributions on finding solutions to certain classes of integrals; not unworthy to many mathematicians in those days. Some classes of integrals were unsolved at the time; many had been solved, but the solution could be simplified or better approximations could be obtained. Certainly in the 1830s, the number of papers on the solving of various classes of integrals were numerous, and Lobatto published his share44. Lobatto also kept track of foreign contributions, which might be illustrated by his reaction to a paper about a certain $n$-dimensional integral, for which he offered a simpler solution in the three-dimensional case45.

Lobatto was really working in the forefront of mathematics at the time. In the 1830s he already wrote on analysis, building on the work by Lorgna and Arbogast. Here he was making use of complex functions and characteristics, and he defined operators on arbitrary functions. Lobatto was not doing really new things, but at that time he was doing state-of-the-art mathematics46. In two small papers he explained why this theory was so interesting. He was investigating differential equations of the form

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \right) y \right) = V$$

with $\partial$ being the operator which transforms a function $y$ into $\frac{\partial y}{\partial x} + ay$47. In a restriction to the three-dimensional case he stated:

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42D. Bierens de Haan, *Bibliographie Néerlandaise historique-scientifique des ouvrages importants dont les auteurs sont nés aux 16e, 17e et 18e siècles, sur les sciences mathématiques et physiques, avec leurs applications*, Rome (1883)


44On this subject Lobatto published in *Crelle’s Journal* the following papers: ‘Note sur l’intégration de la fonction \( \frac{\partial y}{\partial x} \)' 9 (1832), pp. 259-260; ‘Sur l’intégration de la différentielle \( \frac{\partial y}{\partial x} \)’ 10 (1833), pp. 280-287; ‘Note sur les différentielles partielles de la fonction \( \sqrt{x^2 + ax^3 + bx^2 + cx + d} \)’ 11 (1834), pp. 169-172; ‘Sur l’intégration des équations \( \frac{\partial y}{\partial x} \) = xy = 0 et \( \frac{\partial y}{\partial x} \) = 0 par des intégrales définies’ 17 (1837), pp. 363-371


46R. Lobatto, ‘Mémoire sur la Théorie des Caractéristiques’ in: *Nieuwe Verhandelingen der eerste klasse van het Koninklijk Nederlandsch Instituut van Wetenschappen* VI (1837), pp. 1-82. The publisher clearly had trouble with this kind of work: the formulae were full of misprints.

Toute fois, en regardant la disparité des méthodes d’intégration, et la prolixité des calculs qu’elles exigent souvent, ou se convaincra sans peine que la matière est loin d’être épuisée encore, et que les progrès ultérieurs dans cette partie de l’analyse ne peuvent provenir que de l’emploi de nouvelles méthodes tendantes à simplifier et à généraliser en même temps les procédés d’intégration. A cet effet, je pense qu’il deviendrait indispensable d’adopter de nouveaux signes propres à réprésenter l’ensemble de diverses opération analytiques, et à former la base d’une espèce d’algorithme de calcul, applicable à ces opérations.48

Lobatto did not stop his work in this field when he was appointed in Delft49. Although the bulk of his work was on applications of mathematical theories50 and from the quotation above it is clear he also had applications in mind when he was working in the field of analysis, his work in this latter field was very up-to-date. It was only because he was Jewish, that he never obtained a university position51. The work by Lobatto and Bierens de Haan was well known in the Netherlands: they were national celebrities as well, and published many of their results in Dutch journals. From these observations it is clear that although Dutch mathematicians did not arouse great international interest with their work, quite a few of them certainly were aware of developments abroad.

Being educated in the days of De Gelder, the generation of Dutch mathematicians that began working in the 1840s and 1850s, already had been endowed with different notions on rigour. Either they had received their education from the books by De Gelder at Leyden or Groningen university52, or they were raised in a more Leibnizian tradition at Utrecht university or the Military Academy53. The tradition of De Gelder could no longer hold by the end of the 1840s. The only Dutch alternative which mathematicians could fall back on was the one based on infinitesimals. How seriously contemporaries regarded these foundations of calculus might be illustrated by the historical survey by E. van der Ven (1861), who claimed that it was indeed the infinitesimal theory by Lucas Valerius (ca. 1580) that had been the basis for calculus.

49 R. Lobatto, Mémoire sur l’intégration des équations linéaires du première ordre aux différentielles partielles, à quatre variables, Amsterdam: C.G. Post (1854)
52 D.J. Beckers, ‘Lagrange in the Netherlands’ accepted for publication in Historia Mathematica
The calculus in his days he saw as mere refinements of this theory, and he explained Euclidean and Archimedean exhaustion theorems in terms of infinitesimals\textsuperscript{54}. Nowadays, analogously, Weierstraß's and Riemann's ideas on calculus are often invoked while discussing the theorem on the area of a paraboloidal segment by Archimedes\textsuperscript{55}.

6 Conclusions

During the 19th Century, Dutch mathematicians were very well aware of what was going on in other European countries. Their ideas of rigour, however, were different from those of French or German mathematicians. Professors of mathematics at the universities were less involved in fundamental issues than their counterparts abroad. Unlike the situation in France and Germany, thinking about foundational matters was done at the Military Academy and the Dutch Polytechnic institute. In education it was considered to be crucial for the pupil to obtain clear and distinct notions of the concepts he was working with. It was acceptable to base the proof on a good intuition of the concepts.

Education seems to have been the major driving force behind the development of a rigorous calculus. The strive for rigour had a very national character. Both Lobatto and Bierens de Haan deliberately used “new” Dutch words for notions like “limit” and “continuous”, for which more international terms already existed and had been used for decades. Apart from that, they both clearly deviated from other European textbooks. Although Lobatto and Bierens de Haan were very actively involved in international circles, they clearly opted for their own foundational approach.

Summarizing our results we might say that Cauchy’s work found hardly any reception in the Netherlands until Lobatto published his calculus textbook in the 1850s. Compared to England and Germany, the interest in Cauchy’s work in the Netherlands came rather late. Furthermore, it was precisely the old-fashioned side of Cauchy’s book that found its way into Lobatto’s calculus. It would take another decade before Bierens de Haan wrote about calculus in a fashion that showed some Riemannian influence. But he preferred to refer to Cauchy, and even in his textbook infinitesimals played an important role.

Even if I had a solid explanation for this “national taste”, the fact that this “taste” existed would still be striking. Whether it was the result of “an engineering state-of-mind”, educational brainwashing, or national pride: it tells us something about what was regarded as “mathematics” in the Netherlands during this period. The Dutch taste for rigour in calculus from 1850 until well into the second half of the nineteenth century was definitely infinitesimal.

\textsuperscript{54}E. van der Ven, De exhaustiemethode, s.l. (1861)
\textsuperscript{55}C. Boyer, A history of mathematics, New York / London / Sydney (1968), pp. 144–145